

CHAPTER 11

ANGULAR MOMENTUM

In this chapter we are going to discuss the following topics:

- ❑ Angular Momentum
- ❑ Angular Momentum of Rigid Bodies
- ❑ Conservation of Angular Momentum

Recommended Problems:

11, 15, 16, 17, 25, 27, 30, 32, 33, 34, 35, 37, 39, 41, 49, 51, 52, 53, 58, 60

11.2 Angular Momentum of Nonisolated System

The angular momentum $\vec{L}$ of the particle relative to an origin is defined as

$$|\vec{L}| = |\vec{r} \times \vec{p}| = |\vec{r} \times m\vec{v}| = rmv \sin \theta$$

where $\vec{r}$ is the position vector of the particle relative to O . The angular momentum is a vector quantity with a direction perpendicular to the plane formed by $\vec{r}$ and $\vec{v}$, and its sense is determined by the right hand rule discussed in chapter 2.

It is clear that, when the translational velocity of the particle is along a line that passes through an axis, the particle has zero angular momentum with respect to that axis.

The SI unit of L is $\text{kg.m}^2/\text{s}$

Now we have

$$\sum \vec{\tau} = \vec{r} \times \sum \vec{F} \quad \text{and} \quad \sum \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow$$
$$\sum \vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\text{but } \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \Rightarrow$$

$$\sum \vec{\tau} = \frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{p}}) \Rightarrow$$

$$\sum \vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt}$$

Which is the rotational analogue of the linear equation $\sum \vec{F} = \frac{d\vec{\mathbf{p}}}{dt}$

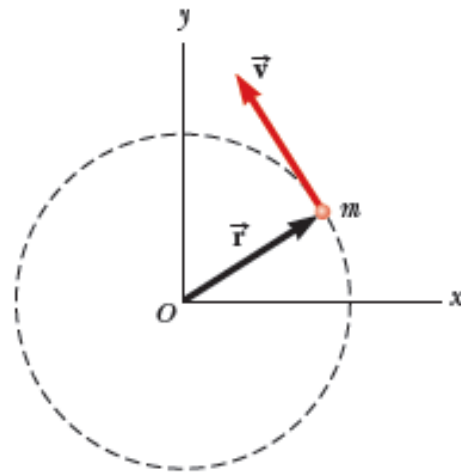
For a system of particles the total angular momentum is

$$\vec{\mathbf{L}} = \sum \vec{\mathbf{L}}_i \Rightarrow \frac{d\vec{\mathbf{L}}}{dt} = \sum \frac{d\vec{\mathbf{L}}_i}{dt} = \sum \vec{\tau}_i$$

But the net torque associated with all the internal forces is zero $\Rightarrow$

$$\sum \vec{\tau}_{ext} = \frac{d\vec{\mathbf{L}}_{tot}}{dt}$$

Example 11.3 A particle moves in the xy plane in a circular path of radius r as shown. Find the magnitude and direction of its angular momentum relative to an axis through O when its velocity is $\vec{v}$.



Solution Using the equation

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \quad \Rightarrow \quad |\vec{L}| = rmv \sin \phi$$

But it is clear that $\vec{r}$ and $\vec{v}$ are perpendicular so that $\phi = \pi/2$, so

$$|\vec{L}| = rmv \sin(\phi/2) = rmv$$

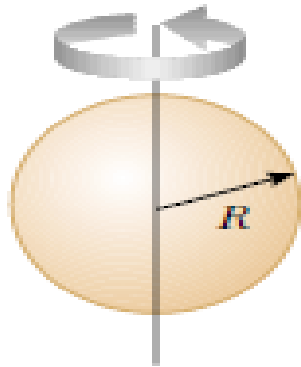
According to the right hand rule, the direction of the angular momentum is out of the page.

