

CHAPTER 4

MOTION IN TWO DIMENSIONS

In this chapter we are going to discuss the following topics:

- ❑ **Position, Velocity, & Acceleration**
- ❑ **2-D Motion with Constant Acceleration**
- ❑ **Projectile Motion**
- ❑ **Circular Motion (Uniform & Non-Uniform)**
- ❑ **Relative Motion**

Recommended Problems:

1, 3, 4, 7, 9, 18, 19, 21, 23, 25, 29, 41, 42, 43, 44, 45, 47, 50, 51, 54, 55, 60, 62, 64, 70, 71, 84, 86

4.1 The position, Velocity, and Acceleration.

Consider a particle moves along the curve shown from initial point i to a final point f .

The initial and the final position of the particle are two-dimensional vectors:

$$\vec{r}_i = x_i \hat{i} + y_i \hat{j}$$

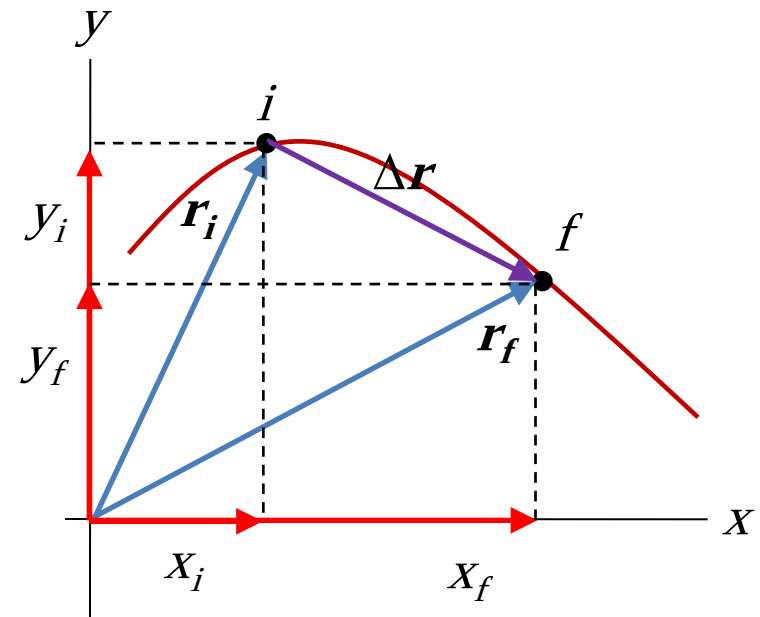
$$\vec{r}_f = x_f \hat{i} + y_f \hat{j}$$

Now the displacement of the particle is

$$\Delta \vec{r} = \vec{r}_f - \vec{r}_i = (x_f - x_i) \hat{i} + (y_f - y_i) \hat{j} = \Delta x \hat{i} + \Delta y \hat{j}$$

Now the average velocity of the particle is

$$\vec{v} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t} \hat{i} + \frac{\Delta y}{\Delta t} \hat{j} = \bar{v}_x \hat{i} + \bar{v}_y \hat{j}$$



The instantaneous velocity is now defined as

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} = v_x \hat{i} + v_y \hat{j}$$

Similarly, the average and the instantaneous acceleration s are now defined as

$$\bar{\vec{a}} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t} \hat{i} + \frac{\Delta v_y}{\Delta t} \hat{j} = \bar{a}_x \hat{i} + \bar{a}_y \hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} = a_x \hat{i} + a_y \hat{j}$$

It is clear now that the planar motion can be considered as a vector sum of two-perpendicular linear motions: one along the x-axis and the other along the y-axis.

4.2 Two Dimensional Motion with Constant Acceleration

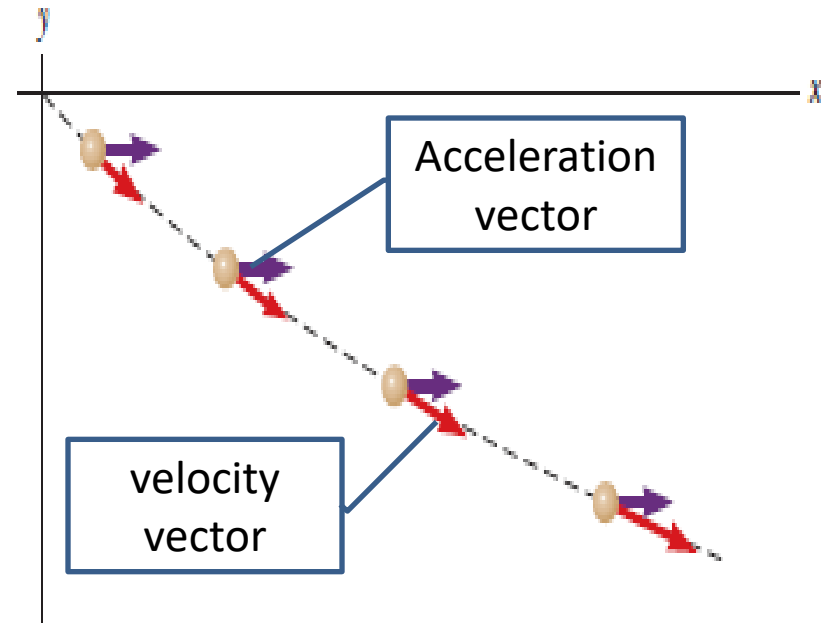
If $\mathbf{a}$ is constant, then a_x and a_y are consequently constants, then we can write, by letting $x_i = y_i = 0$,

x-axis	y-axis	Net
$v_{xf} = v_{xi} + a_x t$	$v_{yf} = v_{yi} + a_y t$	$\vec{v}_f = \vec{v}_i + \vec{a}t$
$x_f = v_{xi}t + \frac{1}{2}a_x t^2$	$y_f = v_{yi}t + \frac{1}{2}a_y t^2$	$\vec{r}_f = \vec{v}_i t + \frac{1}{2}\vec{a}t^2$
$v_{xf}^2 = v_{xi}^2 + 2a_x x_f$	$v_{yf}^2 = v_{yi}^2 + 2a_y y_f$	$\vec{v}_f^2 = \vec{v}_i^2 + 2\vec{a} \cdot \vec{r}$

Remember that the time is common between the two motions, i.e., the time the particle takes to move from one point to the other can be found either from the x-axis, or from the y-axis equations.

Example 4.1 A particle moves in the x - y plane, starting from the origin with an initial velocity having $v_{xi}=20\text{m/s}$, and $v_{yi}=-15\text{m/s}$. The particle experiences an acceleration in the x -direction as $a_x=4.0\text{m/s}^2$.

- Calculate the velocity and the speed of the particle at $t=5.0\text{s}$.
- Calculate the position of the particle at $t=5\text{s}$.



