

Chapter 3

VECTORS

In this chapter we are going to discuss the following topics:

- ❑ **Coordinates Systems**
- ❑ **Vectors & Scalars**
- ❑ **Adding and Subtracting Vectors**
- ❑ **Components of Vectors**
- ❑ **Product of Vectors (Multiplying a Vector by a Scalar, Scalar product, & Vector Product)**

Recommended Problems:

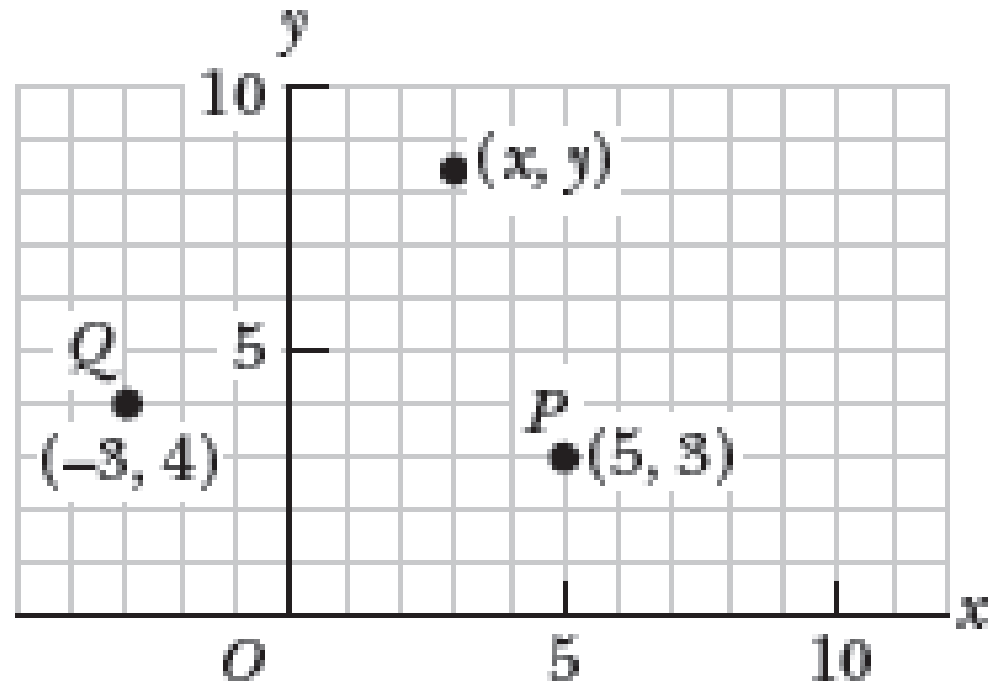
15, 19, 21, 23, 26, 31, 36, 37, 41, 45, 57, 59 (problems from Ch.7)
7, 9, 12, 13 (problem from Ch.11) 1, 3, 7, 10

3.1 Coordinates Systems

In physics it is important to define the position of a point in space. In doing so we need a coordinate system.

Cartesian Coordinates:

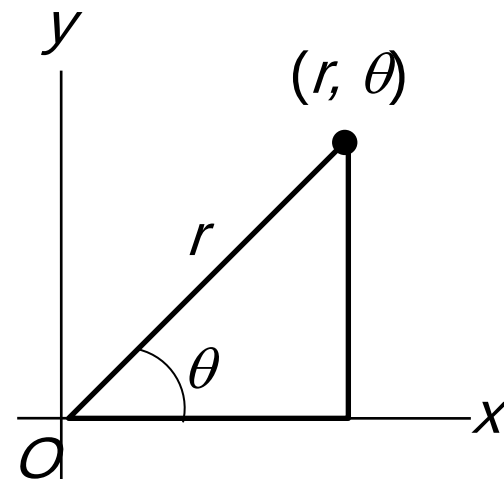
It consists of two perpendicular axes (x-axis, and y-axis) that intersect at a point called the origin. Any point in a plane is assigned two numerical values: x-value which gives the position of the point along the x-axis, and y-value which gives the position of the point along the y-axis.



For 3-dimensional case there is a third axis (z-axis) perpendicular to x-y plane and passing through the origin. The Cartesian coordinates are called the **rectangular coordinates**.

Polar Coordinates:

Some times, it is convenient to specify a point in space by its distance from the origin (r -value) and a value (θ -value) denotes the angle between the x -axis and r . θ is usually measured counterclockwise.



