



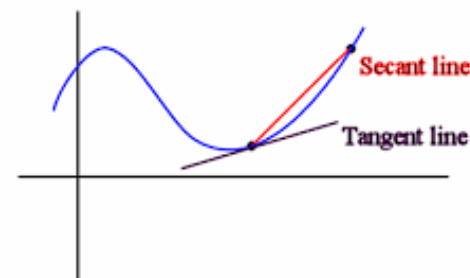
Chapter 3 DIFFERENTIATION

First Semester 2020-2021

أ. د. أيمن هاشم السقا

3.1 Tangents and Derivative at a Point

- In chapter 2 we defined the slope of a curve at a point as the limit of secant slopes.
- This limit, called a derivative, measure the rate at which a function changes and is one of the most important ideas in calculus.



3.2 The Derivative as a Function

Definition

(The derivative of a function at a point)

The derivative of a function f at a point x_0 is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

Remark

The following are all interpretations of $f'(x_0)$:

- (1) $f'(x_0)$ is the derivative of $f(x)$ at $x = x_0$.
- (2) $f'(x_0)$ is the slope of the graph of $y = f(x)$ at $x = x_0$.
- (3) $f'(x_0)$ is the slope of the tangent to the curve $y = f(x)$ at $x = x_0$.
- (4) $f'(x_0)$ is the rate of change of $f(x)$ with respect to x at $x = x_0$.

Definition

(The derivative of a function f as a function f')

The derivative of the function f with respect to the variable x is the function f' given by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists.

Notation

(1) There are many ways to denote the derivative of a function $y = f(x)$. The most common notations are:

$$f'(x) = y' = \frac{d}{dx} f(x) = \frac{dy}{dx} = D_x f(x) = \dot{f}(x) = \dot{y}.$$

(2) The derivative of f at a point x_0 is denoted by:

$$f'(x_0) = \left. \frac{d}{dx} f(x) \right|_{x_0} = \left. \frac{dy}{dx} \right|_{x_0}.$$

Example

Use the definition to find $f'(x)$ for the following functions:

(a) $f(x) = x + 3$.

(b) $f(x) = x/(x + 2)$.

(c) $f(x) = \sqrt{x - 1}$.

Solution

(a) $f(x) = x + 3$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{x+h+3 - (x+3)}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Solution

(b) $f(x) = \frac{x}{x+2}$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{\frac{(x+h)(x+2) - x(x+h+2)}{(x+h+2)(x+2)}}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{x^2 + 2x + hx + 2h - (x^2 + hx + 2x)}{h(x+h+2)(x+2)}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{2h}{h(x+h+2)(x+2)}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{2}{(x+h+2)(x+2)} = \frac{2}{(x+2)^2}$$

Solution

(c) $f(x) = \sqrt{x - 1}$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \left[\frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \right] \left[\frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}} \right]$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{(x+h-1) - (x-1)}{h[\sqrt{x+h-1} + \sqrt{x-1}]}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{h}{h[\sqrt{x+h-1} + \sqrt{x-1}]}$$

$$\bullet f'(x) = \lim_{h \rightarrow 0} \frac{1}{[\sqrt{x+h-1} + \sqrt{x-1}]} = \frac{1}{2\sqrt{x-1}}$$

Differentiable on an interval; one-sided derivatives

Definition

- (1) A function $y = f(x)$ is differentiable at a point x_0 if $f'(x_0)$ exists.
- (2) A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval.
- (3) A function $y = f(x)$ is differentiable on a closed interval $[a, b]$ if it is differentiable on (a, b) and if the right-hand derivative of f at a ,

$$\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h},$$

and the left-hand derivative of f at b ,

$$\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h},$$

exist.

Examples

Example 1

Show that $f(x) = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but not differentiable at $x = 0$.

