



Chapter 1 Functions

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1.1 Functions and Their graphs

- Functions are the major tools for describing the real world in mathematical terms.
- A function can be represented by an equation, a graph, a numerical table, or a verbal description.
- This section reviews these function ideas.

Chapter 1: Functions

- Functions are fundamental to the study of calculus.
- In this chapter we review the definition of functions and their graphs, how they are combined and transformed, and ways they can be classified.
- We review the trigonometric functions.

Functions

Definition

A function from a set D to a set Y is a rule that assigns a single (unique) element of Y to each element of D .

Notations

- 1 We will denote functions by $f, g, \dots$.
- 2 If f is a function from a set D to a set Y and $x \in D$, then the value of x under f is denoted by $f(x)$.
- 3 To say that y is a function of x , we write $y = f(x)$.
 x is called the independent variable
 y is called the dependent variable.

Examples

Example 1

Let $D = \{a, b, c\}$ and let $Y = \{t, s, r\}$ and define $f(a) = r, f(b) = t, f(c) = s$. Then f is a function from D to Y .

Example 2

Let $D = \mathbb{R}$ and let $Y = [0, \infty)$. Then $f(x) = x^2$ is a function from D to Y .

Domain and range of a function

Definition

Let f be a function from D to Y .

- 1 D (the set of all possible input values) is called the domain of the function and is denoted by $Dom(f)$.
- 2 Y is called the codomain of the function f .
- 3 The set of all values $f(x)$ as x varies throughout D is called the range of the function and is denoted by $Ran(f)$.

Remark

The range of a function from D to Y may not include every element in the set Y . In other words, $Ran(f) \subseteq Y$.

Example

Find the range of the function $f(x) = x^2$ from $D = [-2, 2]$ to $\mathbb{R}$.

Solution

- $Ran(f) = \{f(x) : x \in D\}$.
- $x \in Dom(f) = D \Rightarrow x \in [-2, 2] \Rightarrow -2 \leq x \leq 2 \Rightarrow x^2 \in ???$
- $\Rightarrow |x| \leq 2 \Rightarrow 0 \leq x^2 \leq 4$.
- Thus $Ran(f) = [0, 4]$.

The domain convention

- If the domain of a function is not given explicitly, then the domain is the largest set of x -values for which the formula gives real y -values.
- This is the function's natural domain.
- If we want the domain to be restricted in some way, we must say so.

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Remark

In order to find the domain of a given function we must keep the following two restrictions in mind:

- 1 We can not divide by zero.
- 2 If n is an even integer, then we can not take the n -th root of a negative number.

Examples

Example 1

Find the domain and range of the function $f(x) = \frac{x}{x^2 - 4}$.

Solution

- $Dom(f) = ?$
- $Dom(f) = \mathbb{R} - \{x : x^2 - 4 = 0\}$
- $x^2 - 4 = 0 \implies x^2 = 4 \implies x = \pm 2$
- Thus $Dom(f) = \mathbb{R} - \{-2, 2\}$
- $Ran(f) = ?$
- $Ran(f) = \{f(x) : x \in Dom(f)\}$.
- $f(0) = 0, f(1) = -0.3, f(-1) = 0.3$
 $f(1.5) = -0.9, f(-1.5) = 0.9,$
 $f(2.1) = 5.1, f(-2.1) = -5.1, f(2.01) = 50.1$
- Thus $Ran(f) = (-\infty, \infty)$.

Another solution

- $Ran(f) = \{f(x) : x \in Dom(f)\} = \{y : y = f(x) \text{ for some } x \in Dom(f)\}$
- $y = f(x) = \frac{x}{x^2 - 4}$
- We solve for x in terms of y :
- $\implies y(x^2 - 4) = x$
- $\implies yx^2 - x - 4y = 0$.
- If $y = 0$, then $x = 0 \in Dom(f) = \mathbb{R} - \{-2, 2\}$.
- If $y \neq 0$, then $x = \frac{1 \pm \sqrt{1 + 16y^2}}{2y} \in Dom(f) = \mathbb{R} - \{-2, 2\}$.
- Thus no restriction on y and hence $Ran(f) = (-\infty, \infty)$.

Example 2

Example 2

Find the domain and range of the function $f(x) = \sqrt{x^2 - 9}$.

Solution

- $Dom(f) = ?$
- $Dom(f) = \mathbb{R} - \{x : x^2 - 9 < 0\} = \{x : x^2 - 9 \geq 0\}$
- $x^2 - 9 \geq 0 \implies x^2 \geq 9$
- $\implies \sqrt{x^2} \geq \sqrt{9}$
- $\implies |x| \geq 3$
- $\implies x \geq 3 \text{ or } x \leq -3$
- Thus $Dom(f) = (-\infty, -3] \cup [3, \infty) = \mathbb{R} - (-3, 3)$

- $Ran(f) = \{f(x) = \sqrt{x^2 - 9} : x \in Dom(f)\} = ?$
- $f(3) = 0 = f(-3)$
- $f(4) = \sqrt{7} \approx 2.6 = f(-4)$
- $f(5) = 4 = f(-5)$
- $f(10) = \sqrt{91} \approx 9.5 = f(-10)$
- Thus $Ran(f) = [0, \infty)$.
- **Another solution:**
- $Ran(f) = \{f(x) : x \in Dom(f)\} = \{y : y = f(x) \text{ for some } x \in Dom(f)\}$
- $y = f(x) = \sqrt{x^2 - 9} \implies \boxed{y \geq 0}$
- Next we solve for x in terms of y :
- $y^2 = x^2 - 9 \implies x = \pm \sqrt{y^2 + 9}$.
- Thus $x \in Dom(f)$ if $y^2 + 9 \geq 0$.
- Hence $Ran(f) = [0, \infty)$

Example 3

Example 3

Find the domain and range of the function $f(x) = \frac{1}{\sqrt{16-x^2}}$.

Solution

- $Dom(f) = ?$
- $Dom(f) = \mathbb{R} - \{x : 16 - x^2 \leq 0\} = \{x : 16 - x^2 > 0\}$
- $16 - x^2 \leq 0 \Rightarrow 16 \leq x^2$
- $\Rightarrow \sqrt{16} \leq \sqrt{x^2}$
- $\Rightarrow 4 \leq |x|$
- $\Rightarrow x \geq 4 \text{ or } x \leq -4$
- $\Rightarrow x \in (-\infty, -4] \cup [4, \infty)$
- $Dom(f) = \mathbb{R} - \left((-\infty, -4] \cup [4, \infty) \right) = (-4, 4)$

Another method to find $Ran(f)$:

- $Ran(f) = \{f(x) : x \in Dom(f)\} = \{y : y = f(x) \text{ for some } x \in Dom(f)\}$
- $y = f(x) = \frac{1}{\sqrt{16-x^2}} \Rightarrow \boxed{y > 0}$
- Next we solve for x in terms of y :
- $y^2 = \frac{1}{16-x^2}$
- It follows that $16 - x^2 = \frac{1}{y^2} \Rightarrow x^2 = 16 - \frac{1}{y^2} \Rightarrow x = \pm \sqrt{16 - \frac{1}{y^2}}$
- Thus $x \in Dom(f)$ if $16 - \frac{1}{y^2} \geq 0 \Rightarrow 16 \geq \frac{1}{y^2} \Rightarrow y^2 \geq \frac{1}{16} \Rightarrow |y| \geq \frac{1}{4}$
- $\Rightarrow y \geq \frac{1}{4} \text{ or } y \leq -\frac{1}{4} \Rightarrow \boxed{y \in [0.25, \infty)} \text{ or } \boxed{y \in (-\infty, -0.25]}$
- But $y > 0$ gives $Ran(f) = [0.25, \infty)$.

- $Ran(f) = ?$

- $Ran(f) = \left\{ f(x) = \frac{1}{\sqrt{16-x^2}} : x \in Dom(f) = (-4, 4) \right\}$.
- $f(0) = 0.25, f(1) = 0.26, f(2) = 0.28,$
- $f(3) = 0.38, f(3.9) = 1.1, f(3.99) = 3.54, \dots$
- Thus $Ran(f) = [0.25, \infty)$.

Example 4

Find the domain of the function $f(x) = \sqrt{\frac{1+x}{1-x}} - 1$.

Solution

- $f(x)$ is defined if $\frac{1+x}{1-x} - 1 \geq 0$.
- $\frac{1+x}{1-x} - 1 \geq 0$ Common denominator $1-x$
- $\frac{1+x - (1-x)}{1-x} \geq 0$ Simplify
- $\frac{2x}{1-x} \geq 0$ Simplify

- **Find the intervals:**

- The factors of the left-hand side are $2x$ and $1 - x$.
- These are zero when x is 0 and 1.
- These numbers divide the real line into the intervals

$$(-\infty, 0), \quad (0, 1), \quad (1, \infty)$$

- **Make a diagram:**

- We make the following diagram using test points to determine the sign of each factor in each interval.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
sign of $2x$			
sign of $1 - x$			
sign of $\frac{2x}{1 - x}$			

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Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
sign of $2x$	–	+	+
sign of $1 - x$	+	+	–
sign of $\frac{2x}{1 - x}$	–	+	–

- From the diagram we see that the inequality is satisfied on the interval $(0, 1)$.
- Checking the endpoints, we see that 0 satisfies the inequality but 1 does not (because the quotient in the inequality is not defined at 1).
- So the domain is $Dom(f) = [0, 1)$

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
sign of $2x$	–	+	+
sign of $1 - x$	+	+	–
sign of $\frac{2x}{1 - x}$			

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Evaluation of functions

If $a \in Dom(f)$, we call $f(a)$ the value of f at $x = a$.

Example

If $f(x) = x^3 - 3x + 5$, then find $f(2)$ and $f(a + 1)$.