The Cartesian coordinates are related to the polar coordinates as

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\text{and} \quad r = \sqrt{x^2 + y^2},$$

$$\text{with} \quad \tan \theta = \frac{y}{x}$$

These four expressions relating the coordinates (x, y) to the coordinates (r, θ) apply only when θ is defined as shown.

Example 3.1 The Cartesian coordinates of a point in the x - y plane are $(x, y) = (-3.50, -2.50)$ m as shown. Find the polar coordinates of this point.

Solution To find r and θ we have

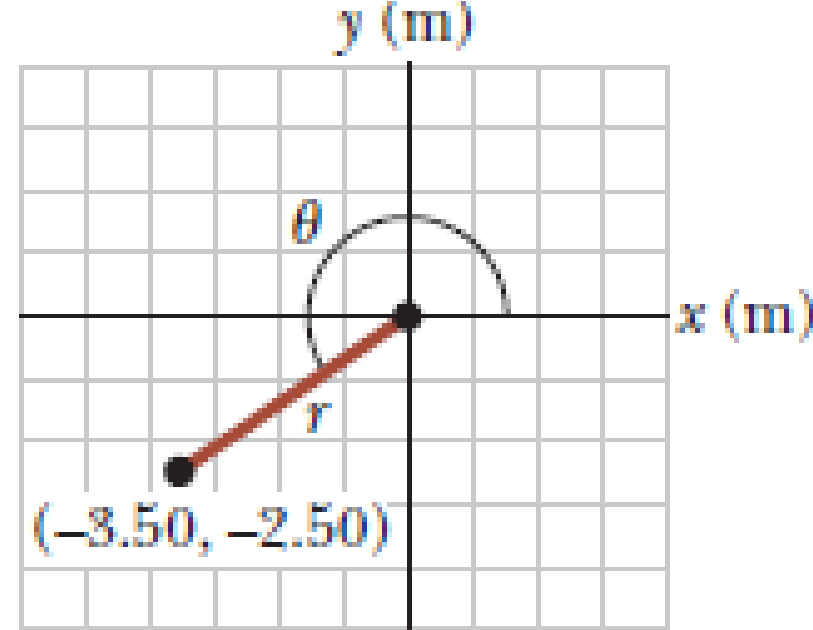
$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50)^2 + (-2.50)^2}$$

$$\text{or } r = 4.30\text{m}$$

$$\text{and } \tan \theta = \frac{y}{x} = \frac{-2.50}{-3.50} = 0.714$$

$$\text{or } \theta = 216^\circ$$

The polar coordinates of the point is $(4.30\text{m}, 216^\circ)$



3.2 Vector and Scalar Quantities

Physical quantities are either vectors or scalars:

Scalar	is the physical quantity that has magnitude only.	e.g., time, volume, mass, density, energy, distance, temperature.
Vector	is the physical quantity that has both magnitude and direction.	e.g., velocity, acceleration, displacement, force, area.

In the book, the vector quantity will be distinguished from the scalar quantity by typing it in boldface with an arrow over the letter, like $\vec{\mathbf{A}}$ or simply $\mathbf{A}$. The magnitude of the vector quantity will be denoted by $|\vec{\mathbf{A}}|$ or simply the italic letter A . The magnitude of a vector is always a positive number.

Geometrically, The vector quantity is represented by a straight line and an arrow at one end of the line.

Tail of the vector  Head of the vector

3.3 Some Properties of Vectors

Equality of Two Vectors Any two vectors are said to be equal if they have the same magnitude and point in the same direction, regardless of their location.

