



Chapter 2

LIMITS AND CONTINUITY

First Semester 2020-2021

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2.1 Rates of change and tangents to curves

- Calculus is a tool to help us understand how functional relationships change, such as the position or speed of a moving object as a function of time, or the changing slope of a curve being traversed by a point moving along it.
- In this section we introduce the ideas of average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve.
- We give precise developments of these important concepts in the next chapter, but for now we use an informal approach so you will see how they lead naturally to the main idea of the chapter, the limit.

Limits

- The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry.
- You will see that limits play a major role in calculus and the study of change.
- We need limits to study how the values of a function $f(x)$ behave when its argument x gets closer and closer to a given value x_0 not necessary in the domain of f .
- This might be the case, for instance, when trying to evaluate a function at x_0 leads to division by zero.

Average and Instantaneous Speed

- A moving body's average speed over any particular time interval is the amount of distance covered during the interval divided by the length of the interval:

$$\text{Average speed} = \frac{\Delta y}{\Delta t}.$$

- We want a way to determine the speed of a falling object at a single instant t_0 , instead of using its average speed over an interval of time.
- To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0 .
- We cannot use this formula to calculate the “instantaneous” speed at the exact moment t_0 by simply substituting $\Delta t = 0$, because we cannot divide by zero.

Average Rates of Change and Secant Lines

Definition

(Average rate of change)

The average rate of change of a function $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h},$$

where $h = x_2 - x_1 \neq 0$.

Remark

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line (the secant line) through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

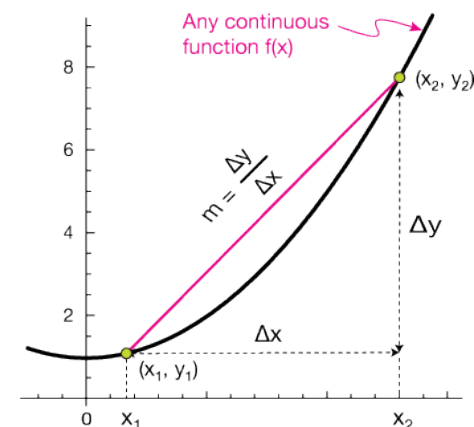


Figure: A secant to the graph $y = f(x)$

Defining the slope of a curve

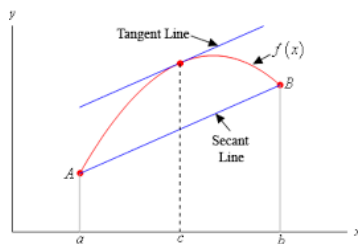


Figure: Tangent and secant lines

The slope of a curve $y = f(x)$ at a point $P(x_1, y_1)$ on the curve is the slope of the tangent line at P .

$$\text{slope} = \lim_{h \rightarrow 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

2.2 Limits of a Function and Limit Laws

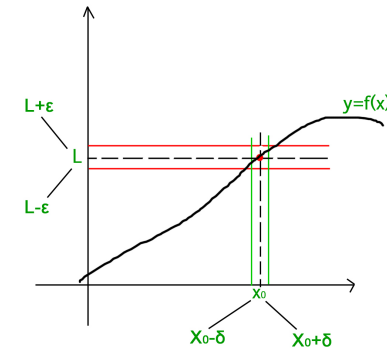
- In this section we give an informal definition of limit and show how we can calculate the values of limits.

Limits of function values

- Frequently when we study a function $y = f(x)$, we are interested in the behavior of the function near a particular point x_0 , but not at x_0 .
- Limits give us an estimate of $f(x)$ when we can't compute a result directly.
- How do we obtain an estimation?
- Zoom into the neighboring points.
- If our prediction is always in-between neighboring points, no matter how much we zoom, that's our estimation.

Informal definition of limit

- Suppose $f(x)$ is defined on an open interval about x_0 , except possibly at x_0 itself.
- If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0 , we say that f approaches the limit L as x approaches x_0 , and write $\lim_{x \rightarrow x_0} f(x) = L$, which is read "the limit of $f(x)$ as x approaches x_0 is L "



Example

Let $f(x) = \frac{x^2 - x - 2}{x - 2}$. How does $f(x)$ behave near $x = 2$?

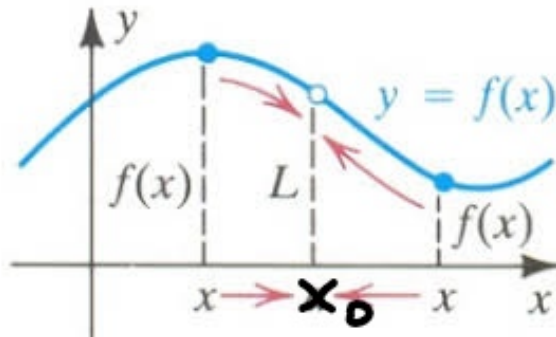
Solution

$\text{Dom}(f) = \mathbb{R} - \{2\}$

x	$f(x)$
1.9	2.9
2.1	3.1
1.99	2.99
2.01	3.01
1.9999	2.9999

As x gets closer and closer to 2, $f(x)$ get closer and closer to 3.

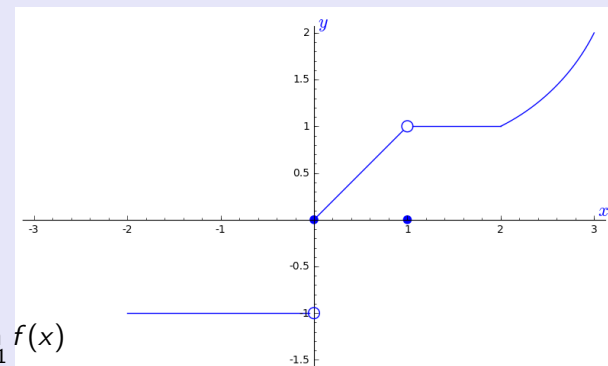
- النهاية عملية التصاق و ليست عملية اندماج
- No distance between $f(x)$ and L when there is no distance between x and x_0 , but $f(x_0)$ is not L .
- Zoom into the neighboring points.
- If our prediction is always in-between neighboring points, no matter how much we zoom, that's our estimation.



Examples

Example 1

For the function $f(x)$ graphed below, find the following limits or explain why they do not exist.



- $\lim_{x \rightarrow 1} f(x)$
- $\lim_{x \rightarrow 2} f(x)$
- $\lim_{x \rightarrow 0} f(x)$

Solution

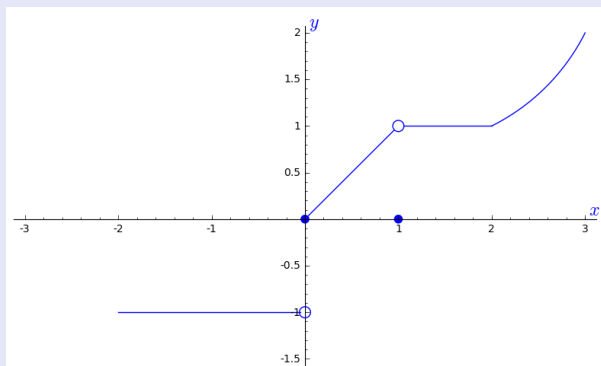


Figure: Graph of $f(x)$

- (a) $\lim_{x \rightarrow 1} f(x) = 1$
 (b) $\lim_{x \rightarrow 2} f(x) = 1$
 (c) $\lim_{x \rightarrow 0} f(x)$ does not exist because $f(x)$ approaches -1 when x approach 0 from the left and $f(x)$ approaches 0 when x approaches 0 from the right.

Example 2

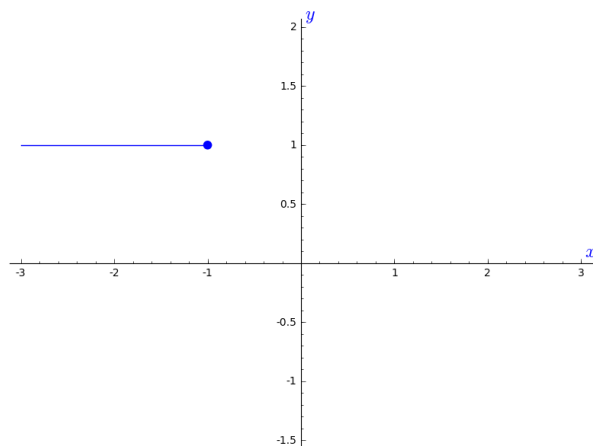
$$\text{Graph } f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1, \end{cases}$$

then find the following limits or explain why they do not exist.

- (a) $\lim_{x \rightarrow -1} f(x)$
 (b) $\lim_{x \rightarrow 0} f(x)$
 (c) $\lim_{x \rightarrow 1} f(x)$

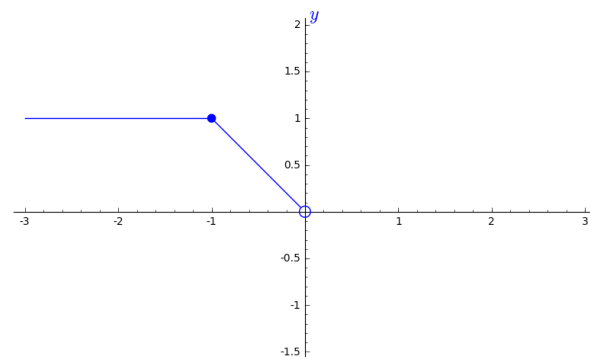
Solution

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1, \end{cases}$$



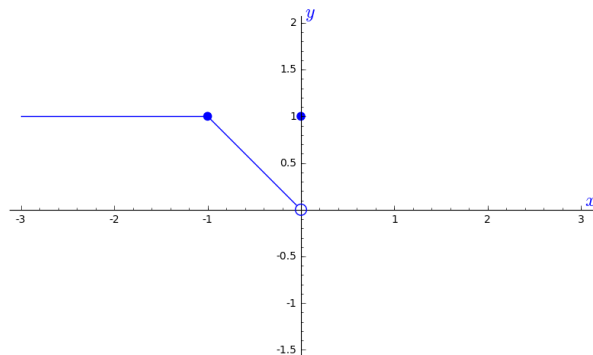
Solution

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1, \end{cases}$$



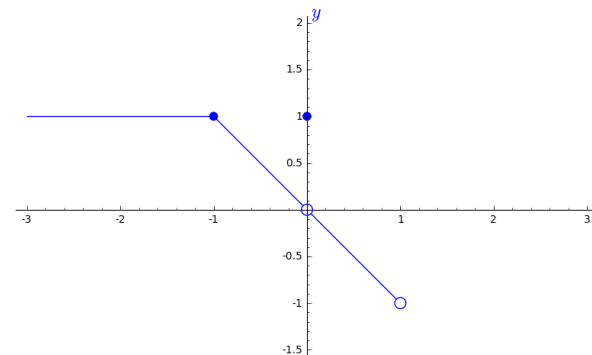
Solution

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1, \end{cases}$$



Solution

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1, \end{cases}$$



Solution

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1, \end{cases}$$

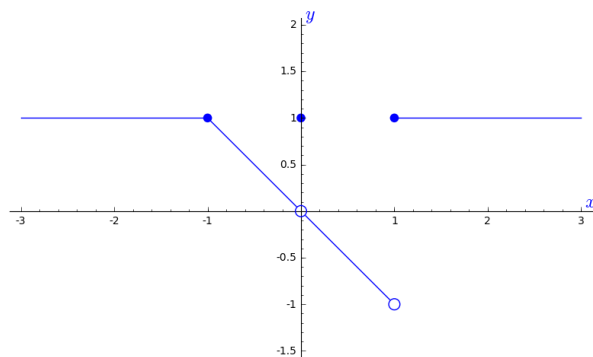
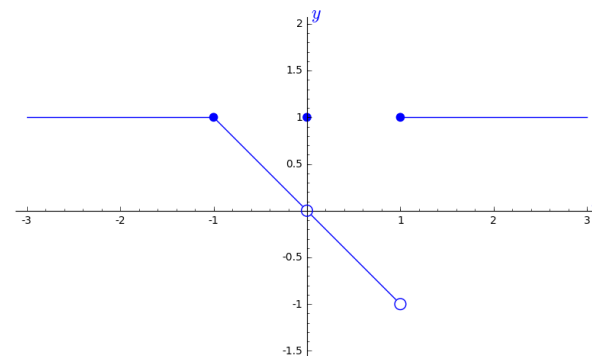


