

CHAPTER 12

STATIC EQUILIBRIUM AND ELASTICITY

In this chapter we are going to discuss the following topics:

- ❑ **Rigid Objects in Equilibrium**
- ❑ **Center of Gravity**
- ❑ **Elastic Properties of Solids**

Recommended Problems:

9, 11, 13, 14, 15, 16, 18, 19, 21, 23, 24, 25, 38, 43, 46, 49, 50, 51, 55, 56

12.1 Rigid Object in Equilibrium

A rigid body is said to be in equilibrium when the two following conditions are satisfied:

1. (Translational Equilibrium) The resultant external force acting on the body must equal zero, i.e.,

$$\sum \vec{F} = 0 \quad \Rightarrow \quad \sum F_x = 0 \quad \sum F_y = 0$$

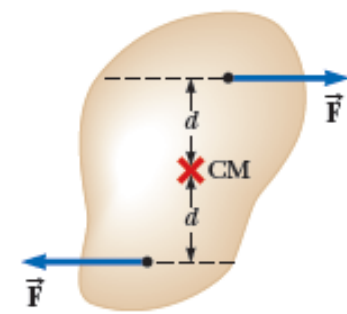
2. (Rotational Equilibrium) The resultant external torque must be zero about any origin, i.e.,

$$\sum \vec{\tau} = 0$$

In the special case of **static equilibrium**, the object in equilibrium is at rest relative to the observer and so has no translational or angular speed $v_{CM}=0$ and $\omega=0$.

• Quick Quiz (1)

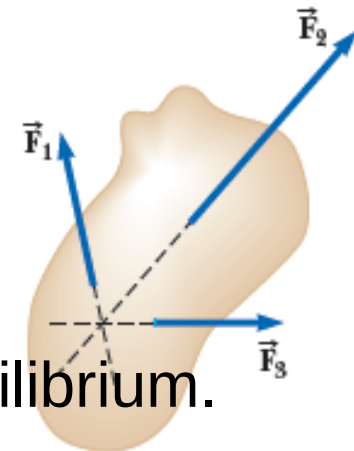
Consider the object subject to the two forces of equal magnitude as shown. Choose the correct statement.



- a) The object is in force equilibrium but not torque equilibrium.
- b) The object is in torque equilibrium but not force equilibrium.
- c) The object is in both torque equilibrium and force equilibrium.
- d) The object is in neither torque equilibrium nor force equilibrium.

• Quick Quiz (2)

Consider the object subject to the three forces as shown. Choose the correct statement.



- a) The object is in force equilibrium but not torque equilibrium.
- b) The object is in torque equilibrium but not force equilibrium.
- c) The object is in both torque equilibrium and force equilibrium.
- d) The object is in neither torque equilibrium nor force equilibrium.

• Quick Quiz (3)

For an object moving in a uniform circular motion, which of the following statements is correct?

- a) The object is in both translational and rotational equilibrium.
- b) The object is in neither translational nor rotational equilibrium.
- c) The object is in translational but not rotational equilibrium.
- d) The object is in rotational but not translational equilibrium.

• Quick Quiz (4)

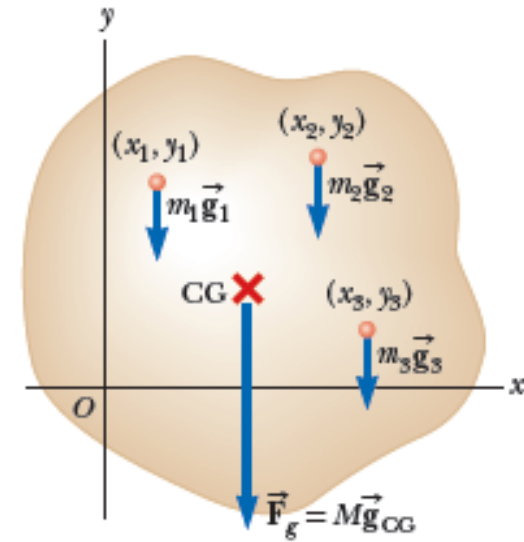
For an object moving in a non-uniform circular motion, which of the following statements is correct?

- a) The object is in both translational and rotational equilibrium.
- b) The object is in neither translational nor rotational equilibrium.
- c) The object is in translational but not rotational equilibrium.
- d) The object is in rotational but not translational equilibrium.

12.2 CENTER OF GRAVITY

It is a point in which all of the weight can be considered as being concentrated at that point.