Solution Taking $t_f=5.0\text{s}$ and using the relations

$$v_{xf} = v_{xi} + a_x t \quad \Rightarrow v_{xf} = 20 + 4.0(5.0) = 40\text{m/s}$$

$$v_{yf} = v_{yi} + a_y t \quad \Rightarrow v_{yf} = -15 + 0(5.0) = -15\text{m/s}$$

$$\Rightarrow \quad \vec{v} = (40\hat{i} - 15\hat{j}) \text{ m/s}$$

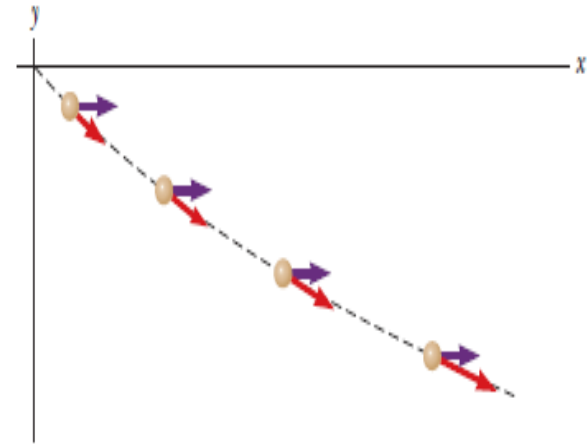
And for the speed we have $\Rightarrow \quad |\vec{v}| = \sqrt{(40)^2 + (-15)^2} = 43 \text{ m/s}$

To find the position we use

$$x_f = v_{xi}t + \frac{1}{2}a_x t^2 = 20(5.0) + \frac{1}{2}(4.0)(5.0)^2 = 150\text{m}$$

$$y_f = v_{yi}t + \frac{1}{2}a_y t^2 = -15(5.0) + \frac{1}{2}(0)(5.0)^2 = -75\text{m}$$

$$\Rightarrow \quad \vec{r} = (150\hat{i} - 75\hat{j})\text{m}$$



• Quick Quiz (1)

Two balls are projected at the same instant from the same height. Ball *A* is projected horizontally and ball *B* is dropped from rest. Assuming the only acceleration acting on the two balls is the acceleration of gravity which is constant and directed directly downward, which ball will reach the ground first?

a) Ball A.

b) They will reach together.

c) Ball B.

d) Ball A will never reach the ground.

4.3 Projectile Motion

If a particle given an initial velocity and then follows a path determined only by gravity we have projectile motion.

As the acceleration of gravity is always straight downward with a magnitude of g , the acceleration of a projectile is always given as

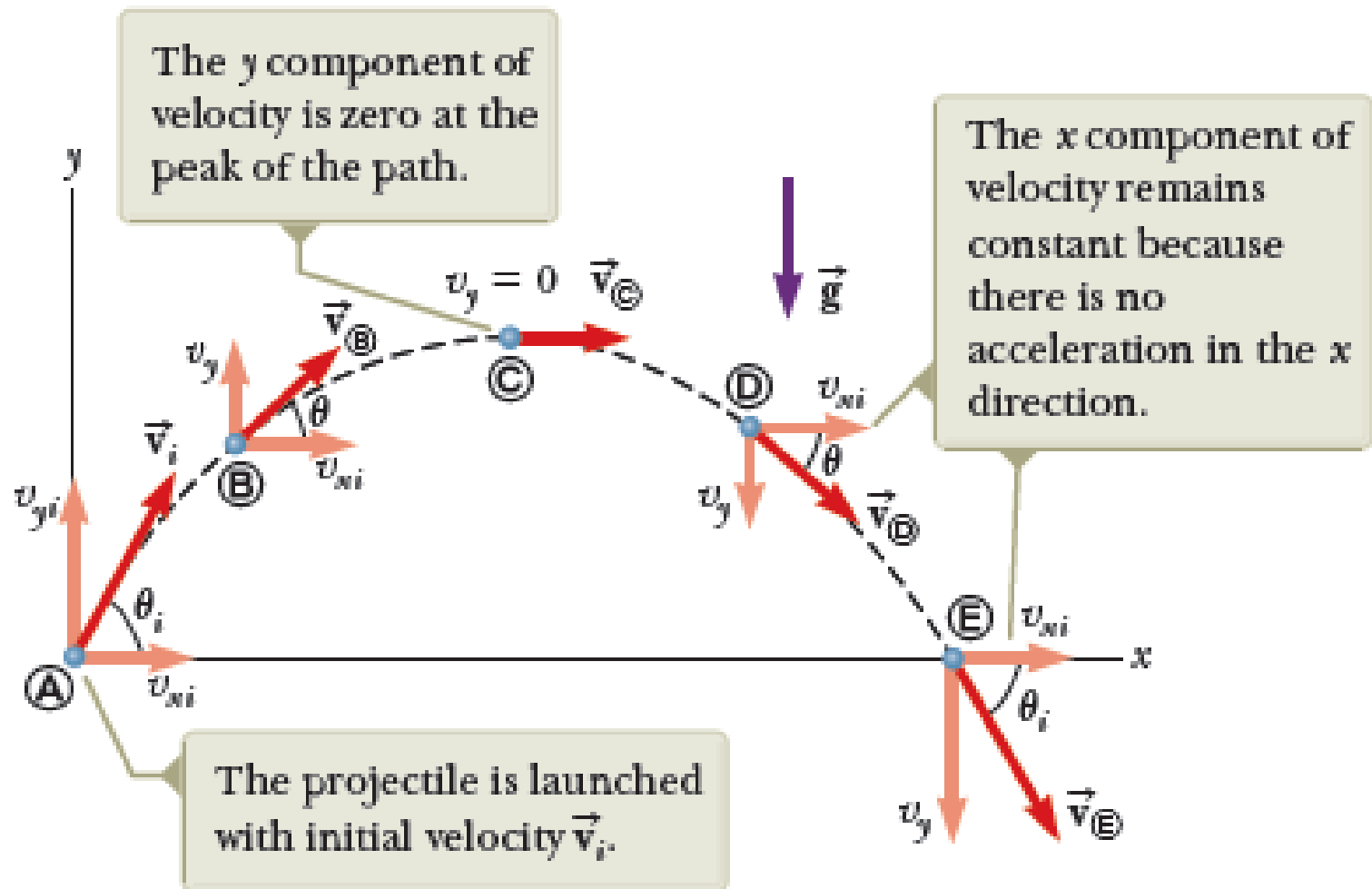
$$\vec{a} = -g\hat{j}$$

This means that the projectile motion can be considered as a vector sum of a vertical motion with $a_y = -g$ (the positive sense is taken upward) and a horizontal motion with $a_x = 0$ (constant velocity). Taking $x_i = x_f = 0$, the formulas govern the motion of a projectile is then

Horizontal motion ($a_x = 0$)	Vertical motion ($a_y = -g$)
$v_{xf} = v_{xi}$	$v_{yf} = v_{yi} - gt$
$x_f = v_{xi}t$	$y_f = v_{yi}t - \frac{1}{2}gt^2$
$v_{xf}^2 = v_{xi}^2$	$v_{yf}^2 = v_{yi}^2 - 2gy_f$

If one substitute for t from x -eq. into the 2nd y -eq. we get the equation of a parabola.

$$y = (\tan \theta)x - \left(\frac{g}{2v_0^2 \cos^2 \theta} \right) x^2$$



Horizontal Range and Maximum Height of a Projectile.

The maximum horizontal distance a projectile covered is called the Range of the projectile, while the height of the maximum point reached by the projectile is called the maximum height. Resolving the initial velocity into its rectangular components we have

$$v_{xi} = v_i \cos \theta$$

and $v_{yi} = v_i \sin \theta$

At the maximum height (h), $v_{yf}=0$, using the equation

$$v_{yf}^2 = v_{yi}^2 - 2gy_f \quad \Rightarrow$$

$$0 = (v_i \sin \theta)^2 - 2gh. \quad \Rightarrow$$

$$h = \frac{v_i^2 \sin^2 \theta}{2g}$$

To find the range we need the time of flight. Noting that for the total time of flight $y_f=0$, so we can use

$$y_f = v_{yi}t - \frac{1}{2}gt^2 \Rightarrow t = \frac{2v_{yi}}{g} = \frac{2v_i \sin \theta}{g} \Rightarrow$$

Now substituting for t in the eq. $x_f = v_{xi}t \Rightarrow$

$$R = v_{xi} \frac{2v_i \sin \theta}{g} = \frac{2v_i^2 \sin \theta \cos \theta}{g}$$