Figure: Graph of $f(x)$

Solution

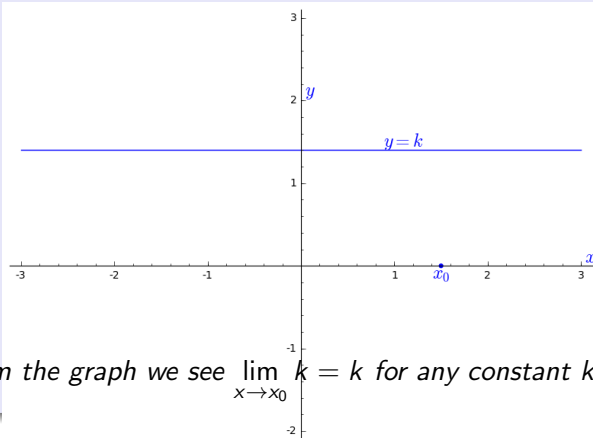


- (a) $\lim_{x \rightarrow -1} f(x) = 1$ Figure: Graph of $f(x)$
- (b) $\lim_{x \rightarrow 0} f(x) = 0$
- (c) $\lim_{x \rightarrow 1} f(x)$ does not exist because $f(x)$ approaches -1 when x approaches 1 from the left and $f(x)$ approaches 1 when x approaches 1 from the right.

Example 3

Show that $\lim_{x \rightarrow x_0} k = k$ for any constant k .

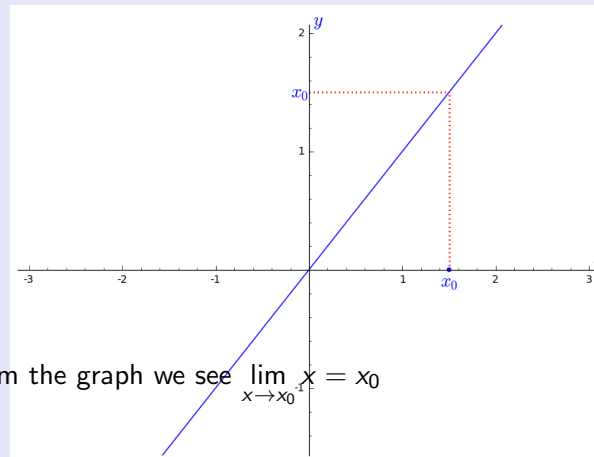
Solution



Example 4

Show that $\lim_{x \rightarrow x_0} x = x_0$.

Solution



2.2 Limits of a Function and Limit Laws (part 2)

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The limit laws

Theorem

If $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = M$, where M and L are real numbers, then

- (1) Sum Rule: $\lim_{x \rightarrow x_0} [f(x) + g(x)] = L + M$
- (2) Difference Rule: $\lim_{x \rightarrow x_0} [f(x) - g(x)] = L - M$
- (3) Constant Multiple Rule: $\lim_{x \rightarrow x_0} [kf(x)] = kL$, for any constant k .
- (4) Product Rule: $\lim_{x \rightarrow x_0} [f(x)g(x)] = LM$
- (5) Quotient Rule: $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided that $M \neq 0$
- (6) Power Rule: $\lim_{x \rightarrow x_0} [f(x)]^n = L^n$, for any positive integer n
- (7) Root Rule: $\lim_{x \rightarrow x_0} \sqrt[n]{f(x)} = \sqrt[n]{L}$, for any positive integer n
(If n is even, we assume that $L > 0$.)

Theorem 2 (Limits of Polynomials)

Theorem

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, then

$$\lim_{x \rightarrow x_0} p(x) = p(x_0) = a_n x_0^n + a_{n-1} x_0^{n-1} + \cdots + a_1 x_0 + a_0.$$

Theorem 3 (Limits of Rational Functions)

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function and $q(x_0) \neq 0$, then

$$\lim_{x \rightarrow x_0} \frac{p(x)}{q(x)} = \frac{p(x_0)}{q(x_0)}.$$

Eliminating zero denominators algebraically

- Let $\lim_{x \rightarrow x_0} f(x) = L$ and $\lim_{x \rightarrow x_0} g(x) = M$, where M and L are real numbers.
- If $M \neq 0$, then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$
- If $M = 0$, then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = ?$
- If the denominator is zero at x_0 , then canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at x_0 .
- In such case we can find limit by substituting $x = x_0$ in the simplified fraction.

Examples

Example 1

Evaluate $\lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x}$.

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x} &= \lim_{x \rightarrow 0} \frac{2 - \sqrt{x+4}}{x} \left(\frac{2 + \sqrt{x+4}}{2 + \sqrt{x+4}} \right) \\ &= \lim_{x \rightarrow 0} \frac{4 - (x+4)}{x(2 + \sqrt{x+4})} \\ &= \lim_{x \rightarrow 0} \frac{-x}{x(2 + \sqrt{x+4})} \\ &= \lim_{x \rightarrow 0} \frac{-1}{2 + \sqrt{x+4}} = \frac{-1}{4} \end{aligned}$$

Example 2

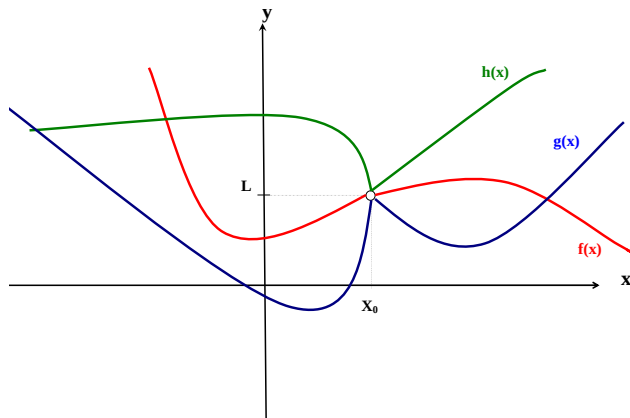
Evaluate $\lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{\sqrt[4]{x} - 2}$.

Solution

$$\begin{aligned} \lim_{x \rightarrow 16} \frac{\sqrt{x} - 4}{\sqrt[4]{x} - 2} &= \lim_{x \rightarrow 16} \frac{(\sqrt[4]{x})^2 - 4}{\sqrt[4]{x} - 2} \\ &= \lim_{x \rightarrow 16} \frac{(\sqrt[4]{x} - 2)(\sqrt[4]{x} + 2)}{\sqrt[4]{x} - 2} \\ &= \lim_{x \rightarrow 16} (\sqrt[4]{x} + 2) = 4 \end{aligned}$$

The Sandwich Theorem

- The following theorem, called the Sandwich theorem, enable us to calculate a variety of limits.



Theorem 4 (The Sandwich Theorem)

Assume that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing x_0 , except possibly at $x = x_0$ itself. Assume also that

$$\lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} h(x) = L.$$

Then $\lim_{x \rightarrow x_0} f(x) = L$.

Remark

The Sandwich Theorem is also called the Squeeze Theorem or Pinching Theorem.

Example 1

Let $2 - x^2 \leq f(x) \leq 2 + x^2$ for all $x \neq 0$. Find $\lim_{x \rightarrow 0} f(x)$.

Solution

- In this example, we can choose $g(x) = 2 - x^2$, $h(x) = 2 + x^2$.
- Then we have $g(x) \leq f(x) \leq h(x)$ for all $x \neq 0$
- Since $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (2 - x^2) = 2$ and $\lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow 0} (2 + x^2) = 2$,
- by Sandwich Theorem, $\lim_{x \rightarrow 0} f(x) = 2$.

Example 2

Let $\lim_{x \rightarrow c} |f(x)| = 0$. Find $\lim_{x \rightarrow c} f(x)$.

Solution

- We will use Sandwich Theorem.
- We have $-|f(x)| \leq f(x) \leq |f(x)|$
- $g(x) = -|f(x)|$, $h(x) = |f(x)|$
- and $\lim_{x \rightarrow c} |f(x)| = 0$ and $\lim_{x \rightarrow c} -|f(x)| = -\lim_{x \rightarrow c} |f(x)| = 0$.
- Thus, by Sandwich Theorem, $\lim_{x \rightarrow c} f(x) = 0$.

Example 3

Use the inequalities $-|x| \leq \sin x \leq |x|$ and $-|x| \leq 1 - \cos x \leq |x|$ to show that $\lim_{x \rightarrow 0} \sin x = 0$ and

$$\lim_{x \rightarrow 0} \cos x = 1$$

Solution

- We use the Sandwich Theorem, $\lim_{x \rightarrow 0} -|x| = 0 = \lim_{x \rightarrow 0} |x|$
- $\Rightarrow \lim_{x \rightarrow 0} \sin x = 0$ and $\lim_{x \rightarrow 0} (1 - \cos x) = 0$.
- The last limit gives

$$0 = \lim_{x \rightarrow 0} 1 - \lim_{x \rightarrow 0} \cos x = 1 - \lim_{x \rightarrow 0} \cos x$$

$$\Rightarrow \lim_{x \rightarrow 0} \cos x = 1$$

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Theorem 5

Assume that $g(x) \leq f(x)$ for all x in some open interval containing x_0 , except possibly at $x = x_0$ itself. Assume also that $\lim_{x \rightarrow x_0} g(x) = L$ and $\lim_{x \rightarrow x_0} f(x) = M$ both exist. Then

$$\lim_{x \rightarrow x_0} g(x) \leq \lim_{x \rightarrow x_0} f(x).$$

Remark

If we replace $\leq$ by $<$, then Theorem(5) is false. For example $0 < x^2$ for all $x \neq 0$, but $\lim_{x \rightarrow 0} 0 = \lim_{x \rightarrow 0} x^2 = 0$.

The Precise Definition of a Limit

- Let us review the precise definition of a limit.
- We replace the vague phrases like “gets arbitrarily close to” in the informal definition with specific conditions that can be applied to any particular example.
- Using the precise definition, we can prove the limit properties given in the preceding section and establish many important limits.
- To show that $\lim_{x \rightarrow x_0} f(x) = L$, we need to show that the gap between $f(x)$ and L can be made “as small as we choose” if x is kept “close enough” to x_0 .

Precise definition of limit

Definition

Let $f(x)$ be defined on an open interval about x_0 except possibly at x_0 itself. We say that the limit of $f(x)$ as x approaches x_0 is the number L , and write

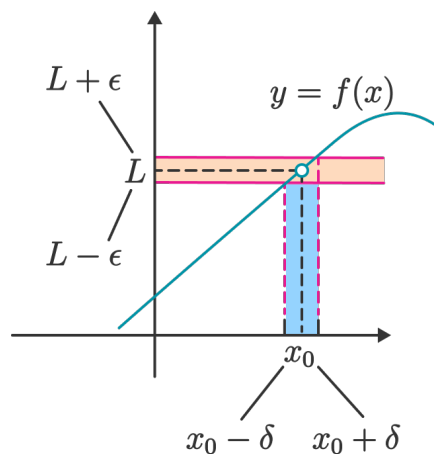
$$\lim_{x \rightarrow x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

- $|f(x) - L| < \epsilon \iff L - \epsilon < f(x) < L + \epsilon \iff f(x) \in (L - \epsilon, L + \epsilon)$
- $0 < |x - x_0| < \delta \iff x_0 - \delta < x < x_0 + \delta \text{ and } x \neq x_0 \iff x \in (x_0 - \delta, x_0 + \delta) - \{x_0\}$

- $f(x) \in (L - \epsilon, L + \epsilon)$ whenever $x \in (x_0 - \delta, x_0 + \delta) - \{x_0\}$



Remark

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct.