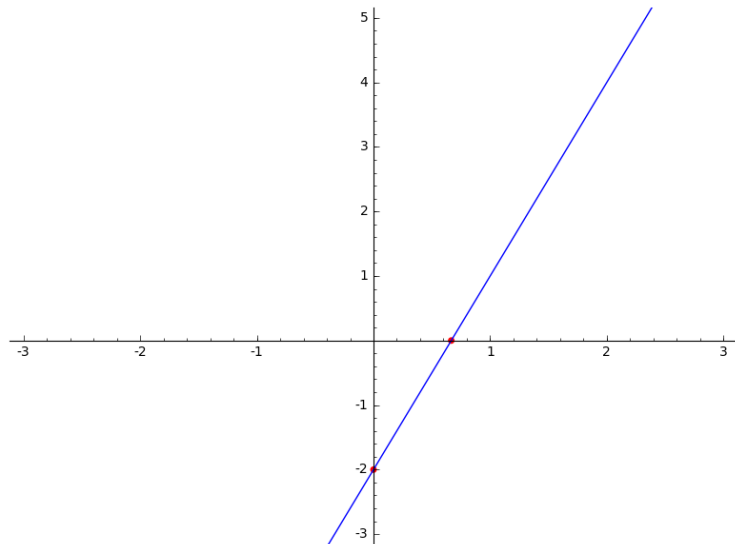
Solution

- $f(2) = (2)^3 - 3(2) + 5 = 8 - 6 + 5 = 7$
- $f(a + 1) = (a + 1)^3 - 3(a + 1) + 5 = a^3 + 3a^2 + 3a + 1 - 3a - 3 + 5 = a^3 + 3a^2 + 3$

Graphs of functions

- The graph of a function f is the graph of the equation $y = f(x)$; that is the set of all points (x, y) whose coordinates satisfy the equation $y = f(x)$.
- In order to graph functions, we make a table of xy -pairs that satisfy the equation $y = f(x)$, then we plot the points (x, y) whose coordinates appear in the table, and draw a smooth curve through the plotted points.

The graph of $f(x) = 3x - 2$



Example 1

Example 1

Graph $f(x) = 3x - 2$.

Solution

- The graph of $y = f(x)$ is a straight line.
- So to draw it we need two points.
- For example,
- the x -intercept can be found by substituting $y = 0 \implies x = \frac{2}{3}$
- and the y -intercept can be found by substituting $x = 0 \implies y = -2$.

Example 2

Example 2

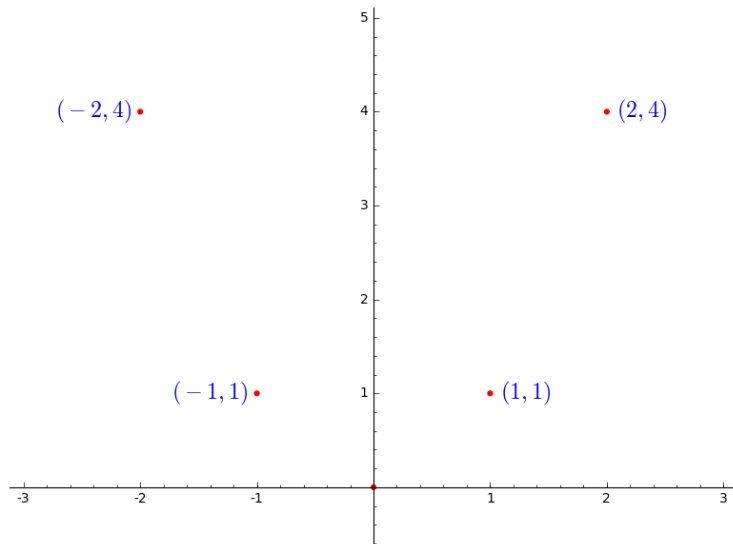
Graph $f(x) = x^2$.

- (1) Find the domain: $Dom(f) = \mathbb{R}$.
- (2) Make a table of xy -pairs that satisfy the equation $y = x^2$:

x	$y = x^2$
0	0
1	1
-1	1
2	4
-2	4

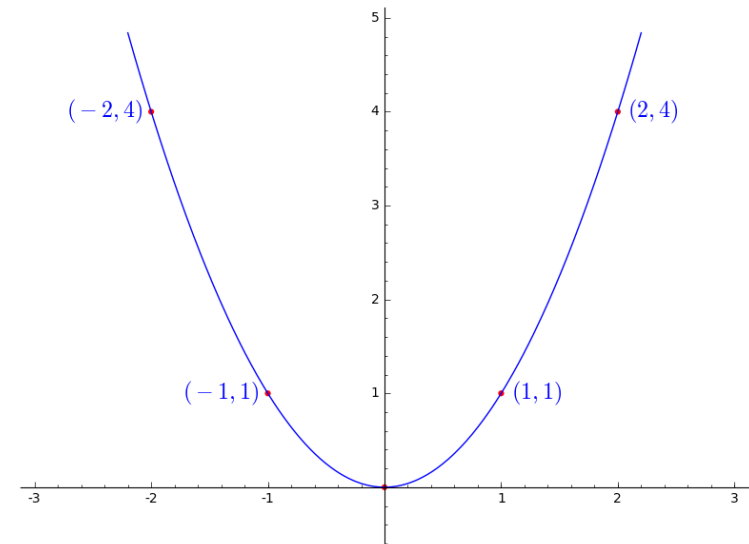
Example 2

(3) Plot the points (x, y) whose coordinates appear in the table:



Example 2

(4) Draw a smooth curve through the plotted points:



Example 3

Example 3

Graph $f(x) = \sqrt{x}$.

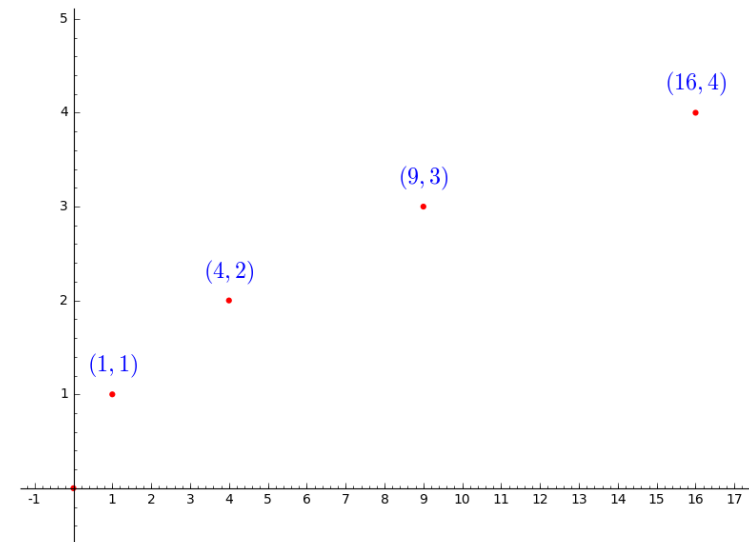
Solution

- (1) Find the domain $\text{Dom}(f) = [0, \infty)$.
- (2) Make a table of xy -pairs that satisfy the equation $y = \sqrt{x}$:

x	$y = \sqrt{x}$
0	0
1	1
4	2
9	3
16	4

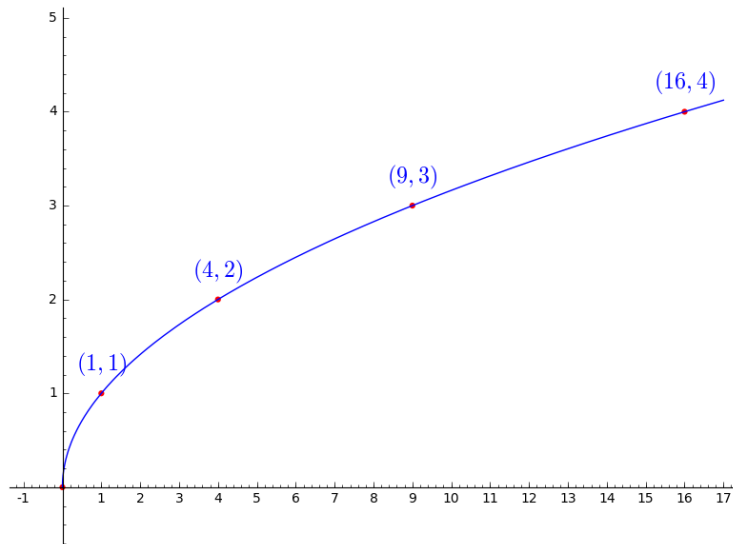
Example 3

(3) Plot the points (x, y) whose coordinates appear in the table:



Example 3

(4) Draw a smooth curve through the plotted points:



Example 4

Example 4

Graph $f(x) = \frac{1}{x}$.

Solution

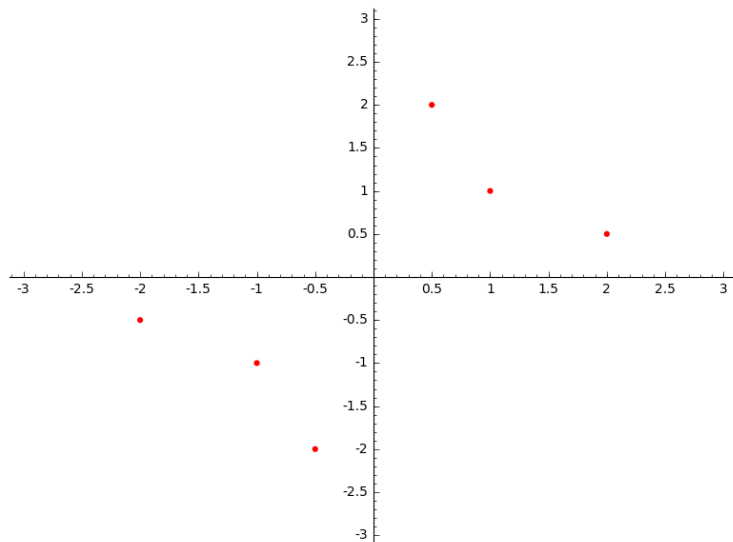
(1) Find the domain $\text{Dom}(f) = \mathbb{R} - \{0\}$.