Knowing that $2 \sin \theta \cos \theta = \sin 2\theta \Rightarrow$

$$R = \frac{v_i^2 \sin 2\theta}{g}$$

It is clear that the maximum value of R is $R = v_i^2 / g$

This is because the maximum value of $\sin 2\theta$ is 1, and this occurs at $2\theta=90^\circ$ so the range is a maximum when $\theta=45^\circ$

• Quick Quiz (2)

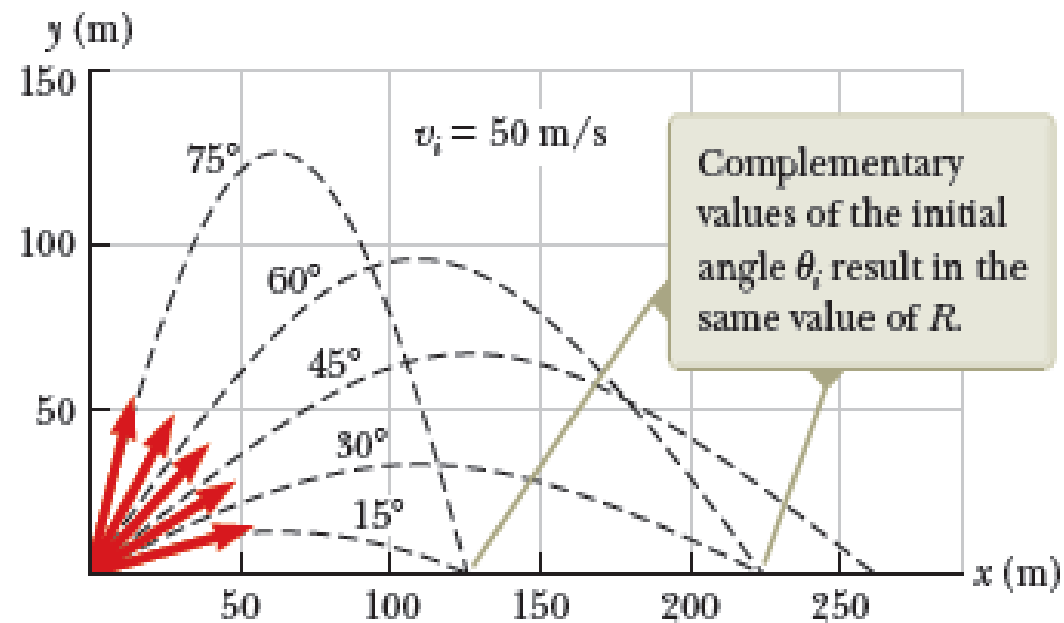
A projectile launched from the origin with the same initial speed but at various angles of projection. For which of the launch angles the time of flight is the largest.

a) For 15°

c) For 75°

b) For 45°

d) All are the same



• Quick Quiz (3)

In a projectile motion, at what point along the path are the velocity and acceleration vectors for the projectile perpendicular to each other?

a) the launch point.

b) The highest point

c) The landing point

d) nowhere

• Quick Quiz (4)

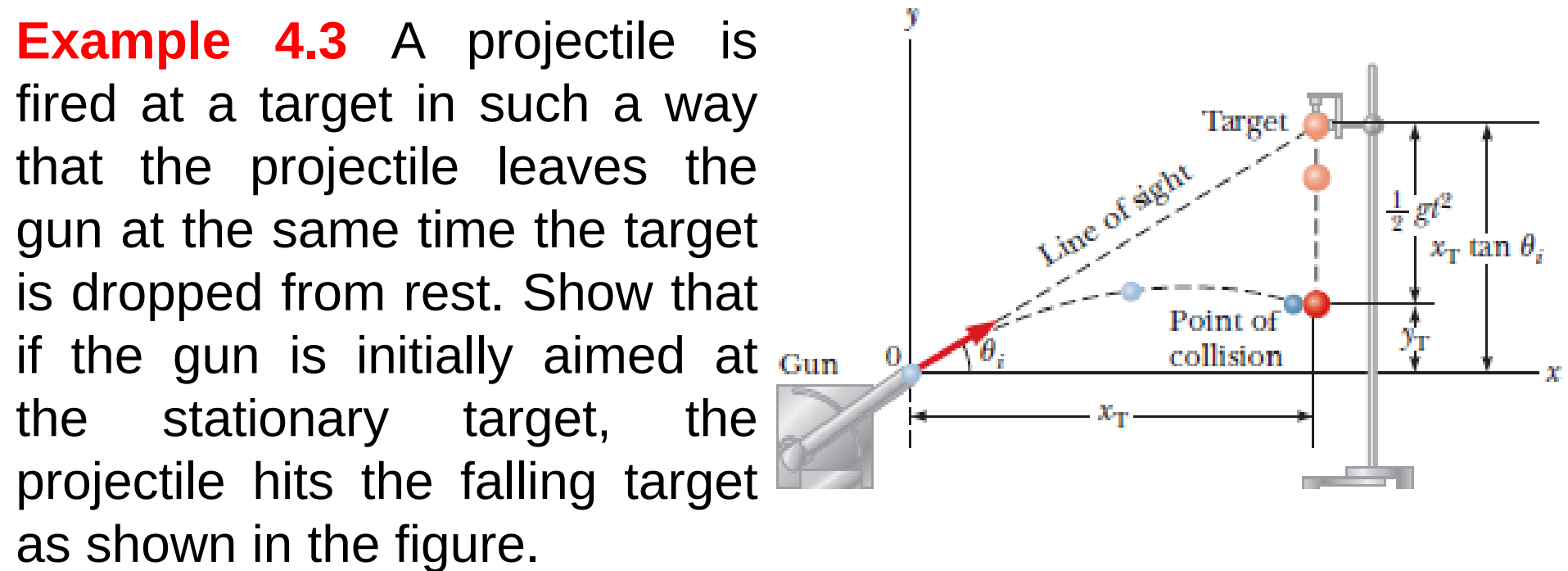
In a projectile motion, at what point along the path are the velocity and acceleration vectors for the projectile parallel to each other?

a) the launch point.

b) The highest point

c) The landing point

d) nowhere



Solution Let θ be the angle the aiming line (Line of sight), and α be the angle the initial velocity of the projectile make with the horizontal. We will show that if $\theta = \alpha$ the projectile hits the target.

The target undergoes free falling motion with $v_{Ti} = 0$ from the point x_{Ti} and y_{Ti}

$$y_{Tf} = y_{Ti} + v_{Ti}t - \frac{1}{2}gt^2 = y_{Ti} - \frac{1}{2}gt^2$$

As it is clear from the figure we have $y_{Ti} = x_{Ti} \tan \theta = x_{Tf} \tan \theta \Rightarrow$

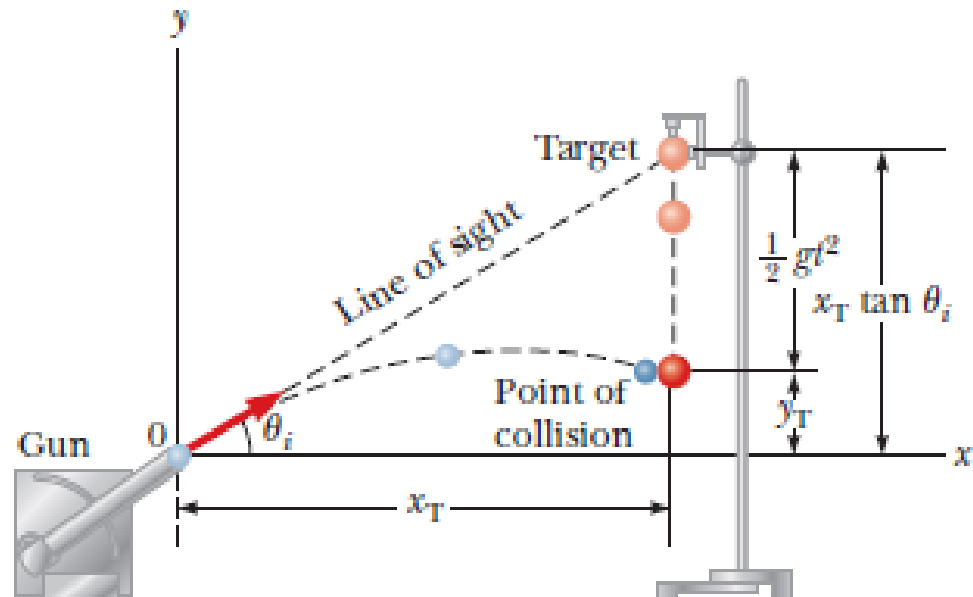
$$y_{Tf} = x_T \tan \theta - \frac{1}{2} g t^2 \quad (1)$$