Consider an object of arbitrary shape lying in the xy plane as shown. Suppose the object is divided into a large number of particles of masses $m_1, m_2, m_3, \dots$ having coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$



The net torque acting on the object about an axis through O is

$$\sum \vec{\tau}_g = m_1 g_1 x_1 + m_2 g_2 x_2 + m_3 g_3 x_3 + \dots = \sum_i m_i g_i x_i$$

$$\text{But } x_{CM} = \frac{1}{M} \sum_i m_i x_i \quad \Rightarrow \quad \sum \vec{\tau}_g = M g x_{CM}$$

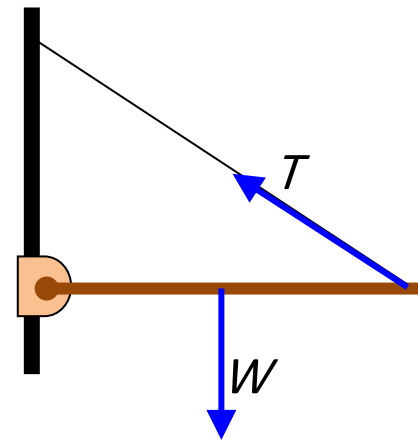
Provided that g is uniform over the object, i.e., for homogeneous, symmetric objects the center of gravity coincides with the geometric center of the object.

Problem solving strategy:

- (i) Make a sketch of the system under consideration.
- (ii) Draw a free-body diagram for the system showing all the external forces acting on it. Try to label each of these forces. Note that the internal forces acting from one part of the system on the others must not included in the diagram
- (iii) Resolve all forces into their rectangular components, choosing an appropriate coordinate system
- (iv) Choose a convenient origin for calculating the net torque on the object. Remember that the choice of the origin for the torque equation is arbitrary; therefore choose an origin that will simplify your calculation as much as possible.
- (v) Applying the equations of equilibrium you get a set of linear equations with several unknowns. Simultaneous solutions of these equations give us the unknowns.

• Quick Quiz (5)

A horizontal beam of weight W is supported by a hinge and cable as shown. The force exerted on the beam by the hinge has a horizontal component that must be:



a) zero.

b) non zero and left.

c) non zero and right.

d) cannot be determined.

• Quick Quiz (6)

Referring to the previous question, the force exerted on the beam by the hinge has a vertical component that must be:

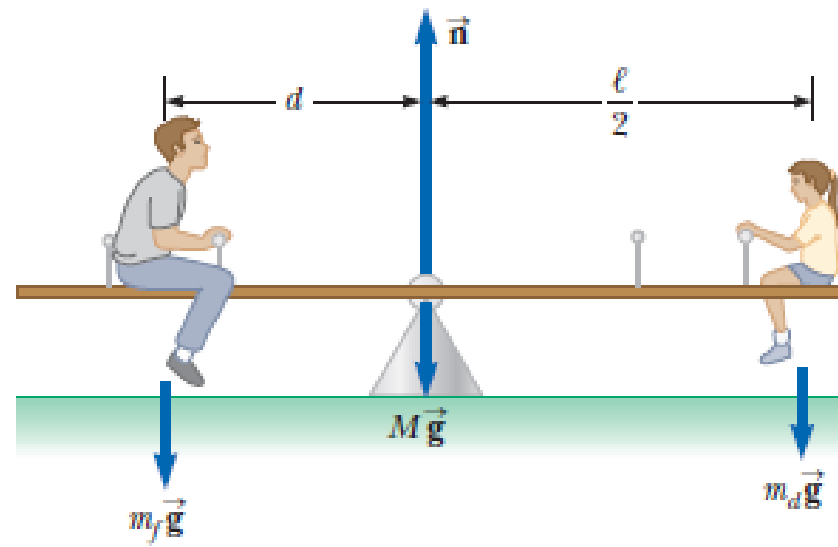
a) zero.

b) non zero and up.

c) non zero and down.

d) cannot be determined.

Example 12.1 A seesaw consisting of a uniform board of mass M and length L , supports at rest a father and daughter with masses m_f and m_d , respectively, as shown. The support (called the *fulcrum*) is under the center of gravity of the board, the father is a distance d from the center, and the daughter is a distance $L/2$ from the center.



- Find the upward force n exerted on the board by the support,
- Determine where the father should sit to balance the system.