(2) Make a table of xy -pairs that satisfy the equation $y = \frac{1}{x}$:

x	$y = \frac{1}{x}$
1	1
-1	-1
2	0.5
-2	-0.5
0.5	2
-0.5	-2

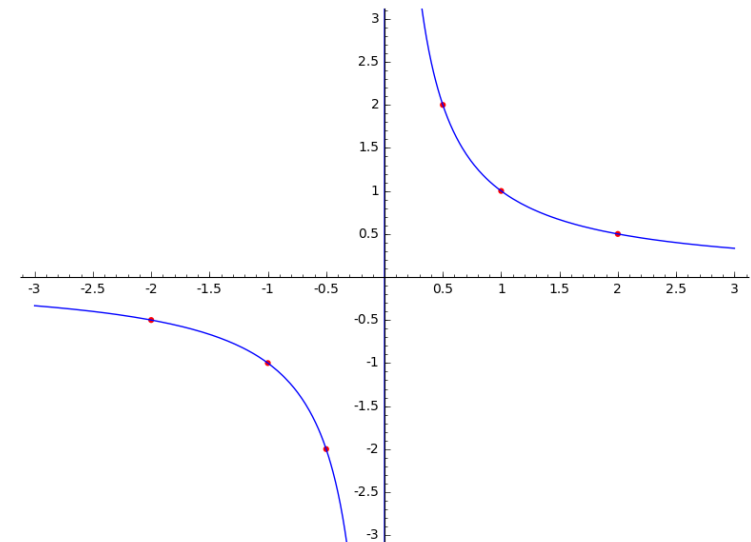
Example 4

(3) Plot the points (x, y) whose coordinates appear in the table:



Example 4

(4) Draw a smooth curve through the plotted points:



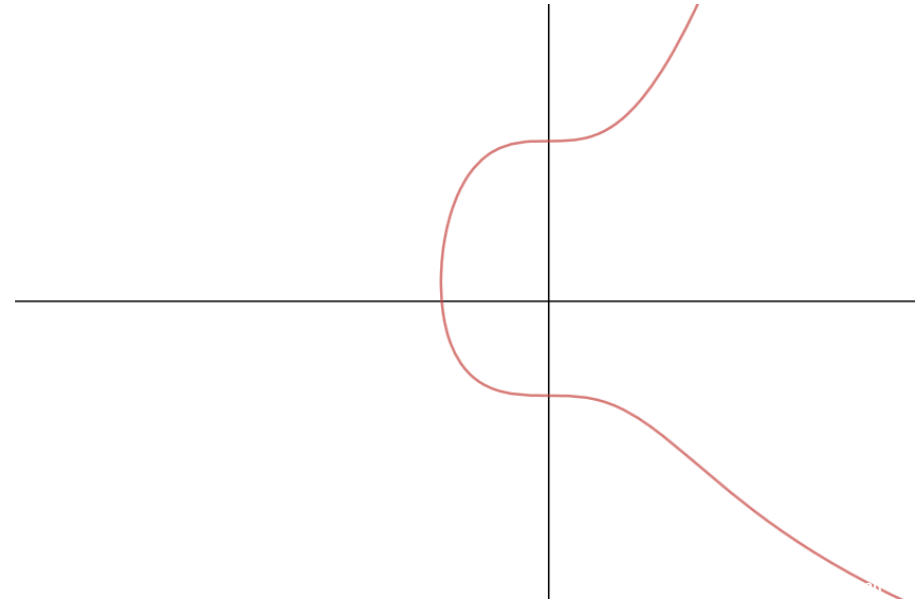
The vertical line test for a function

Any vertical line can not cross the graph of a function at more than one point.

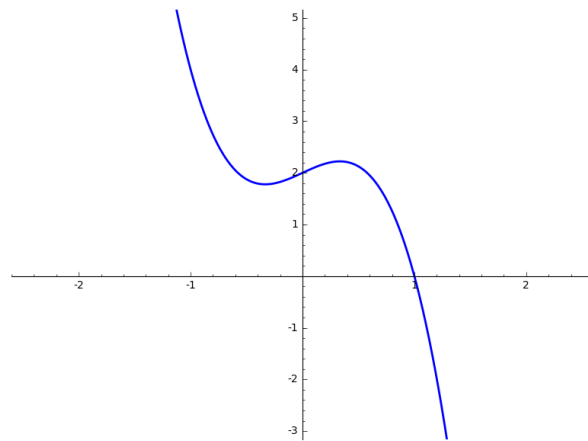
Example

Which of the graphs are graphs of functions of x , and which are not? Give reasons for your answers.

(a)



(b)



Solution

- (a) It is not a graph of a function since there is a vertical line that can cross the graph at more than one point.
- (b) It is a graph of a function since any vertical line that can NOT cross the graph at more than one point.

Piecewise-defined functions

Some functions are defined by different formulas to different parts of their domains.

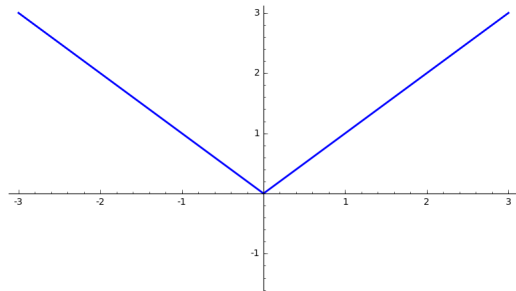
Example (Absolute value function)

The absolute value function:

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

$$\text{Dom}(f) = \mathbb{R} \text{ and } \text{Ran}(f) = [0, \infty)$$

$$|-5| = 5, \quad |3| = 3, \quad |x| = ?$$



Remarks

- ❶ $|x| \geq 0$ for every real number x and $|x| = 0$ if and only if $x = 0$.
- ❷ Since $\sqrt{a} \geq 0$, we may define $\sqrt{x^2} = |x|$.
- ❸ Geometrically, $|x - y|$ represents the distance between x and y . In particular $|x|$ represents the distance between x and 0.
- ❹ $|-x| = |x|$.
- ❺ $|xy| = |x||y|$.
- ❻ $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
- ❼ $|x + y| \leq |x| + |y|$. (The triangle inequality)

Example 1

$$\text{Graph } f(x) = x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Solution

We graph every part of the function on its interval.

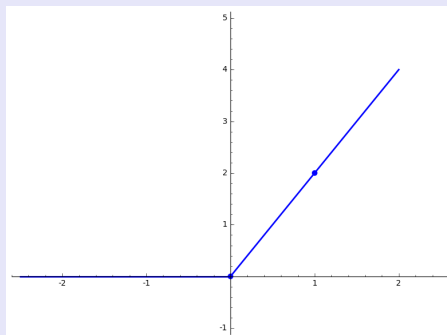
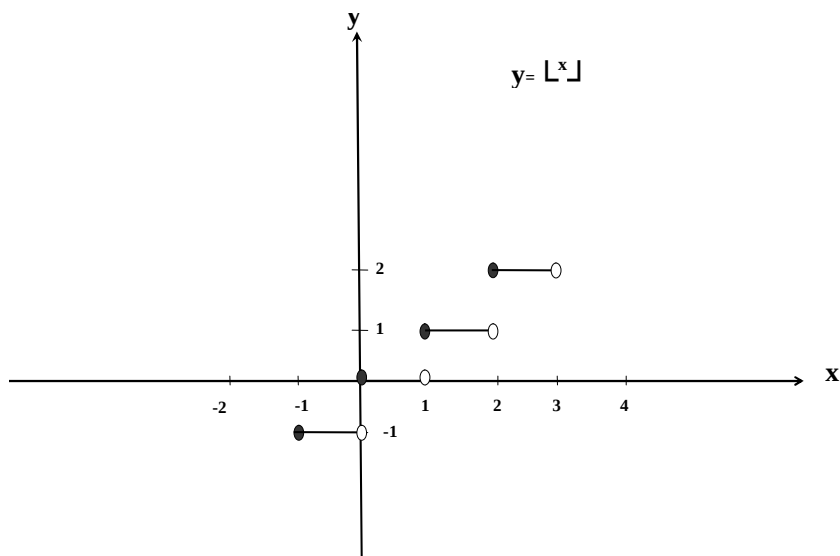


Figure: The graph of $f(x) = x + |x|$

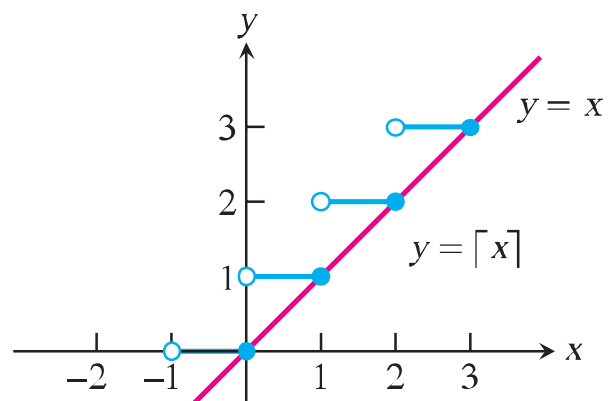
Example 2 (The greatest integer function)

- $f(x) = \lfloor x \rfloor$ is the greatest integer less than or equal to x .
- That is $\lfloor x \rfloor = n$ such that $n \leq x < n + 1$.
- $\text{Dom}(f) = \mathbb{R}$ and $\text{Ran}(f) = \mathbb{Z}$ the set of all integers.
- $\lfloor 2 \rfloor = 2, \quad \lfloor 2.5 \rfloor = 2, \quad \lfloor -2.5 \rfloor = -3$



Example 3 (The least integer function)

- $f(x) = \lceil x \rceil$ is the least (smallest) integer greater than or equal to x .
- That is $\lceil x \rceil = n$ such that $n - 1 < x \leq n$.
- $\text{Dom}(f) = \mathbb{R}$ and $\text{Ran}(f) = \mathbb{Z}$ the set of all integers.
- $\lceil -4 \rceil = -4$, $\lceil 2.5 \rceil = 3$, $\lceil -2.5 \rceil = -2$



Example 4

Solve the equation $\lceil x \rceil = 4$.

Solution

In this example $n = 4$ and by definition $n - 1 < x \leq n$. Thus the solution of the equation is $3 < x \leq 4$

Increasing and decreasing functions

Definition

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I .

- ① f is said to be increasing on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
- ② f is said to be decreasing on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Remarks

- ① It is important to realize that the definitions of increasing and decreasing functions must be satisfied for every pair of points x_1 and x_2 in I with $x_1 < x_2$.
- ② Since we use $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is strictly increasing or decreasing on I .
- ③ The graph of an increasing function rises or climbs as you move from left to right while the graph of a decreasing function descends or falls as you move from left to right.

Examples

Example 1

Show that the function $f(x) = 2x + 1$ is increasing on $\mathbb{R}$ and graph it.

Solution

- Let $x_1, x_2 \in \mathbb{R}$ and assume $x_1 < x_2$.
- We want to show that $f(x_1) < f(x_2)$.
- Starting with $x_1 < x_2$.

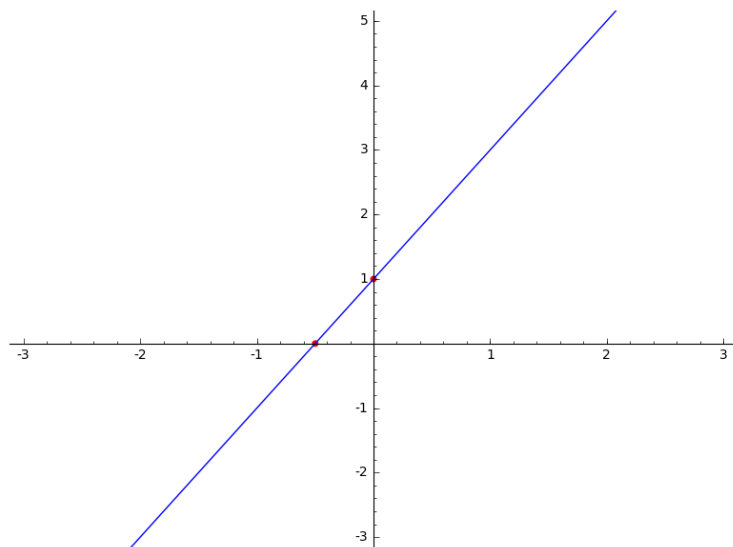
$$x_1 < x_2 \quad \text{multiply both sides by } 2$$

$$\Rightarrow 2x_1 < 2x_2 \quad \text{add 1 to both sides}$$

$$\Rightarrow 2x_1 + 1 < 2x_2 + 1$$

46

- $\Rightarrow f(x_1) < f(x_2)$
- Thus $f(x)$ is increasing on $\mathbb{R}$.