For the projectile, its horizontal and vertical positions are

$$x_{Pf} = x_{Pi} + v_{Pxi} t = v_i \cos \alpha t$$

$$y_{Pf} = y_{Pi} + v_{Pyi} t - \frac{1}{2} g t^2 \quad \Rightarrow$$

$$y_{Pf} = v_i \sin \alpha t - \frac{1}{2} g t^2$$



From the x -equation we have $t = \frac{x_{Pf}}{v_i \cos \alpha}$

Substituting for t into the y -equation we get

$$y_{Pf} = v_i \sin \alpha \left(\frac{x_{Pf}}{v_i \cos \alpha} \right) - \frac{1}{2} g t^2 \quad \Rightarrow$$

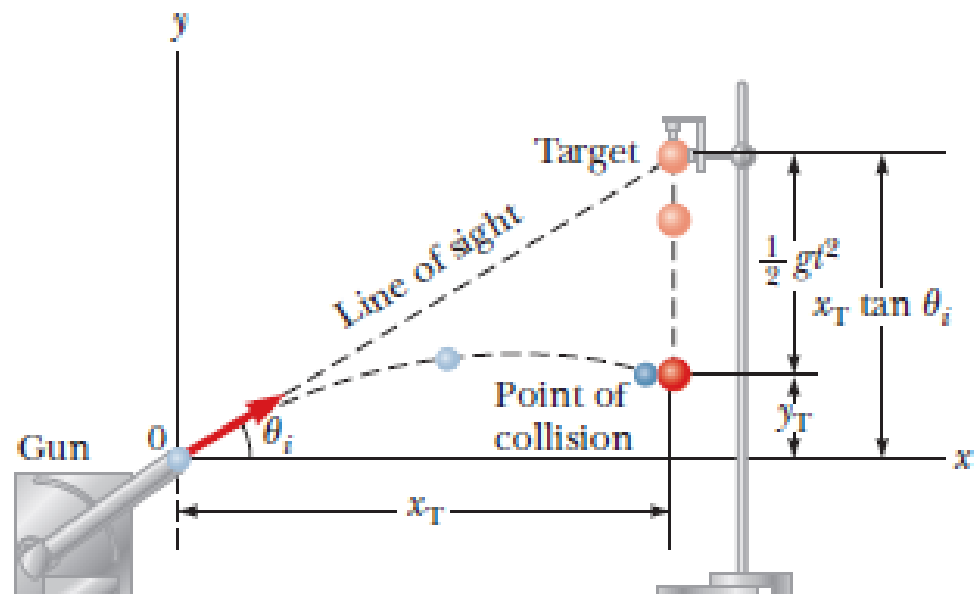
$$y_{Pf} = x_{Pf} \tan \alpha - \frac{1}{2} g t^2 \quad (2)$$

$$y_{Tf} = x_{Ti} \tan \theta - \frac{1}{2} g t^2 \quad (1)$$

$$y_{Pf} = x_{Pf} \tan \alpha - \frac{1}{2} g t^2 \quad (2)$$

For the projectile to hit the target they must be at the same position, i.e.,

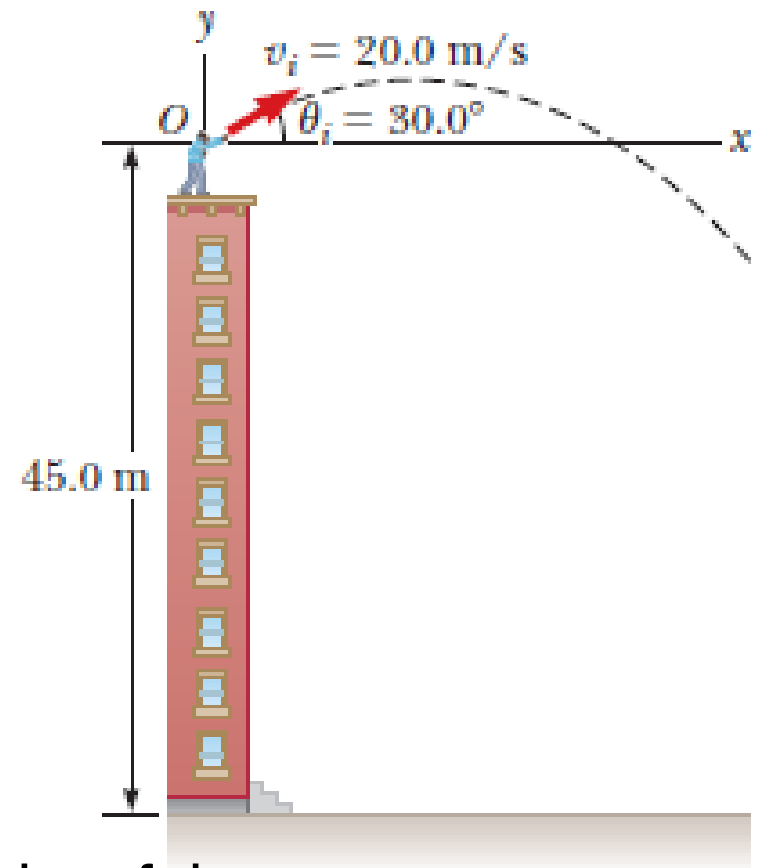
$$x_{Pf} = x_T \quad \text{and} \quad y_{Pf} = y_{Tf} \quad (3)$$



Comparing Eq.(1), Eq.(2), and Eq.(3) we conclude that for the projectile to hit the target then θ *must be equal to* α .

Example 4.4 A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s as shown. The height from which the stone is thrown is 45.0 m above the ground.

- How long does it take the stone to reach the ground?
- What is the speed of the stone just before it strikes the ground?



Solution First resolving the initial velocity of the stone to get

$$v_{xi} = v_i \cos \theta = 20.0 \cos 30.0^\circ = 17.3 \text{ m/s}$$

$$v_{yi} = v_i \sin \theta = 20.0 \sin 30.0^\circ = 10.0 \text{ m/s}$$

- To find the time when the stone reaches the ground we use

$$y_f = v_{yi}t - \frac{1}{2}gt^2 \quad \Rightarrow$$

$$-45.0 = 10.0t - \frac{1}{2}(9.80)t^2 \quad \Rightarrow \quad t = 4.22\text{ s}$$

b) First let us find the y-component of the velocity as the stone strikes the ground. We use