11.3 Angular Momentum of a Rotating Rigid Body

Consider a rigid body that rotates about the

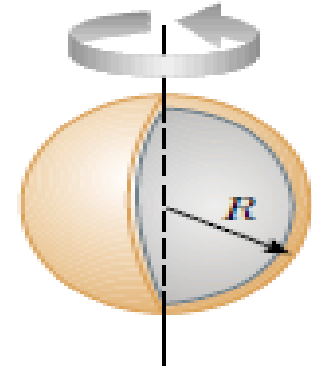
Solid sphere

$$I_{\text{CM}} = \frac{2}{5} MR^2$$



Thin spherical shell

$$I_{\text{CM}} = \frac{2}{3} MR^2$$



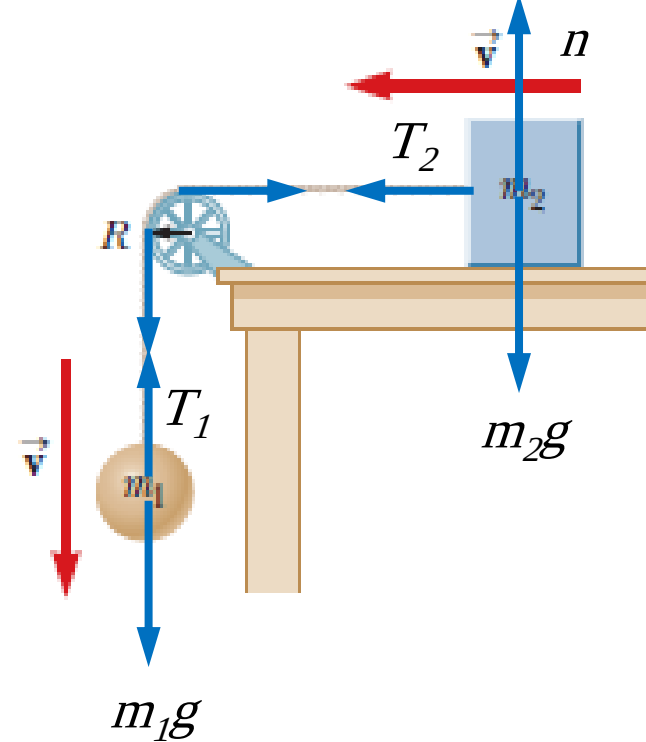
$$L = \sum_i L_i = \omega \left(\sum_i m_i r_i^2 \right) \Rightarrow \boxed{L = I\omega}$$

• Quick Quiz (1)

A hollow sphere and a solid sphere have the same mass and radius. They both rotate with the same angular speed. Which one has the higher angular momentum?

- a) The hollow sphere.
- b) The solid sphere
- c) Both have the same L
- d) Impossible to determine.

Example 11.4 A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley as shown. The radius of the pulley is R , and the mass of the thin rim is M . The spokes of the pulley have negligible mass. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.



Solution Let us first solve the problem using Newton's law. The forces acting on the blocks and the pulley are shown.

For m_1 we use $\sum F_y = ma \Rightarrow m_1g - T_1 = m_1a$ (1)

For m_2 we use $\sum F_x = ma \Rightarrow T_2 = m_2a$ (2)

For the pulley we use $\sum \tau = I\alpha = I\frac{a}{R} \Rightarrow (T_1 - T_2)R = I\frac{a}{R}$ (3)

Adding Eqs. (1 & 2) we get

$$T_1 - T_2 = m_1 g - (m_1 + m_2)a \quad (4)$$

From Eqs. (3 & 4) we get

$$I \frac{a}{R^2} = m_1 g - (m_1 + m_2)a \quad \Rightarrow \quad a = \frac{m_1 g}{\frac{I}{R^2} + m_1 + m_2}$$

As the spokes of the pulley have negligible mass, we treat the pulley as a hoop such that $I = MR^2 \quad \Rightarrow$

$$a = \frac{m_1 g}{M + m_1 + m_2}$$

Now to apply the concept of angular momentum, we consider the system consists of the sphere, the block and the pulley which is a nonisolated system due to the gravitational force acting on the sphere. We want to use

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt}$$

Here the tensions are internal forces.