Example 2

Graph the function $f(x) = 4 - x$ and determine whether it is increasing or decreasing on $\mathbb{R}$.

Solution

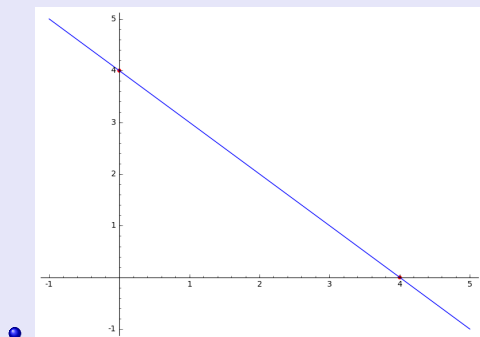


Figure: The graph of $f(x) = 4 - x$

- Note that the graph descends as we move from left to right.
- Thus $f(x)$ is decreasing on $\mathbb{R}$.

Example 3

Consider the function $f(x) = x^2$.

- Graph the function.
- Determine whether it is increasing or decreasing on $\mathbb{R}$.
- Find the interval over which it is increasing and the interval over which it is decreasing.

Solution

- The graph of $y = x^2$

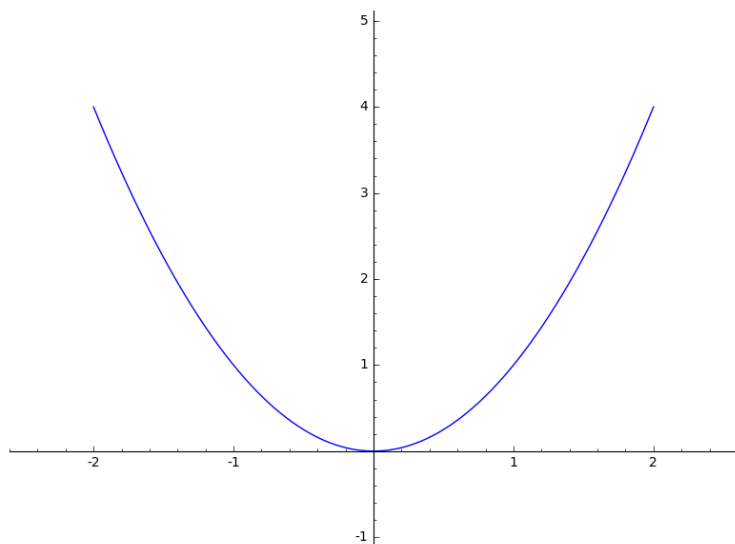


Figure: The graph of $f(x) = x^2$

- Determine whether it is increasing or decreasing on $\mathbb{R}$.
 - Note that as we move from left to right, the graph rises on $[0, \infty)$ and descends on $(-\infty, 0]$.
 - Thus the function $f(x) = x^2$ is neither increasing nor decreasing on $\mathbb{R}$.
- Find the interval over which it is increasing and the interval over which it is decreasing.
 - Note that the graph rises as we move from left to right on $[0, \infty)$.
 - Thus $f(x)$ is increasing on $[0, \infty)$.
 - The graph descends as we move from left to right on $(-\infty, 0]$.
 - Thus $f(x)$ is decreasing on $(-\infty, 0]$.

Example 4 (Solution)

Example 4

Graph the function $f(x) = \frac{1}{|x|}$ and specify the interval over which it is increasing and the interval over which it is decreasing.

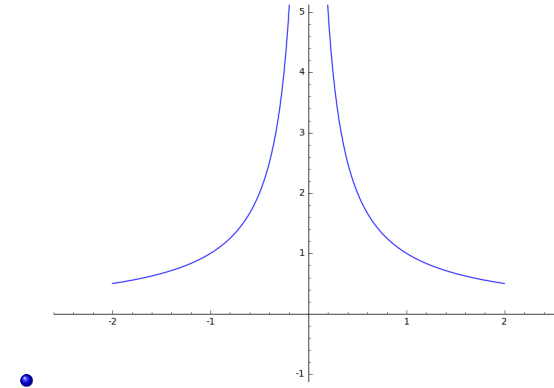


Figure: The graph of $f(x) = 1/|x|$

- From the graph we see that f is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$.

Even and odd functions

Definition

- 1 A function $y = f(x)$ is an even function if $f(-x) = f(x)$ for every $x \in \text{Dom}(f)$.
- 2 A function $y = f(x)$ is an odd function if $f(-x) = -f(x)$ for every $x \in \text{Dom}(f)$.

Example

Determine whether each of the following functions are odd, even, or neither. Give reason for your answer.

- (a) $f(x) = 2x^2 - 3$
- (b) $f(x) = 7 - 5x$
- (c) $f(x) = 2x^3$.
- (d) $f(x) = \frac{x^3}{|x| + 1}$.

Solution

(a) $f(x) = 2x^2 - 3$

- We calculate

$$f(-x) = 2(-x)^2 - 3$$

- Simplify:

$$f(-x) = 2(-x)^2 - 3 = 2x^2 - 3$$

- Compare:

$$f(-x) = 2x^2 - 3 = f(x)$$

- Thus $f(x)$ is even.

Solution

(b) $f(x) = 7 - 5x$

- We calculate

$$f(-x) = 7 - 5(-x)$$

- Simplify:

$$f(-x) = 7 - 5(-x) = 7 + 5x$$

- Compare:

$$f(-x) = 7 + 5x \neq f(x)$$

- and

$$-f(x) = -(7 - 5x) = -7 + 5x$$

-

$$\implies f(-x) \neq -f(x)$$

- Thus $f(x)$ is neither odd nor even.

Solution

(c) $f(x) = 2x^3$.

- We calculate

$$f(-x) = 2(-x)^3$$

- Simplify:

$$f(-x) = -2x^3$$

- Compare:

$$f(-x) = -2x^3 = -f(x)$$

- Thus $f(x)$ is odd.

Solution

(d) $f(x) = \frac{x^3}{|x| + 1}$.

- We calculate

$$f(-x) = \frac{(-x)^3}{|-x| + 1}$$

- Simplify:

$$f(-x) = \frac{-x^3}{|x| + 1}$$

- Compare:

$$f(-x) = -f(x)$$

- Thus $f(x)$ is odd.

Symmetry of even and odd functions

Remark

The graph of an even function is symmetric about the y -axis and the graph of an odd function is symmetric about the origin $(0, 0)$.

Common Functions

A variety of important types of functions are frequently encountered in calculus. We briefly describe them here.

Polynomials and Rational Functions

Definition

(Polynomials)

- (1) Let n be a nonnegative integer and let $a_n, a_{n-1}, \dots, a_1, a_0$ be real numbers (constants). Then a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

is a polynomial.

- (2) The constants $a_n, a_{n-1}, \dots, a_1, a_0$ are the coefficients of the polynomial.
- (3) If $a_n \neq 0$, then n is called the degree of the polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.

Remarks

- ❶ If $p(x)$ is a polynomial function, then $\text{Dom}(p) = \mathbb{R}$.
- ❷ A polynomial of degree 0, $p(x) = a_0$, is also called a constant function.
- ❸ A polynomial of degree 1, $p(x) = a_1 x + a_0$, is also called a linear function.
- ❹ A polynomial of degree 2, $p(x) = a_2 x^2 + a_1 x + a_0$, is also called a quadratic function.

Example 1

- $f(x) = 2x^4 - x^2 + 3x - 5$ is a polynomial function of degree 4.
- $p(x) = 5$ is a polynomial of degree 0.

Example 2

Example

Find the roots of the polynomial $f(x) = x^2 - 5x + 6$

Solution

- To factor a trinomial of the form $x^2 + bx + c$, we note that
-

$$(x + r)(x + s) = x^2 + (r + s)x + rs$$

- So we need to choose numbers r and s so that $r + s = b$ and $rs = c$
- Thus

$$f(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$$

- and the roots are $x = 2$ and $x = 3$.

Rational functions

Definition

A rational function is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

Remark

The domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ is $\mathbb{R} - \{x : q(x) = 0\}$.

Example 1

- $f(x) = \frac{x^2 + 4x + 25}{x - 1}$ is a rational function.
- $g(x) = \frac{4x^2 - 6}{2x^3 - x^2 + 5x}$ is a rational function.
- $f(x) = -x^2$ is a rational function.

Example 2

Example

Find the domain of the rational function $f(x) = \frac{x^2 - 9}{x^2 - 2x - 3}$ and simplify it.

Solution

- Since f is a rational function,
 $\text{Dom}(f) = \mathbb{R} - \{x : x^2 - 2x - 3 = 0\}$

-

$$x^2 - 2x - 3 = 0 \implies (x - 3)(x + 1) = 0 \implies x = 3 \text{ or } x = -1.$$

- Thus $\text{Dom}(f) = \mathbb{R} - \{-1, 3\}$.

- Simplification:

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{(x - 3)(x + 3)}{(x - 3)(x + 1)} = \frac{(x + 3)}{(x + 1)}$$

- Be careful: $\text{Dom}(f) = \mathbb{R} - \{-1, 3\}$

Power functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function.

Remark

- (1) If $a = n$ is a positive integer, then $\text{Dom}(f) = \mathbb{R}$.
 - If n is even, then $\text{Ran}(f) = [0, \infty)$ and if n is odd, then $\text{Ran}(f) = \mathbb{R}$.
- (2) If $a = n$ a negative integer, then $\text{Dom}(f) = \mathbb{R} - \{0\}$.
- (3) If $a = \frac{n}{m}$ is a rational number, then we apply the domain convention to find $\text{Dom}(f)$.

Example

- The functions

$$f(x) = x^3,$$

$$g(x) = x^{2/3},$$

and

$$h(x) = x^{-2}$$

are power functions.

- $\text{Dom}(f) = \mathbb{R}$,
- $\text{Dom}(g) = \mathbb{R}$,
- $\text{Dom}(h) = \mathbb{R} - \{0\}$.

Algebraic functions

Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) is an algebraic function.

Trigonometric functions

The six basic trigonometric functions are reviewed in section 1.3.

Exponential and logarithmic functions

We study exponential and logarithmic functions in Calculus B.