$$v_{yf} = v_{yi} - gt = 10.0 - 9.80(4.22) = -31.3\text{ m/s}$$

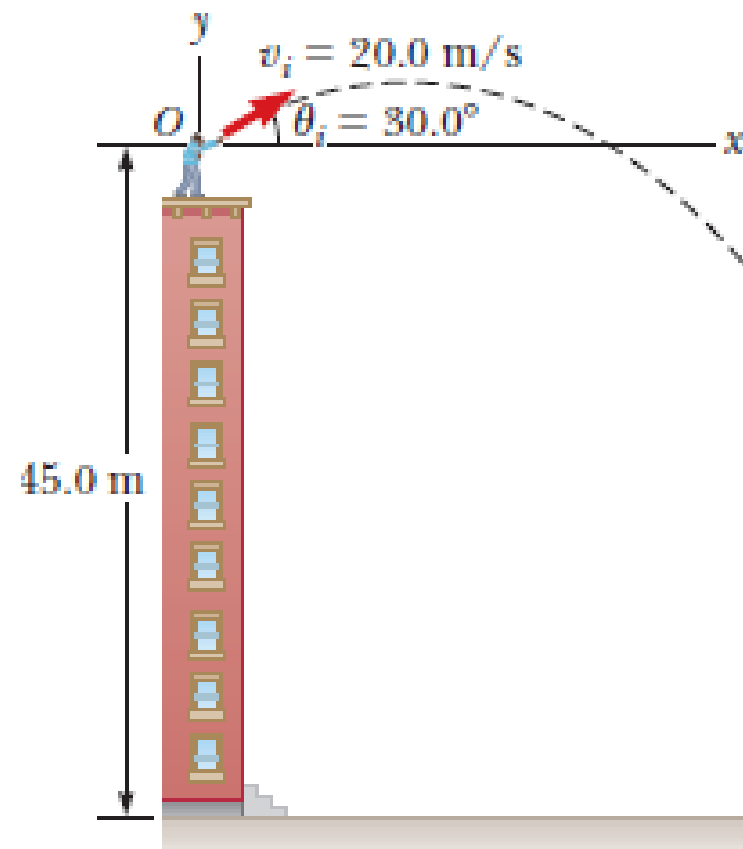
Down

As the x-component of the velocity is constant, the velocity of the stone as it strikes the ground is

$$\vec{v}_f = v_{xf}\hat{i} + v_{yf}\hat{j} = (17.3\hat{i} - 31.3\hat{j})\text{ m/s}$$

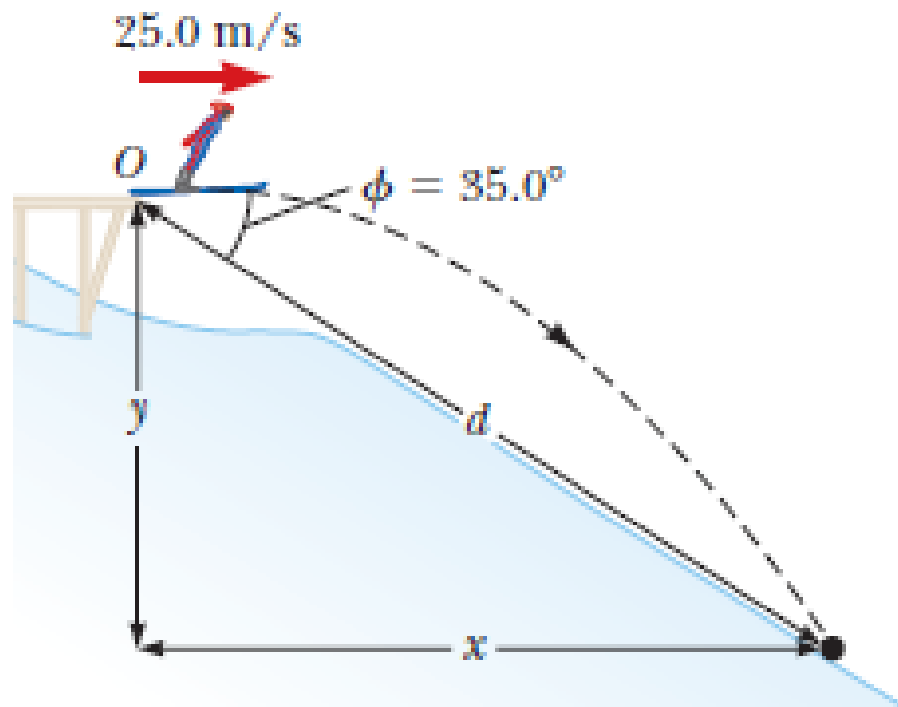
For the speed we have

$$|\vec{v}_f| = \sqrt{(17.3)^2 + (-31.3)^2} = 35.8\text{ m/s}$$



Example 4.5 A ski jumper leaves the ski track moving horizontally with a speed of 25.0 m/s as shown. The landing incline below her falls off with a slope of 35.0° . Where does she land on the incline?

Solution It is clear that $v_{xi}=25.0$ m/s, and $v_{yi}=0$.



let d be the distance traveled along the incline. Now The coordinates of the landing point is then

$$x_f = d \cos \phi \quad \text{and} \quad y_f = -d \sin \phi$$

$$\text{Using } x_f = v_{xi}t = 25.0t \quad \Rightarrow d \cos 35.0^\circ = 25.0t$$

$$\text{And using } y_f = v_{yi}t - \frac{1}{2}gt^2 = \quad \Rightarrow -d \sin 35.0^\circ = -\frac{1}{2}9.80t^2$$

Substituting for t from the 1st equation into the 2nd one we get

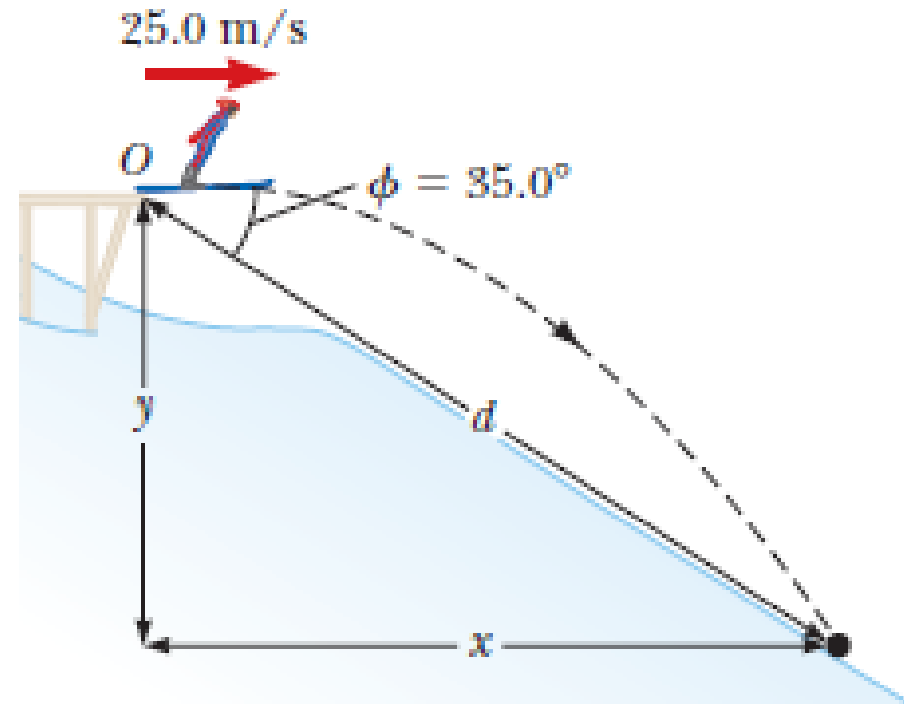
$$t = \frac{d \cos 35.0^\circ}{25.0} \Rightarrow$$

$$d \sin 35.0^\circ = 4.90 \left(\frac{d \cos 35.0^\circ}{25.0} \right)^2 \Rightarrow$$

$$d = \frac{(25.0)^2 \sin 35.0^\circ}{(4.90) \cos^2 35.0^\circ} = 109 \text{ m} \Rightarrow$$