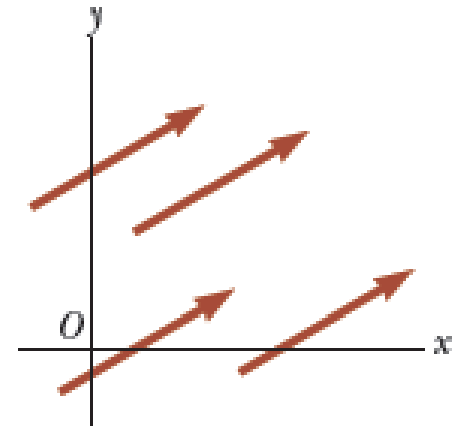
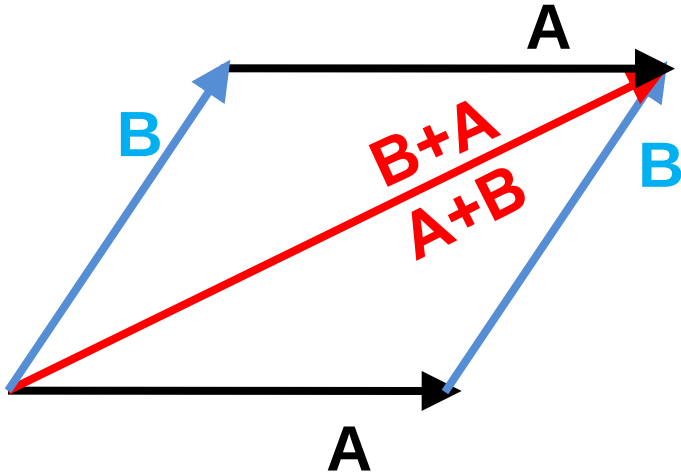


Figure 3.5 These four vectors are equal because they have equal lengths and point in the same direction.

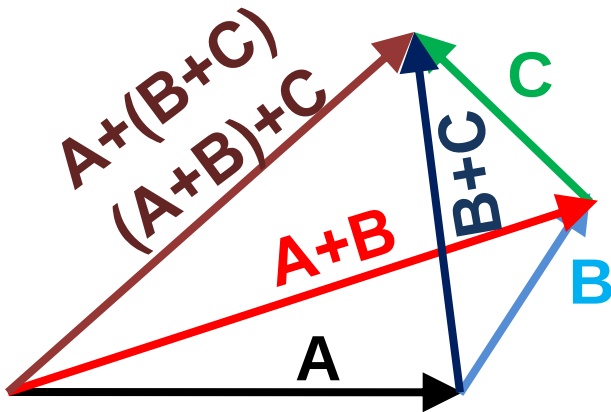
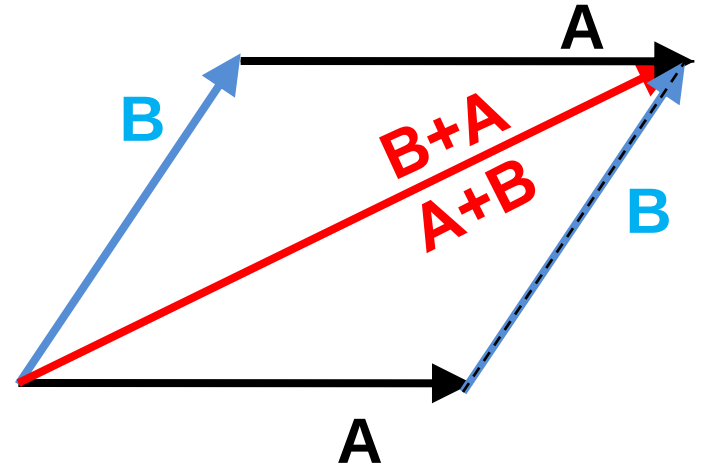


Adding vectors (graphical method)

triangle method.



parallelogram method.

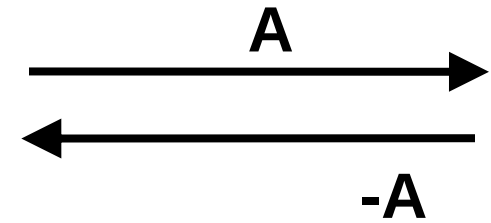


$$A+B = B+A \quad (\text{commutative})$$

$$(A+B)+C = A+(B+C) \quad (\text{associative})$$

Negative of a Vectors

The vector $-\vec{A}$ is a vector with the same magnitude as the vector $\vec{A}$ but points in opposite direction.