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1.2 Combining Functions; Shifting and Scaling Graphs

First Semester 2020-2021

أ.د. أيمن هاشم السقا

1.2 Combining Function; Shifting and Scaling Graphs

- Many functions arise as combinations of other functions.
- In this section we study the main ways functions are combined or transformed to form new functions.

Sums, Differences, Products, and Quotients

Like numbers, we can add, subtract, multiply, and divide functions. That is, if f and g are two functions, then

- 1 $(f + g)(x) = f(x) + g(x).$
- 2 $(f - g)(x) = f(x) - g(x).$
- 3 $(fg)(x) = f(x)g(x).$
- 4 $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{provided } g(x) \neq 0.$

Remarks

- 1 $\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g).$
- 2 $\text{Dom}(f - g) = \text{Dom}(f) \cap \text{Dom}(g).$
- 3 $\text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g).$
- 4 $\text{Dom}(f/g) = [\text{Dom}(f) \cap \text{Dom}(g)] - \{x : g(x) = 0\}.$

Examples

Example 1

If $f(x) = \frac{1}{x}$ and $g(x) = x^2 - 1$, then find $\text{Dom}(f + g)$, $\text{Dom}(f - g)$, $\text{Dom}(fg)$, $\text{Dom}(f/g)$.

Solution

- We have $\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g) = \text{Dom}(f - g) = \text{Dom}(fg)$
- We find $\text{Dom}(f) = \mathbb{R} - \{0\}$ and $\text{Dom}(g) = \mathbb{R}.$
- Thus
$$\text{Dom}(f + g) = \text{Dom}(f - g) = \text{Dom}(fg)$$
$$= [\mathbb{R} - \{0\}] \cap \mathbb{R} = \mathbb{R} - \{0\}$$
- $$\text{Dom}(f/g) = [\text{Dom}(f) \cap \text{Dom}(g)] - \{x : g(x) = 0\}$$
$$= [\mathbb{R} - \{0\}] - \{-1, 1\} = \mathbb{R} - \{0, 1, -1\}$$

Example 2 (Exam)

Let $f(x) = x + 4$ and $g(x) = \frac{1}{\sqrt{x+4}}$. Find $(fg)(x)$ and $\text{Dom}(fg)$.

Solution

- $$(fg)(x) = f(x)g(x) = (x + 4) \frac{1}{\sqrt{x+4}} = \sqrt{x+4}$$
- and
$$\text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g)$$
- But $\text{Dom}(f) = \mathbb{R}$
- and $\text{Dom}(g) = \{x : x + 4 > 0\} = (-4, \infty).$
- Thus
$$\text{Dom}(fg) = \mathbb{R} \cap (-4, \infty) = (-4, \infty).$$

Composite Functions

Composition is another method for combining functions.

Definition

If f and g are functions, then the composite function $f \circ g$ (f circle g) is defined by $(f \circ g)(x) = f(g(x))$.

Remark

$\text{Dom}(f \circ g) = \{x \in \text{Dom}(g) : g(x) \in \text{Dom}(f)\}$.

Examples

Example 1

Let $f(x) = |x - 1|$, $g(x) = \sqrt{x - 3}$. Find $(f \circ g)(7)$, $(g \circ f)(7)$, $(f \circ f)(-1)$.

- $(f \circ g)(7) = f(g(7)) = f(2) = 1$
- $(g \circ f)(7) = g(f(7)) = g(6) = \sqrt{3}$
- $(f \circ f)(-1) = f(f(-1)) = f(2) = 1$

Example 2 (Exam)

Let $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = x^2 - 4x$. Find $(f \circ g)(x)$ and $\text{Dom}(f \circ g)$.

Solution

- Firstly, we calculate

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 4x) = \frac{1}{\sqrt{x^2 - 4x}}$$

- Next we calculate

$$\text{Dom}(f \circ g) = \{x \in \text{Dom}(g) : g(x) \in \text{Dom}(f)\}$$

- $= \{x \in \mathbb{R} : x^2 - 4x \in (0, \infty)\}$

- We simplify

$$x^2 - 4x \in (0, \infty) \implies x^2 - 4x > 0 \implies x(x - 4) > 0$$

- Find the intervals:**

- The factors of the left-hand side are x and $x - 4$.
- These are zero when x is 0 and 4.
- These numbers divide the real line into the intervals

$$(-\infty, 0), \quad (0, 4), \quad (4, \infty)$$

- Make a diagram (table):**

- We make the following diagram using test points to determine the sign of each factor in each interval.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
sign of x	−	+	+
sign of $x - 4$	−	−	+
sign of $x(x - 4)$	+	−	+

- From the diagram we see that the inequality is satisfied on the intervals $(-\infty, 0)$ and $(4, \infty)$.

- No need for checking the endpoints.
- So the solution of the inequality $x(x - 4) > 0$ is $(-\infty, 0) \cup (4, \infty)$.
- Thus $\text{Dom}(f \circ g) = \{x \in \mathbb{R} : x \in (-\infty, 0) \cup (4, \infty)\}$
- $\text{Dom}(f \circ g) = \mathbb{R} \cap [(-\infty, 0) \cup (4, \infty)] = (-\infty, 0) \cup (4, \infty)$.

Example 3 (Exam)

Let $f(x) = \sqrt{x - 4}$ and $g(x) = x^2$. Find $(g \circ f)(x)$ and $\text{Dom}(g \circ f)$.

Solution

- Firstly, we calculate
 $(g \circ f)(x) = g(f(x)) = g(\sqrt{x - 4}) = (\sqrt{x - 4})^2 = x - 4$
- Next we calculate

$$\text{Dom}(g \circ f) = \{x \in \text{Dom}(f) : f(x) \in \text{Dom}(g)\}$$

- $$= \{x \in [4, \infty) : \sqrt{x - 4} \in \mathbb{R}\}$$
- We simplify $\sqrt{x - 4} \in \mathbb{R} \implies x \in [4, \infty)$
- Thus $\text{Dom}(g \circ f) = \{x \in [4, \infty) : x \in [4, \infty)\}$
- $= [4, \infty) \cap [4, \infty) = [4, \infty)$.

Example 4

Let $f(x) = (x - 1)^{1/4}$. Find $(f \circ f)(x)$ and $\text{Dom}(f \circ f)$.

Solution

- We calculate
 $(f \circ f)(x) = f(f(x)) = f((x - 1)^{1/4}) = [(x - 1)^{1/4} - 1]^{1/4}$
- Next we calculate
 $\text{Dom}(f \circ f) = \{x \in \text{Dom}(f) : f(x) \in \text{Dom}(f)\}$
- $= \{x \in [1, \infty) : (x - 1)^{1/4} \in [1, \infty)\}$
- We simplify $(x - 1)^{1/4} \in [1, \infty) \implies (x - 1)^{1/4} \geq 1$
- $\implies x - 1 \geq 1 \implies x \geq 2 \implies x \in [2, \infty)$
- $\text{Dom}(f \circ f) = \{x \in [1, \infty) : x \in [2, \infty)\}$
- $= [1, \infty) \cap [2, \infty) = [2, \infty)$.

Shifting a Graph of a Function

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally.

Remarks

- To find the point at which a graph of a function $y = f(x)$ intersects the y -axis we substitute $x = 0$ into the equation $y = f(x)$.
- To find the points at which a graph of a function intersects the x -axis, we solve $f(x) = 0$.

Shift Formulas

- (1) **Vertical shifts:** $y = f(x) + k$ or $y - k = f(x)$ shifts the graph of $y = f(x)$ up k units if $k > 0$ and down $|k|$ units if $k < 0$.
- (2) **Horizontal shifts:** $y = f(x - h)$ shifts the graph of $y = f(x)$ right h units if $h > 0$ and left $|h|$ units if $h < 0$.

Example 1

Use the graph of $y = x^2$ and shifts to graph the following functions:

- (a) $y = x^2 - 2$
- (b) $y = (x - 3)^2$.

Solution

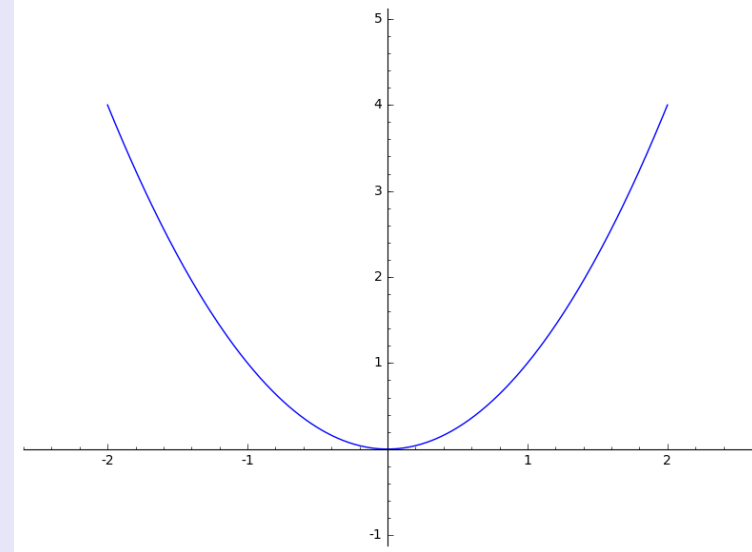


Figure: The graph of $f(x) = x^2$

(a) $y = x^2 - 2$

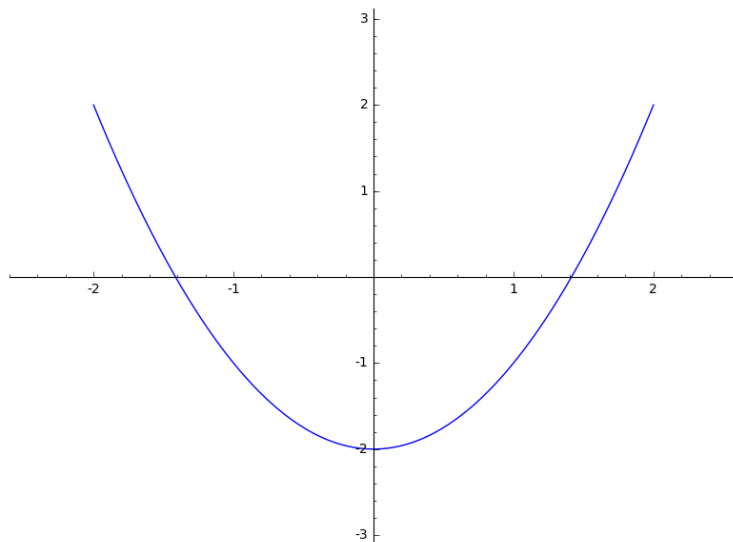


Figure: The graph of $f(x) = x^2 - 2$

(b) $y = (x - 3)^2$

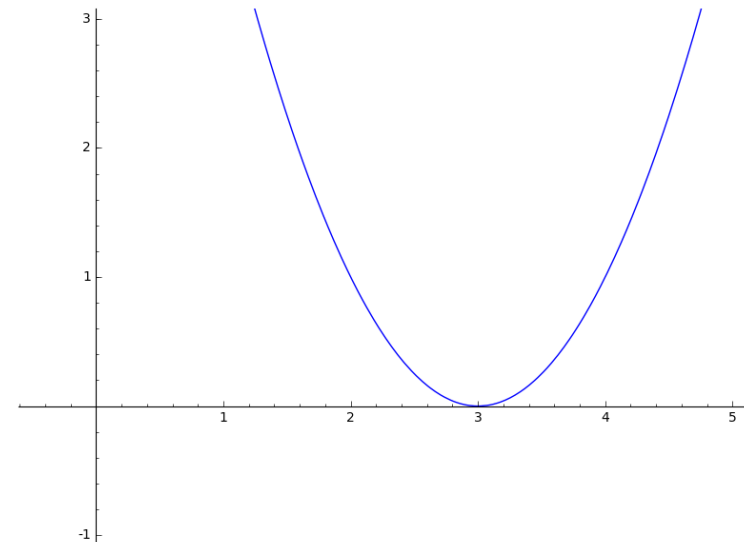


Figure: The graph of $f(x) = (x - 3)^2$

Example 2

Find the equation for the graph obtained by shifting the graph of $y = |x|$ up 2 units and left 1 unit. Then graph the original and shifted graphs together.

