

CHAPTER 9

LINEAR MOMENTUM AND COLLISIONS

In this chapter we are going to discuss the following topics:

- ❑ **Linear Momentum**
- ❑ **Conservation of Linear Momentum**
- ❑ **Impulse**
- ❑ **Collision (in 1-D & 2-D)**
- ❑ **Center of Mass**
- ❑ **Systems of Particles**

Recommended Problems:

5, 7, 11, 22, 23, 25, 30, 31, 33, 35, 37, 41, 45, 49, 55, 67, 68, 69, 70, 71, 73, 75, 77, 80, 81, 89

9.1 Linear Momentum

The linear momentum $\vec{p}$ of a particle of mass m moving with velocity $\vec{v}$ is defined as

$$\vec{p} = m\vec{v}$$

It is clear that the momentum is a vector quantity with it SI unit of kg.m/s.

From Newton's second law, and assuming m to be constant, we have

$$\sum \vec{F} = m\vec{a} = m \frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} \Rightarrow$$

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

General form of Newton's second law.

Note that the kinetic energy is expressed in terms of momentum as.

$$K = \frac{1}{2}mv^2 = \frac{1}{2} \frac{m^2v^2}{m} = \frac{p^2}{2m}$$

• Quick Quiz (1)

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{p^2}{2m}$$

Two objects of m_1 and m_2 have equal kinetic energies. How do the magnitudes of their momenta compare?

a) $p_1 = p_2$

b) $p_1 > p_2$

c) $p_1 < p_2$

d) No enough information.

• Quick Quiz (2)

Your physical education teacher throws a baseball to you at a certain speed and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. Rank the following choices from easiest to hardest to catch. The medicine ball should be thrown with

a) the same speed as the baseball,

b) the same momentum as the baseball,

c) the same kinetic energy as the baseball,

b,
then c,
and then a

9.2 Linear Momentum of an Isolated System

For isolated system the net external force is zero so we have

$$\sum \vec{F} = \frac{d\vec{p}}{dt} = 0$$

This means that the total momentum of an isolated system must be constant.

For a system of two objects, the total momentum is

$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 = \text{constant}$$

$$\text{or } \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

In components form the last equation can be written as

$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx} \quad p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy}$$

If only one component of the net external force acting on a system is zero, then the component of the linear momentum of the system along that axis is constant only.

Remark: Avoid applying conservation of momentum to a single particle.

Example 9.1 A 60-kg archer stands at rest on frictionless ice and fires a 0.030-kg arrow horizontally at 85 m/s as shown. With what velocity does the archer move across the ice after firing the arrow?

Solution Our system here is the archer (including the bow) and the arrow. The system is isolated since the force of gravity and the normal force cancel each other. So we write

$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx} \quad \Rightarrow$$

$$0 + 0 = m_1 v_{1f} + m_2 v_{2f} \quad \Rightarrow v_{1f} = -\frac{m_2}{m_1} v_{2f}$$

$$\text{Or } v_{1f} = -\left(\frac{0.030}{60.0}\right)85 = -0.042 \text{ m/s}$$



9.3 Linear Momentum of a Non-isolated System and Impulse

For a nonisolated system Newton's 2nd law reads

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad (\sum \vec{F})dt = d\vec{p}$$

Integrating the last Eq. $\Rightarrow$

$$\int (\sum \vec{F})dt = \int d\vec{p} = \Delta\vec{p}$$

Defining the impulse of the force as

$$\vec{I} = \int (\sum \vec{F})dt$$

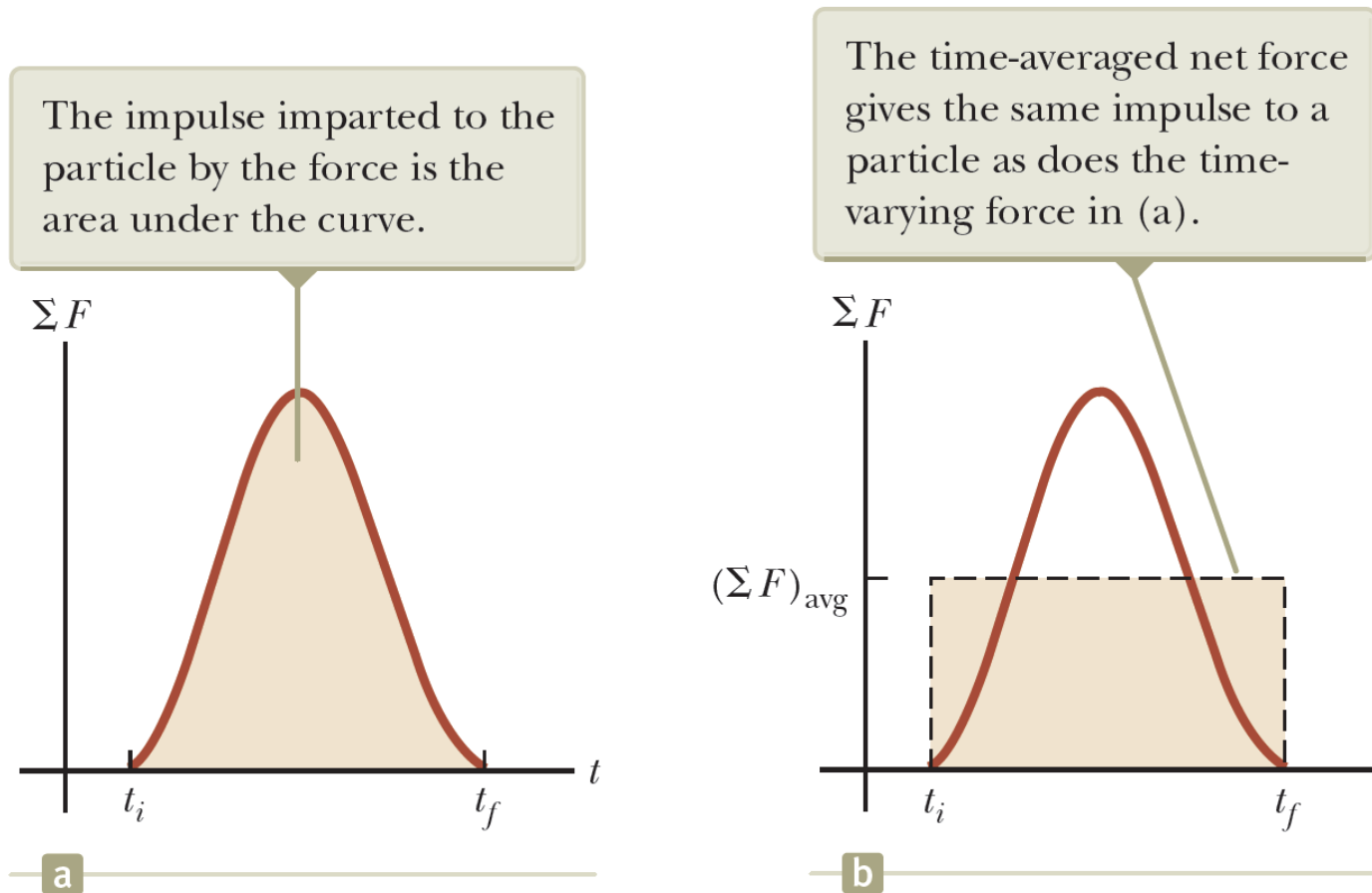
We get $\vec{I} = \Delta\vec{p} = \vec{F}\Delta t$

As the force varies with time, we can define the average force as

$$\boxed{\vec{I} = \Delta\vec{p}}$$

It is clear now that the impulse, like the linear momentum, is a vector quantity and with the same unit.

The impulse is defined as the area under the force-time curve.



Remarks:

- 1- Impulsive forces are external forces acting on the system.
- 2- For isolated systems there is no impulses

• Quick Quiz (3)

A piece of clay and a ball of the same mass are thrown at a wall with the same velocity. The clay sticks to the wall and the ball bounces off. Which one experiences the larger impulse?

- a) The ball.
- b) The clay.
- c) Both have the same impulse.
- d) Cannot be determined.

Example 9.3 A car of mass 1500 kg crashed into a wall as shown. The initial velocity is $v_i = 15.0$ m/s to the left, and the final velocity $v_f = 2.6$ m/s to the right.