Now the total angular momentum of the system about the axis of the pulley is

$$L_{tot} = m_1 v R + m_2 v R + I \omega$$

But $I = MR^2$, and $\omega = v/R \Rightarrow$

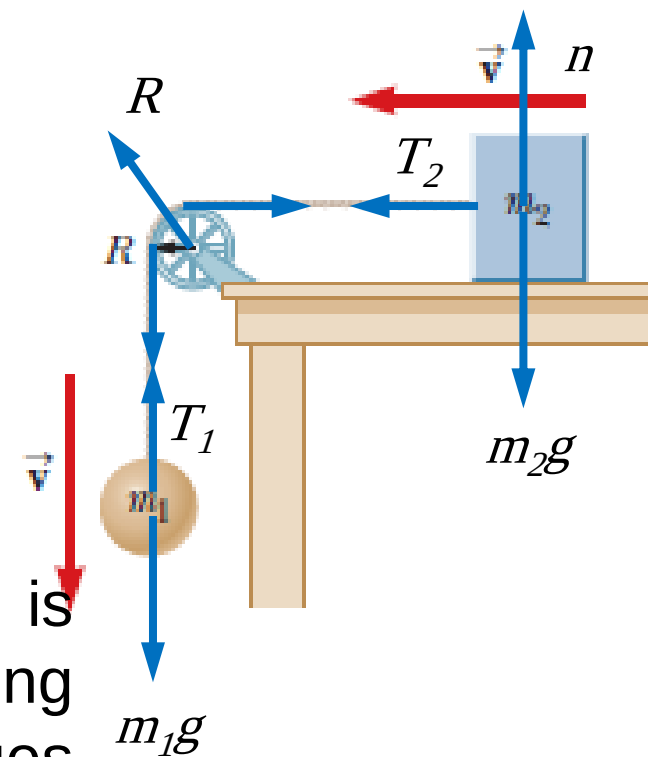
$$L_{tot} = m_1 v R + m_2 v R + I \omega = (m_1 + m_2 + M) v R$$

Knowing that the torque done by $m_2 g$ is canceled by that done by normal force acting by the surface on m_2 and also the torques done by tensions are internal so we write

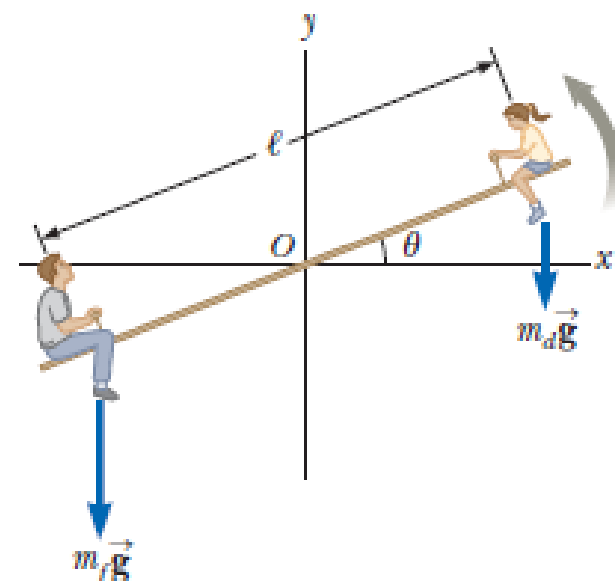
$$\sum \vec{\tau} = m_1 g R$$

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow m_1 g R = (m_1 + m_2 + M) R \frac{dv}{dt} \Rightarrow$$

$$m_1 g R = (m_1 + m_2 + M) R a \Rightarrow a = \frac{m_1 g}{(m_1 + m_2 + M)}$$



Example 11.6 A father of mass m_f and his daughter of mass m_d sit on opposite ends of a seesaw at equal distances from the pivot at the center as shown. The seesaw is modeled as a rigid rod of mass M and length l , and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed ω .



- Find the total angular momentum of the system.
- Find the angular acceleration of the system when the seesaw makes an angle θ with the horizontal.