Solution All the forces acting on the system (the rod plus the two persons) are shown. Note that the forces acting by the board on the two persons are not included, why? Now using

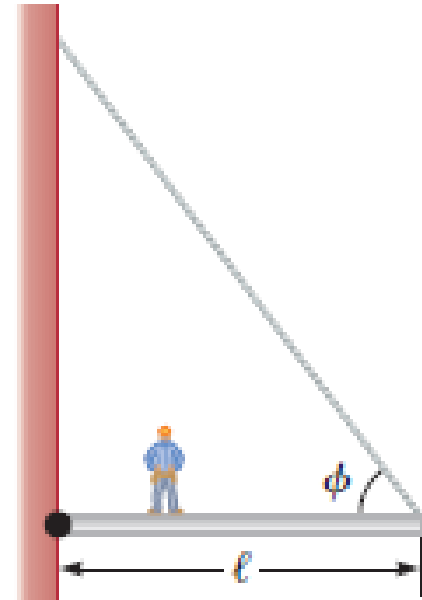
$$\sum F_y = 0 \Rightarrow n - Mg - m_f g - m_d g = 0 \Rightarrow$$

$$n = (M + m_f + m_d)g$$

Now finding the torque about the center of gravity and using

$$\sum \vec{\tau} = 0 \quad \Rightarrow \quad m_f d - m_d (L/2) = 0 \quad \Rightarrow \quad d = \left(\frac{m_d}{m_f} \right) \frac{L}{2}$$

Example 12.2 A uniform horizontal beam of length 8.00 m and weight $W_b = 200$ N is attached to a wall by a pin connection. Its far end is supported by a cable that makes an angle of $\phi = 53^\circ$ with the beam. A person of weight $W_p = 600$ N stands 2.00 m from the wall. Find the tension in the cable and the force exerted by the wall on the beam.



Solution The free-body diagram of the system is shown. Applying the conditions of equilibrium we obtain

$$\sum F_x = 0 \Rightarrow$$

$$R_x = T \cos 53^\circ \quad (1)$$

$$\sum F_y = 0 \Rightarrow$$

$$R_y + T \sin 53^\circ = 800 \text{ N} \quad (2)$$

$$\sum \tau_o = 0 \Rightarrow$$

$$T \sin 53^\circ \times 8 - 600 \times 2 - 200 \times 4 = 0 \Rightarrow$$

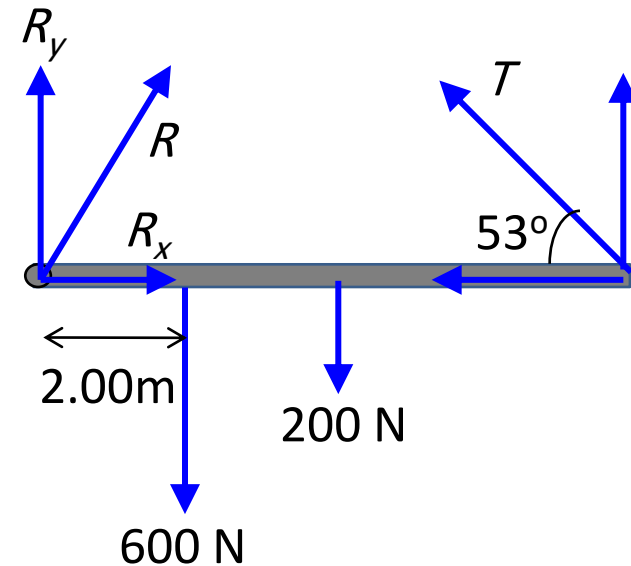
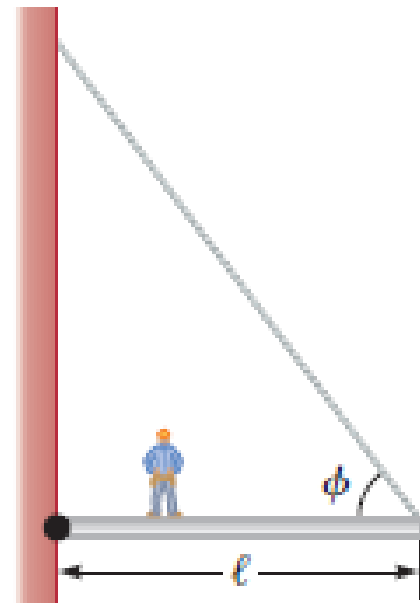
$$T = 313 \text{ N}$$