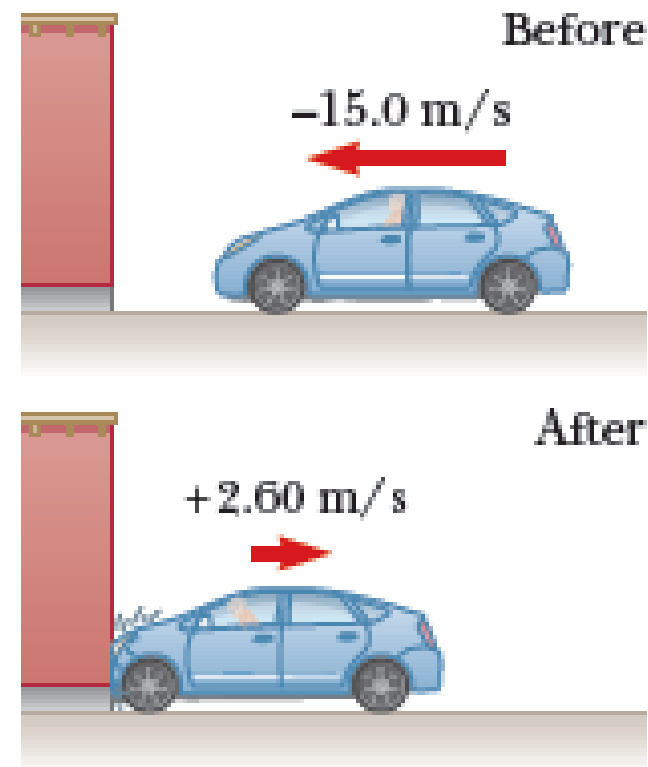
- a) Find the impulse due to the collision.
b) If the collision lasts 0.150 s, find the average force exerted on the car.

Solution a) Let us take the positive sense to be to the right. The impulse I is

$$\vec{I} = \Delta\vec{p} = m(\vec{v}_f - \vec{v}_i) = 1500(2.6\hat{i} - (-15.0\hat{i})) = 2.64 \times 10^4 \hat{i} \text{ kg.m/s}$$

b) Now we have,

$$\vec{F} = \frac{\vec{I}}{\Delta t} = \frac{2.64 \times 10^4 \hat{i}}{0.150} = 1.76 \times 10^5 \hat{i} \text{ N}$$



• Quick Quiz (4)

For a car made a collision which of the following exerts the maximum impulse on the passenger?

- a) The dashboard.
- b) The seat belt.
- c) The air bag.
- d) All exert the same impulse.

• Quick Quiz (5)

Referring to the previous quiz, which deliver the minimum force

- a) The dashboard.
- b) The seat belt.
- c) The air bag.
- d) All exert the same force.

9.4 Collision in One Dimension

The collision process represents the event of two particles coming close to each other for a short time, producing impulse forces on each other.

Considering the collided particles as one system, the forces produced by the collision is internal forces and we have an isolated system. Therefore, the total momentum of the system just before the collision equals the total momentum of the system just after the collision. That is

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

For collision in one dimension we have only one component for velocities and the conservation of linear momentum becomes

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Remark: In applying the above equation we have to take into account the signs of the speeds. v is positive if the body moves to the right, and is negative if the body moves to the left.

According to the conservation of kinetic energy, two types of collision will be considered:

Elastic collision

the kinetic energy of the system is conserved

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{v}_f$$

Inelastic collision

the kinetic energy of the system is not-conserved

Perfectly Inelastic collision

The two collided particles stick and move together after collision

Let us consider an elastic collision between two particles. One particle of mass m_1 moving with speed v_{1i} and the other particle of mass m_2 moving with speed $v_{2i} \Rightarrow$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \Rightarrow$$

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i}) \quad (1)$$

Because the collision is elastic we can write

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad \Rightarrow$$

$$m_1 (v_{1i}^2 - v_{1f}^2) = m_2 (v_{2f}^2 - v_{2i}^2) \quad \Rightarrow$$

$$m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2 (v_{2f} - v_{2i})(v_{2f} + v_{2i}) \quad (2)$$

Dividing Equation (2) by equation (1) yield

$$(v_{1i} + v_{1f}) = (v_{2f} + v_{2i}) \quad \Rightarrow$$

$$(v_{12})_i = (v_{12})_f \quad (3)$$

From equation (3) we conclude that the relative velocity of the two particles before collision, $(v_{12})_i$ equals the negative of the relative velocity of the two particles after collision, $(v_{12})_f$.

Now solving Equation (1) and Equation (3), for v_{1f} and v_{2f} we get

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i}$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) v_{2i} \quad v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i}$$

Let us consider few special cases concerning the last two equations:

- (i) If $m_1 = m_2$ we obtain $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, if the two masses are equal the particles exchange their velocities.
- (ii) If m_2 is initially at rest and is very light compared with m_1 we get $v_{1f} \approx v_{1i}$ and $v_{2f} = 2v_{1i}$. That is, when a very heavy particle collides with a light particle, initially at rest, the heavy particle continues its motion unchanged after collision, while the lighter particle moves with a speed equal to twice the initial speed of the heavy particle.
- (iii) If m_2 is initially at rest and is very heavy compared with m_1 we get $v_{1f} \approx -v_{1i}$ and $v_{2f} \ll v_{1i}$. That is, when a very light particle collides with a very heavy particle, initially at rest, the velocity of the light particle will be reversed after collision, while the heavy particle will remain approximately at rest.

Example 9.5 An 1800-kg car stopped at a traffic light is struck from the rear by a 900-kg car. The two cars become entangled, moving along the same path as that of the originally moving car. If the smaller car were moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution Since the two cars stick together we have a perfectly inelastic collision so we write with the heavy car m_1 initially at rest

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \quad \Rightarrow$$

$$1800(0) + 900(20.0) = (1800 + 900) v_f \quad \Rightarrow$$

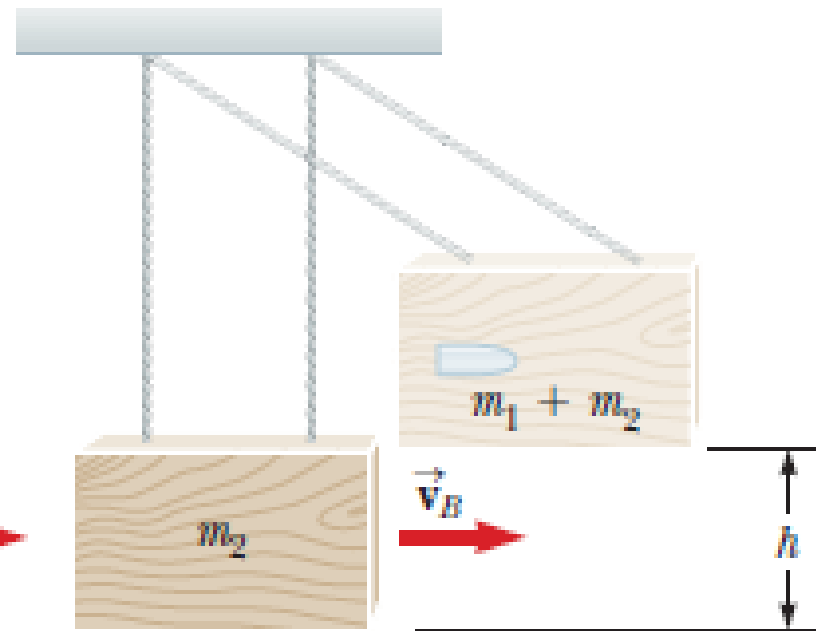
$$v_f = \frac{900(20.0)}{2700} = 6.67 \text{ m/s}$$

Example 9.6 The ballistic pendulum (shown) is an apparatus used to measure the speed of a fast-moving projectile such as a bullet. A projectile of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The projectile embeds in the block, and the entire system swings through a height h . Find the speed of the projectile in terms of h .

Solution Again we are dealing with a perfectly inelastic collision with $v_{2i}=0$, so we have

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \Rightarrow$$

$$v_{1i} = \frac{(m_1 + m_2)}{m_1} v_B$$



To find v_B we have to apply the conservation of mechanical energy principle, $E_i = E_f$, with the initial point is the point just after impact and the final point is the maximum height of the combination.