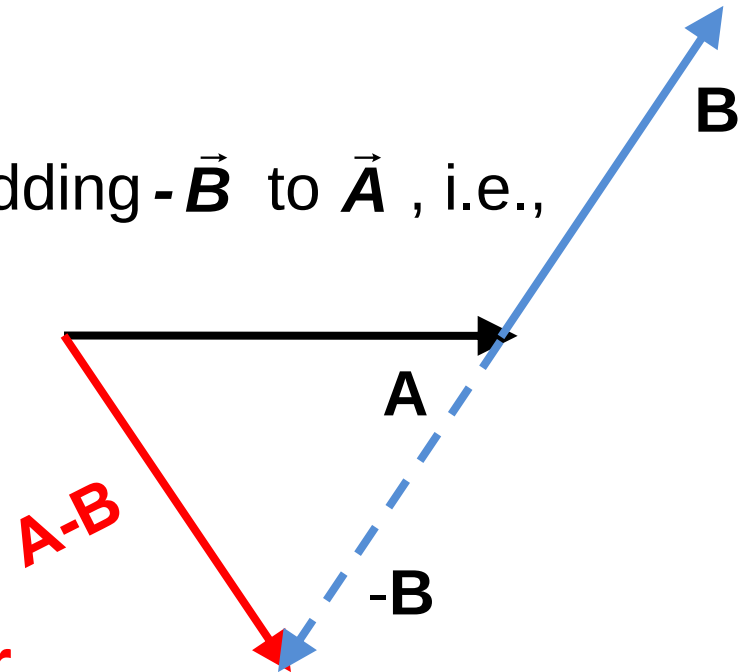


Subtracting Vectors

Subtracting $\vec{B}$ from $\vec{A}$ is the same as adding $-\vec{B}$ to $\vec{A}$, i.e.,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Or $\vec{A} - \vec{B}$ is the vector when added to $\vec{B}$ must give $\vec{A}$



Multiplying a Vector by a Scalar

If $\vec{A}$ is multiplied by a positive scalar λ , the product $\lambda\vec{A}$ is a vector with the same direction as $\vec{A}$ and magnitude λA . If λ is negative the vector $\lambda\vec{A}$ points in the opposite direction of $\vec{A}$.

• Quick Quiz (1)

The magnitudes of two vectors **A** and **B** are $A=12$ units and $B=8$ units. Which pair of numbers represents the *largest* and *smallest* possible values for the magnitude of the resultant vector $\mathbf{R}=\mathbf{A}+\mathbf{B}$?

a) 14.4 units, 4 units

b) 12 units, 8 units

c) 20 units, 4 units

d) Non of them

• Quick Quiz (2)

Under what condition does the resultant vector $\mathbf{A} + \mathbf{B}$ have a magnitude equals to $A + B$?

a) **A** and **B** are parallel and in the same direction.

b) **A** and **B** are parallel and in opposite directions.

c) **A** and **B** are perpendicular.

d) Under no condition.

Components of Vector and Unit Vectors

A **unit vector** is a vector, with specific direction, having a magnitude of 1 without any units or dimensions.

In Cartesian coordinates three unit vectors are adopted to specify the positive directions of the three-axes.

$\hat{i}$ is directed along the positive x -axis

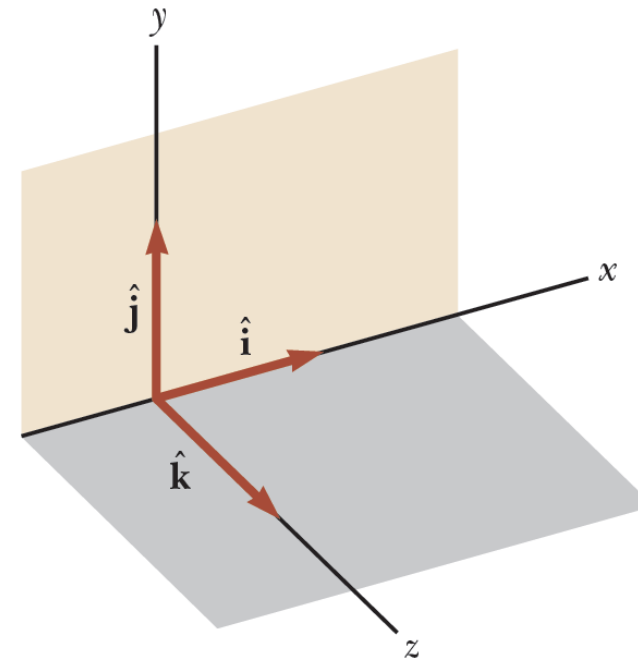
$\hat{j}$ is directed along the positive y -axis

$\hat{k}$ is directed along the positive z -axis

This means that if a vector $\mathbf{A}$ is directed along the positive x -axis with a magnitude of A , this vector can be written as

$$\vec{\mathbf{A}} = A\hat{i}$$

The minus sign in front of any vector indicates the opposite direction of that vector, i.e. $-\hat{i}$ refers to the negative x -axis, and so for the other two unit vectors.