Substituting for T into Eqs.(1&2) we get

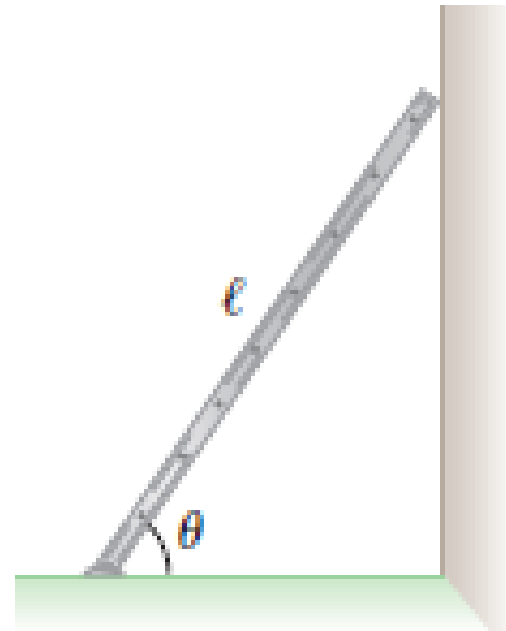
$$R_x = 188 \text{ N} \quad R_y = 550 \text{ N}$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(188)^2 + (550)^2} = 581 \text{ N}$$

$$\tan \theta = \frac{R_y}{R_x} = \frac{550}{188} \Rightarrow \theta = 71.1^\circ$$



Example 12.3 A uniform ladder of length l rests against a smooth vertical wall as shown. The mass of the ladder is m , and the coefficient of static friction between the ladder and the ground is 0.4. Find the minimum angle $\theta_{\min}$ at which the ladder does not slip.



Solution The free-body diagram of the system is shown. Applying the conditions of equilibrium we get

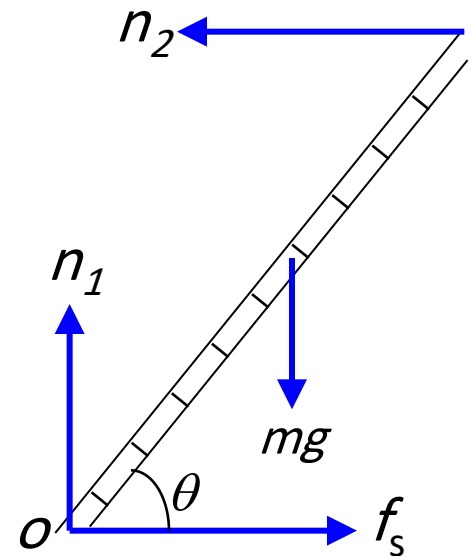
$$\sum F_x = 0 \Rightarrow f_s = n_2 \quad (1)$$

$$\sum F_y = 0 \Rightarrow n_1 - mg = 0 \quad (2)$$

$$\sum \tau_o = 0 \Rightarrow$$

$$n_2 l \sin \theta - mg(l/2)(\cos \theta) = 0 \Rightarrow$$

$$\frac{\sin \theta}{\cos \theta} = \tan \theta = \frac{mg}{2n_2} \quad (3)$$



When the ladder is on the verge of slipping, the frictional force is a maximum, i.e.,

$$f_s = \mu_s n_1 \quad (4)$$

From Eq.(2) we have

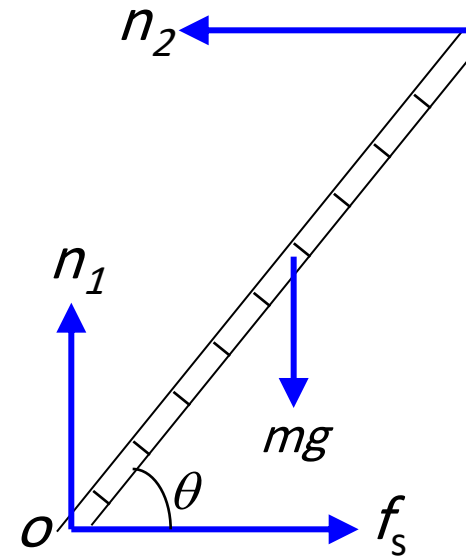
$$n_1 = mg \quad \Rightarrow \quad f_s = \mu_s n_1 = \mu_s mg$$

From Eq.(1) we have $n_2 = f_s = \mu_s mg$

Substituting for n_2 into Eq.(3) we get

$$\tan \theta = \frac{mg}{2\mu_s mg} = \frac{1}{2\mu_s} \Rightarrow$$

$$\theta_{\min} = \tan^{-1}\left(\frac{1}{2\mu_s}\right) = \tan^{-1}\left(\frac{1}{2(0.40)}\right) = 51^\circ$$



Notice that the angle depends only on the coefficient of friction, not on the mass or length of the ladder.

What happen to this minimum $\theta_{\min}$ if a man of mass M climbs a distance d up the ladder as shown.