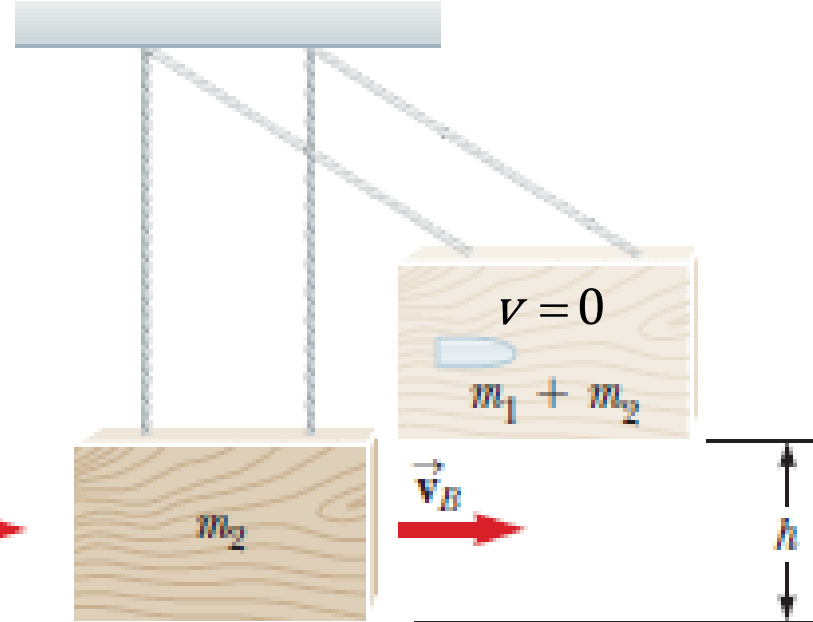
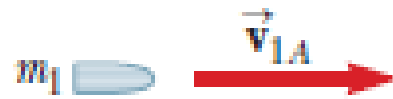
Taking the point of impact as the level of zero potential energy, and knowing that the speed at the maximum height is zero we get

$$E_i = \frac{1}{2}(m_1 + m_2)v_b^2$$

$$E_f = (m_1 + m_2)gh$$

$$\Rightarrow \frac{1}{2}(m_1 + m_2)v_B^2 = (m_1 + m_2)gh \quad \Rightarrow$$

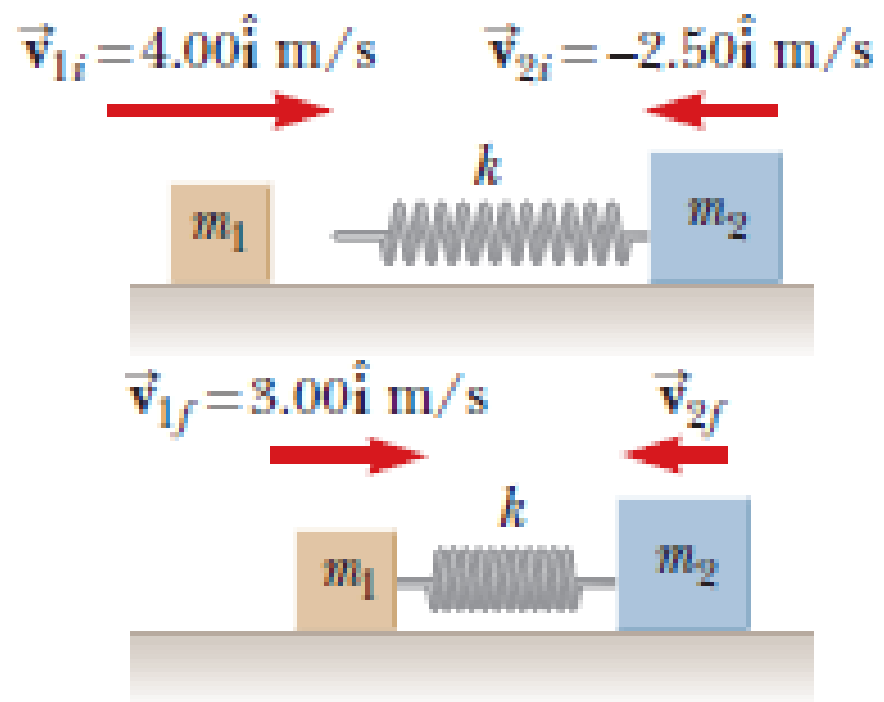
$$\Rightarrow v_B = \sqrt{2gh}$$



Substituting for v_B into the first equation to get

$$v_{1i} = \frac{(m_1 + m_2)}{m_1} \sqrt{2gh}$$

Example 9.7 A block of mass $m_1=1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless, horizontal track collides with a light spring attached to a second block of mass $m_2=2.10$ kg initially moving to the left with a speed of 2.50 m/s as shown. The spring constant is 600 N/m.



- Find the velocities of the two blocks after the collision.
- Determine the velocity of block 2 at the instant block 1 is moving to the right at 3.00 m/s.
- Determine the distance the spring is compressed at that instant.

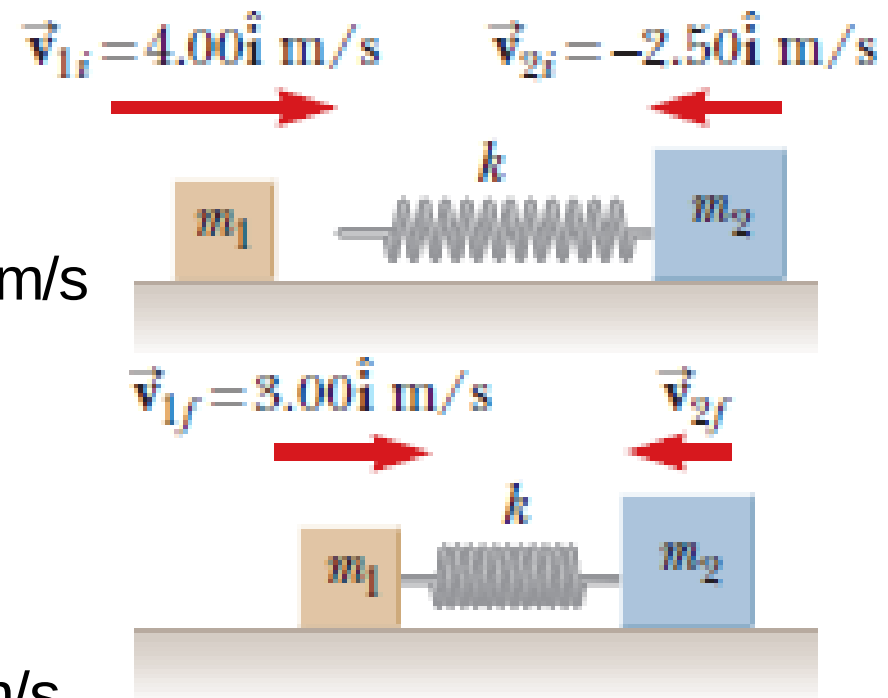
Solution a) After collision the two blocks will separate with no elastic potential energy stored in the system so the kinetic energy will be conserved and so the collision is elastic. So we have

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) v_{2i}$$

$$= \left(\frac{-0.50}{3.70} \right) 4.00 + \left(\frac{4.20}{3.70} \right) (-2.50) = -3.38 \text{ m/s}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i}$$

$$= \left(\frac{3.20}{3.70} \right) 4.00 + \left(\frac{0.5}{3.70} \right) (-2.50) = 3.12 \text{ m/s}$$



b) Applying the conservation of momentum principle we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \Rightarrow$$

$$1.60(4.00) + 2.10(-2.50) = 1.60(3.00) + 2.10v_{2f} \Rightarrow$$

$$v_{2f} = -1.74 \text{ m/s}$$

c) Since the system is isolated we apply the conservation of mechanical energy , $E_i = E_f$, to get