Components of Vector

Consider the vector $\mathbf{A}$ in the x - y plane.

It is clear that the vector $\mathbf{A}$ can be represented by the sum of two vectors, one parallel to the x -axis ($A_x \hat{i}$), and the other parallel to the y -axis ($A_y \hat{j}$), i.e.,

$$\vec{\mathbf{A}} = A_x \hat{i} + A_y \hat{j}$$

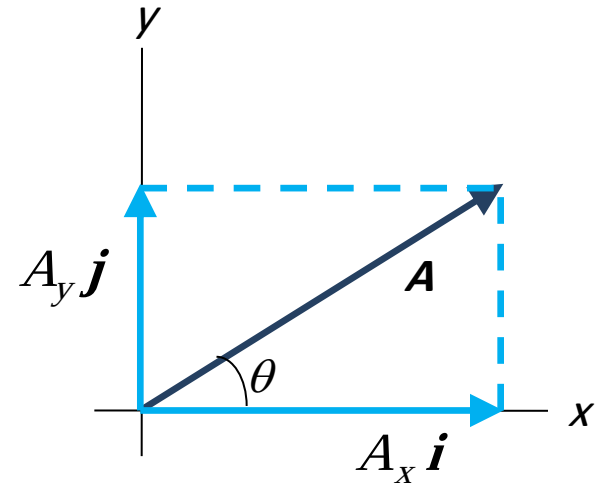
A_x and A_y are called, respectively the x - and the y -components of the vector $\mathbf{A}$.

It is clear from the figure that

$$A_x = A \cos \theta \quad \text{and} \quad A_y = A \sin \theta$$

The magnitude of the vector is given by

$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$$



To find the direction of a vector it is enough to determine the angle θ the vector makes with a specific axis. The angle a vector makes with the x-axis is given by

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

In three dimensions any vector $\mathbf{A}$ can be resolved into three components as.

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$$

Where A_z is called the z-components of the vector $\mathbf{A}$.

Note that if the vector $\mathbf{A}$ is zero then all its components are zero.

Adding Vectors

Now to add two vectors analytically proceed as follow:

(i) Resolve each vector into its components according to suitable coordinate axes.

(i) Add, algebraically, the x -components of the individual vectors to obtain the x -component of the resultant vector. Do the same thing for the other components, i.e., if

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\text{and } \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

Then the resultant vector **R** is

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} + (A_z + B_z) \hat{k}$$

• Quick Quiz (3)

For which of the following vectors is the magnitude of the vector equal to one of the components of the vector?

a) $\vec{A} = 2\hat{i} + 5\hat{j} + \hat{k}$

b) $\vec{A} = 2\hat{i} + 5\hat{j}$

c) $\vec{A} = -3\hat{j}$

d) $\vec{A} = 5\hat{k}$

Example 3.3 Find the sum of the 2-displacement vectors

$$\vec{A} = (2.0\hat{i} + 2.0\hat{j})\text{m} \quad \vec{B} = (2.0\hat{i} - 4.0\hat{j})\text{m}$$

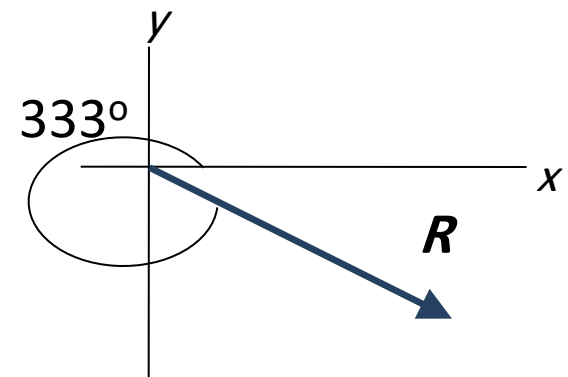
Solution From the given information we have

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \Rightarrow$$

$$\vec{R} = (2.0 + 2.0)\hat{i} + (2.0 - 4.0)\hat{j} = (4\hat{i} - 2.0\hat{j})\text{m}$$

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0)^2 + (-2.0)^2} = 4.5\text{m}$$

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{-2.0}{4.0}\right) = -27^\circ$$



Example 3.4 A particle undergoes three consecutive displacements

$$\vec{A} = (15\hat{i} + 30\hat{j} + 12\hat{k}) \text{ cm} \quad \vec{B} = (23\hat{i} - 14\hat{j} - 5.0\hat{k}) \text{ cm} \quad \vec{C} = (-13\hat{i} + 15\hat{j}) \text{ cm}$$

Find the resultant displacement and its magnitude.