Solution We have to add the weight of the man to the free-body diagram of the system. Applying the conditions of equilibrium we get

$$\sum F_x = 0 \Rightarrow f_s = n_2 \quad (1)$$

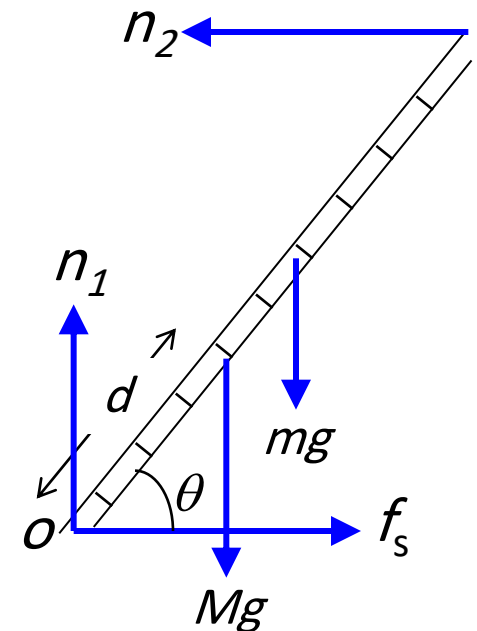
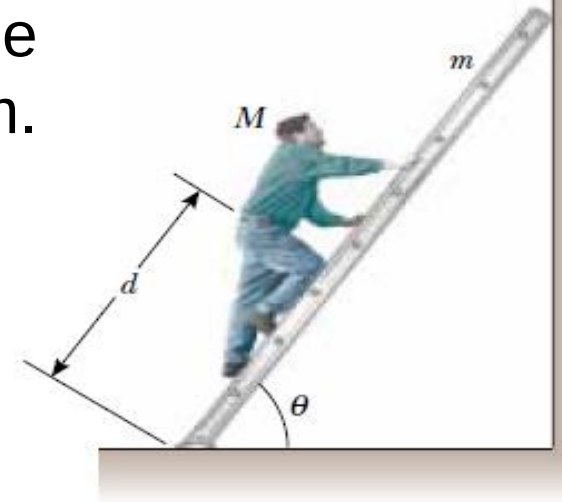
$$\sum F_y = 0 \Rightarrow n_1 - mg - Mg = 0 \quad (2)$$

$$\sum \tau_o = 0 \Rightarrow$$

$$n_2 l \sin \theta - mg(l/2)(\cos \theta) - Mgd \cos \theta = 0 \quad (3)$$

From Eq.(2) we have $n_1 = (m + M)g$

and $n_2 = f_s = \mu_s n_1 = \mu_s (m + M)g$



From Eq.(3) we have

$$\tan \theta = \frac{[m + M(2d/l)]}{2\mu_s[m + M]}$$

Using the last expression we conclude that

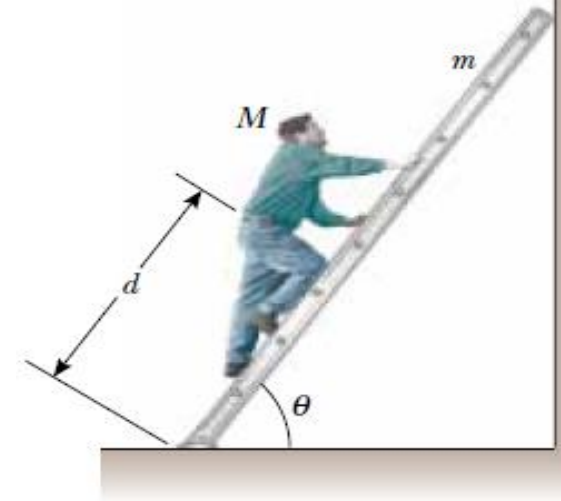
(i) If $d=0.5l$ we have

$$\tan \theta = \frac{[m + M]}{2\mu_s[m + M]} = \frac{1}{2\mu_s}$$

So if $d=0.5l$ the angle will be the same as before without any change and independent of M .

(ii) If $d < 0.5l$, say $d=0.4l$, we have $\tan \theta = \frac{[m + 0.8M]}{2\mu_s[m + M]} < \frac{1}{2\mu_s}$

So if $d < 0.5l$ the angle will be decreased, i.e. the ladder will be more safe with the existence of M . The heavier the man is the ladder is less likely to slip.