$$x_f = d \cos \phi = 109 \cos 35.0^\circ = 89.3 \text{ m}$$

$$y_f = -d \sin \phi = -109 \sin 35.0^\circ = -62.5 \text{ m}$$



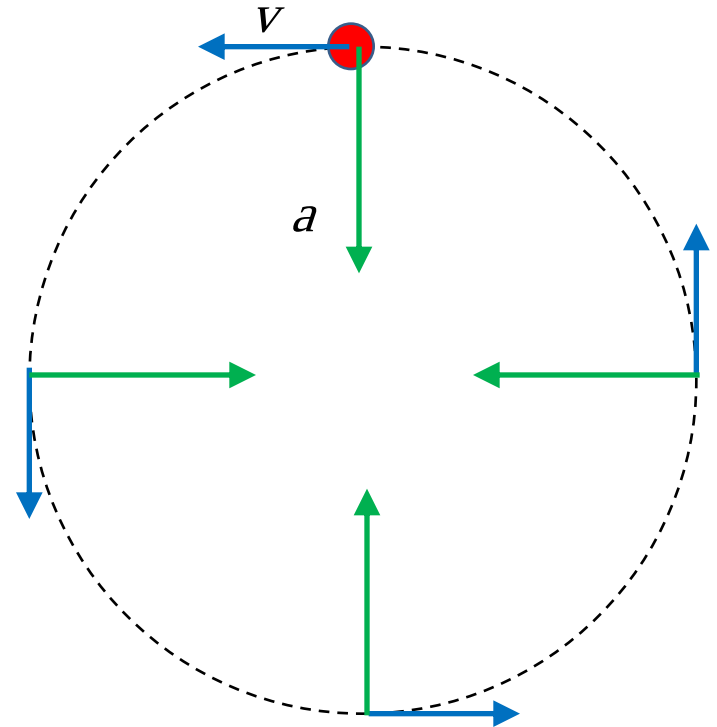
4.4 Particle in Uniform Circular Motion

When an object moves on a circular path with constant speed v , the object is said to be in a uniform circular motion.

As the velocity is always tangent to the path followed, the direction of the velocity in uniform circular motion always changing.

Although the speed is constant, we have an acceleration due to the change of the direction of the velocity. This acceleration is directed toward the center (**centripetal acceleration**). and is given as

$$a_c = \frac{v^2}{r}$$



The period T , which is defined as the time needed for the particle to complete one revolution, can be found from

$$T = \frac{2\pi r}{v}$$

The reciprocal of the period is called the frequency f , which is defined as the number of revolutions in one second.

Because one revolution equals to 2π radians, the angular speed, measured in radian/s is given by

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{v}{r}$$

The centripetal acceleration in terms of the angular speed can be written as

$$a_c = \frac{v^2}{r} = \omega^2 r$$

Example 4.6 a) The earth revolves around the sun in an orbit (assuming it is a circular) of radius 1.496×10^{11} m and complete one revolution in 1 year.

b) Find the acceleration of the moon toward the earth if the moon revolves about the earth in an orbit of average radius 3.845×10^8 km and makes one revolution in 27.32 days (**Sidereal Period**).

Solution a) The period of earth about sun is

$$T = 1 \times 365.25 \times 24 \times 3600 = 3.156 \times 10^7 \text{ s.}$$

The angular speed of the earth is $\omega = \frac{2\pi}{3.156 \times 10^7} = 1.991 \times 10^{-7} \text{ rad/s}$

$$\text{or } \omega = 0.986 \text{ degree/day}$$

The centripetal acceleration is now

$$a_c = \omega^2 r = (1.991 \times 10^{-7})^2 (1.496 \times 10^{11}) = 5.930 \times 10^{-3} \text{ m/s}^2$$

b) The period of Moon about Earth is

$$T = 27.32 \times 24 \times 3600 = 2.360 \times 10^6 \text{ s.}$$

The angular speed of the moon is $\omega = \frac{2\pi}{2.360 \times 10^7} = 2.662 \times 10^{-6} \text{ rad/s}$

$$\text{or } \omega = 13.18 \text{ degree/day}$$

The centripetal acceleration is now

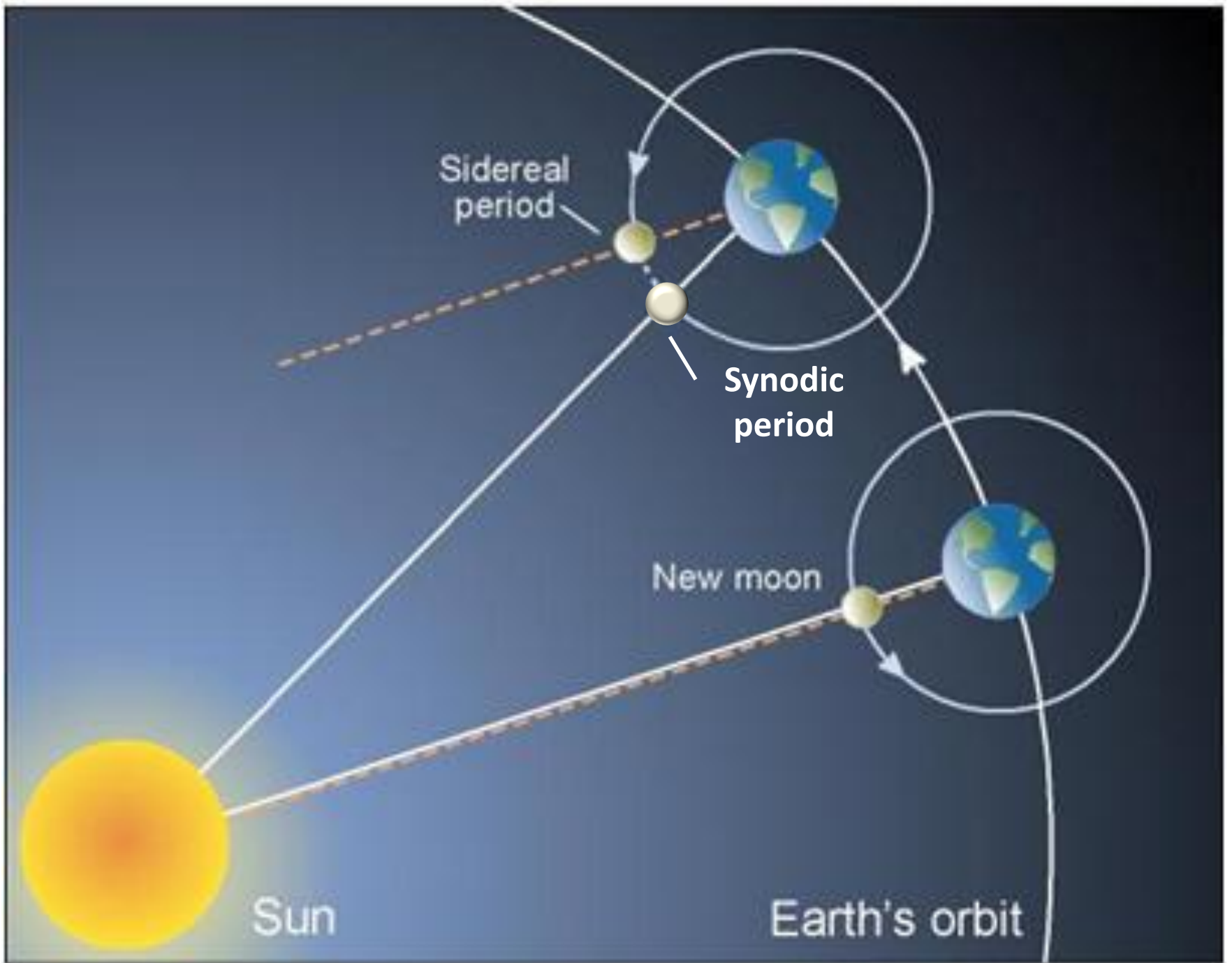
$$a_c = \omega^2 r = (2.662 \times 10^{-6})^2 (3.845 \times 10^8) = 2.725 \times 10^{-3} \text{ m/s}^2$$