Solution

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} = (A_x + B_x + C_x)\hat{i} + (A_y + B_y + C_y)\hat{j} + (A_z + B_z + C_z)\hat{k} \Rightarrow$$

$$\vec{R} = (15 + 23 - 13)\hat{i} + (30 - 14 + 15)\hat{j} + (12 - 5.0 + 0)\hat{k} \Rightarrow$$

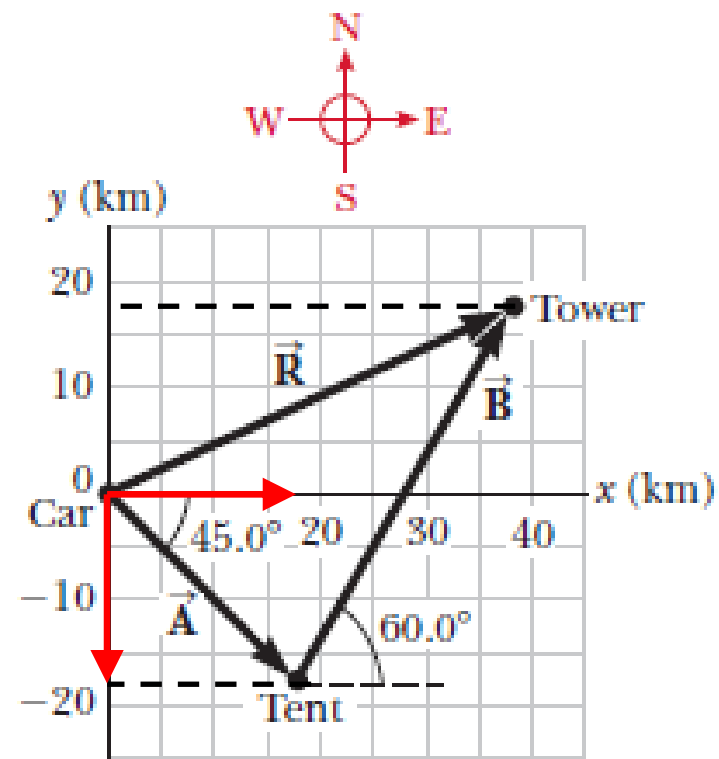
$$\vec{R} = (25\hat{i} + 31\hat{j} + 7.0\hat{k}) \text{ cm}$$

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$|\vec{R}| = \sqrt{(25.0)^2 + (31)^2 + (7.0)^2} = 40 \text{ cm}$$

Example 3.5 A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

- Determine the components of each displacement.
- Find the resultant displacement in terms of unit vectors.