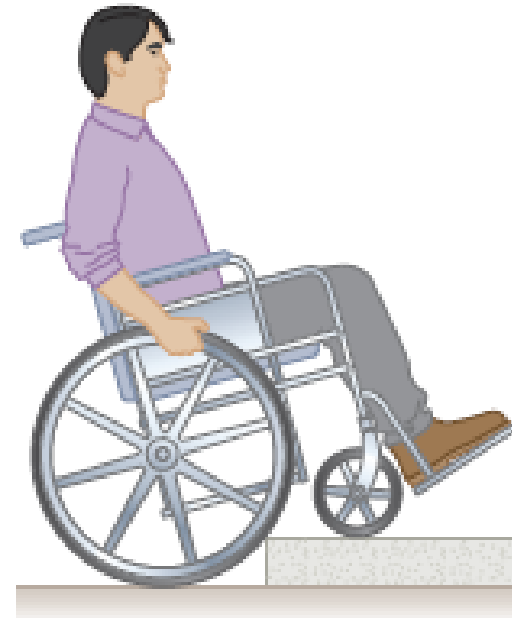
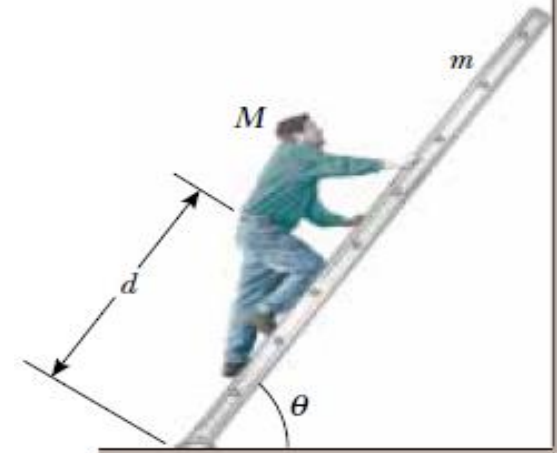
(ii) If $d > 0.5l$, say $d = 1.2l$, we have

$$\tan \theta = \frac{[m + 1.2M]}{2\mu_s [m + M]} > \frac{1}{2\mu_s}$$

(ii) If $d > 0.5l$ the angle will be increased, i.e. the ladder will be less safe with the existence of M . The heavier the man is the ladder is more likely to slip.

Example 12.4 Estimate the magnitude of the force $\mathbf{F}$ a person must apply to a wheelchair's main wheel to roll up over a sidewalk curb as shown. This main wheel that comes in contact with the curb has a radius r , and the height of the curb is h .

Solution Because the wheel is in the verge of leaving the ground, there is no reaction force acting on the wheel by the ground.



Solution Let us first draw the free-body diagram of the system.

Applying the torque equilibrium condition we get

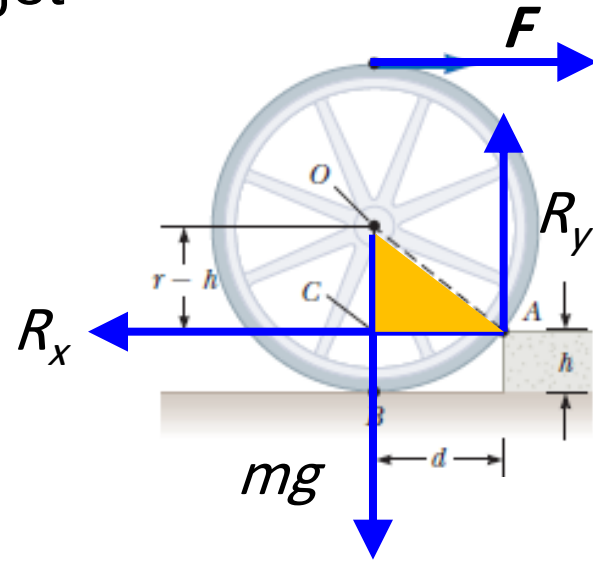
$$\sum \tau_A = 0 \Rightarrow$$

$$mgd - F(2r - h) = 0$$

$$\text{But } d = \sqrt{r^2 - (r - h)^2} \Rightarrow$$

$$mg\sqrt{r^2 - (r - h)^2} - F(2r - h) = 0 \Rightarrow$$

$$F = \frac{mg\sqrt{2rh - h^2}}{2r - h} = mg\sqrt{\frac{h}{2r - h}}$$



Let's consider that each wheel supports a weight of 700 N, and choose a wheel radius of $r=30$ cm. We also pick a curb height of $h=10$ cm. Then we get

$$F = (700)\sqrt{\frac{0.1}{2(0.3) - 0.1}} = 3.0 \times 10^2 \text{ N}$$

12.4 Elastic Properties of Solids

In reality, all objects are deformable under the influence of external forces. That is, it is possible to change the shape or the size (or both) of an object by applying external forces.