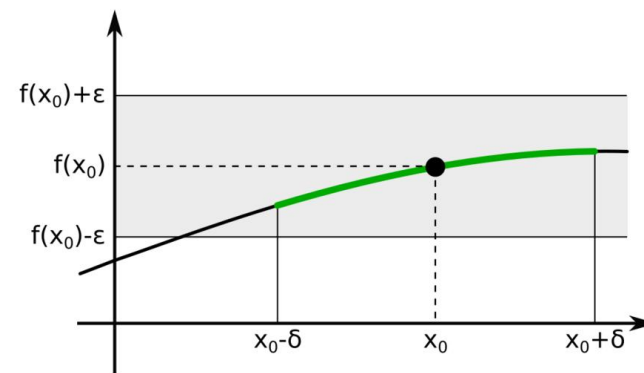


2.4 One-Sided Limits

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2.4 One-Sided Limits

- In this section we extend the concept of limit to one-sided limits, which are limits as x approaches the number x_0 from the right-hand side or the left-hand side only.
- To have a limit L as x approaches a number x_0 , a function f must be defined on both sides of x_0 and its values $f(x)$ must approach L as x approaches x_0 from either side.



- Because of this, ordinary limits are called two-sided limits.
- If f fails to have a two-side limit at x_0 , it may still have a one-side limit.

Example 1

Let $f(x) = \sqrt{x}$. Then what can we say about $\lim_{x \rightarrow 0} f(x)$?

Solution

The limit does not exist since $f(x)$ is not defined for $x < 0$.

Example 2

Let $f(x) = \begin{cases} g(x), & x < x_0 \\ h(x), & x > x_0. \end{cases}$. Then how can we find $\lim_{x \rightarrow x_0} f(x)$?

Solution

We can not evaluate $\lim_{x \rightarrow x_0} f(x)$ in one step since $f(x)$ have deferent formulas for $x > x_0$ and $x < x_0$.

Right-hand and left-hand limits

Definition

- (1) We say that a function $f(x)$ has a right-hand limit L at x_0 , and write

$$\lim_{x \rightarrow x_0^+} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x ,

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \epsilon.$$

Definition

- (2) We say that a function $f(x)$ has a left-hand limit L at x_0 , and write

$$\lim_{x \rightarrow x_0^-} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x ,

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \epsilon.$$

Remark

One-sided limits have all the limit properties listed in Theorem 1-Theorem 5.

Example 1

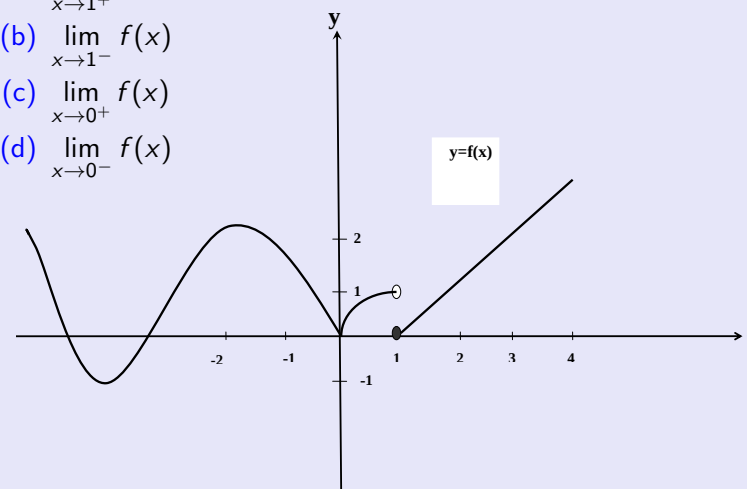
For the function $f(x)$ graphed below, find the following limits.

(a) $\lim_{x \rightarrow 1^+} f(x)$

(b) $\lim_{x \rightarrow 1^-} f(x)$

(c) $\lim_{x \rightarrow 0^+} f(x)$

(d) $\lim_{x \rightarrow 0^-} f(x)$



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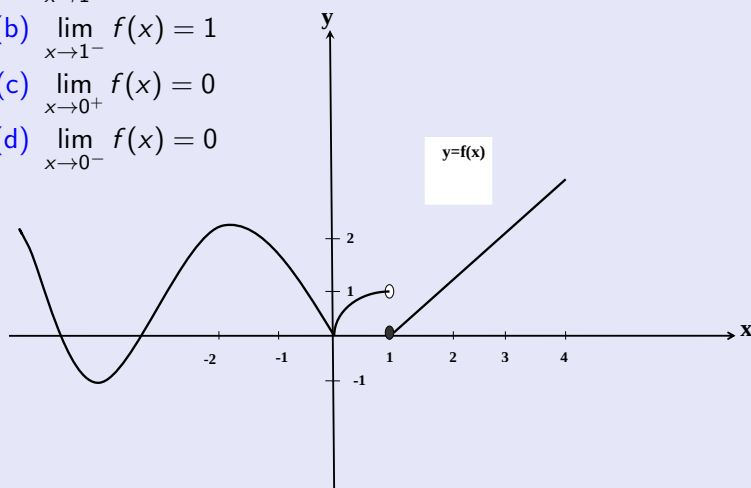
Solution

(a) $\lim_{x \rightarrow 1^+} f(x) = 0$

(b) $\lim_{x \rightarrow 1^-} f(x) = 1$

(c) $\lim_{x \rightarrow 0^+} f(x) = 0$

(d) $\lim_{x \rightarrow 0^-} f(x) = 0$



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Example 2

Find $\lim_{x \rightarrow 3^+} \sqrt{x^2 - 9}$.

Solution

$$\lim_{x \rightarrow 3^+} \sqrt{x^2 - 9} = 0.$$

Example 3

Find $\lim_{x \rightarrow 2^-} \frac{x-2}{|x-2|}$.

Solution

- $\lim_{x \rightarrow 2^-} \frac{x-2}{|x-2|} \quad \begin{matrix} 0 \\ 0 \end{matrix}$
- $x \rightarrow 2^-$
- $\Rightarrow x < 2$
- $\Rightarrow x - 2 < 0 \Rightarrow |x - 2| = -(x - 2)$
- Thus $\lim_{x \rightarrow 2^-} \frac{x-2}{|x-2|} = \lim_{x \rightarrow 2^-} \frac{x-2}{-(x-2)} = -1$

Example 4

Let $f(x) = \begin{cases} x, & x < -1 \\ 1-x, & x > -1. \end{cases}$
Find $\lim_{x \rightarrow -1^+} f(x)$ and $\lim_{x \rightarrow -1^-} f(x)$

Solution

- $\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (1-x) = 2$
- $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} x = -1$

Example 5

Find $\lim_{h \rightarrow 0^-} \frac{\sqrt{7} - \sqrt{h^2 + 4h + 7}}{h}$.

Solution

- $\lim_{h \rightarrow 0^-} \frac{\sqrt{7} - \sqrt{h^2 + 4h + 7}}{h} \quad \begin{matrix} 0 \\ 0 \end{matrix}$
- $= \lim_{h \rightarrow 0^-} \frac{\sqrt{7} - \sqrt{h^2 + 4h + 7}}{h} \left(\frac{\sqrt{7} + \sqrt{h^2 + 4h + 7}}{\sqrt{7} + \sqrt{h^2 + 4h + 7}} \right)$
- $= \lim_{h \rightarrow 0^-} \frac{7 - (h^2 + 4h + 7)}{h(\sqrt{7} + \sqrt{h^2 + 4h + 7})} = \lim_{h \rightarrow 0^-} \frac{-(h^2 + 4h)}{h(\sqrt{7} + \sqrt{h^2 + 4h + 7})}$
- $= \lim_{h \rightarrow 0^-} \frac{-(h+4)}{(\sqrt{7} + \sqrt{h^2 + 4h + 7})} = \frac{-4}{2\sqrt{7}} = \frac{-2}{\sqrt{7}}$

Example 6

Find $\lim_{x \rightarrow 4^-} \frac{\lfloor x \rfloor}{x}$.

Solution

- In order to calculate $\lfloor x \rfloor$ we have to find an integer n such that $n \leq x < n+1$.
- Since $x \rightarrow 4^-$, we have $x < 4$.
- Since x is close to 4 and $x < 4$, we may take $3 \leq x < 4$.
- Thus $\lfloor x \rfloor = 3$ and hence

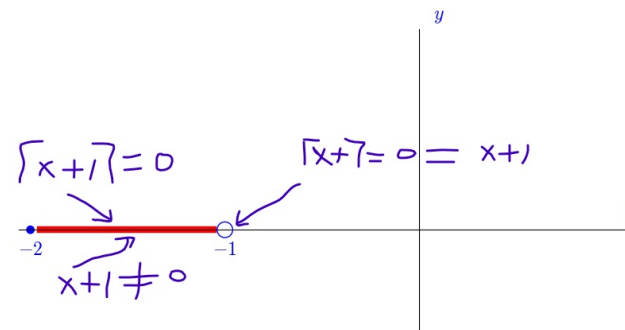
$$\lim_{x \rightarrow 4^-} \frac{\lfloor x \rfloor}{x} = \frac{3}{4}.$$

Example 7

Find $\lim_{x \rightarrow -1^-} \frac{\lceil x+1 \rceil}{x+1}$.

Solution

- In order to calculate $\lceil x+1 \rceil$ we have to find an integer n such that $n-1 < x+1 \leq n$.
- Since $x \rightarrow -1^-$, we have $x < -1$ and hence $x+1 < 0$.
- It follows that $-1 < x+1 < 0 \leq 0$.
- Therefore, $\lim_{x \rightarrow -1^-} \lceil x+1 \rceil = 0$.
- and hence $\lim_{x \rightarrow -1^-} \frac{\lceil x+1 \rceil}{x+1} = \frac{0}{0}$
- What we can do?



- We note that $\lim_{x \rightarrow -1^-} \lceil x+1 \rceil = 0$ much faster than $\lim_{x \rightarrow -1^-} (x+1) = 0$.
- Therefore, $\lim_{x \rightarrow -1^-} \frac{\lceil x+1 \rceil}{x+1} = 0$.

Relation between one-sided limits and two-sided limits

Theorem 6

$\lim_{x \rightarrow x_0} f(x) = L$ if and only if $\lim_{x \rightarrow x_0^+} f(x) = L = \lim_{x \rightarrow x_0^-} f(x)$.

Example 1

If $\lim_{x \rightarrow 1} f(x) = 5$, then find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$

Solution

Since $\lim_{x \rightarrow 1} f(x) = 5$, we have $\lim_{x \rightarrow 1^+} f(x) = 5$ and $\lim_{x \rightarrow 1^-} f(x) = 5$.

Example 2

If $\lim_{x \rightarrow 4^+} f(x) = -9$ and $\lim_{x \rightarrow 4^-} f(x) = -9$, then find $\lim_{x \rightarrow 4} f(x)$.