Solution

- Shifting the graph up 2 units: replace y by $y - 2$
 $\Rightarrow y - 2 = |x|$ or $y = |x| + 2$
- Shifting the graph left 1 unit: replace x by $x + 1$
 $\Rightarrow y = |x + 1| + 2$

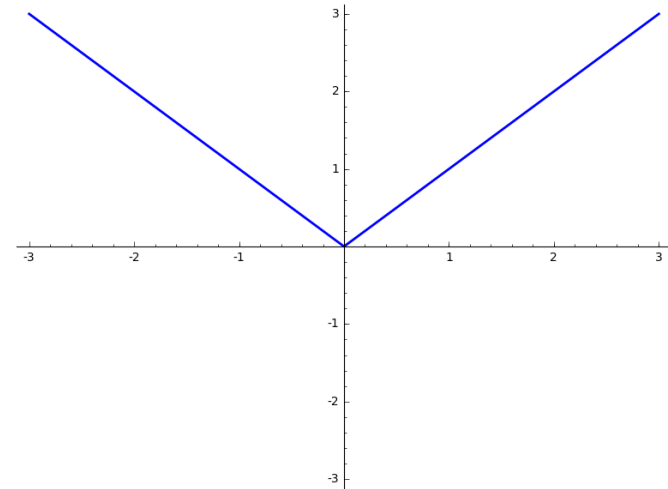


Figure: The graph of $f(x) = |x|$

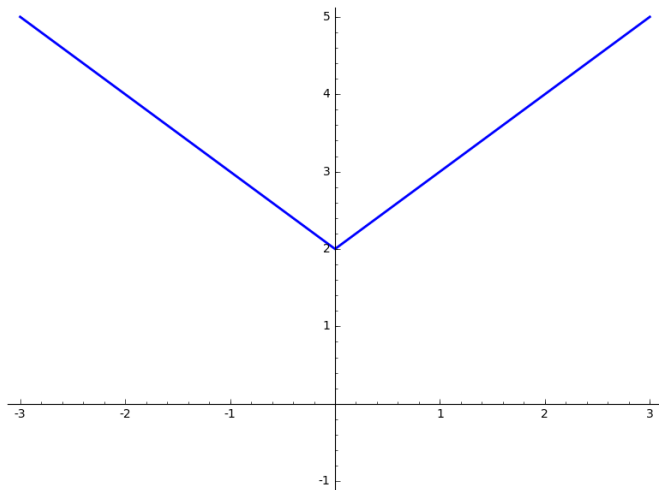


Figure: The graph of $f(x) = |x| + 2$

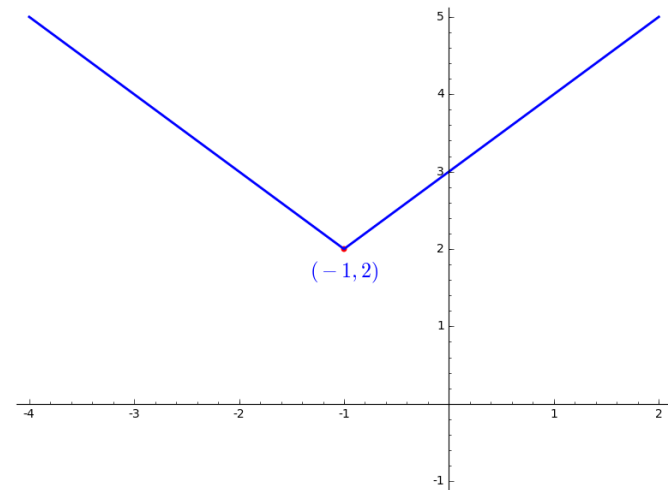


Figure: The graph of $f(x) = |x + 1| + 2$

Example 3

Graph $f(x) = \sqrt{x-5} + 3$.

Solution

- The function f is obtained from $y = \sqrt{x}$
- x is replaced by $x - 5$
- and y is replaced by $y - 3$
($y = \sqrt{x-5} \rightarrow y - 3 = \sqrt{x-5}$)
- Thus the graph of $f(x) = \sqrt{x-5} + 3$ can be obtained by shifting the graph of $y = \sqrt{x}$, 5 units to the right and 3 units up.

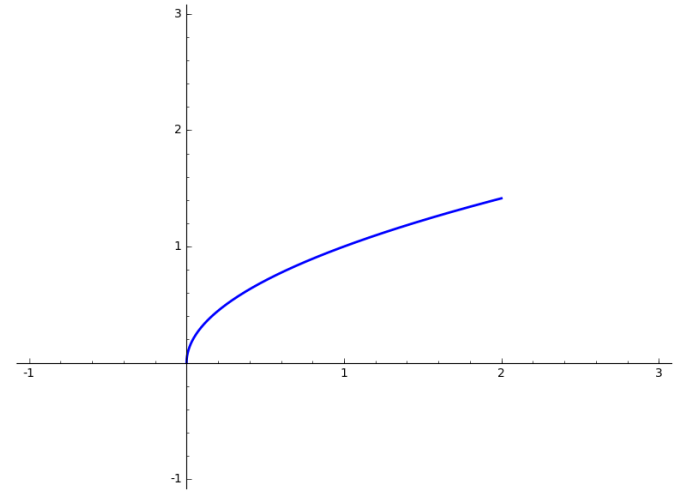


Figure: The graph of $f(x) = \sqrt{x}$

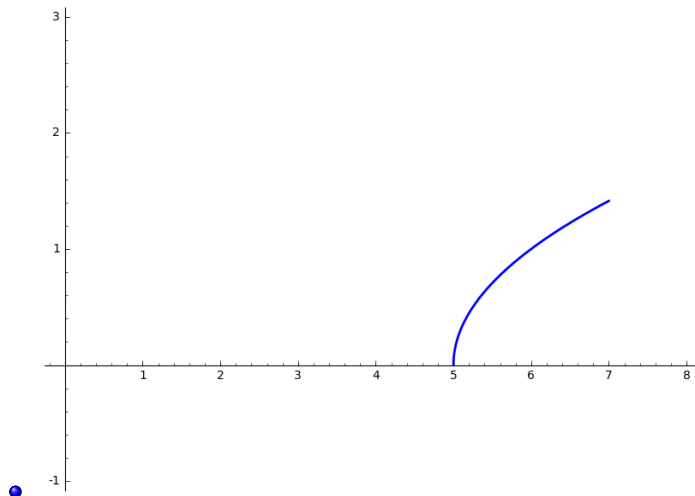


Figure: The graph of $f(x) = \sqrt{x-5}$

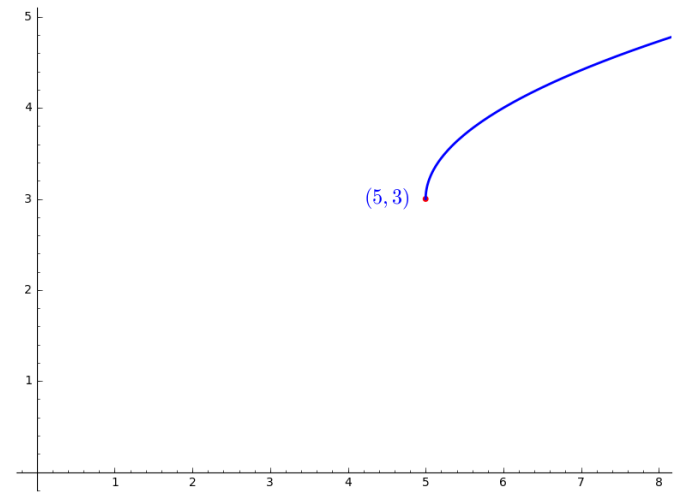


Figure: The graph of $f(x) = \sqrt{x-5} + 3$

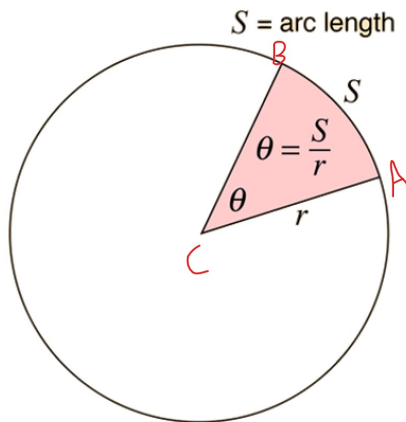


1.3 Trigonometric Functions

First Semester 2020-2021

أ. د. أيمن هاشم السقا

- The number of radians in the central angle ACB within a circle of radius r is defined as the number of radians units contained in the arc s subtended by the central angle.

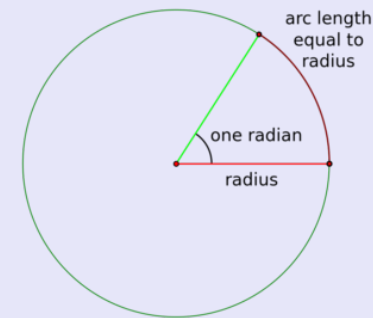


1.3 Trigonometric Functions

- This section reviews radian measure, basic trigonometric (circular) functions, periodicity, and basic trigonometric identities.

Angles

- Angles are measured in degrees or radians.
- One radian is the measure of the central angle whose arc length is equal to the radius of the circle.