Solution

- Recall $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$
- If $x_0 \in (-\infty, 0)$, then $f(x) = -x$ and hence $f'(x_0) = -1$.
- Thus $f(x)$ is differentiable on $(-\infty, 0)$
- If $x_0 \in (0, \infty)$, then $f(x) = x$ and hence $f'(x_0) = 1$.
- Thus $f(x)$ is differentiable on $(0, \infty)$
- If $x_0 = 0$, then
$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$$

Solution

- $\lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$
- $\lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$
- Thus $f'(0)$ does not exist.

Example 2

The function $f(x) = \sqrt{x}$ is not differentiable at $x = 0$ since the right-hand derivative of $f(x)$ at $x = 0$

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{\sqrt{h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty \text{ does not exist.}$$

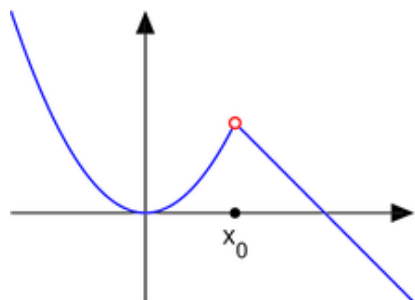
$$\text{Dom}(f) = [0, \infty)$$

When does a function not have a derivative at a point?

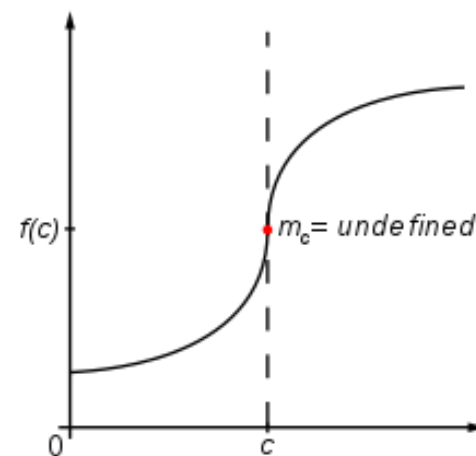
- A function has a derivative at a point x_0 if the slopes of secant lines through $P(x_0, f(x_0))$ and a nearby point $Q(x_0 + h, f(x_0 + h))$ on the graph approach a finite limit as Q approaches P .
- Whenever the secant fail to take up a limiting position or become vertical as Q approaches P , the derivative does not exist.
- Thus, differentiability is a “smoothness” condition on the graph of f .
- A function f can fail to have a derivative at a point for many reasons.

Some of these reasons are as follows:

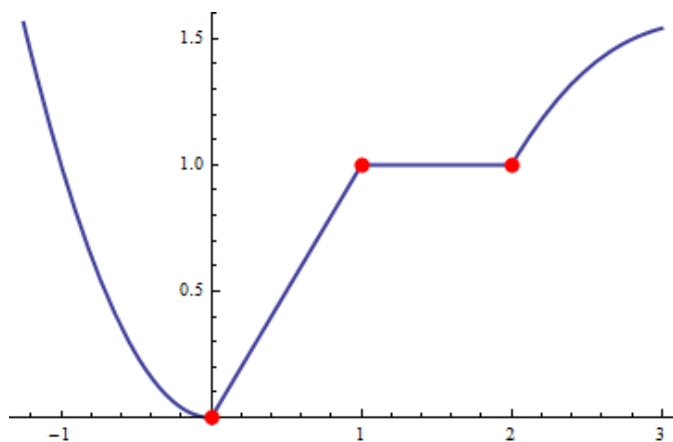
- (1) If f has a discontinuity at x_0 .