Solution a) Our system consists of the seesaw, the father and the daughter. The moment of inertia of the system about the pivot is

$$I = m_f (L/2)^2 + m_d (L/2)^2 + \frac{1}{12} M L^2 = \frac{L^2}{4} \left(m_f + m_d + \frac{M}{3} \right) \quad \Rightarrow$$

$$L = I \omega = \frac{L^2}{4} \left(m_f + m_d + \frac{M}{3} \right) \omega$$

b) To find the angular acceleration of the system we use

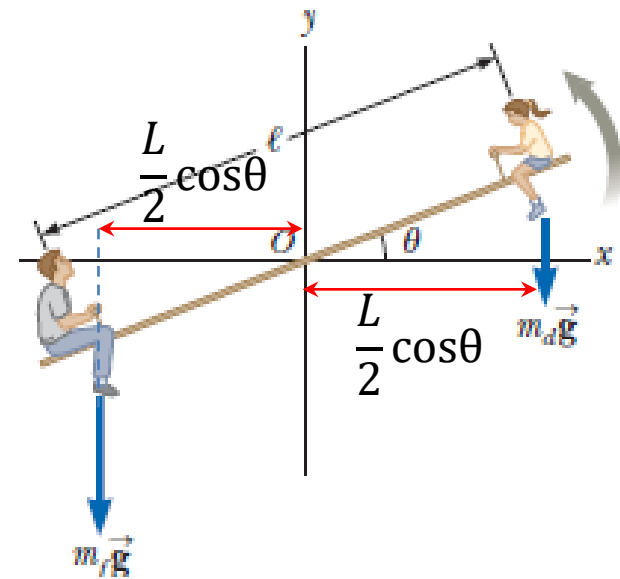
$$\sum \tau_{ext} = \frac{dL}{dt} = I \frac{d\omega}{dt} = I\alpha$$

But the only external forces that contribute to the torque are the weights of both the father and the daughter. So we write

$$\sum \tau_{ext} = m_f g (L/2) \cos \theta - m_d g (L/2) \cos \theta$$

$$= (m_f - m_d) g (L/2) \cos \theta$$

$$\alpha = \frac{\sum \tau_{ext}}{I} = \frac{2(m_f - m_d) g \cos \theta}{L \left[(M/3) + m_f + m_d \right]}$$



11.4 Conservation of Angular Momentum

For an isolated system the net external force is zero and so the net external torque is also zero. Recalling the equation

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}_{tot}}{dt}$$

For isolated systems the above equation gives

$$\frac{d\vec{L}_{tot}}{dt} = 0 \quad \Rightarrow \quad \vec{L}_{tot} = \text{constant} \quad \text{or}$$

$$\vec{L}_i = \vec{L}_f \quad \Rightarrow$$

$$I_i \omega_i = I_f \omega_f$$

We conclude that the total angular momentum of a system is constant in both magnitude and direction if the net external torque acting on the system is zero, i.e., if the system is isolated.

• Quick Quiz (2)

A ball, attached to one end of a string, is whirled in a circle around the other end of the string. If the ball moves at a constant linear speed, which of the following statements is correct concerning the linear momentum $\mathbf{P}$, and the angular momentum $\mathbf{L}$?

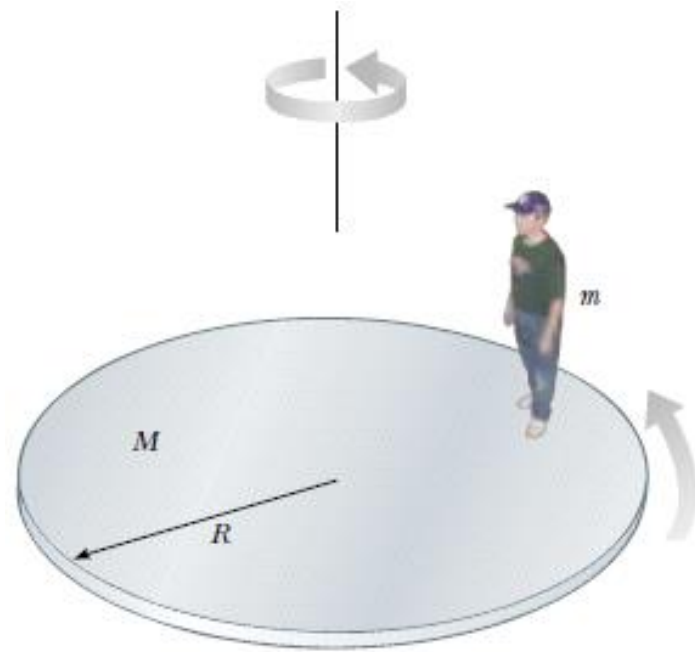
- a) $\mathbf{P}$ is constant, while $\mathbf{L}$ is not.
- b) $\mathbf{L}$ is constant, while $\mathbf{P}$ is not.
- c) Both $\mathbf{P}$, and $\mathbf{L}$ are constant.
- d) Neither $\mathbf{P}$, nor $\mathbf{L}$ is constant.