The deformation of solids are discussed in terms of two concepts:

Stress: is the external force per unit cross-sectional area acting on an object.

Strain: is a measure of the degree of deformation.

It is found that, for sufficiently small stress, the stress is proportional to the strain; the constant of proportionality is called the ***elastic modulus, i.e,***

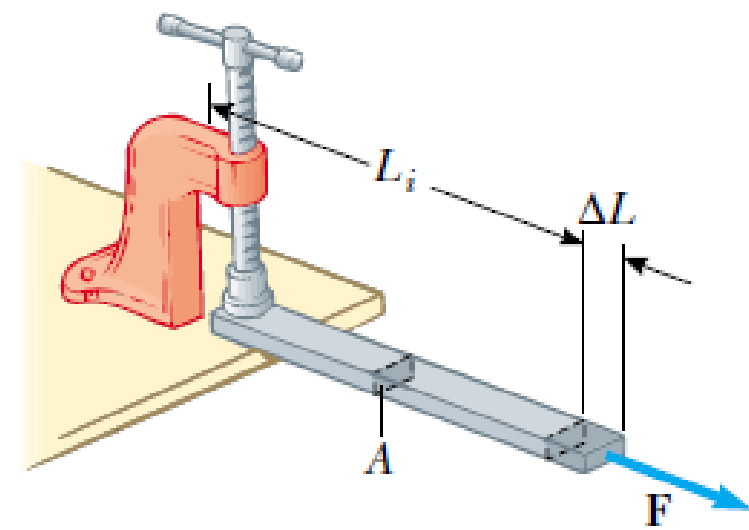
$$\text{Elastic Modulus} = \frac{\text{Stress}}{\text{Strain}}$$

We shall consider three types of deformation and define an elastic modulus for each:

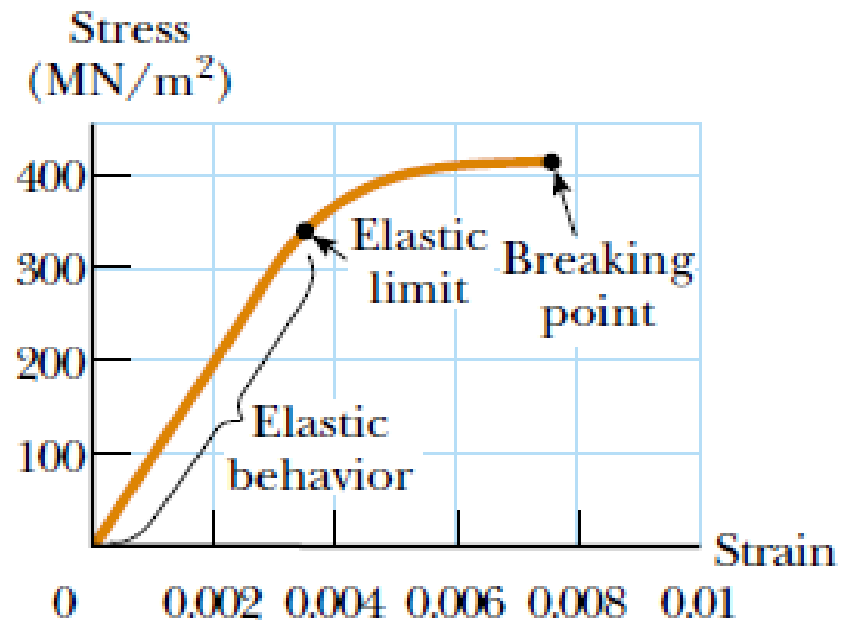
Young's Modulus: Elasticity in Length

Young's modulus measures the resistance of a solid to a change in its length, i.e.,

$$Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{F/A}{\Delta L/L}$$



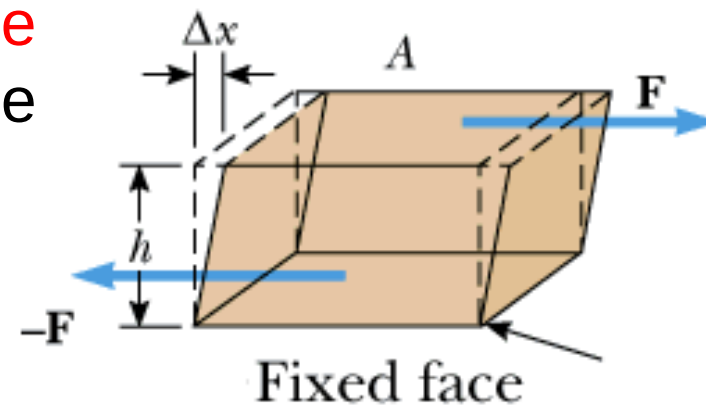
The figure shows that when the stress exceeds the elastic limit, the object is permanently distorted and will not return to its original shape after the stress is removed. As the stress is increased even further, the material will ultimately break.