Solution

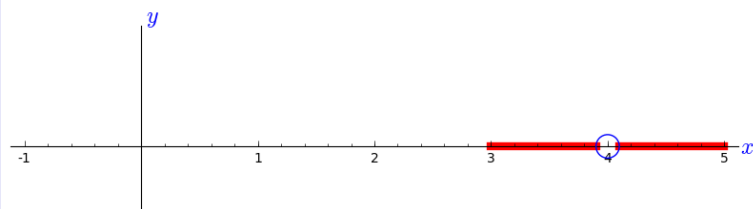
Since $\lim_{x \rightarrow 4^+} f(x) = -9$ and $\lim_{x \rightarrow 4^-} f(x) = -9$, we have $\lim_{x \rightarrow 4} f(x) = -9$.

Example 3

Find $\lim_{x \rightarrow 4} \lceil x - 3 \rceil$.

Solution

- We cannot evaluate by direct substitution.
- In order to calculate $\lceil x - 3 \rceil$ we have to find an integer n such that $n - 1 < x - 3 \leq n$.
- Since $x \rightarrow 4$, we have to study the values of $f(x) = \lceil x - 3 \rceil$ when $x > 4$ and $x < 4$.



- When $x > 4$, we have $4 < x \implies 1 < x - 3$
- $\implies 1 < x - 3 \leq 2$
- Thus $\lim_{x \rightarrow 4^+} \lceil x - 3 \rceil = 2$
- When $x < 4$, we have $x < 4 \implies x - 3 < 1$
- $\implies 0 < x - 3 < 1 \leq 1$.
- Thus $\lim_{x \rightarrow 4^-} \lceil x - 3 \rceil = 1$
- Therefore, $\lim_{x \rightarrow 4} \lceil x - 3 \rceil$ does not exist.

Example 4

Find $\lim_{x \rightarrow 5} \frac{x|x - 5|}{x - 5}$.

Solution

- The denominator $x - 5 \rightarrow 0$ as $x \rightarrow 5$.
- In order to eliminate the zero denominator we have to simplify $|x - 5|$.

• But

$$|x - 5| = \begin{cases} x - 5, & x - 5 \geq 0 \\ -(x - 5), & x - 5 < 0 \end{cases}$$

$$= \begin{cases} x - 5, & x \geq 5 \\ -(x - 5), & x < 5 \end{cases}$$

- Now $\lim_{x \rightarrow 5^+} \frac{x|x - 5|}{x - 5} = \lim_{x \rightarrow 5^+} \frac{x(x - 5)}{x - 5} = 5$
- and $\lim_{x \rightarrow 5^-} \frac{x|x - 5|}{x - 5} = \lim_{x \rightarrow 5^-} \frac{-x(x - 5)}{x - 5} = -5$.
- Therefore, $\lim_{x \rightarrow 5} \frac{x|x - 5|}{x - 5}$ does not exist.

Example 5

Find $\lim_{x \rightarrow 2} f(x)$, if $f(x) = \begin{cases} x^2, & x < 2 \\ \sqrt{7x+2}, & x > 2. \end{cases}$

Solution

- We have to calculate $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \sqrt{7x+2} = 4$
- and $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x^2 = 4$
- Therefore, $\lim_{x \rightarrow 2} f(x) = 4$.



2.4 One-Sided Limits (part 2)

First Semester 2020-2021
أ. د. أيمن هاشم السقا

Limits Involving $(\sin x)/x$

Theorem 7

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Corollary

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

Proof.

- $\lim_{x \rightarrow 0} \frac{x}{\sin x} = \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} = 1$
- $\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1$

Examples

Example 1

Show that $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$.

Solution

- $\frac{0}{0}$
- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \left[\frac{1 + \cos x}{1 + \cos x} \right]$
- $= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)}$
- $= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right] \left[\frac{\sin x}{(1 + \cos x)} \right] = (1)(0) = 0$

Example 2

Find $\lim_{x \rightarrow 0} \frac{\tan(5x)}{x}$.

Solution

- $\frac{0}{0}$
- $\lim_{x \rightarrow 0} \frac{\tan(5x)}{x} = \lim_{x \rightarrow 0} \frac{5 \tan(5x)}{5x}$.
- $= \lim_{x \rightarrow 0} 5 \frac{\tan(5x)}{5x} = 5$.

Example 3

Find $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - 1}{\sin x}$.

Solution

- $\frac{0}{0}$
- Let $y = \sin x$.
- Then as $x \rightarrow 0 \implies y \rightarrow 0$
- and $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - 1}{\sin x}$ can be written as
- $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - 1}{\sin x} = \lim_{y \rightarrow 0} \frac{\cos(y) - 1}{y}$
- $= \lim_{y \rightarrow 0} \frac{\cos(y) - 1}{y} = 0$

Example 4

Find $\lim_{t \rightarrow 0} \frac{t^2 \cos t}{\sin^2(2t)}$.

Solution

- $\frac{0}{0}$
- $\lim_{t \rightarrow 0} \frac{t^2 \cos t}{\sin^2(2t)} = \lim_{t \rightarrow 0} (\cos t) \left(\frac{t}{\sin(2t)} \right)^2$
- $= \lim_{t \rightarrow 0} (\cos t) \left(\frac{2t}{2 \sin(2t)} \right)^2$
- $= \lim_{t \rightarrow 0} \left(\frac{\cos t}{4} \right) \left(\frac{2t}{\sin(2t)} \right)^2 = \frac{1}{4}$

Example 5 (Exam)

Find $\lim_{x \rightarrow 0} \frac{x(1 + \tan^2 x)}{\sin(3x)}$.

Solution

- $\frac{0}{0}$
- $\lim_{x \rightarrow 0} \frac{x(1 + \tan^2 x)}{\sin(3x)} = \lim_{x \rightarrow 0} (1 + \tan^2 x) \frac{3x}{3 \sin(3x)}$
- $= \lim_{x \rightarrow 0} \left(\frac{1 + \tan^2 x}{3} \right) \left(\frac{3x}{\sin(3x)} \right) = \frac{1}{3}$

Example 6

Find $\lim_{x \rightarrow 0} \frac{3x + \sin(4x)}{5x}$.

Solution

- $\frac{0}{0}$
- $\lim_{x \rightarrow 0} \frac{3x + \sin(4x)}{5x} = \lim_{x \rightarrow 0} \left(\frac{3x}{5x} + \frac{\sin(4x)}{5x} \right)$
- $= \lim_{x \rightarrow 0} \left(\frac{3}{5} + \frac{4 \sin(4x)}{5 \cdot 4x} \right) = \frac{3}{5} + \frac{4}{5} = \frac{7}{5}$

Example 7

Find $\lim_{x \rightarrow 2} \frac{\sin(x-2)}{\sqrt{x+3} - \sqrt{2x+1}}$.

Solution

- $\lim_{x \rightarrow 2} \frac{\sin(x-2)}{\sqrt{x+3} - \sqrt{2x+1}} \quad \frac{0}{0}$
- $\lim_{x \rightarrow 2} \frac{\sin(x-2)}{\sqrt{x+3} - \sqrt{2x+1}} =$
- $\lim_{x \rightarrow 2} \frac{\sin(x-2)}{\sqrt{x+3} - \sqrt{2x+1}} \left(\frac{\sqrt{x+3} + \sqrt{2x+1}}{\sqrt{x+3} + \sqrt{2x+1}} \right)$
- $= \lim_{x \rightarrow 2} \frac{\sin(x-2) [\sqrt{x+3} + \sqrt{2x+1}]}{(x+3) - (2x+1)}$

- $\lim_{x \rightarrow 2} \frac{\sin(x-2)}{\sqrt{x+3} - \sqrt{2x+1}}$
- $= \lim_{x \rightarrow 2} \frac{\sin(x-2) [\sqrt{x+3} + \sqrt{2x+1}]}{(x+3) - (2x+1)}$
- $= \lim_{x \rightarrow 2} \frac{\sin(x-2) [\sqrt{x+3} + \sqrt{2x+1}]}{-(x-2)}$
- $= \lim_{x \rightarrow 2} - \left[\frac{\sin(x-2)}{(x-2)} \right] [\sqrt{x+3} + \sqrt{2x+1}]$
- $= -[\sqrt{5} + \sqrt{5}] = -2\sqrt{5}$

Example 8

Find $\lim_{x \rightarrow 0} \frac{\sin(x^3 + 2x^2 - 5x)}{2x}$.

Solution

- $\lim_{x \rightarrow 0} \frac{\sin(x^3 + 2x^2 - 5x)}{2x} \quad \frac{0}{0}$
- $= \lim_{x \rightarrow 0} \frac{\sin(x^3 + 2x^2 - 5x)}{2x} \left(\frac{x^3 + 2x^2 - 5x}{x^3 + 2x^2 - 5x} \right)$
- $= \lim_{x \rightarrow 0} \frac{\sin(x^3 + 2x^2 - 5x)}{(x^3 + 2x^2 - 5x)} \left(\frac{x^3 + 2x^2 - 5x}{2x} \right)$
- $= \lim_{x \rightarrow 0} \frac{x^3 + 2x^2 - 5x}{2} = \frac{-5}{2}$

Example 9

Find $\lim_{x \rightarrow 8} \frac{[1 - \cos(\sqrt[3]{x} - 2)]^2}{(x - 8)^2}$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 8} \frac{[1 - \cos(\sqrt[3]{x} - 2)]^2}{(x - 8)^2} \quad \boxed{\frac{0}{0}} \\ &= \lim_{x \rightarrow 8} \frac{[1 - \cos(\sqrt[3]{x} - 2)]^2 (\sqrt[3]{x} - 2)^2}{(\sqrt[3]{x} - 2)^2 (x - 8)^2} \\ &= \lim_{x \rightarrow 8} \left[\frac{1 - \cos(\sqrt[3]{x} - 2)}{(\sqrt[3]{x} - 2)} \right]^2 \left[\frac{(\sqrt[3]{x} - 2)}{(x - 8)} \right]^2 \end{aligned}$$

$$= \lim_{x \rightarrow 8} \left[\frac{1 - \cos(\sqrt[3]{x} - 2)}{(\sqrt[3]{x} - 2)} \right]^2 \left[\frac{(\sqrt[3]{x} - 2)}{(x - 8)} \right]^2$$

$$\boxed{a^3 - b^3 = (a - b)(a^2 + ab + b^2)} \quad (a = \sqrt[3]{x}, \quad b = 2)$$

$$= \lim_{x \rightarrow 8} \left[\frac{1 - \cos(\sqrt[3]{x} - 2)}{(\sqrt[3]{x} - 2)} \right]^2 \left[\frac{(\sqrt[3]{x} - 2)}{(\sqrt[3]{x} - 2)((\sqrt[3]{x})^2 + 2\sqrt[3]{x} + 4)} \right]^2$$

$$= \lim_{x \rightarrow 8} \left[\frac{1 - \cos(\sqrt[3]{x} - 2)}{(\sqrt[3]{x} - 2)} \right]^2 \left[\frac{1}{((\sqrt[3]{x})^2 + 2\sqrt[3]{x} + 4)} \right]^2$$

$$= (0)^2 \left[\frac{1}{((\sqrt[3]{8})^2 + 2\sqrt[3]{8} + 4)} \right]^2 = 0$$

Example 10

Find $\lim_{x \rightarrow 0} \tan(9x) \csc(2x)$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \tan(9x) \csc(2x) \\ &= \lim_{x \rightarrow 0} \frac{\sin(9x)}{\cos(9x)} \frac{1}{\sin(2x)} \quad \boxed{\frac{0}{0}} \\ &= \lim_{x \rightarrow 0} \left[\frac{1}{\cos(9x)} \right] \left[\frac{\sin(9x)}{\sin(2x)} \right] \frac{18x}{18x} \\ &= \lim_{x \rightarrow 0} \left[\frac{1}{\cos(9x)} \right] \left[\frac{\sin(9x)}{9x} \right] \left[\frac{2x}{\sin(2x)} \right] \left[\frac{9}{2} \right] = \frac{9}{2} \end{aligned}$$

Example 11

Find $\lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x}$.