• Quick Quiz (3)

When a man on a frictionless rotating stool extends his arms horizontally to increase his moment of inertia, his angular momentum will:

- a) increase.
- b) decrease
- c) not change.
- d) cannot be determined.

Example 11.8 A horizontal platform in the shape of a circular disc rotates freely in a horizontal plane about a frictionless vertical axle. The platform has a mass $M=100$ kg and radius $R=2.0$ m. A boy of mass $m=60$ kg, walks slowly from the rim of the disc toward its center. If the angular speed of the system is 2.0 rad/s when the boy is at the rim, what is the angular speed when he reaches a point $r=0.50$ m from the center?



Solution Let us call the moment of inertia of the disc I_d and the moment of inertia of the boy I_b . The initial moment of inertia of the system is, therefore

$$I_i = (I_d + I_b)_i = \frac{1}{2}MR^2 + mr_i^2 = \frac{100}{2}(2.0)^2 + 60(2.0)^2 = 440 \text{ Kg.m}^2$$

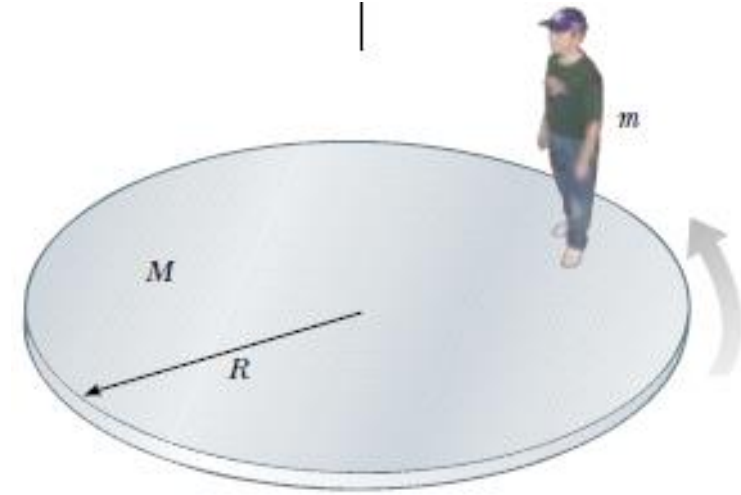
$$I_f = (I_d + I_b)_f = \frac{1}{2}MR^2 + mr_f^2 = \frac{100}{2}(2.0)^2 + 60(0.5)^2 = 215 \text{ Kg.m}^2$$

Applying the conservation of angular momentum law:

$$\vec{L}_i = \vec{L}_f \Rightarrow I_i \omega_i = I_f \omega_f \Rightarrow$$

$$\omega_f = \frac{I_i}{I_f} \omega_i \Rightarrow$$

$$\omega_f = \frac{440}{215} 2.0 = 4.1 \text{ rad/s}$$



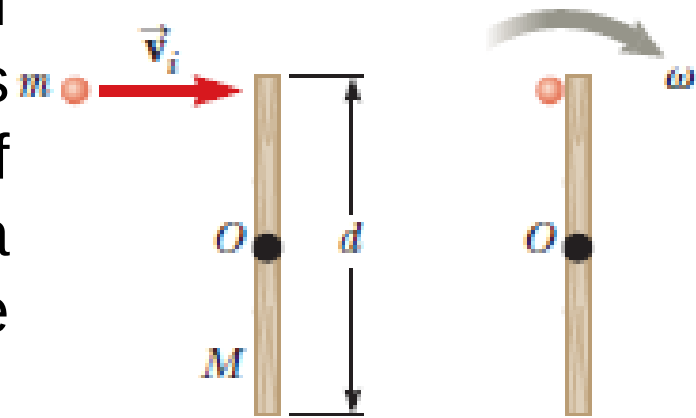
Let us calculate the kinetic energy of the system in the two cases.