Shear Modulus: Elasticity in Shape

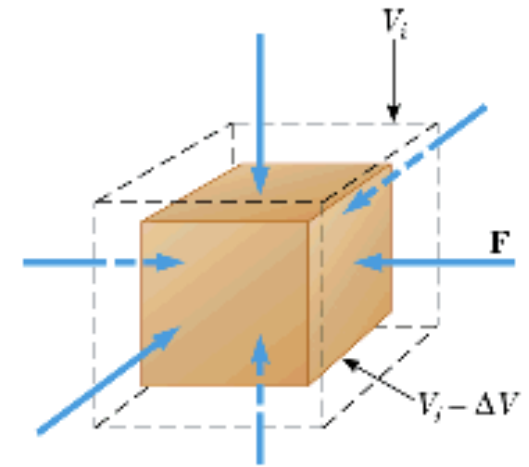
Shear's modulus measures the resistance of a solid to a change in its shape, i.e.,

$$S = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\Delta x/h}$$



Bulk Modulus: Volume Elasticity Bulk's modulus measures the resistance of a solid to a change in its volume, i.e.,

$$B = \frac{\text{bulk stress}}{\text{bulk strain}} = -\frac{F/A}{\Delta V/V} = -\frac{\Delta P}{\Delta V/V}$$



With ΔP is the change in pressure. The negative sign in the last equation is to ensure that B is a positive number. Because an increase in pressure causes a decrease in volume (negative ΔV) and vice versa.

• Quick Quiz (7)

A block of iron is sliding across a horizontal floor. The friction force between the sliding block and the floor causes the block to deform.

a) Young's modulus.

b) Shear modulus.

c) Bulk modulus.

d) Non of them.

• Quick Quiz (8)

A trapeze artist swings through a circular arc. At the bottom of the swing, the wires supporting the trapeze are longer than when the trapeze artist simply hangs from the trapeze due to the increased tension in them.

a) Young's modulus.

b) Shear modulus.

c) Bulk modulus.

d) Non of them.

- Quick Quiz (9)

A spacecraft carries a *steel sphere* to a planet on which atmospheric pressure is much higher than on the Earth. The higher pressure causes the radius of the sphere to decrease.

a) Young's modulus.

b) Shear modulus.

c) Bulk modulus.

d) Non of them.

Example 12.5 A mass of 100 kg is supported by a wire of length 2 m and cross-sectional area 0.1 cm^2 . The wire is stretched by 0.22 cm. Find the tensile stress, the tensile strain, and the Young's modulus for the wire from this information.

Solution

$$\text{Tensile stress} = \frac{F}{A} = \frac{Mg}{A} = \frac{100 \times 9.8}{0.1 \times 10^{-4}} = 9.8 \times 10^7 \text{ N/m}^2$$

$$\text{Tensile strain} = \frac{\Delta L}{L} = \frac{0.22 \times 10^{-2}}{2} = 0.11 \times 10^{-2}$$

$$\Rightarrow Y = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{9.8 \times 10^7}{0.11 \times 10^{-2}} = 8.9 \times 10^{10} \text{ N/m}^2$$

Example 12.6 A solid brass sphere of volume 0.5 m^3 is initially surrounded by air with pressure $1.0 \times 10^5 \text{ N/m}^2$. The sphere is lowered to a depth in the ocean where the water pressure is equal to $2.0 \times 10^7 \text{ N/m}^2$. The bulk modulus of brass is equal to $6.1 \times 10^{10} \text{ N/m}^2$. By how much does the volume of the sphere change once it is submerged?

Solution

$$B = - \frac{\Delta P}{\Delta V / V} \quad \Rightarrow$$

$$\Delta V = - \frac{(\Delta P)V}{B}$$

$$= - \frac{0.50(2.0 \times 10^7 - 1.0 \times 10^5)}{6.1 \times 10^{10}} = -1.6 \times 10^{-4} \text{ m}^3$$

The negative sign indicates a decrease in volume.