Solution

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} \quad \boxed{\frac{0}{0}} \\ &= \lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} \left[\frac{1 + \cos x}{1 + \cos x} \right] \\ &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{1 - \cos^2 x} \\ &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{\sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{x(1 + \cos x)}{\sin x} = \lim_{x \rightarrow 0} \frac{x}{\sin x} (1 + \cos x) = 2 \end{aligned}$$



2.5 Continuity

First Semester 2020-2021

أ.د. أيمن هاشم السقا

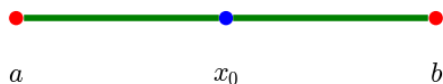
2.5 Continuity

- Continuity relates two concepts: $f(x_0)$ and $\lim_{x \rightarrow x_0} f(x)$.
- Intuitively, a function $y = f(x)$ whose graph can be sketched over its domain in one continuous motion without lifting the pencil is an example of a continuous function.
- In this section we investigate more precisely what it means for a function to be continuous.
- Since most functions of a real variable have domains that are intervals or unions of separate intervals, it is natural to restrict our study of continuity to functions with these domains.

Continuity at a Point

Definition

- (1) A function $y = f(x)$ is continuous at an interior point x_0 of its domain if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.
- (2) A function $y = f(x)$ is continuous at a left endpoint a of its domain if $\lim_{x \rightarrow a^+} f(x) = f(a)$.
- (3) A function $y = f(x)$ is continuous at a right endpoint b of its domain if $\lim_{x \rightarrow b^-} f(x) = f(b)$.



Examples

Example 1

Let $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1. \end{cases}$
Is $f(x)$ continuous at $x = 1$? Explain.

Solution

- In this example $\text{Dom}(f) = \mathbb{R}$.
- It follows that $x = 1$ is an interior point.
- Thus, f is continuous at $x = 1$ if it satisfies the following three conditions:
 - 1 $f(1)$ is defined.
 - 2 $\lim_{x \rightarrow 1} f(x) = L$ exists.
 - 3 $L = f(1)$.

$$f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1. \end{cases}$$

- ① $f(1) = 2$ is defined.
- ② $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2$
- ③ $L = 2 = f(1)$.
- Therefore, $f(x)$ is continuous at $x = 1$.

- $\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} \sqrt{9 - x^2} = 0 = f(-3)$
- Thus $f(x)$ is continuous at $x = -3$.
- $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \sqrt{9 - x^2} = 0 = f(3)$
- Thus $f(x)$ is continuous at $x = 3$.

Example 2

Let $f(x) = \sqrt{9 - x^2}$. Is f continuous at $x = \pm 3$?

Solution

- In this example $\text{Dom}(f) = [-3, 3]$.
- Thus $x = -3$ is a left endpoint and $x = 3$ is a right endpoint.
- It follows that f is continuous at $x = -3$ if
- $\lim_{x \rightarrow -3^+} f(x) = f(-3)$
- and f is continuous at $x = 3$ if
- $\lim_{x \rightarrow 3^-} f(x) = f(3)$.
- Let us test these conditions.

Example 3

Let $f(x) = \frac{1}{x}$. Is f continuous at $x = 0$?

Solution

- f is NOT continuous at $x = 0$ since $f(0)$ is not defined.

Remark

If a function is not continuous at a point x_0 , we say that f is discontinuous at x_0 and x_0 is a point of discontinuity of f .

Continuity on Intervals

Definition

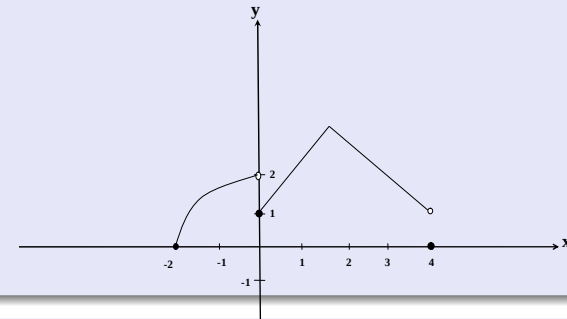
A function $y = f(x)$ is continuous on an interval if it is continuous at every point of the interval.

Example 1

The function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example 2

Say whether the function graphed below is continuous on $[-2, 4]$. If not, where does it fail to be continuous and why?



Solution

- The function is continuous on $[-2, 4]$ except at $x = 0, 4$.
- $\lim_{x \rightarrow 0} f(x)$ does not exist.
- $\lim_{x \rightarrow 4^-} f(x) = 1 \neq f(4) = 0$

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Definition

(Continuous functions)

A function $y = f(x)$ is continuous if it is continuous at every point of its domain.

Example

$f(x) = x$ is continuous.

Theorem 8 (Properties of continuous functions)

If the functions f and g are continuous at $x = x_0$ and k is a constant, then

- (1) $f \pm g$, fg , and kf are continuous at x_0 .
- (2) f/g is continuous at x_0 provided $g(x_0) \neq 0$.
- (3) $f^{n/m}$ is continuous at x_0 provided it is defined on an open interval containing x_0 , where n and m are positive integers.

Example 1 (Polynomials are continuous)

If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial, then p is continuous.

Example 2 (Rational functions are continuous)

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, then f is continuous at every point of its domain.

Continuity of composite functions

Example 3 (The sine function is continuous)

- We will show that $\lim_{x \rightarrow x_0} \sin x = \sin x_0$
- Let $h = x - x_0$. Then $x = x_0 + h$ and
- $\lim_{x \rightarrow x_0} \sin x = \lim_{h \rightarrow 0} \sin(x_0 + h)$
- $= \lim_{h \rightarrow 0} [\sin x_0 \cos h + \sin h \cos x_0] = \sin x_0$.
- Thus, $\sin x$ is continuous at every point in $\text{Dom}(\sin x) = \mathbb{R}$.

Example 4 (The cosine function is continuous)

$\cos x$ is continuous.

Example 5 (The absolute value function is continuous)

$f(x) = |x|$ is continuous.

Theorem 9 (Composite of continuous functions is continuous)

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .

Example 1

Show that $f(x) = \sqrt{x^2 - 1}$ is continuous on its domain.

Solution

- We have $\text{Dom}(f) = (-\infty, -1] \cup [1, \infty)$.
- For any $x_0 \in \text{Dom}(f)$, $g(x) = x^2 - 1$ is continuous at x_0 since g is continuous on $\mathbb{R}$.
- Moreover, $h(x) = \sqrt{x}$ is continuous at $g(x_0)$ since $g(x_0) \in [0, \infty)$ and h is continuous on $[0, \infty)$.
- Thus $f = h \circ g$ is continuous on its domain.

Theorem 10 (Limits of continuous functions)

Theorem 10

If g is continuous at the point L and $\lim_{x \rightarrow x_0} f(x) = L$, then

$$\lim_{x \rightarrow x_0} g(f(x)) = g(L) = g\left(\lim_{x \rightarrow x_0} f(x)\right).$$

Example

Evaluate $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan x)\right)$.

Solution

- By Theorem 10, $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan x)\right)$
- $= \sin\left(\frac{\pi}{2} \cos(\tan 0)\right)$
- $= \sin\left(\frac{\pi}{2} \cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Example 2

At what points is the function $f(x) = \frac{\sin(1 - x^2)}{x + 4}$ continuous?

Solution

- The function $f(x)$ can be written as $f(x) = \frac{h(x)}{g(x)}$, where $h(x) = \sin(1 - x^2)$ and $g(x) = x + 4$.
- $h(x)$ is continuous on $\mathbb{R}$ since it is a composition of the two continuous functions, the sine function and a polynomial.
- $g(x)$ is continuous on $\mathbb{R}$ since it is a polynomial.
- Thus $f(x) = \frac{h(x)}{g(x)}$ is continuous for all x provided $g(x) \neq 0$.
- Therefore, f is continuous at any $x \in \mathbb{R} - \{-4\}$.



2.5 Continuity (part 2)

First Semester 2020-2021

أ. د. أيمن هاشم السقا

Continuity of composite functions

Theorem 9 (Composite of continuous functions is continuous)

If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .

Example 1

Show that $f(x) = \sqrt{x^2 - 1}$ is continuous on its domain.

Solution

- We have $\text{Dom}(f) = (-\infty, -1] \cup [1, \infty)$.
- For any $x_0 \in \text{Dom}(f)$, $g(x) = x^2 - 1$ is continuous at x_0 since g is continuous on $\mathbb{R}$.
- Moreover, $h(x) = \sqrt{x}$ is continuous at $g(x_0)$ since $g(x_0) \in [0, \infty)$ and h is continuous on $[0, \infty)$.
- Thus $f = h \circ g$ is continuous on its domain.

Theorem 10 (Limits of continuous functions)

Theorem 10

If g is continuous at the point L and $\lim_{x \rightarrow x_0} f(x) = L$, then

$$\lim_{x \rightarrow x_0} g(f(x)) = g(L) = g\left(\lim_{x \rightarrow x_0} f(x)\right).$$

Example

Evaluate $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan x)\right)$.

Solution

- By Theorem 10, $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan x)\right)$
- $= \sin\left(\frac{\pi}{2} \cos(\tan 0)\right)$
- $= \sin\left(\frac{\pi}{2} \cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Example 2

At what points is the function $f(x) = \frac{\sin(1 - x^2)}{x + 4}$ continuous?

Solution

- The function $f(x)$ can be written as $f(x) = \frac{h(x)}{g(x)}$, where $h(x) = \sin(1 - x^2)$ and $g(x) = x + 4$.
- $h(x)$ is continuous on $\mathbb{R}$ since it is a composition of the two continuous functions, the sine function and a polynomial.
- $g(x)$ is continuous on $\mathbb{R}$ since it is a polynomial.
- Thus $f(x) = \frac{h(x)}{g(x)}$ is continuous for all x provided $g(x) \neq 0$.
- Therefore, f is continuous at any $x \in \mathbb{R} - \{-4\}$.

Continuous Extension to a Point

- If $f(x_0)$ is not defined and $\lim_{x \rightarrow x_0} f(x) = L$, then we say that f has a removable discontinuity at $x = x_0$.
- Moreover the function

$$F(x) = \begin{cases} f(x), & x \neq x_0 \\ L, & x = x_0 \end{cases}$$

is called the continuous extension of f to the point $x = x_0$.

- We can fill the hole

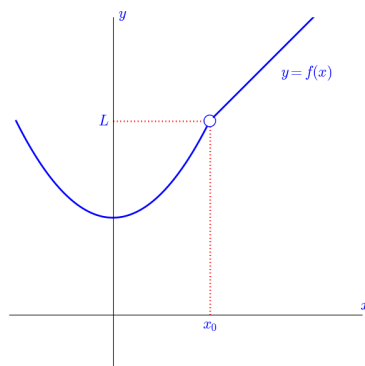


Figure: Removable discontinuity

$$\begin{aligned} L &= \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{x}{\sqrt{x+1}-1} \\ L &= \lim_{x \rightarrow 0} \left[\frac{x}{\sqrt{x+1}-1} \right] \left[\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \right] \\ L &= \lim_{x \rightarrow 0} \frac{x[\sqrt{x+1}+1]}{x+1-1} \\ L &= \lim_{x \rightarrow 0} \frac{x[\sqrt{x+1}+1]}{x} \\ L &= \lim_{x \rightarrow 0} [\sqrt{x+1}+1] = 2 \end{aligned}$$

- Thus the continuous extension of $f(x)$ is

$$F(x) = \begin{cases} \frac{x}{\sqrt{x+1}-1}, & x \neq 0 \\ 2, & x = 0, \end{cases}$$

- Note that $\text{Dom}(F) = [-1, \infty)$.