Note that in the period of 27.32 days the earth moves about

$$\theta = 0.986 \times 27.32 = 26.94^\circ$$

So the moon needs to move this additional angle to complete one **Synodic Period** this will take the moon a time of

$$t = \frac{\theta}{\omega} = \frac{26.94^\circ}{13.18} = 2.044 \text{ day}$$



SIDEREAL AND SYNODIC MONTH

Orbit of Moon
around Earth

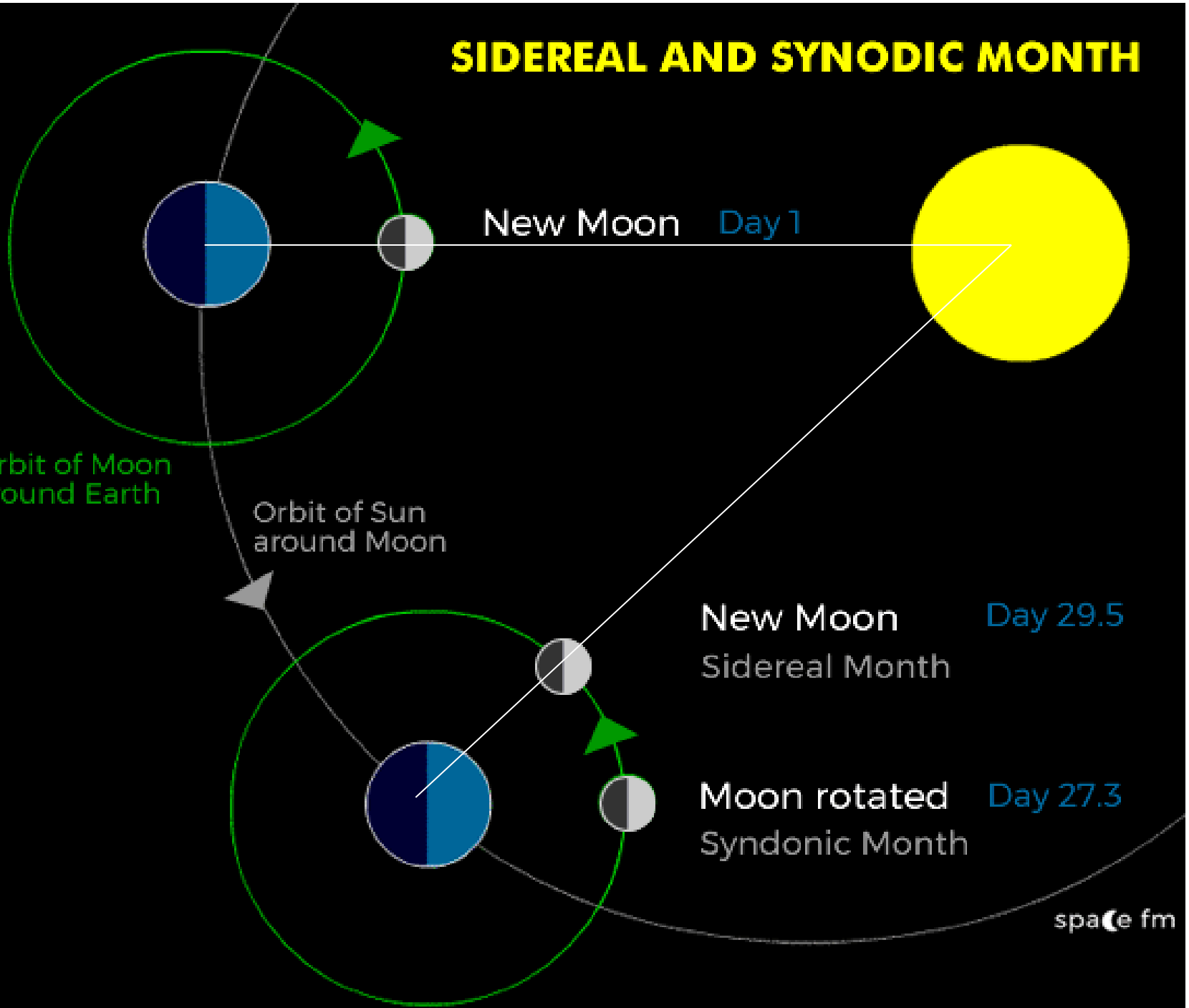
Orbit of Sun
around Moon

New Moon Day 1

New Moon Day 29.5
Sidereal Month

Moon rotated Day 27.3
Syndonic Month

space fm



4.5 Tangential and Radial Acceleration

If the speed of the particle changes with time, a tangential acceleration will result. The total acceleration of the particle is now

$$\vec{a} = \vec{a}_r + \vec{a}_t$$

$$\text{with } a_r = -a_c = -\frac{v^2}{r} \text{ and } a_t = \frac{d|\vec{v}|}{dt} \quad \Rightarrow \quad \vec{a} = -a_c \hat{r} + a_t \hat{\theta}$$

with $\hat{r}$ and $\hat{\theta}$: are unit vectors directed away from the center and counterclockwise, respectively.

The negative sign indicates that the radial acceleration is toward the center.

• Quick Quiz (5)

For a particle moves along a circular path, In which of the following cases are its acceleration and velocity vectors parallel?

- a) When the speed is constant.
- b) When the speed changes
- c) Always
- d) Never

Example 4.6 A car leaves a stop sign and exhibits a constant acceleration of 0.300 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like an arc of a circle of radius 500 m . At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s . What are the magnitude and direction of the total acceleration vector for the car at this instant?

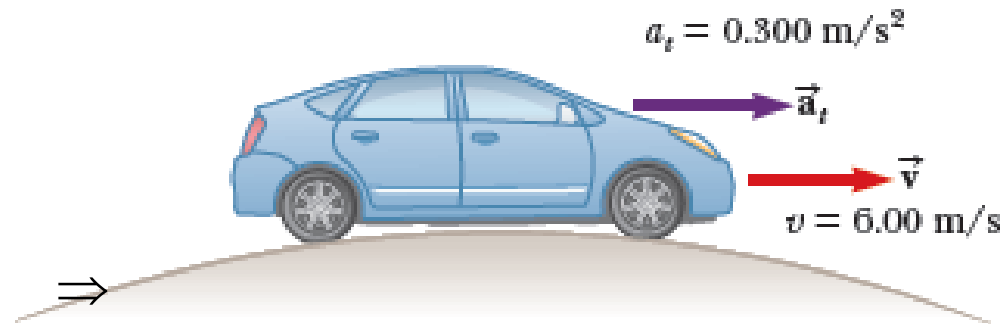
Solution a) The radial acceleration can be found as

$$a_r = -\frac{v^2}{r} = \frac{(6.00)^2}{500} = 0.072 \text{ m/s}^2$$

$$\vec{a} = (-0.072\hat{r} + 0.300\hat{\theta})\text{m/s}^2 \quad \Rightarrow$$

$$|\vec{a}| = \sqrt{a_r^2 + a_t^2} = \sqrt{(-0.072)^2 + (0.300)^2} = 0.309 \text{ m/s}^2$$

$$\phi = \tan^{-1}\left(\frac{a_r}{a_t}\right) = \tan^{-1}\left(\frac{-0.072}{0.300}\right) = -13.5^\circ$$



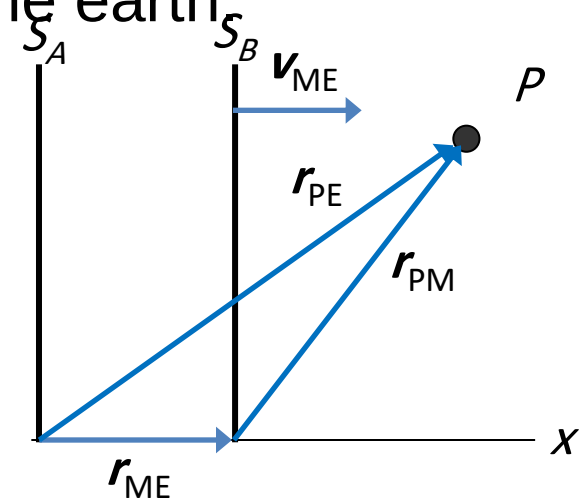
4.6 Relative Motion

Suppose that two observers want to describe the motion of a particle P . The first observer A in a frame S_A is stationary with respect to the earth, while the second observer B in frame S_B is moving with constant velocity $\mathbf{v}_{BE}$ relative to the earth.