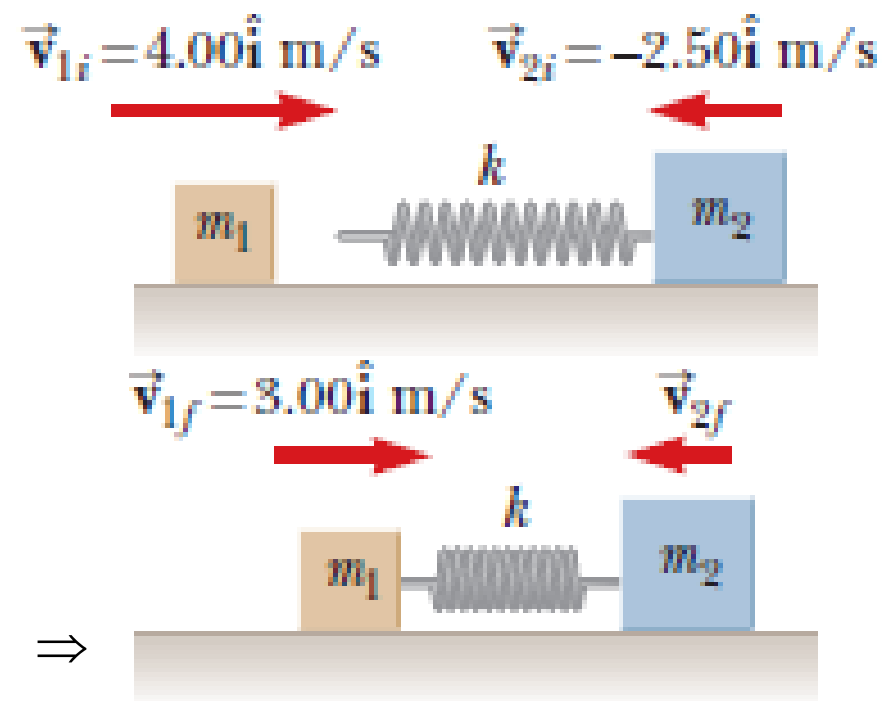
$$E_i = \left(\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 \right)$$

$$E_f = \left(\frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \right) + \frac{1}{2} k x^2$$

$$(m_1 v_{1i}^2 + m_2 v_{2i}^2) = (m_1 v_{1f}^2 + m_2 v_{2f}^2) + k x^2$$

$$x^2 = \frac{1}{k} \left[m_1 (v_{1i}^2 - v_{1f}^2) + m_2 (v_{2i}^2 - v_{2f}^2) \right] \Rightarrow$$

$$x^2 = \frac{1}{600} [1.6(16 - 9) + 2.1(6.25 - 3.0)] \Rightarrow x = 0.173 \text{ m}$$



9.5 Collisions in Two Dimensions

If the collided particles move in two dimensions either before, or after collision, the equation of conservation of linear momentum is now resolved into two rectangular components. That is

$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx}$$

$$p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy}$$

For elastic collision, and since the kinetic energy is scalar, the equation of conservation of kinetic energy is not changed for two dimensional collision, but now the speed is

$$v^2 = v_x^2 + v_y^2$$

Example 9.8 A 1500-kg car traveling east with 25.0 m/s collides at an intersection with a 2500-kg truck traveling north at 20.0 m/s as shown. Find the magnitude and direction of the velocity of the wreckage after collision, if both vehicles stick together after collision.

Solution Conservation of momentum in x -direction, gives

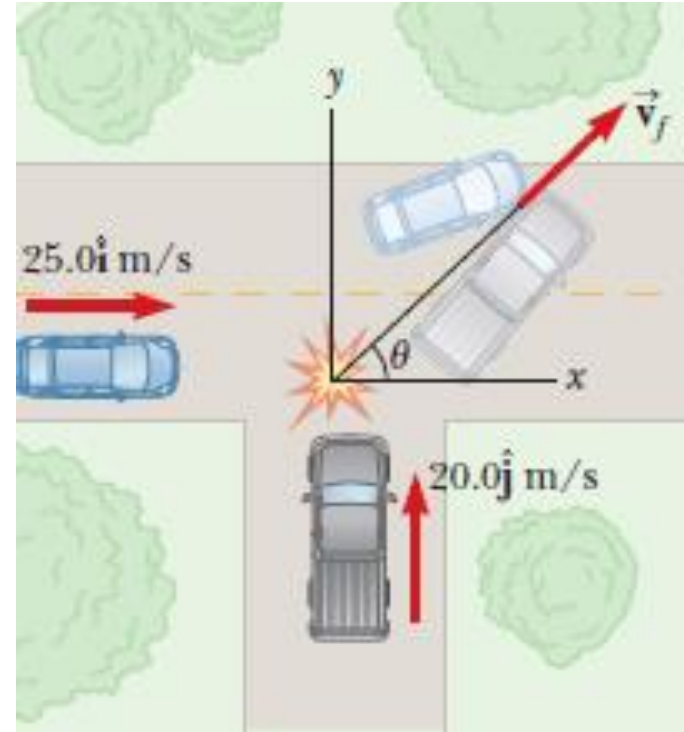
$$p_{1ix} + p_{2ix} = p_{1fx} + p_{2fx} \quad \Rightarrow$$

$$m_1 v_{1i} + m_2(0) = (m_1 + m_2) v_f \cos \theta \quad \Rightarrow$$

$$1500 \times 25.0 = (1500 + 2500) v_f \cos \theta \quad \Rightarrow \quad 37500 = 4000 v_f \cos \theta \quad (1)$$

And in y -direction, gives $p_{1iy} + p_{2iy} = p_{1fy} + p_{2fy} \quad \Rightarrow$

$$m_1(0) + m_2 v_{2i} = (m_1 + m_2) v_f \sin \theta \quad \Rightarrow$$



$$2500 \times 20.0 = (1500 + 2500)v_f \sin \theta \Rightarrow$$

$$5000 = 4000v_f \sin \theta \quad (2)$$

$$37500 = 4000v_f \cos \theta \quad (1)$$

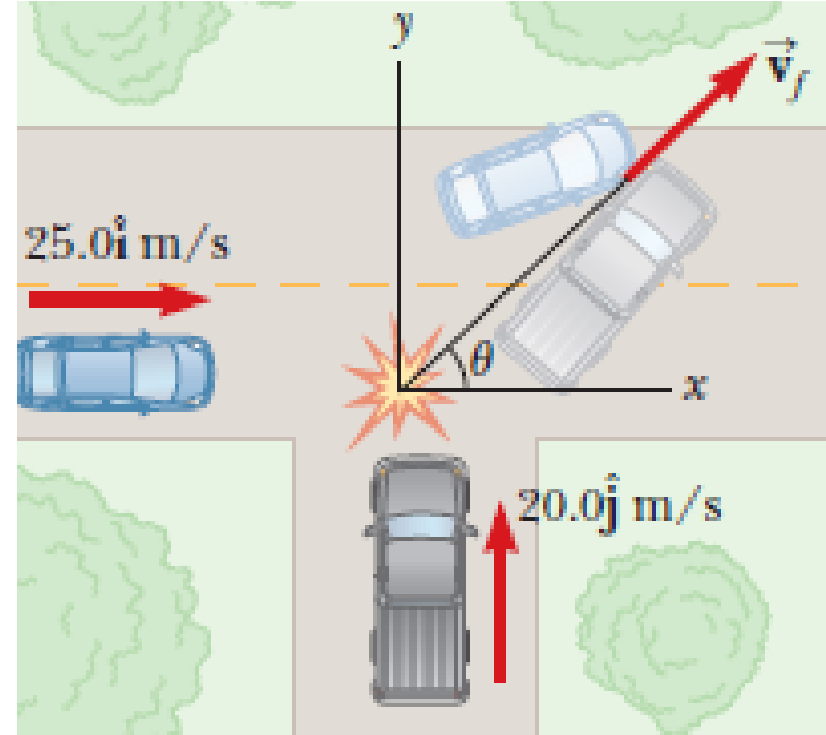
Dividing Eq. (2) by Eq.(1) $\Rightarrow$

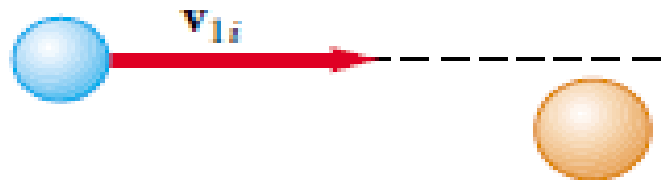
$$\frac{50000}{37500} = \tan \theta \Rightarrow$$