Solution a) The components of the first displacement are

$$A_x = A \cos \theta = 25.0 \cos(-45^\circ) = 17.7 \text{ km}$$

$$A_y = A \sin \theta = 25.0 \sin(-45^\circ) = -17.7 \text{ km}$$

$$\Rightarrow \vec{A} = (17.7\hat{i} - 17.7\hat{j}) \text{ km}$$

And the components of the second displacement are

$$B_x = B \cos \theta = 40.0 \cos(60^\circ) = 20.0 \text{ km}$$

$$B_y = B \sin \theta = 40.0 \sin(60^\circ) = 34.6 \text{ km}$$

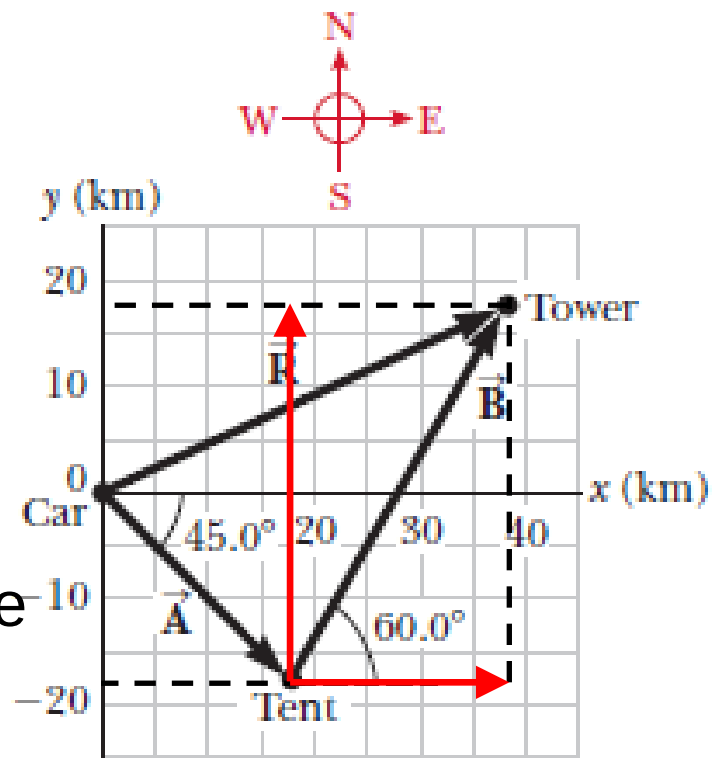
$$\Rightarrow \vec{B} = (20.0\hat{i} + 34.6\hat{j}) \text{ km}$$

b) For the resultant displacement we have

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \Rightarrow$$

$$\vec{R} = (20.0 + 17.7)\hat{i} + (34.6 - 17.7)\hat{j} \Rightarrow$$

$$\vec{R} = (37.7\hat{i} + 17.0\hat{j}) \text{ km}$$



To find the direction of the resultant displacement we have

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{17.7}{37.7}\right) = 24.3^\circ$$

Product of Vectors

1- Multiplying a vector by a scalar:

If λ is a scalar and $\mathbf{A}$ is a vector then

$$\lambda \vec{\mathbf{A}} = \lambda A_x \hat{\mathbf{i}} + \lambda A_y \hat{\mathbf{j}} + \lambda A_z \hat{\mathbf{k}}$$

Note that the direction of $\lambda \mathbf{A}$ is the same as the direction of $\mathbf{A}$ if λ is +ve and opposite to $\mathbf{A}$ if λ is -ve.

2- Scalar (dot) product:

The scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \cdot \mathbf{B}$, is a **scalar** quantity defined by,

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = |\vec{\mathbf{A}}| |\vec{\mathbf{B}}| \cos \theta$$

with θ is the smallest angle between the two vectors.

Since $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are unit vectors perpendicular to each other, and using the last equation we can write

$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= A_x B_x \hat{i} \cdot \hat{i} + \cancel{A_x B_y \hat{i} \cdot \hat{j}} + \cancel{A_x B_z \hat{i} \cdot \hat{k}} + \cancel{A_y B_x \hat{j} \cdot \hat{i}} + A_y B_y \hat{j} \cdot \hat{j} + \cancel{A_y B_z \hat{j} \cdot \hat{k}} + \cancel{A_z B_x \hat{k} \cdot \hat{i}} + \cancel{A_z B_y \hat{k} \cdot \hat{j}} + A_z B_z \hat{k} \cdot \hat{k}$$

$$\Rightarrow \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Example 7.2 You are given the vectors $\vec{A} = 2\hat{i} + 3\hat{j}$ $\vec{B} = -\hat{i} + 2\hat{j}$

- Find the dot product between the two vectors.
- Find the angle between the two vectors.

Solution a) $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = 2(-1) + 3(2) = -2 + 6 = 4$

b) To find the angle between the two vectors we have

$$\vec{A} = 2\hat{i} + 3\hat{j} \quad \vec{B} = -\hat{i} + 2\hat{j}$$

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \quad \Rightarrow \cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$\text{But } |\vec{A}| = \sqrt{2^2 + 3^2} = \sqrt{13} \quad \text{and} \quad |\vec{B}| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$\Rightarrow \cos \theta = \frac{4}{\sqrt{13 \times 5}} = 0.5 \quad \Rightarrow \theta = 60^\circ$$

• Quick Quiz (4)

Which of the following statements is true?