If the two reference frames coincide at $t=0$, then after a time t , the displacement of B relative to A will be $\mathbf{r}_{BE} = \mathbf{v}_{BE} t$

The displacement of the particle w.r.t. to the earth (A) is denoted by $\mathbf{r}_{PE}$,

the displacement of the particle w.r.t. B is denoted by $\mathbf{r}_{PB}$



From the figure we can write $\vec{\mathbf{r}}_{PE} = \vec{\mathbf{r}}_{PB} + \vec{\mathbf{r}}_{BE} = \vec{\mathbf{r}}_{PM} + \vec{\mathbf{v}}_{BE}t$

Differentiate w.r.t. t we get $\vec{\mathbf{v}}_{PE} = \vec{\mathbf{v}}_{PB} + \vec{\mathbf{v}}_{BE}$

Differentiate again w.r.t. t we get $\vec{\mathbf{a}}_{PE} = \vec{\mathbf{a}}_{PM}$

Example 4.8 A boat crossing a wide river moves with a speed of 10.0 km/h relative to the water. The water in the river has a uniform speed of 5.00 km/h due east relative to the Earth.

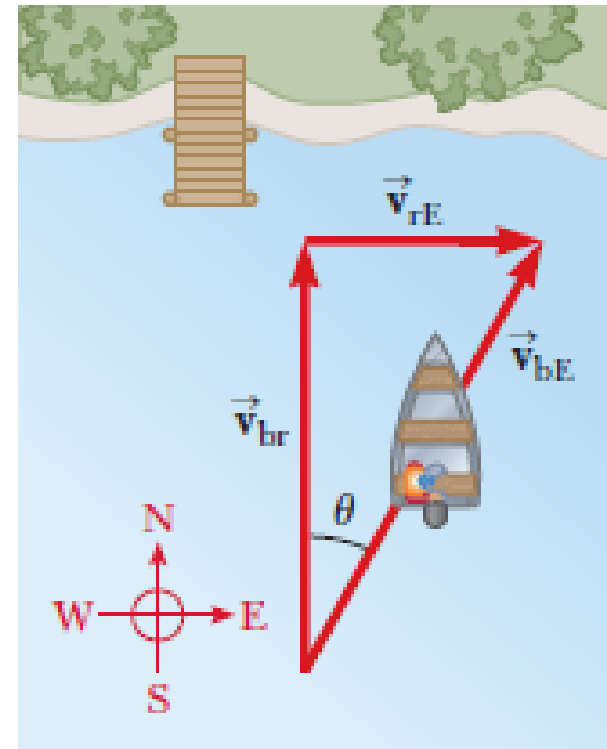
(a) If the boat heads due north, determine the velocity of the boat relative to an observer standing on either bank.

Solution Let $\mathbf{v}_{WE}$, $\mathbf{v}_{BE}$, and $\mathbf{v}_{BW}$ be the velocity of the water relative to the earth, the velocity of the boat relative to the earth, and the velocity of the boat relative to the water, respectively.

$$\mathbf{v}_{BE} = \mathbf{v}_{BW} + \mathbf{v}_{WE} = (5.00\hat{i} + 10.0\hat{j})\text{km/hr}$$

$$|\mathbf{v}_{BE}| = \sqrt{(5.00)^2 + (10.0)^2} = 11.2\text{ km/hr}$$

$$\theta = \tan^{-1}\left(\frac{v_x}{v_y}\right) = \tan^{-1}\left(\frac{5.00}{10.0}\right) = 26.6^\circ$$



So the boat is moving at a speed of 11.2 km/h in the direction 26.6° east of north relative to the Earth.

(b) If the boat travels with the same speed of 10.0 km/h relative to the river and is to travel due north as shown, what should its heading be?

Solution For the boat to travel due north means that

$$\vec{v}_{BE} = v_{BE} \hat{j}$$

$$\text{Now } \vec{v}_{BE} = \vec{v}_{BW} + \vec{v}_{WE} \quad \Rightarrow$$

$$\text{Now } \vec{v}_{BW} = \vec{v}_{BE} - \vec{v}_{WE} = v_{BE} \hat{j} - 5.00 \hat{i} \quad \Rightarrow$$

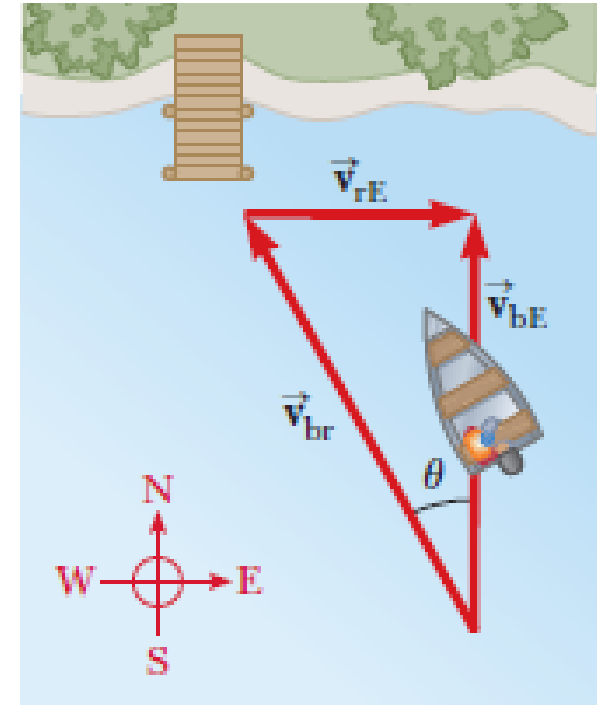
$$\text{But } |\vec{v}_{BW}| = 10.0 \text{ km/hr} \quad \Rightarrow$$

$$10.0 = \sqrt{(v_{BE})^2 + (-5.00)^2} \Rightarrow$$

$$v_{BE} = \sqrt{75} = 8.66 \text{ km/hr} \quad \Rightarrow$$

$$\text{Now } \vec{v}_{BW} = (8.66 \hat{j} - 5.00 \hat{i}) \text{ km/hr}$$

$$\theta = \tan^{-1} \left(\frac{v_x}{v_y} \right) = \tan^{-1} \left(\frac{5.00}{8.66} \right) = 30.0^\circ$$



• Quick Quiz (6)

A boy, driving a bike with constant speed, throws a ball vertically upward relative to himself. The ball falls:

- a) on the bike.
- b) in front of the bike.
- c) behind the bike.
- d) can't be known.

• Quick Quiz (7)

Suppose you are running at constant velocity and you wish to throw a ball such that you will catch it as it comes back down. In what direction should you throw the ball relative to you?

- a) in the forward direction.
- b) in the backward direction.
- c) Straight up.
- d) At some angle with the ground.