$$\theta = \tan^{-1}(1.33) = 53.1^\circ$$

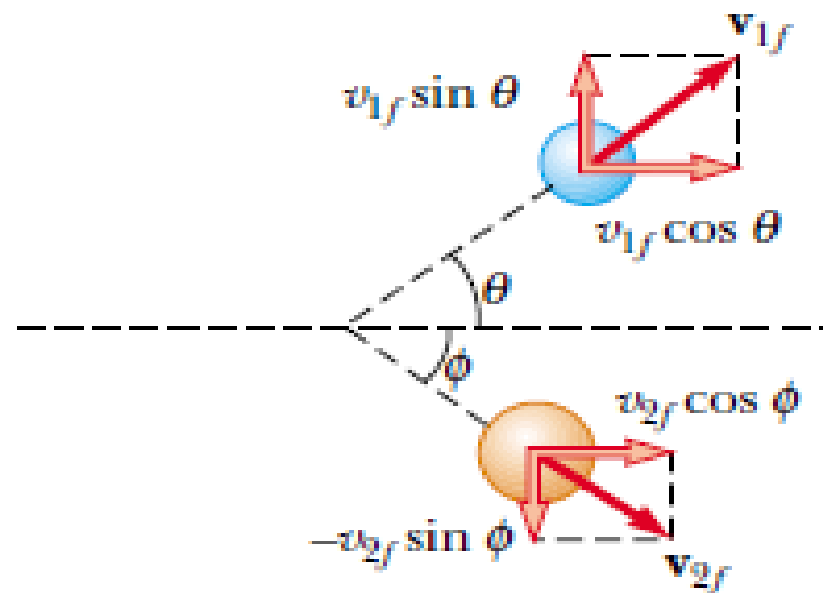
Substituting in Eq. (2) $\Rightarrow$

$$v_f = \frac{50}{4 \sin 53.1^\circ} = 15.6 \text{ m/s}$$



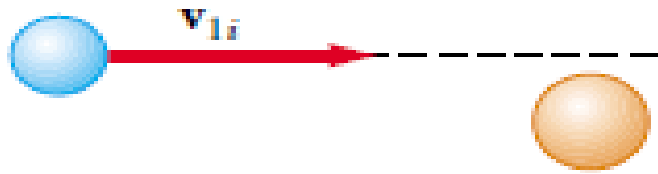


(a) Before the collision

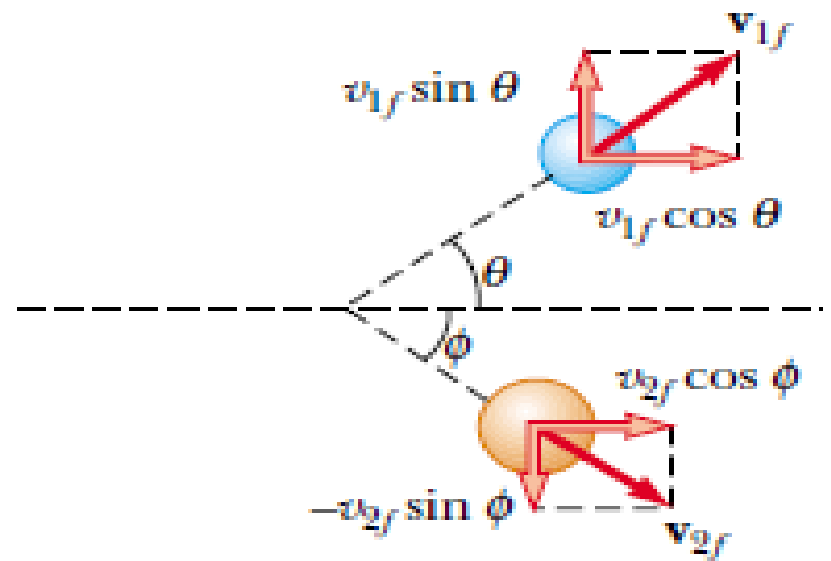


(b) After the collision

Example 9.9 A proton collides elastically with another proton that is initially at rest. The incoming proton has an initial speed $v_{1i} = 3.5 \times 10^5$ m/s. After collision one proton moves off at an angle $\theta = 37^\circ$ to the original direction of motion and the second deflects at an angle ϕ to the same axis, as shown. Find the final speeds of the two protons and the angle ϕ .



(a) Before the collision



(b) After the collision

Solution Conservation of momentum in x -direction, gives

$$mv_{1i} = mv_{1f} \cos \theta + mv_{2f} \cos \phi \quad \Rightarrow$$

$$v_{1i} = v_{1f} \cos \theta + v_{2f} \cos \phi \quad (1)$$

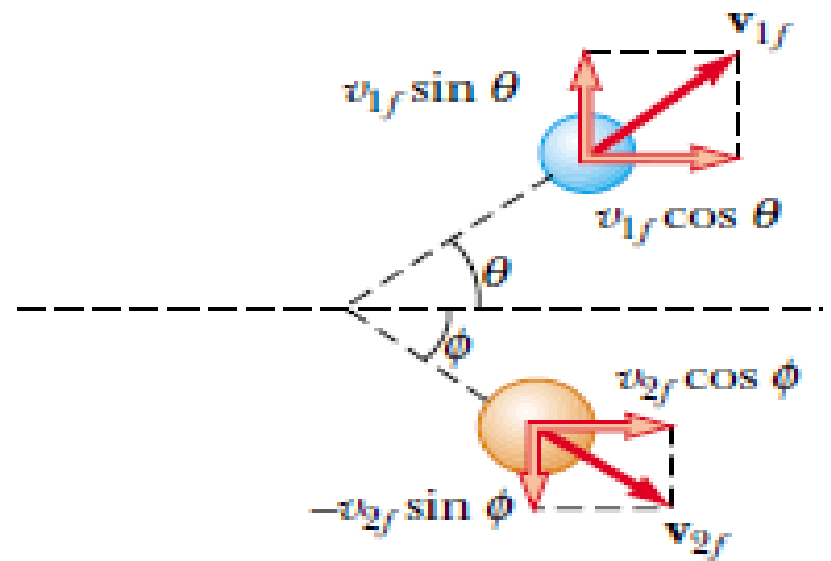
And in y -direction, gives $0 = mv_{1f} \sin \theta - mv_{2f} \sin \phi \quad \Rightarrow$

$$0 = v_{1f} \sin \theta - v_{2f} \sin \phi \quad (2)$$

The conservation of kinetic energy gives



(a) Before the collision



(b) After the collision

$$\begin{aligned}\frac{1}{2} m v_{1i}^2 &= \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2 \Rightarrow \\ v_{1i}^2 &= v_{1f}^2 + v_{2f}^2\end{aligned}\quad (3)$$

Squaring the two sides of equations (1) and (2) gives, respectively

$$v_{1i}^2 = v_{1f}^2 \cos^2 \theta + v_{2f}^2 \cos^2 \phi + 2v_{1f} v_{2f} \cos \theta \cos \phi \quad (4)$$

$$0 = v_{1f}^2 \sin^2 \theta + v_{2f}^2 \sin^2 \phi - 2v_{1f} v_{2f} \sin \theta \sin \phi \quad (5)$$

Now adding equations (4) and (5) gives $v_{1i}^2 = v_{1f}^2 + v_{2f}^2$ (3)

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}[\cos(\theta + \phi)] \quad (6)$$

Subtracting Equation (3) from Equation (6) gives

$$0 = 2v_{1f}v_{2f}[\cos(\theta + \phi)] \Rightarrow$$

$$\cos(\theta + \phi) = 0 \Rightarrow$$

$$\theta + \phi = 90.0^\circ \Rightarrow \phi = 53.0^\circ$$

Substituting the values of θ , ϕ , and v_{1i} into Equations (1) and (2) and solving the two equations for v_{1f} and v_{2f} we obtain

$$v_{1f} = 2.80 \times 10^5 \text{ m/s}$$

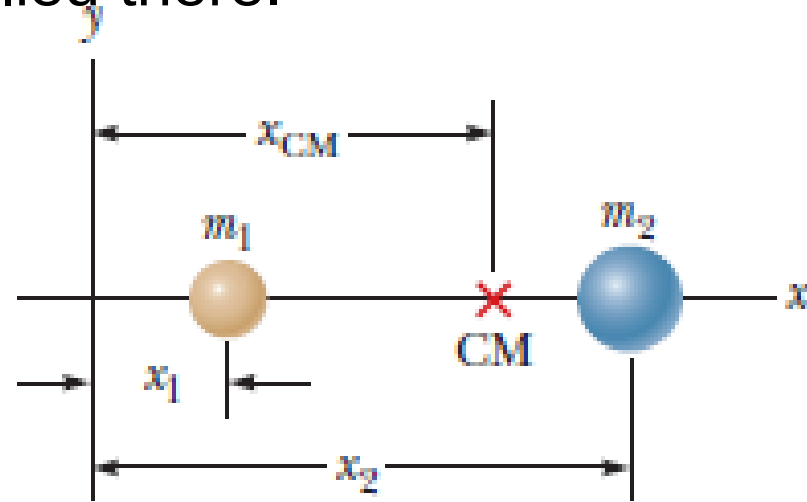
$$v_{2f} = 2.11 \times 10^5 \text{ m/s}$$

9.6 Center of Mass

The center of mass of a system of particles or a rigid body is the **point** at which all of the mass are considered to be concentrated there and all external forces were applied there.