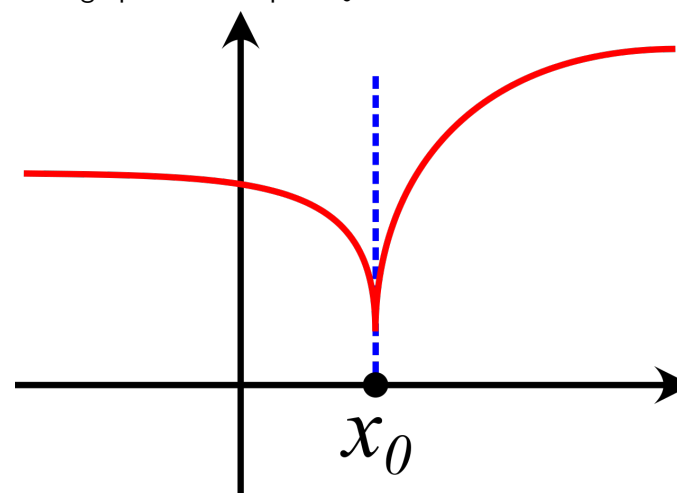
- (2) If $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \pm\infty$, then the graph has a vertical tangent at x_0 .



- (3) If $\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = L \neq \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = M$, then the graph has a corner at x_0 .



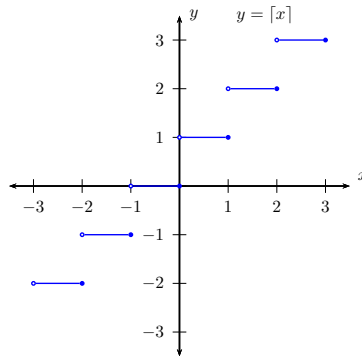
- (4) If $\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \pm\infty \neq \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = \mp\infty$, then the graph has a cusp at x_0 .



Examples

Example 1

Find the points at which f is not differentiable if $f(x) = \lceil x \rceil$.

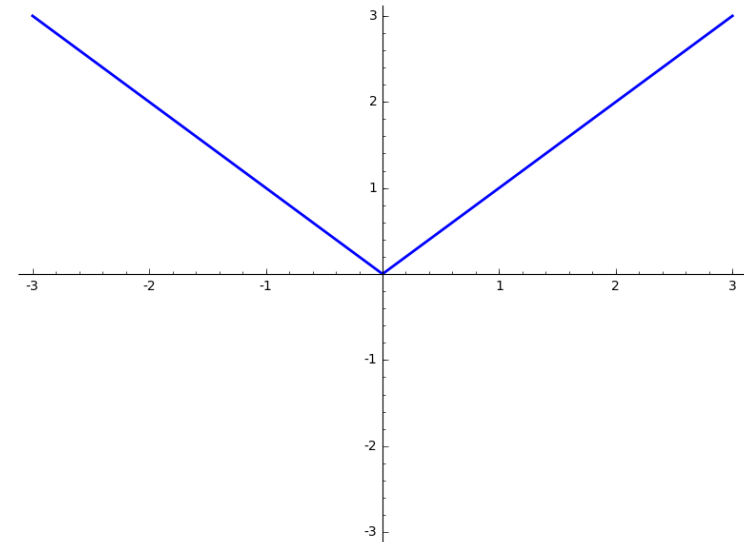


Solution

- f is not differentiable at $x = 0, \pm 1, \pm 2, \dots$ since f is discontinuous at these points.

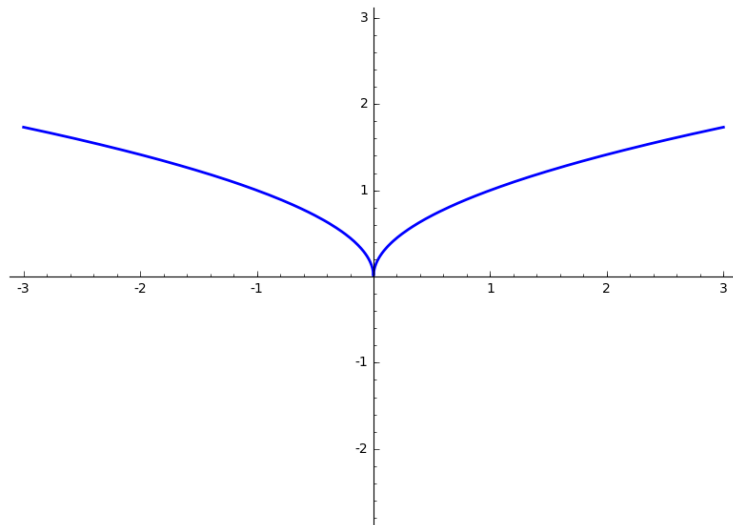
Example 2

$f(x) = |x|$ is not differentiable at $x = 0$ since its graph has a corner at $x = 0$.