- That is the radian measure θ of the angle ACB is given by

$$\theta = \frac{s}{r}$$

- If $r = 1$, we see that the radian measure θ of the central angle ACB is the length of the circular arc AB that the angle cut from the unit circle.

Remarks

- (1) Since one complete revolution of the unit circle is 360° or 2π radian, we have
 π radians $= 180^\circ$. Thus 1 radian $\approx 57.3^\circ$ and $1^\circ \approx 0.017$ radians.
- (2) To convert degrees to radians multiply by $\pi/180$.
- (3) To convert radians to degrees multiply by $180/\pi$.

Example 1

$$\pi/2 \text{ radians} = 90^\circ$$

$$\pi/6 \text{ radians} = 30^\circ$$

Example 2

A central angle in a circle of radius 8 is subtended by an arc of length 10π . Find the angle's radian and degree measures.

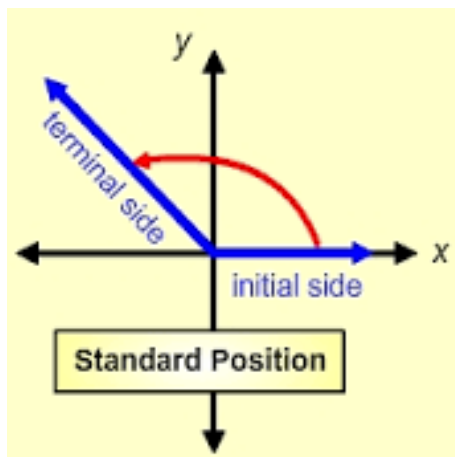
Solution

- In this example we have $s = 10\pi$ and $r = 8$.
- Thus the angle's measure in radians is $\theta = \frac{s}{r} = \frac{10\pi}{8} = \frac{5\pi}{4}$ radians.
- The angle's measure in degrees is $\theta = \frac{5\pi}{4} \times \frac{180}{\pi} = 225^\circ$.

Standard position

Definition

An angle in the xy -plane is in standard position if its vertex lies at the origin and its initial ray lies along the positive x -axis.



Angle convention

Remark

In this course it is assumed that all angles are measured in radians unless degrees or some other units is stated explicitly.

The six basic trigonometric (circular) functions

You are probably familiar with the definitions of the trigonometric functions of an acute angle in terms of the sides of a right triangle. We extend this definition to any angle in the following way:

$$\sin \theta = \frac{y}{r}$$

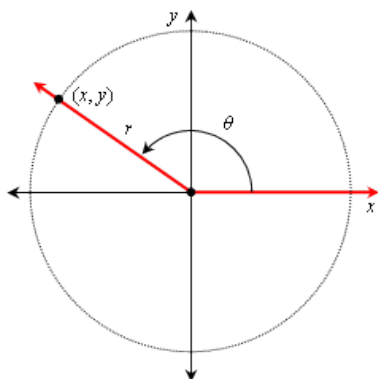
$$\sec \theta = \frac{r}{x}$$

$$\cos \theta = \frac{x}{r}$$

$$\csc \theta = \frac{r}{y}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$



• We extend this definition to any angle in the following way:

- 1 Sine: $\sin \theta = \frac{y}{r}$
- 2 Cosine: $\cos \theta = \frac{x}{r}$
- 3 Tangent: $\tan \theta = \frac{y}{x}$
- 4 Cotangent: $\cot \theta = \frac{x}{y}$
- 5 Secant: $\sec \theta = \frac{r}{x}$
- 6 Cosecant: $\csc \theta = \frac{r}{y}$

Remarks

- 1 Since $r > 0$, $\sin \theta$ and $\cos \theta$ are always defined.
- 2 $\tan \theta$ and $\sec \theta$ are not defined when $x = 0$; that is when $\theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$
- 3 $\cot \theta$ and $\csc \theta$ are not defined when $y = 0$; that is when $\theta = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \dots$

106

Relations between the trigonometric functions

$$1 \quad \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$2 \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$3 \quad \sec \theta = \frac{1}{\cos \theta}$$

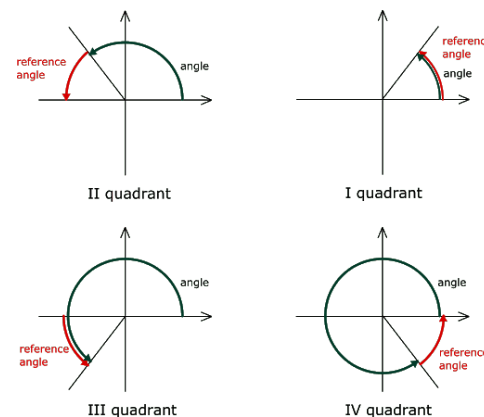
$$4 \quad \csc \theta = \frac{1}{\sin \theta}$$

Finding the Exact Values of the Trigonometric Functions for Any Angle

Definition

(Reference angle)

For any angle θ , the acute angle θ^* formed by the terminal side of θ and the x-axis is called the reference angle for θ .



Relations between the values of trigonometric function at θ and θ^*

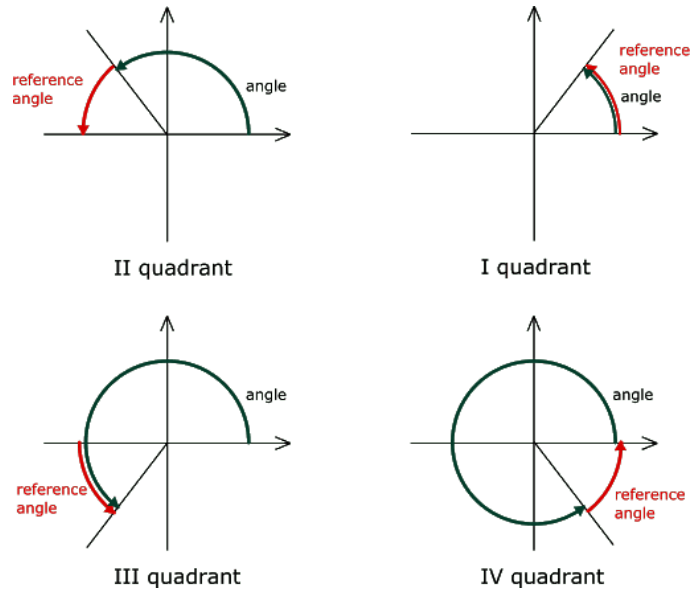
Theorem

If θ^* is the reference angle of an angle θ , then

- (1) $\sin^2(\theta) = \sin^2(\theta^*)$
- (2) $\cos^2(\theta) = \cos^2(\theta^*)$
- (3) $\tan^2(\theta) = \tan^2(\theta^*)$
- (4) $\cot^2(\theta) = \cot^2(\theta^*)$
- (5) $\sec^2(\theta) = \sec^2(\theta^*)$
- (6) $\csc^2(\theta) = \csc^2(\theta^*)$

Remark

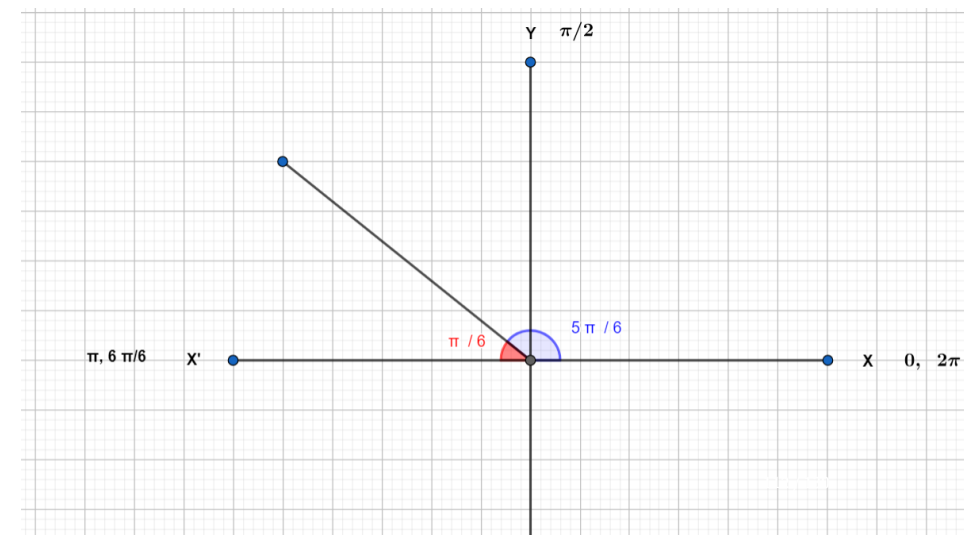
The above Theorem says that if the exact value of a trigonometric function for the reference angle θ^* is $f(\theta^*)$, then the exact value of trigonometric function for the angle θ is $\pm f(\theta^*)$, where the signs are chosen according to the quadrant in which the angle θ lies.



Solution

(a) $\sin\left(\frac{5\pi}{6}\right) = ?$

• $\theta = \frac{5\pi}{6} = 150^\circ \Rightarrow \theta^* = \frac{\pi}{6}$



Example

Find the exact value of each of the following trigonometric functions using reference angles.

(a) $\sin\left(\frac{5\pi}{6}\right)$

(b) $\tan\left(\frac{7\pi}{4}\right)$

Solution

Solution

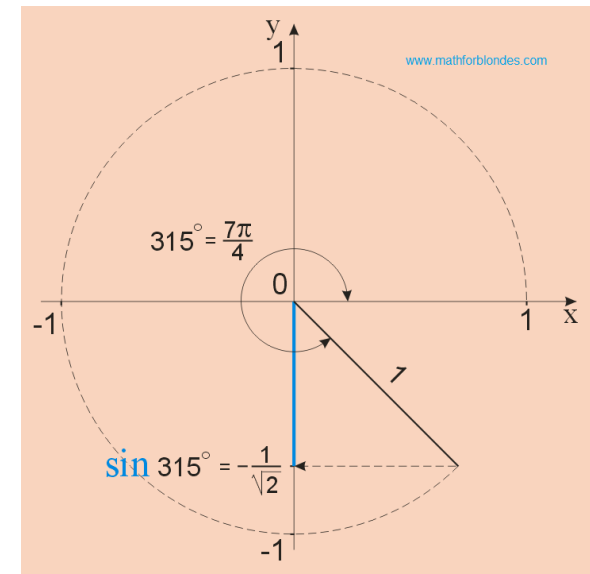
- $\theta = \frac{5\pi}{6} = 150^\circ \Rightarrow \theta^* = \frac{\pi}{6}$
- But $\sin(\theta^*) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.
- Thus $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$ or $-\frac{1}{2}$.
- Since θ lies in the second quadrant, $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

- But $\tan(\theta^*) = \tan\left(\frac{\pi}{4}\right) = 1$.
- Thus $\tan\left(\frac{7\pi}{4}\right) = 1$ or -1 .
- Since θ lies in the fourth quadrant, $\tan\left(\frac{7\pi}{4}\right) = -1$.