Consider a system of two masses m_1 and m_2 located along the x -axis as shown. The position x_{CM} of the center of mass of these two masses is defined to be

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



It is clear that the center of mass is closer to the mass having the larger mass.

For a system of n -particles $m_1, m_2, \dots, m_n$, the center of mass is

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n}$$

$$\text{or } x_{CM} = \frac{1}{M} \sum_{i=1}^n m_i x_i$$

Where $M = m_1 + m_1 + \dots + m_n$ is the total mas of the system.

If the system is distributed on three dimensions, the y and the z coordinates of the center of mass are similarly defined by

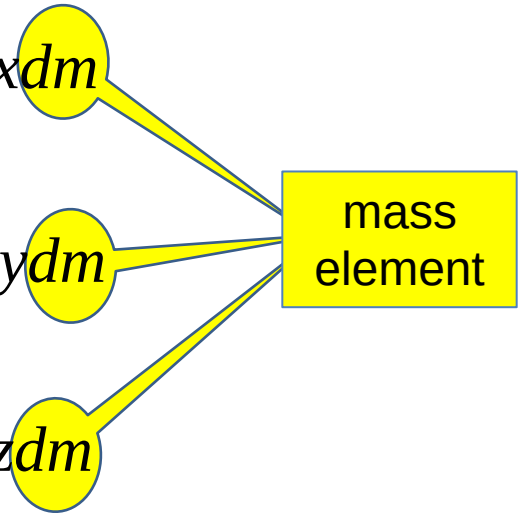
$$y_{CM} = \frac{1}{M} \sum_{i=1}^n m_i y_i$$

$$z_{CM} = \frac{1}{M} \sum_{i=1}^n m_i z_i$$

In vector notation, the position vector of the center of mass $\mathbf{r}_{cm}$ can be expressed as

$$\vec{\mathbf{r}}_{CM} = x_{CM} \hat{\mathbf{i}} + y_{CM} \hat{\mathbf{j}} + z_{CM} \hat{\mathbf{k}} = \frac{1}{M} \sum_{i=1}^n m_i \vec{\mathbf{r}}_i$$

For a rigid body (continuous mass distribution) we treat the body as consisting of so large number of small elements each of mass dm such that the sums become integrals $\Rightarrow$

$$\begin{aligned}
 x_{CM} &= \frac{1}{M} \int x dm \\
 y_{CM} &= \frac{1}{M} \int y dm \\
 z_{CM} &= \frac{1}{M} \int z dm
 \end{aligned}$$


With x, y , and z are the coordinates of the element dm . For the vector position of the center of mass we write

$$\vec{r}_{cm} = \frac{1}{M} \int \vec{r} dm$$

Center of Gravity: It is a point in which all of the weight can be considered as being acted at that point. If $\mathbf{g}$ is constant over the mass distribution, the center of gravity coincides with the center of mass.

• Quick Quiz (6)

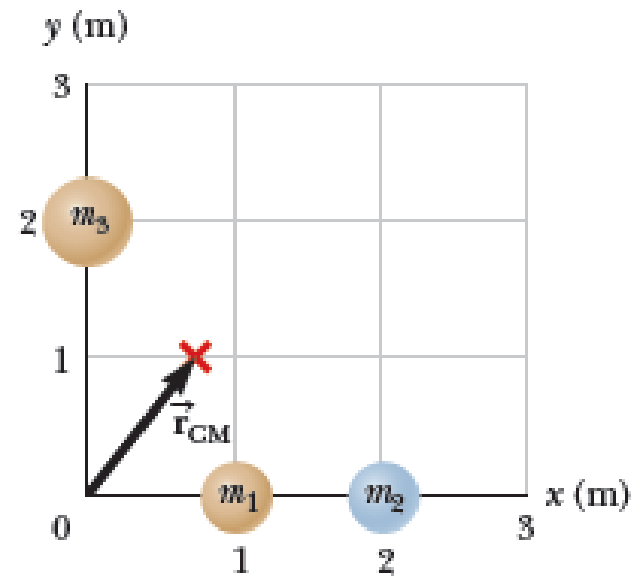
A baseball bat is cut at the location of its center of mass as shown. The piece with the smaller mass is:



- a) The piece on the left.
- b) The piece on the right.
- c) Both have the same mass.
- d) Cannot be determined.

Example 9.10 A system consists of three particles located as shown. Find the center of mass of the system. The masses of the particles are $m_1=m_2=1.0$ kg and $m_3=2.0$ kg.

Solution The x -component of the center of mass is



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{1.0 \times 1.0 + 1.0 \times 2.0 + 2.0 \times 0}{1.0 + 1.0 + 2.0} = \frac{3.0}{4.0} = 0.75 \text{ m}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{1.0 \times 0 + 1.0 \times 0 + 2.0 \times 2.0}{1.0 + 1.0 + 2.0} = \frac{4.0}{4.0} = 1.0 \text{ m}$$

$$\vec{r}_{CM} = (0.75\hat{i} + 1.0\hat{j})\text{m}$$

Example 9.11 Find the center of mass of a rod of mass M and length L .

- a) If its mass density λ is uniform,
- b) If its mass density is not uniform but varies with x as $\lambda = \alpha x$

Solution a) Let the rod be located along the x -axis such that $y_{CM} = z_{CM} = 0$.

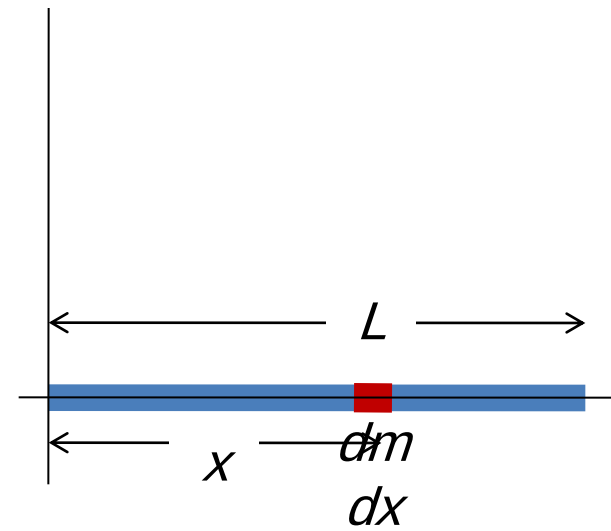
Let us take a small element of mass dm and length dx .

$$\text{Now } x_{CM} = \frac{1}{M} \int x dm$$

To find a relation between the mass dm and the variable x we define the linear mass density λ (mass per unit length), as

$$\lambda = \frac{dm}{dx} = \frac{M}{L} \Rightarrow dm = \lambda dx$$

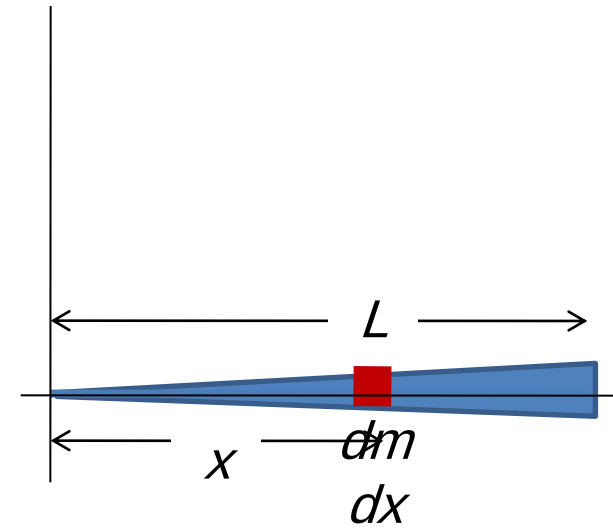
$$x_{CM} = \frac{\lambda}{M} \int_0^L x dx \Rightarrow x_{CM} = \frac{\lambda}{M} \left[\frac{x^2}{2} \right]_0^L = \frac{\lambda L^2}{2M} = \left(\frac{M}{L} \right) \frac{L^2}{2M} = \frac{L}{2}$$



b) For non-uniform mass density and from the relation given we conclude that the rod is more massive as x increases, so we have

$$x_{\text{CM}} = \frac{1}{M} \int_0^L \lambda x dx = \frac{\alpha}{M} \int_0^L x^2 dx \quad \Rightarrow$$

$$x_{\text{CM}} = \left. \frac{\alpha}{M} \frac{x^3}{3} \right]_0^L = \frac{\alpha L^3}{3M}$$