Example 1

$$\text{Let } f(x) = \frac{x}{\sqrt{x+1}-1}.$$

Define $f(0)$ in a way that extends f to be continuous at $x = 0$.

Solution

- First of all, $\text{Dom}(f) = [-1, \infty) - \{0\}$.
- We want to find the continuous extension

$$F(x) = \begin{cases} f(x), & x \neq 0 \\ L, & x = 0, \end{cases}$$

- where $L = \lim_{x \rightarrow 0} f(x)$

Example 2

$$\text{Let } f(x) = \begin{cases} x^2 + x - 1, & x < 1 \\ a, & x \geq 1. \end{cases}$$

Find the value of a such that f is continuous at every x .

Solution

- We have $\text{Dom}(f) = \mathbb{R}$.
- For $x \in (-\infty, 1)$, $f(x) = x^2 + x - 1$ is a polynomial and hence is continuous.
- Thus $f(x)$ is continuous on $(-\infty, 1)$.
- For $x \in (1, \infty)$, $f(x) = a$ is a polynomial and hence is continuous.
- Thus $f(x)$ is continuous on $(1, \infty)$.
- It remains to check continuity at $x = 1$.

$$\bullet f(x) = \begin{cases} x^2 + x - 1, & x < 1 \\ a, & x \geq 1. \end{cases}$$

① $f(1) = a$ is defined.

② $\lim_{x \rightarrow 1} f(x) = L$

③ $L = f(1)$.

$$\bullet \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} a = a$$

$$\bullet \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + x - 1) = 1$$

$$\bullet \lim_{x \rightarrow 1} f(x) \text{ exists if and only if } \boxed{a = 1}.$$

Example 3

For what values of a and b is $f(x)$ continuous at every x ?

$$f(x) = \begin{cases} ax + b, & x \leq 0 \\ x^2 + a + 2b, & 0 < x \leq 2 \\ 2x + 1, & x > 2. \end{cases}$$

Solution

- We have $\text{Dom}(f) = \mathbb{R}$.
- For $x \in (-\infty, 0)$, $f(x) = ax + b$ is a polynomial and hence is continuous.
- Thus $f(x)$ is continuous on $(-\infty, 0)$.
- For $x \in (0, 2)$, $f(x) = x^2 + a + 2b$ is a polynomial and hence is continuous.
- Thus $f(x)$ is continuous on $(0, 2)$.

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$$f(x) = \begin{cases} ax + b, & x \leq 0 \\ x^2 + a + 2b, & 0 < x \leq 2 \\ 2x + 1, & x > 2. \end{cases}$$

- For $x \in (2, \infty)$, $f(x) = 2x + 1$ is a polynomial and hence is continuous.
- Thus $f(x)$ is continuous on $(2, \infty)$.
- It remains to check continuity at $x = 0$ and $x = 2$.
- Continuity at $x = 0$
 - ① $f(0) = b$ is defined.
 - ② $\lim_{x \rightarrow 0} f(x) = L$
 - ③ $L = f(0)$.
- $L = \lim_{x \rightarrow 0} f(x) = ?$
- $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (ax + b) = b$
- $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x^2 + a + 2b) = a + 2b$
- $\lim_{x \rightarrow 0} f(x) = L$ exists if and only if $b = a + 2b \implies \boxed{b = -a}$

$$f(x) = \begin{cases} ax + b, & x \leq 0 \\ x^2 + a + 2b, & 0 < x \leq 2 \\ 2x + 1, & x > 2. \end{cases}$$

- Continuity at $x = 2$
 - ① $f(2) = 4 + a + 2b$ is defined.
 - ② $\lim_{x \rightarrow 2} f(x) = L$
 - ③ $L = f(2)$.
- $L = \lim_{x \rightarrow 2} f(x) = ?$
- $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 + a + 2b) = 4 + a + 2b$
- $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (2x + 1) = 5$
- $\lim_{x \rightarrow 2} f(x) = L$ exists if and only if $4 + a + 2b = 5 \implies -a = 1 \implies \boxed{a = -1}$
- $b = -a \implies \boxed{b = 1}$

Intermediate Value Theorem for Continuous Functions

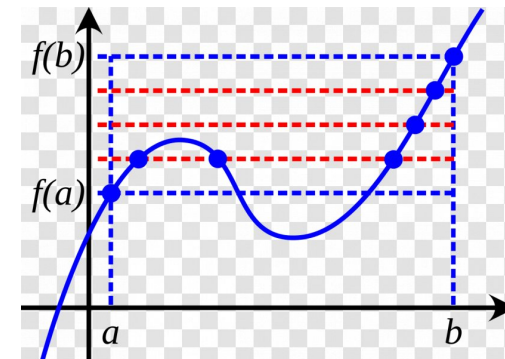
- Continuous functions have properties that make them useful in mathematics and its applications.
- One of these is the intermediate value property: If a function takes on two values, it also takes on all values in between.

Real World Example 1

- The temperature at various times of the day is a good example of a continuous function.
- Let's say that at 6am, it is 26 degrees outside, and by noon, it is 35 degrees.
- By the intermediate value theorem, at some time between 6am and noon, the temperature outside must have been exactly 31.7 degrees.
- We can pick any value between 26 and 35 and be confident that that exact temperature was attained sometime between 6am and noon.

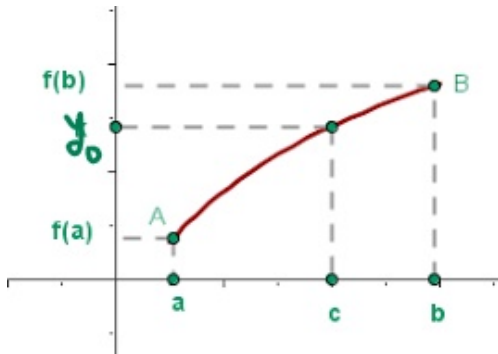
Real World Example 2

- Ali measures his height one day (day 0) and he is 150 cm tall.
- Ali then measures his height 100 days later and finds he is now 160 cm tall.
- If we were to graph his height as a function of every day of this year, the function would naturally be continuous.
- The closed interval would have these boundaries:
 $f(a) = f(0\text{days}) = 150\text{cm}$, $f(b) = f(100\text{days}) = 160\text{cm}$.
- Using such a graph, you could be asked to find the date when Ali was exactly 156cm tall.
- This scenario satisfies the hypotheses of the intermediate value theorem; Ali's growth is continuous over the closed interval of day 0 to day 100, and there is a value $d = 156\text{cm}$ between $f(0) = 150$ and $f(100) = 160$.
- According to the intermediate value theorem any height between 150cm and 160cm can be chosen and there will be a certain day in this closed interval where Ali was this height.



Theorem 11 (The Intermediate Value Theorem)

Suppose that f is continuous on a closed interval $[a, b]$ and let y_0 be any number between $f(a)$ and $f(b)$. Then $y_0 = f(c)$ for some $c \in [a, b]$.



Example

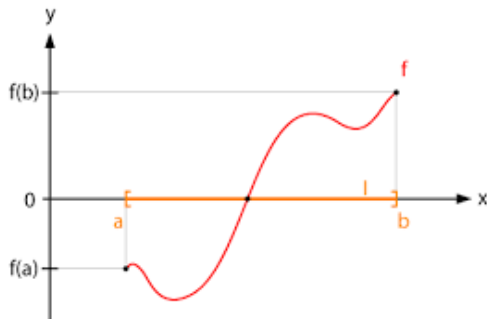
Let $f(x) = x^3$. Is there a number $c \in [0, 3]$ such that $f(c) = 7$?

Solution

- In this example $f(x)$ is continuous on $[0, 3]$
- $f(0) = 0$ and $f(3) = 27$
- Since $f(c) = 7$ is a number between $f(0) = 0$ and $f(3) = 27$, by The Intermediate Value Theorem, there is a number $c \in [0, 3]$ such that $f(c) = 7$.
- Note that $f(c) = 7 \implies c^3 = 7 \implies c = \sqrt[3]{7} \approx 1.79 \in [0, 3]$

Remarks

- 1 The graph of a continuous function can not have any breaks. It will be connected, a single unbroken curve.
- 2 A solution of the equation $f(x) = 0$ is called a root of equation or a zero of the function. The Intermediate Value Theorem tell us that if f is continuous and change sign on an interval, then f has a zero in that interval (Bolzano Theorem).



Example

Show that $f(x) = \frac{x^4 - 2x - 1}{x^2 + 1}$ has a zero in the interval $[1, 2]$.

Solution

- $f(x)$ is continuous on $[1, 2]$
- ($f(x)$ is a rational function and $x^2 + 1 \neq 0$ for all x)
- $f(1) = \frac{1 - 2 - 1}{1 + 1} = -1 < 0$
- $f(2) = \frac{16 - 4 - 1}{4 + 1} = \frac{11}{5} > 0$
- By Bolzano Theorem, $f(x)$ has a zero in $(1, 2)$.

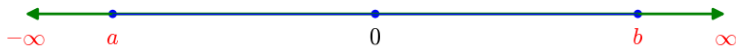


2.6 Limits Involving Infinity; Asymptotes of Graphs (part 1)

First Semester 2019-2020
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Finite Limits as $x \rightarrow \pm\infty$

- If $(-\infty, a) \subseteq \text{Dom}(f)$ or $(b, \infty) \subseteq \text{Dom}(f)$, then we use $\lim_{x \rightarrow \pm\infty} f(x)$ to study the behavior of f when x very large and positive or very large and negative.



2.6 Limits Involving Infinity; Asymptotes of Graphs

- The symbol for infinity (∞) does not represent a real number.
- We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds.
- In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or as $x \rightarrow \pm\infty$:

$$\lim_{x \rightarrow \pm\infty} f(x) = L.$$

- We further extend the concept of limit to infinite limits, which are not limits as before, but rather a new use of the term limit:

$$\lim_{x \rightarrow x_0} f(x) = \pm\infty.$$

- Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude.
- We use limits involving infinity to analyze the graphs of functions having horizontal or vertical asymptotes.

Definition

- (1) We say that $f(x)$ has the limit L as x approaches infinity and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \Rightarrow |f(x) - L| < \epsilon.$$

- (2) We say that $f(x)$ has the limit L as x approaches minus infinity and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

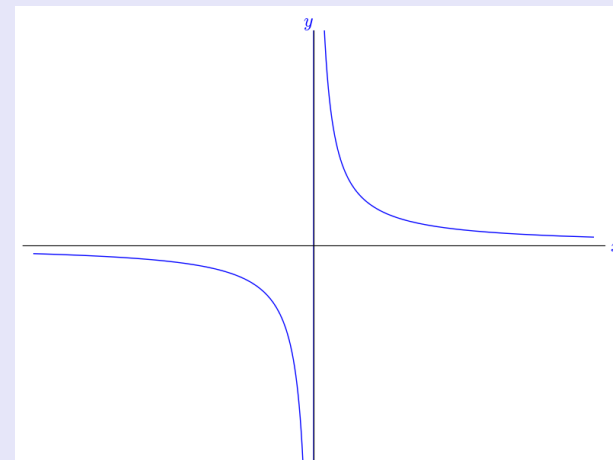
$$x < M \Rightarrow |f(x) - L| < \epsilon.$$

Theorem 12

All the limit laws in Theorem 1 are true when we replace $\lim_{x \rightarrow x_0}$ by $\lim_{x \rightarrow \infty}$ or $\lim_{x \rightarrow -\infty}$. That is, the variable x may approach a finite number x_0 or $\pm\infty$.