- a) $\vec{A} \cdot \vec{B}$ is larger than AB
- b) $\vec{A} \cdot \vec{B}$ is smaller than AB
- c) $\vec{A} \cdot \vec{B}$ could be larger than or smaller than AB
- d) $\vec{A} \cdot \vec{B}$ could be equal to AB

3- Vector (cross) product:

The vector product of two vectors $\mathbf{A}$ and $\mathbf{B}$, written as $\mathbf{A} \times \mathbf{B}$, is a third vector $\mathbf{C}$ with a magnitude given by

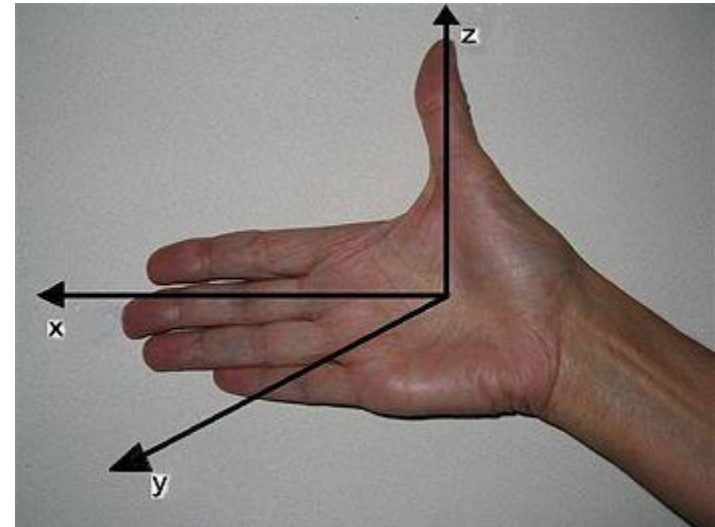
$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

The vector $\mathbf{C}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. Since there are two directions perpendicular to a given plane, the **right hand rule** is used to decide to which direction the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is directed.

The rule states that the four fingers of the right hand are pointed along $\mathbf{A}$ and then curled toward $\mathbf{B}$ through the smaller angle between $\mathbf{A}$ and $\mathbf{B}$. The thumb then gives the direction of $\mathbf{C}$.

Using this rule it is clear that.

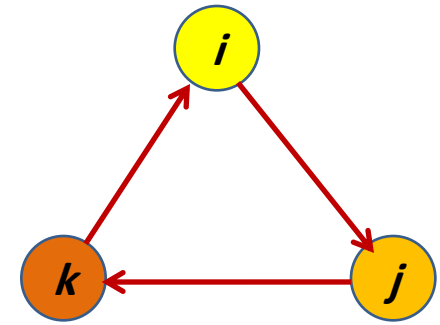
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$



By the same rule one can verify that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}$$



$$\vec{A} \times \vec{B} = (A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \times (B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k})$$

$$= A_x B_x \mathbf{i} \times \mathbf{i} + A_x B_y \mathbf{i} \times \mathbf{j} + A_x B_z \mathbf{i} \times \mathbf{k}$$

$$+ A_y B_x \mathbf{j} \times \mathbf{i} + A_y B_y \mathbf{j} \times \mathbf{j} + A_y B_z \mathbf{j} \times \mathbf{k}$$

$$+ A_z B_x \mathbf{k} \times \mathbf{i} + A_z B_y \mathbf{k} \times \mathbf{j} + A_z B_z \mathbf{k} \times \mathbf{k}$$

$$= (A_y B_z - A_z B_y) \mathbf{i} + (A_z B_x - A_x B_z) \mathbf{j} + (A_x B_y - A_y B_x) \mathbf{k}$$

Or equivalently we have

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

Example 11.1 You are given the vectors $\vec{A} = 2\hat{i} + 3\hat{j}$ $\vec{B} = -\hat{i} + 2\hat{j}$

- a) Find the cross product between the two vectors.
- b) Verify that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Solution a) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ -1 & 2 & 0 \end{vmatrix}$

$$= (3 \times 0 - 0 \times 2) \hat{i} + (0 \times -1 - 2 \times 0) \hat{j} + (2 \times 2 - 3 \times -1) \hat{k} = 7\hat{k}$$

$$\text{b)} \quad \vec{B} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}$$

$$= (2 \times 0 - 0 \times 3)\hat{i} + (0 \times 2 - (-1 \times 0))\hat{j} + (-1 \times 3 - 2 \times 2)\hat{k} = -7\hat{k}$$

• Quick Quiz (5)

Which of the following is always zero?

a) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{A}$

b) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A}$

c) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{B}$

d) $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{A}$