To find the value of α we use the fact that

$$M = \int dm = \int \lambda dx = \alpha \int_0^L x dx = \frac{1}{2} \alpha L^2$$

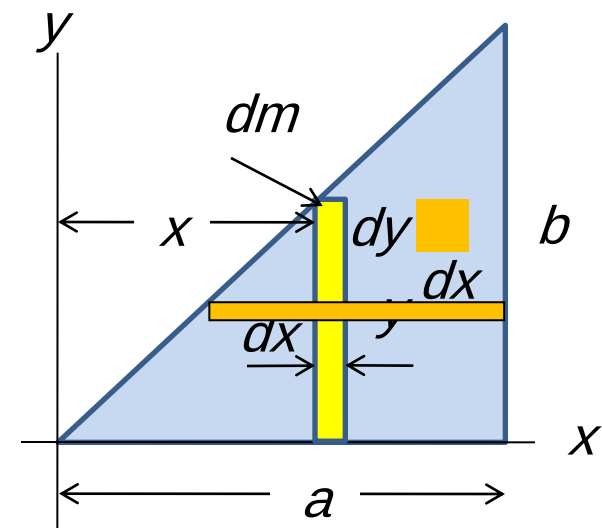
$$x_{\text{CM}} = \frac{L^3}{3M} \left(\frac{2M}{L^2} \right) = \frac{2}{3} L$$

Which is expected result since the center of mass must be shifted toward the massive part of the rod.

Example 9.12 Find the center of mass of a right triangle and lengths a and b as shown.

Solution It is clear now that the center of mass has two components. For the x -component we have

$$\text{Now } x_{CM} = \frac{1}{M} \int x dm$$



As an element we choose a vertical rectangle of length y and width dx , as shown with mass dm .

$$dm = \sigma dA = \sigma y dx \Rightarrow x_{CM} = \frac{\sigma}{M} \int_0^a y x dx$$

To find a relation between x and y we have $y/b = x/a \Rightarrow y = (b/a)x$

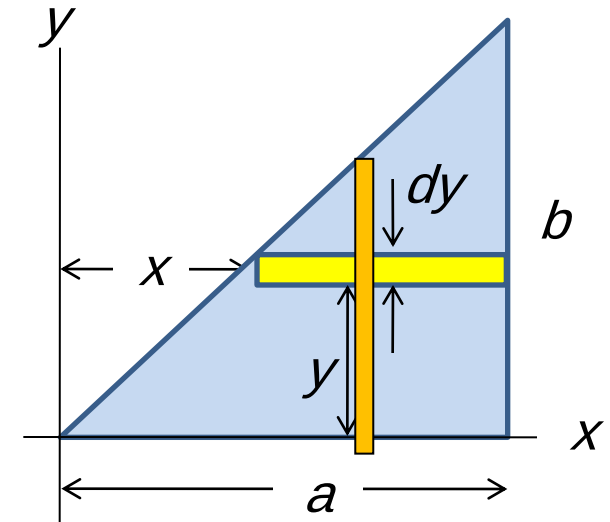
$$x_{CM} = \frac{b\sigma}{aM} \int_0^a x^2 dx \Rightarrow x_{CM} = \left. \frac{b\sigma}{aM} \frac{x^3}{3} \right|_0^a = \frac{b\sigma}{aM} \frac{a^3}{3} = \left(\frac{M}{\frac{1}{2}ab} \right) \frac{ba^2}{3M} = \frac{2}{3}a$$

For the y-component we choose as our element a horizontal rectangle as shown

$$\text{Now } y_{CM} = \frac{1}{M} \int y dm$$

$$dm = \sigma dA = \sigma(a - x) dy \quad \Rightarrow$$

$$y_{CM} = \frac{\sigma}{M} \int_0^b y(a - x) dy = \frac{a\sigma}{M} \int_0^b y dy - \frac{\sigma}{M} \int_0^b yx dy$$



$$\text{Now } (b - y)/b = (a - x)/a \Rightarrow x = (a/b)y$$

$$y_{CM} = \frac{a\sigma}{M} \int_0^b y dy - \frac{a\sigma}{bM} \int_0^b y^2 dy = \frac{a\sigma}{M} \left(\frac{1}{2} b^2 \right) - \frac{a\sigma}{bM} \left(\frac{1}{3} b^3 \right) = \frac{ab^2\sigma}{6M}$$

$$\text{But } \sigma = \frac{M}{\frac{1}{2}ab} \Rightarrow y_{CM} = \frac{1}{3}b$$

9.7 Systems of Many Particles

It is known that
$$\vec{\mathbf{r}}_{CM} = \frac{1}{M} \sum_{i=1}^n m_i \vec{\mathbf{r}}_i \Rightarrow M \vec{\mathbf{r}}_{CM} = \sum_{i=1}^n m_i \vec{\mathbf{r}}_i$$

Differentiating the last equation with respect to time gives

$$M \vec{\mathbf{v}}_{CM} = \sum_{i=1}^n m_i \vec{\mathbf{v}}_i = \sum_{i=1}^n \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_{tot}$$

Differentiating again with respect to time gives

$$M \vec{\mathbf{a}}_{CM} = \sum_{i=1}^n m_i \vec{\mathbf{a}}_i = \sum_{i=1}^n \vec{\mathbf{F}}_i$$

with $\vec{\mathbf{F}}_i$ is the vector sum of all the forces acting on the i^{th} particle. From Newton's 3rd law the internal forces form action-reaction pairs so that they cancel out. So, the sum of $\vec{\mathbf{F}}_i$ is the vector sum of all the external forces $\vec{\mathbf{F}}_{\text{ext}}$ that act on the system.

$$M \vec{\mathbf{a}}_{CM} = \sum \vec{\mathbf{F}}_{\text{ext}}$$

$$M\vec{a}_{CM} = \sum \vec{F}_{\text{ext}} = \frac{d\vec{p}_{\text{tot}}}{dt}$$

Like any vector equation, the last equation can be written as three equations corresponding to the components of $\mathbf{a}_{\text{cm}}$ and $\mathbf{F}_{\text{ext}}$ along the coordinate axes, that is,

$$\sum F_x^{\text{ext}} = Ma_x^{\text{cm}} = M \frac{dv_x^{\text{cm}}}{dt}$$

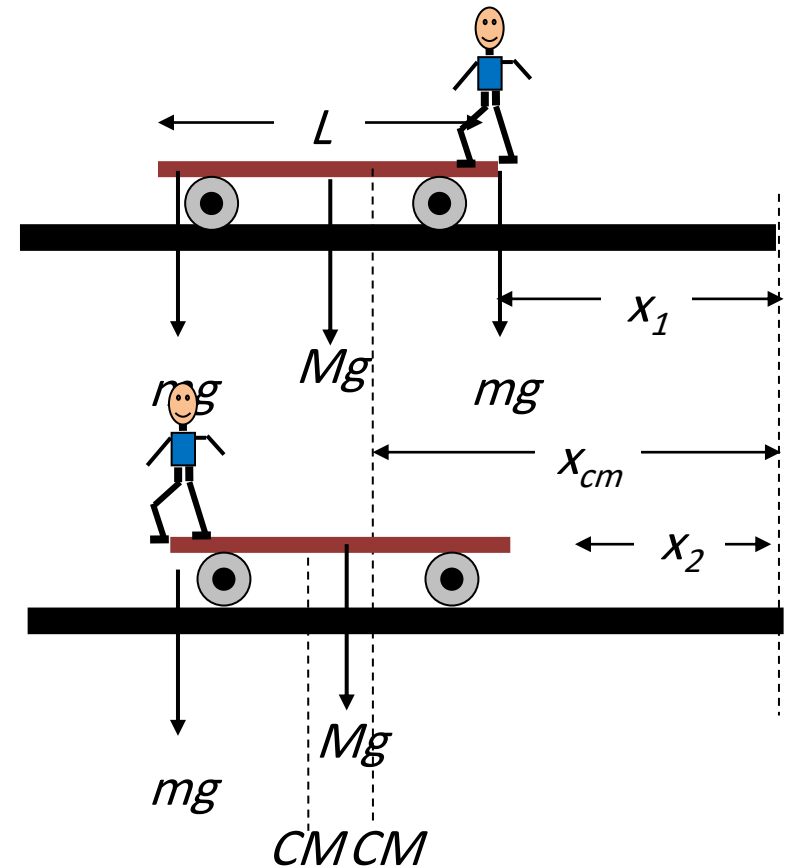
$$\sum F_y^{\text{ext}} = Ma_y^{\text{cm}} = M \frac{dv_y^{\text{cm}}}{dt}$$

$$\sum F_z^{\text{ext}} = Ma_z^{\text{cm}} = M \frac{dv_z^{\text{cm}}}{dt}$$

The last equations tell that if no net external force is acting on a system in one direction, the acceleration of its center of mass in that direction is zero and thus the velocity of the center of mass of the system remains unchanged.