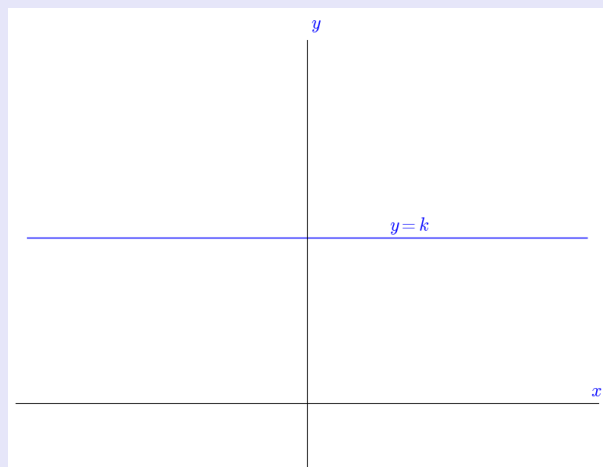
Example 1 (Basic limits)

- $\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$



Example 1 (Basic limits)

- If k is a constant, then $\lim_{x \rightarrow \pm\infty} k = k$



Example 2

Find $\lim_{x \rightarrow \infty} \frac{3}{x^2}$.

Solution

- $\lim_{x \rightarrow \infty} \frac{3}{x^2} = 3 \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)^2 = 0$

Example 3

Find $\lim_{x \rightarrow \infty} \frac{7x^2 - 9x + 2}{4x^2 + 6x - 5}$.

Solution

$$\begin{aligned} \bullet \lim_{x \rightarrow \infty} \frac{7x^2 - 9x + 2}{4x^2 + 6x - 5} &= \lim_{x \rightarrow \infty} \frac{x^2 \left(\frac{7x^2}{x^2} - \frac{9x}{x^2} + \frac{2}{x^2} \right)}{x^2 \left(\frac{4x^2}{x^2} + \frac{6x}{x^2} - \frac{5}{x^2} \right)} \\ \bullet &= \lim_{x \rightarrow \infty} \frac{\left(7 - \frac{9}{x} + \frac{2}{x^2} \right)}{\left(4 + \frac{6}{x} - \frac{5}{x^2} \right)} = \boxed{\frac{7}{4}} \end{aligned}$$

Example 4

Find $\lim_{x \rightarrow -\infty} \frac{-5x^4 + x^2 - x + 2}{3x^4 - x^3 - 9x + 15}$.

Solution

$$\begin{aligned} \bullet \lim_{x \rightarrow -\infty} \frac{-5x^4 + x^2 - x + 2}{3x^4 - x^3 - 9x + 15} &= \lim_{x \rightarrow -\infty} \frac{x^4 \left(-5 + \frac{1}{x^2} - \frac{1}{x^3} + \frac{2}{x^4} \right)}{x^4 \left(3 - \frac{1}{x} - \frac{9}{x^3} + \frac{15}{x^4} \right)} \\ \bullet &= \lim_{x \rightarrow -\infty} \frac{\left(-5 + \frac{1}{x^2} - \frac{1}{x^3} + \frac{2}{x^4} \right)}{\left(3 - \frac{1}{x} - \frac{9}{x^3} + \frac{15}{x^4} \right)} = \boxed{\frac{-5}{3}} \end{aligned}$$

Example 5

Find $\lim_{x \rightarrow -\infty} \frac{(6x^2 + 1)(3 - x)}{2x^3 - 4x + 5}$.

Solution

$$\begin{aligned} \bullet \lim_{x \rightarrow -\infty} \frac{(6x^2 + 1)(3 - x)}{2x^3 - 4x + 5} &= \lim_{x \rightarrow -\infty} \frac{x^3 \left(6 + \frac{1}{x^2} \right) \left(\frac{3}{x} - 1 \right)}{x^3 \left(2 - \frac{4}{x^2} + \frac{5}{x^3} \right)} \\ \bullet &= \lim_{x \rightarrow -\infty} \frac{\left(6 + \frac{1}{x^2} \right) \left(\frac{3}{x} - 1 \right)}{\left(2 - \frac{4}{x^2} + \frac{5}{x^3} \right)} = \frac{(6)(-1)}{2} = \boxed{-3} \end{aligned}$$

Example 6

Find $\lim_{x \rightarrow \infty} \frac{\sqrt{x} - \sqrt[3]{x}}{\sqrt{x} + x^{-1}}$.

Solution

$$\begin{aligned} \bullet \lim_{x \rightarrow \infty} \frac{\sqrt{x} - \sqrt[3]{x}}{\sqrt{x} + x^{-1}} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x} \left(1 - \frac{\sqrt[3]{x}}{\sqrt{x}} \right)}{\sqrt{x} \left(1 + \frac{x^{-1}}{\sqrt{x}} \right)} \\ \bullet &= \lim_{x \rightarrow \infty} \frac{\left(1 - \frac{1}{x^{\frac{1}{2} - \frac{1}{3}}} \right)}{\left(1 + \frac{1}{x^{\frac{1}{2} + 1}} \right)} = \lim_{x \rightarrow \infty} \frac{\left(1 - \frac{1}{x^{\frac{1}{6}}} \right)}{\left(1 + \frac{1}{x^{\frac{3}{2}}} \right)} \\ \bullet &= \boxed{1} \end{aligned}$$

Example 7

Find $\lim_{x \rightarrow -\infty} \frac{8x - 5x^3}{\sqrt{4x^6 - 100}}$.

Solution

$$\bullet \lim_{x \rightarrow -\infty} \frac{8x - 5x^3}{\sqrt{4x^6 - 100}} = \lim_{x \rightarrow -\infty} \frac{x^3 \left(\frac{8}{x^2} - 5 \right)}{\sqrt{x^6 \left(4 - \frac{100}{x^6} \right)}}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{x^3 \left(\frac{8}{x^2} - 5 \right)}{|x^3| \sqrt{\left(4 - \frac{100}{x^6} \right)}}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{x^3 \left(\frac{8}{x^2} - 5 \right)}{-x^3 \sqrt{\left(4 - \frac{100}{x^6} \right)}}$$

$$\boxed{x \rightarrow -\infty \Rightarrow x < 0 \\ \Rightarrow x^3 < 0}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{x^3 \left(\frac{8}{x^2} - 5 \right)}{-x^3 \sqrt{\left(4 - \frac{100}{x^6} \right)}}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{-\left(\frac{8}{x^2} - 5 \right)}{\sqrt{\left(4 - \frac{100}{x^6} \right)}}$$

$$\bullet = \frac{-(-5)}{\sqrt{4}} = \boxed{\frac{5}{2}}$$

$$\boxed{x \rightarrow -\infty \Rightarrow x < 0 \\ \Rightarrow x^3 < 0}$$

Limits of trigonometric functions as $x \rightarrow \pm\infty$

Theorem 12 (Two basic limits)

$$(1) \lim_{x \rightarrow \pm\infty} \frac{\sin x}{x} = 0.$$

$$(2) \lim_{x \rightarrow \pm\infty} \frac{\cos x}{x} = 0.$$

Proof.

$$(1) |\sin x| \leq 1 \Rightarrow -1 \leq \sin x \leq 1$$

$$\bullet \Rightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

□

Example 1

Find $\lim_{x \rightarrow -\infty} \frac{2x - \cos x}{x + \sin(2x)}$.

Solution

$$\bullet \lim_{x \rightarrow -\infty} \frac{2x - \cos x}{x + \sin(2x)} = \lim_{x \rightarrow -\infty} \frac{x \left(2 - \frac{\cos x}{x} \right)}{x \left(1 + \frac{\sin(2x)}{x} \right)}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{\left(2 - \frac{\cos x}{x} \right)}{\left(1 + \frac{\sin(2x)}{x} \right)} = \boxed{2}$$

Example 2 (Exam)

Find $\lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$.

Solution

$$\begin{aligned} \bullet \lim_{x \rightarrow \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} &= \lim_{x \rightarrow \infty} \frac{x \left(1 + \frac{\sin x}{x} + \frac{2\sqrt{x}}{x}\right)}{x \left(1 + \frac{\sin x}{x}\right)} \\ \bullet &= \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{\sin x}{x} + \frac{2}{\sqrt{x}}\right)}{\left(1 + \frac{\sin x}{x}\right)} = \boxed{1} \end{aligned}$$

Example 3

Find $\lim_{x \rightarrow \infty} x \sin \left(\frac{1}{x} \right)$.

Solution

$$\begin{aligned} \bullet \lim_{x \rightarrow \infty} x \sin \left(\frac{1}{x} \right) &= \lim_{x \rightarrow \infty} \frac{\sin \left(\frac{1}{x} \right)}{\frac{1}{x}} \quad \boxed{x \rightarrow \infty \implies \theta = \frac{1}{x} \rightarrow 0} \\ \bullet &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \boxed{1} \end{aligned}$$



2.6 Limits Involving Infinity; Asymptotes of Graphs (part 2)

First Semester 2019-2020
أ.د. أيمن هاشم السقا

Infinite Limits

- We use infinite limits to describe the behavior of functions whose values become arbitrarily large and positive or arbitrary small and negative:

$$\boxed{\lim_{x \rightarrow x_0} f(x) = \pm \infty.}$$

Example

- As $x \rightarrow 0^+$, $f(x) = \frac{1}{x}$ becomes arbitrary large and positive.
- To describe this we write $\boxed{\lim_{x \rightarrow 0^+} f(x) = \infty.}$
- Similarly, as $x \rightarrow 0^-$, $f(x) = \frac{1}{x}$ becomes arbitrary large and negative.
- To describe this we write $\boxed{\lim_{x \rightarrow 0^-} f(x) = -\infty.}$

Definition

- (1) We say that $f(x)$ approaches infinity as x approaches x_0 and write

$$\lim_{x \rightarrow x_0} f(x) = \infty$$

if, for every number $B > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow f(x) > B.$$

- (2) We say that $f(x)$ approaches minus infinity as x approaches x_0 and write

$$\lim_{x \rightarrow x_0} f(x) = -\infty$$

if, for every number $B > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow f(x) < -B.$$

Example 1

Find $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$

Solution

- $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} \quad \frac{1}{0} \rightarrow \infty \text{ or } -\infty$

- $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2}$

- $\lim_{x \rightarrow 1} \frac{1}{(x-1)^2} = \infty$

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Example 2

Find $\lim_{x \rightarrow 0} \frac{x-3}{x^{2/5}}$

Solution

- $\lim_{x \rightarrow 0} \frac{x-3}{x^{2/5}} \quad \frac{-3}{0} \rightarrow \infty \text{ or } -\infty$

- $\lim_{x \rightarrow 0} \frac{x-3}{x^{2/5}} \rightarrow \frac{-3}{0^+}$

- $\lim_{x \rightarrow 0} \frac{x-3}{x^{2/5}} = -\infty$

Example 3

Find $\lim_{x \rightarrow -3^-} \frac{2x}{3x+9}$

Solution

- $\lim_{x \rightarrow -3^-} \frac{2x}{3x+9} \quad \frac{-6}{0} \rightarrow \infty \text{ or } -\infty$

- $\lim_{x \rightarrow -3^-} \frac{2x}{3x+9} \rightarrow \frac{-6}{-}$

- $x \rightarrow -3^- \Rightarrow x < -3 \Rightarrow x+3 < 0 \Rightarrow 3x+9 < 0$