$$K_i = \frac{1}{2} I_i \omega_i^2 = \frac{1}{2} (440) (2.0)^2 = 880 \text{ J}$$

$$K_f = \frac{1}{2} I_f \omega_f^2 = \frac{1}{2} (215) (4.1)^2 = 1.80 \times 10^3 \text{ J}$$

This extra kinetic energy comes from the muscular activity performed by the student to move himself closer to the center of rotation.

Example (Problem 51) A projectile of mass m moves with a speed v_i . It strikes and sticks to the end of a stationary rod of mass M , length d , and pivoted about a frictionless axle perpendicular to the page through O as shown.



- Find the angular speed of the system after the collision.
- Determine the fractional change of kinetic energy due to the collision.

Solution a) Applying the conservation of angular momentum we get

$$L_i = L_f$$

$$\text{But } L_i = L_{pi} + L_{Ri} = mv_i(d/2) + 0 = mv_i(d/2)$$

$$\text{and } L_f = (I_{pf} + I_{Rf})\omega = \left[m(d/2)^2 + \frac{1}{12}Md^2 \right]\omega = \left[\frac{d^2(M+3m)}{12} \right]\omega$$

$$\Rightarrow mv_i \frac{d}{2} = \left[\frac{d^2(M+3m)}{12} \right]\omega \Rightarrow \omega = \frac{6mv_i}{d(M+3m)}$$

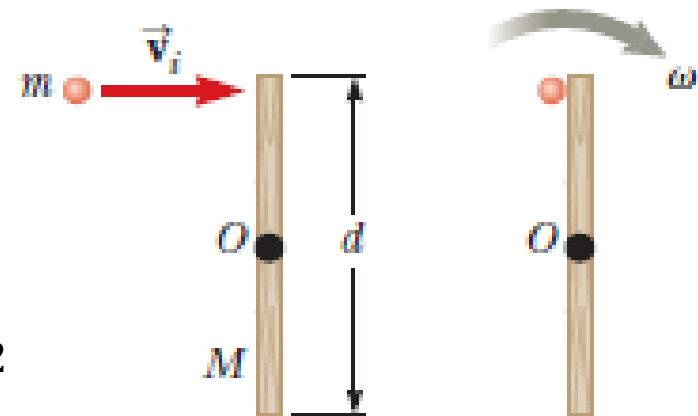
b) let us first find the initial and the final kinetic energy.

$$K_i = \frac{1}{2}mv_i^2$$

$$K_f = \frac{1}{2}(I_{pf} + I_{Rf})\omega^2 = \frac{1}{2}\left[m(d/2)^2 + \frac{1}{12}Md^2\right]\omega^2 = \frac{3m^2v_i^2}{2(M+3m)}$$

$$|\Delta K| = K_i - K_f = \frac{1}{2}mv_i^2 - \frac{3m^2v_i^2}{2(M+3m)} = \frac{mMv_i^2}{2(M+3m)}$$

$$\text{But the fractional change} = \frac{\Delta K}{K_i} = \frac{\frac{mMv_i^2}{2(M+3m)}}{\frac{1}{2}mv_i^2} = \frac{M}{M+3m}$$



To summaries we have

Quantities	Translational	Rotational
Displacement	x	θ
Velocity	v	ω
Acceleration	a	α
Inertia	m	I
Causing motion	F	τ
Momentum	p	L

Translational Motion

$$\sum F = ma$$

$$p = mv$$

$$\sum F = \frac{dp}{dt}$$

Rotational Motion

$$\sum \tau = I\alpha$$

$$L = I\omega$$

$$\sum \tau = \frac{dL}{dt}$$