For example consider the system of a man standing on a car that can move along a smooth horizontal track. If the system is initially at rest and the man starts to walk from one end toward the other end of the car, describe the motion of the car.

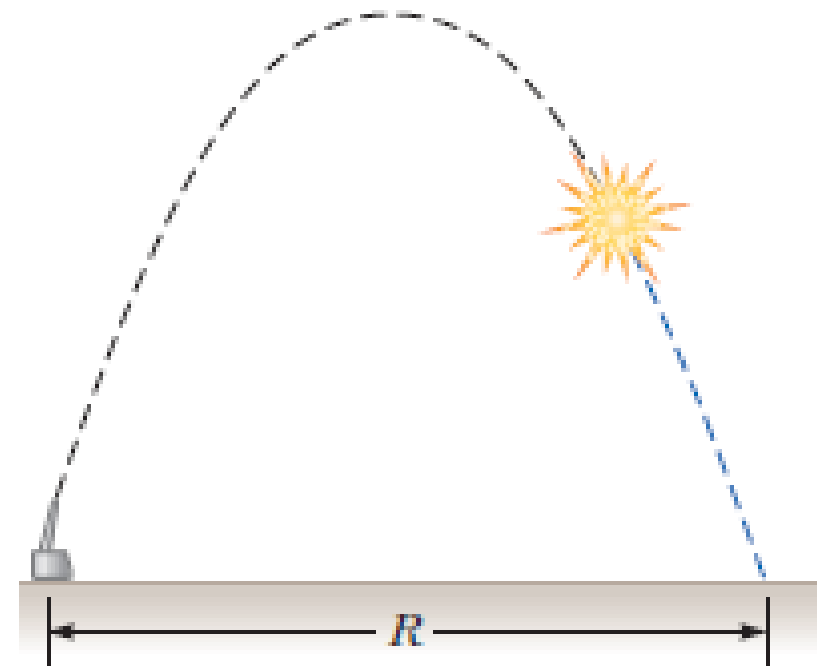
Here there is no external force acting on the man-car system along the horizontal direction. This means that the position of the center of mass of the man-car system must be un-changed. So the car has to move in the opposite direction to keep the position of the center of mass unchanged.



Example 9.13 A projectile fired into the air suddenly explodes into several fragments.

a) What can be said about the motion of the center of mass of the system made up of all the fragments after the explosion?

b) Suppose the projectile explodes and splits into two pieces of equal mass. One piece lands at a distance $2R$ to the right of the launch point. Where does the other piece land?



Solution a) As the forces due to the explosion are internal, they do not affect the motion of the center of mass. Since the only external force acting on the system is the force of gravity, the center of mass follows the same parabolic path the projectile would have followed if no explosion had occurred with horizontal range R indicated in the figure.

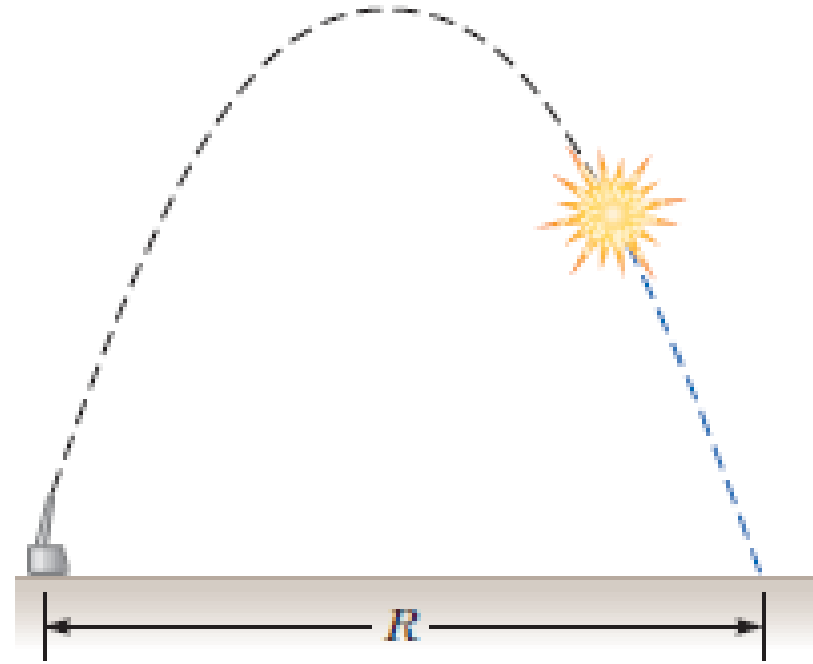
a) Since $x_{\text{CM}} = R$ and using

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{M}$$

with $x_1 = 2R$ and $m_1 = m_2 = \frac{1}{2}M \Rightarrow$

$$R = \frac{\frac{1}{2}M(2R + x_2)}{M} \Rightarrow$$

$$R = R + \frac{1}{2}x_2 \Rightarrow x_2 = 0$$



b) This means that the other piece must land at the launching point .

• Quick Quiz (7)

A cruise ship is moving at constant speed through the water. The vacationers decide to try to speed up the cruise ship by gathering at the front and running together toward the back of the ship. While they are running toward the stern, is the speed of the ship:

- a) higher than it was before,
- b) lower than it was before,
- c) unchanged, or
- d) Impossible to determine.

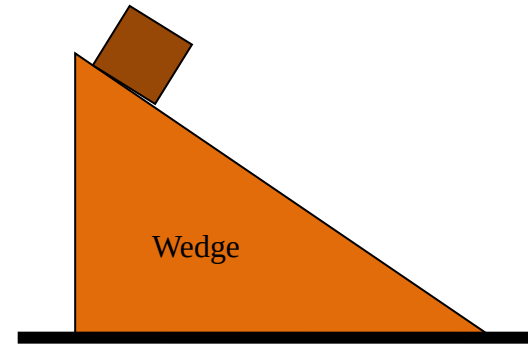
• Quick Quiz (8)

Referring to the previous question, The vacationers stop running when they reach the back of the ship. After they have all stopped running, is the speed of the ship:

- a) higher than it was before,
- b) lower than it was before,
- c) unchanged, or
- d) Impossible to determine.

• Quick Quiz (9)

A large wedge rests on a smooth horizontal surface. A block starts from rest at the top and slides down the inclined, rough surface of the wedge as shown in the figure. During the motion of the block, the center of mass of the block-wedge system:



- a) Does not move.
- b) Moves horizontally.
- c) Moves vertically.
- d) Non of them.

• Quick Quiz (10)

Referring to the previous question, as the block slides down the surface, the wedge will:

- a) Not move.
- b) Move to the left.
- c) Move to the right.
- d) Move up.

Example 9.14 A rocket is fired vertically upward. At the instant it reaches an altitude of 1000 m and a speed of $v_i=300$ m/s, it explodes into three fragments having equal masses. One fragment moves upward with a speed of $v_1=450$ m/s following the explosion. The second fragment has a speed of $v_2=240$ m/s and is moving east right after the explosion. What is the velocity of the third fragment immediately after the explosion?

Solution a) Since the force of explosion is very large compared with the gravitational force we treat the system as an isolated system such that the momentum is conserved, i.e.,

$$\vec{p}_i = \vec{p}_f \quad \Rightarrow \quad M\vec{v}_i = \frac{1}{3}M\vec{v}_1 + \frac{1}{3}M\vec{v}_2 + \frac{1}{3}M\vec{v}_3$$

$$\vec{v}_3 = 3\vec{v}_i - \vec{v}_1 - \vec{v}_2 = 3(300\hat{j}) - 450\hat{j} - 240\hat{i} = (-240\hat{i} + 450\hat{j})\text{m/s}$$

Note that the kinetic energy of the system increase after collision contrary to the inelastic collision where the kinetic energy decreases after impact.