- $\lim_{x \rightarrow -3^-} \frac{2x}{3x+9} \rightarrow \frac{-6}{0^-}$

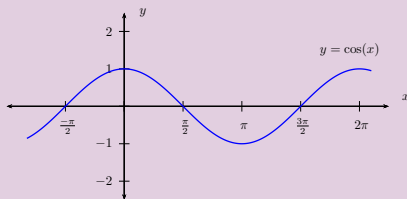
- $\lim_{x \rightarrow -3^-} \frac{2x}{3x+9} = \infty$

Example 4

Find $\lim_{x \rightarrow (\frac{\pi}{2})^+} \sec x$

Solution

$$\bullet \lim_{x \rightarrow (\frac{\pi}{2})^+} \sec x = \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{1}{\cos x} \quad \frac{1}{0} \rightarrow \infty \text{ or } -\infty$$



$$\bullet \lim_{x \rightarrow (\frac{\pi}{2})^+} \sec x = \lim_{x \rightarrow (\frac{\pi}{2})^+} \frac{1}{\cos x} \rightarrow \frac{1}{0^-} = -\infty$$

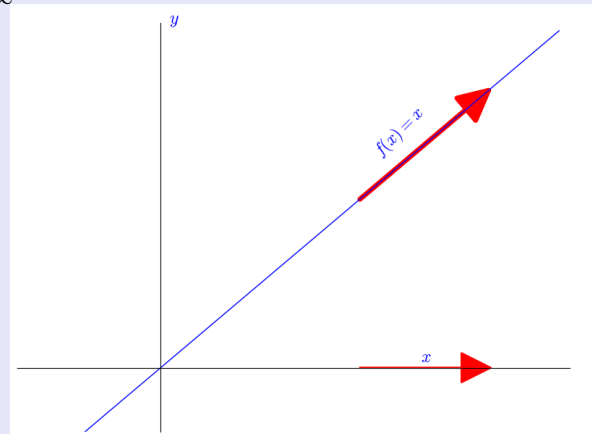
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Example 5

Find $\lim_{x \rightarrow \infty} x$

Solution

$$\bullet \lim_{x \rightarrow \infty} x = \infty$$



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Example 6

Find $\lim_{x \rightarrow -\infty} [\sqrt{x^4 + 4} - x^2]$.

Solution

$$\bullet \lim_{x \rightarrow -\infty} [\sqrt{x^4 + 4} - x^2] \quad \infty - \infty$$

$$\bullet = \lim_{x \rightarrow -\infty} [\sqrt{x^4 + 4} - x^2] \left[\frac{\sqrt{x^4 + 4} + x^2}{\sqrt{x^4 + 4} + x^2} \right]$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{x^4 + 4 - x^4}{\sqrt{x^4 + 4} + x^2}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{4}{\sqrt{x^4 + 4} + x^2}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{4}{\sqrt{x^4 (1 + \frac{4}{x^4})} + x^2}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{4}{|x^2| \sqrt{(1 + \frac{4}{x^4})} + x^2}$$

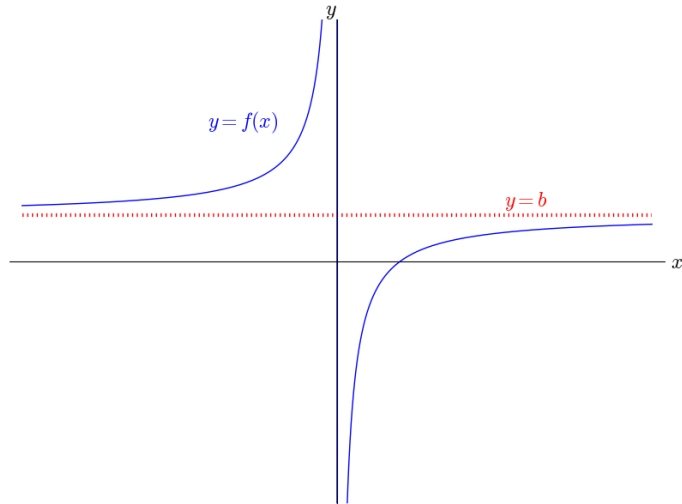
$$\bullet = \lim_{x \rightarrow -\infty} \frac{4}{x^2 \sqrt{(1 + \frac{4}{x^4})} + x^2}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{4}{x^2 [\sqrt{(1 + \frac{4}{x^4})} + 1]}$$

$$\bullet = \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^2}}{[\sqrt{(1 + \frac{4}{x^4})} + 1]} = \frac{0}{2} = 0$$

Horizontal Asymptotes

- If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an asymptote of the graph.



Horizontal Asymptotes

Definition

A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

Example 1

Find the horizontal asymptotes of the graph of $f(x) = \frac{1}{x}$.

Solution

- We calculate $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.
- Thus the line $y = 0$ (x -axis) is a horizontal asymptote of the graph of $f(x) = \frac{1}{x}$.

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Example 2

Find the horizontal asymptotes of the graph of $f(x) = \frac{x-1}{|x|+1}$.

Solution

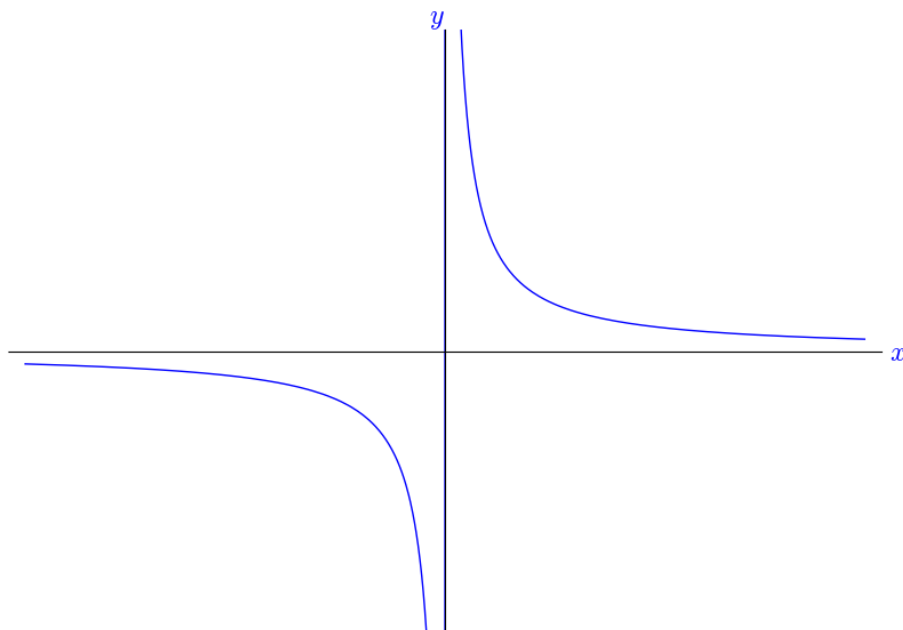
- We calculate

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x-1}{|x|+1} = \lim_{x \rightarrow \infty} \frac{x-1}{x+1}$$

-

$$= \lim_{x \rightarrow \infty} \frac{x(1 - \frac{1}{x})}{x(1 + \frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{(1 - \frac{1}{x})}{(1 + \frac{1}{x})} = 1$$

- Thus $y = 1$ is a horizontal asymptote.



- We calculate

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x-1}{|x|+1} = \lim_{x \rightarrow -\infty} \frac{x-1}{-x+1}$$

•

$$= \lim_{x \rightarrow -\infty} \frac{x(1 - \frac{1}{x})}{x(-1 + \frac{1}{x})} = \lim_{x \rightarrow -\infty} \frac{(1 - \frac{1}{x})}{(-1 + \frac{1}{x})} = -1$$

- Thus $y = -1$ is a horizontal asymptote.

- $$x+1 \overline{) x^2 + x - 1}$$

Oblique Asymptotes

- If a function does not have horizontal asymptotes, then we look for oblique asymptotes.
- We find an equation for the asymptote by dividing numerator by denominator to express $f(x)$ as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

Example

Find the oblique asymptotes of the graph of $f(x) = \frac{x^2 + x - 1}{x + 1}$.

Solution

- Note that $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{x^2 + x - 1}{x + 1} = \pm\infty$
- Thus the graph of $f(x)$ Does Not have horizontal asymptotes.
- Using long division
$$x+1 \overline{) x^2 + x - 1}$$

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- $$\begin{array}{r} x \\ x+1 \overline{) x^2 + x - 1} \\ \underline{x^2 + x} \\ -1 \end{array}$$

- This implies $f(x) = \frac{x^2 + x - 1}{x + 1} = x - \frac{1}{x + 1}$

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$$\begin{array}{r} 5 \\ 2 \overline{) 11} \\ \underline{10} \\ 1 \end{array} \Rightarrow \frac{11}{2} = 5 + \frac{1}{2}$$

- Therefore, $y = x$ is an oblique asymptote.

Vertical Asymptotes

- If the distance between the graph of a function and some fixed vertical line approaches zero as a point moves vertically along the graph and away from the origin, we say that the graph approaches the vertical line asymptotically and that the line is vertical asymptote of the graph.

Definition

A line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

Example 1

Find the vertical asymptotes of $f(x) = \frac{1}{x}$.

Solution

- First of all $\text{Dom}(f) = \mathbb{R} - \{0\}$
- Thus $f(x)$ May have a vertical asymptote at $x = 0$.
- We test:

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{x} = \frac{1}{0^+} = \infty$$

- Thus $x = 0$ is a vertical asymptote.

Example 2

Find all asymptotes of the graph of $f(x) = \frac{x+5}{x^2-3x-4}$.

Solution

Horizontal Asymptotes:

- We calculate

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x+5}{x^2-3x-4}$$

$$= \lim_{x \rightarrow \infty} \frac{x(1 + \frac{5}{x})}{x^2(1 - \frac{3}{x} - \frac{4}{x^2})}$$

$$= \lim_{x \rightarrow \infty} \frac{(1 + \frac{5}{x})}{x(1 - \frac{3}{x} - \frac{4}{x^2})}$$

$$= \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) \frac{(1 + \frac{5}{x})}{(1 - \frac{3}{x} - \frac{4}{x^2})} = (0)(1) = 0$$

- Thus $y = 0$ is a horizontal asymptote and hence there is no oblique asymptote.

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x+5}{x^2-3x-4}$$

$$= \lim_{x \rightarrow -\infty} \frac{x(1 + \frac{5}{x})}{x^2(1 - \frac{3}{x} - \frac{4}{x^2})}$$

$$= \lim_{x \rightarrow -\infty} \frac{(1 + \frac{5}{x})}{x(1 - \frac{3}{x} - \frac{4}{x^2})}$$

$$= \lim_{x \rightarrow -\infty} \left(\frac{1}{x}\right) \frac{(1 + \frac{5}{x})}{(1 - \frac{3}{x} - \frac{4}{x^2})} = (0)(1) = 0$$

- **Vertical Asymptotes**

- First of all

$$Dom(f) = \mathbb{R} - \{x : x^2 - 3x - 4 = 0\} = \mathbb{R} - \{4, -1\}$$

- Testing $x = 4$:

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \frac{x+5}{x^2-3x-4} \rightarrow \frac{9}{0}$$

$$\lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \frac{x+5}{(x-4)(x+1)} \rightarrow \frac{9}{(0^+)(5)} = \infty$$

- Thus $x = 4$ is a vertical asymptote.

- Testing $x = -1$:

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \frac{x+5}{x^2-3x-4} \rightarrow \frac{4}{0}$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \frac{x+5}{(x-4)(x+1)} \rightarrow \frac{4}{(-5)(0^+)} = -\infty$$

- Thus $x = -1$ is a vertical asymptote.

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What is the most difficult part of this section?