Solution

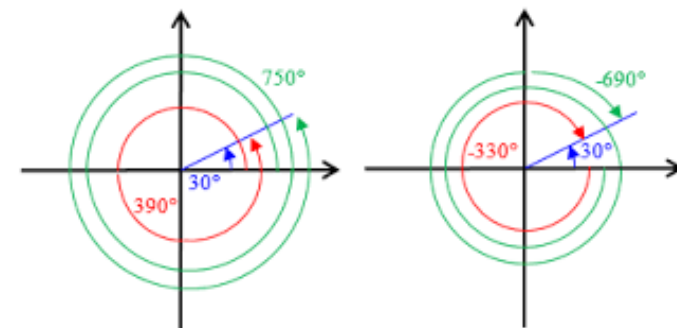
(b) $\tan\left(\frac{7\pi}{4}\right) = ?$

• $\theta = \frac{7\pi}{4} = 315^\circ \Rightarrow \theta^* = \frac{\pi}{4}$



Periodicity and Graphs of the Trigonometric Functions

- When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide.
- Thus the two angles have the same trigonometric function values.
- We describe this repeating behavior by saying that the six basic trigonometric functions are periodic.



Periodic functions

Definition

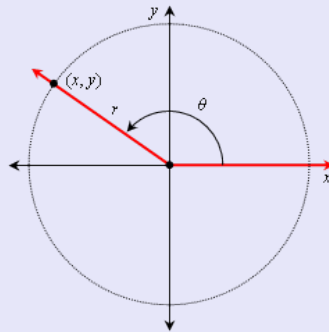
A function $f(x)$ is periodic if there is a positive number p such that $f(x + p) = f(x)$ for every $x \in \text{Dom}(f)$. The smallest such value of p is the period of f .

Periods of trigonometric functions

$$\sin \theta = \frac{y}{r} \quad \sec \theta = \frac{r}{x}$$

$$\cos \theta = \frac{x}{r} \quad \csc \theta = \frac{r}{y}$$

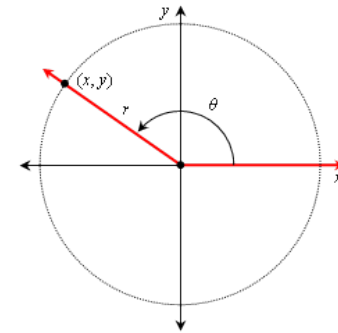
$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$



$$\sin \theta = \frac{y}{r} \quad \sec \theta = \frac{r}{x}$$

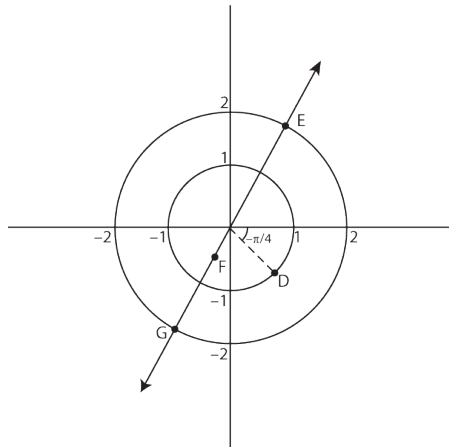
$$\cos \theta = \frac{x}{r} \quad \csc \theta = \frac{r}{y}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$



$$\bullet \quad \boxed{\frac{x}{y} = \frac{-x}{-y}} = \frac{2x}{2y} = \dots$$

Periods of trigonometric functions



- If point (x, y) lies on a line $y = ax$, then the point $(-x, -y)$ also lies on the same line.

$$y = ax \implies -y = a(-x)$$

Periods of trigonometric functions

Remarks

- (1) $\tan x$ and $\cot x$ have period $p = \pi$.
- (2) $\cos x$, $\sin x$, $\sec x$, and $\csc x$ have period $p = 2\pi$.
- (3) If $f(x)$ has period p , then $\boxed{f(ax)}$ has period $\frac{p}{|a|}$.
- (4) If $f(x)$ has period p , then

$$f(x - k), \quad f(x) + h, \quad -f(x), \quad f(-x), \quad af(x)$$

have period p .

The graph of $y = \sin x$

Domain of $\sin x$ is $\mathbb{R}$.

Range of $\sin x$ is $[-1, 1]$.

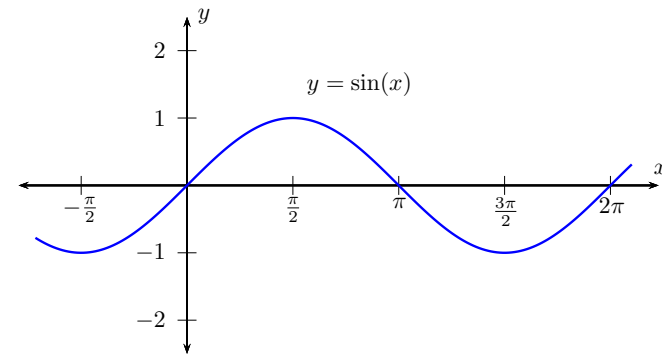


Figure: The sine function

Example

Find the period of each of the following functions:

- (1) $\sin(x - \frac{\pi}{2})$.
- (2) $\tan(\frac{x}{2})$.
- (3) $6 \sec(\frac{2x}{3} - \frac{\pi}{3}) - 2$.

Solution

- (a) The period of $f(x) = \sin(x - \frac{\pi}{2})$ is $p = \frac{2\pi}{|a|} = 2\pi$.
- (b) The period of $f(x) = \tan(\frac{x}{2})$ is $p = \frac{\pi}{|a|} = \frac{\pi}{1/2} = 2\pi$.
- (c) The period of $f(x) = 6 \sec(\frac{2x}{3} - \frac{\pi}{3}) - 2$ is $p = \frac{2\pi}{|a|} = \frac{2\pi}{2/3} = 3\pi$.

The graph of $y = \cos x$

Domain of $\cos x$ is $\mathbb{R}$.

Range of $\cos x$ is $[-1, 1]$.

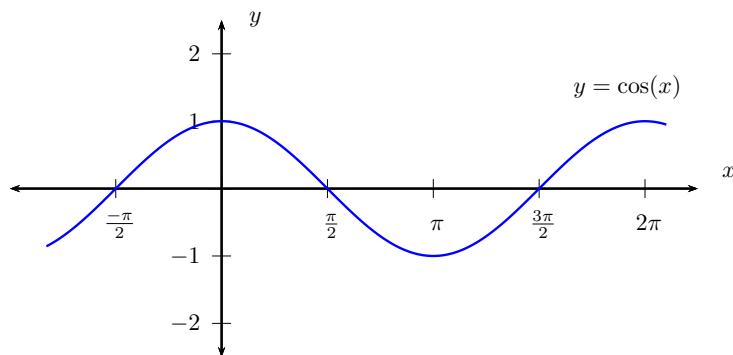


Figure: The Cosine function

The graph of $y = \tan x$

Domain of $\tan x$ is $\mathbb{R} - \{(2n+1)\pi/2 : n \in \mathbb{Z}\}$.

Range of $\tan x$ is $\mathbb{R}$.

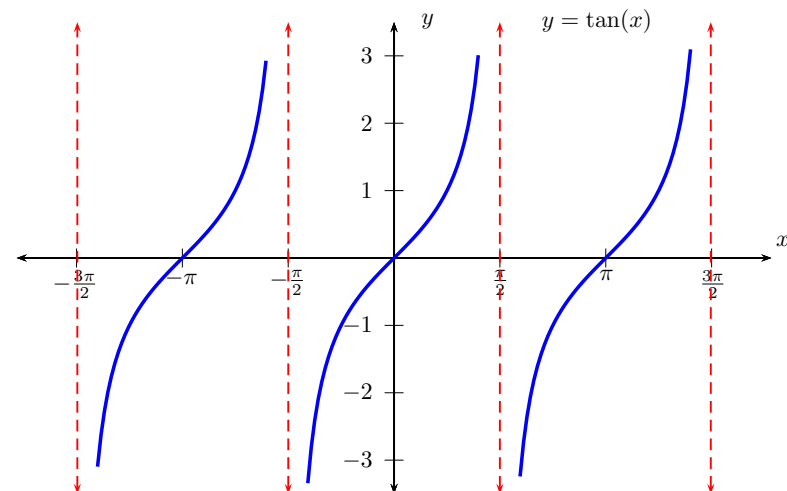


Figure: The Tangent function

Even and odd

Remark

- (1) $\cos x$ and $\sec x$, are even functions.
- (2) $\sin x$, $\tan x$, $\cot x$, and $\csc x$ are odd functions.

Example

If f is an even function and $g(x) = \frac{\sqrt[3]{x}}{\sec x - f(x)}$, then determine whether g is odd, even, or neither.

Solution

- We calculate $g(-x) = \frac{\sqrt[3]{-x}}{\sec(-x) - f(-x)}$
- $g(-x) = \frac{-\sqrt[3]{x}}{\sec(x) - f(x)} = -g(x)$
- Thus g is odd.

Trigonometric Identities

There are many equations, called trigonometric identities, that relates the basic trigonometric functions. The most important ones are:

- (1) $\cos^2 x + \sin^2 x = 1$
- (2) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
- (3) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

Example 1

Show that $\cos^2 x = \frac{1 + \cos(2x)}{2}$.

Solution

- Using the second identity we find

$$\cos(2x) = \cos(x + x)$$

-

$$\cos(2x) = \cos^2 x - \sin^2 x$$

-

$$\cos(2x) = \cos^2 x - [1 - \cos^2 x]$$

-

$$\cos(2x) = 2\cos^2 x - 1$$

- Solving for $\cos^2 x$, we find $\cos^2 x = \frac{1 + \cos(2x)}{2}$.

Example 2

Show that $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

Solution

- Using the third identity we find

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2}\right)\cos x - \sin x \cos\left(\frac{\pi}{2}\right)$$

-

$$\sin\left(\frac{\pi}{2} - x\right) = (1)(\cos x) - (\sin x)(0) = \cos x$$

If a , b , and c are sides of a triangle ABC and θ is the angle opposite c , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

The law of cosines generalize the Pythagorean theorem. If $\theta = \pi/2$, then $\cos \theta = 0$ and hence we have $c^2 = a^2 + b^2$.

تم بحمد الله



مَا أَصْعَبُ جُزْءٍ فِي هَذَا الْفَصْلِ؟
What is the most difficult part of this Chapter?