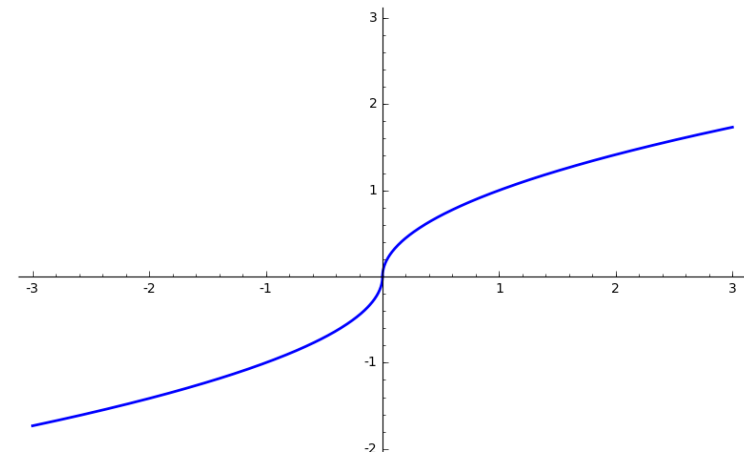
Example 3

$f(x) = \sqrt{|x|}$ is not differentiable at $x = 0$ since its graph has a cusp at $x = 0$.



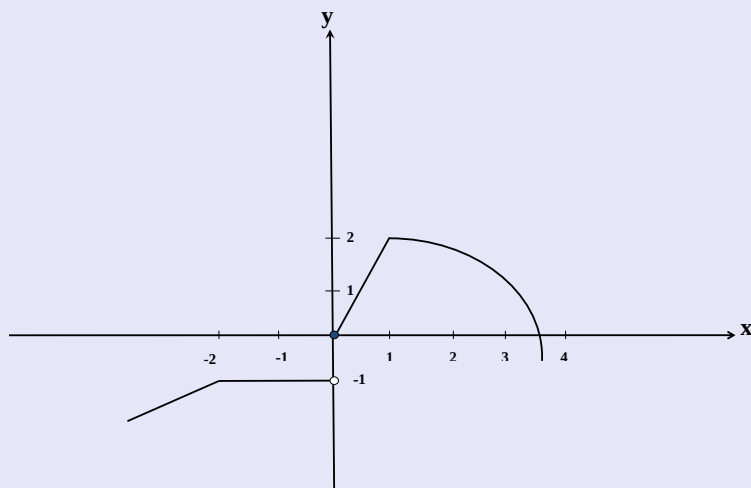
Example 4

$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$ is not differentiable at $x = 0$ since its graph has a vertical tangent at $x = 0$.

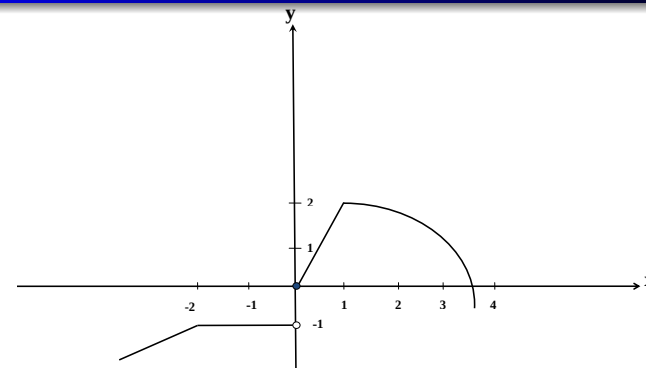


Example 5

If f has the graph in the accompanying figure, then find the points at which f is not differentiable. Give reasons for your answer.



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Solution

- f is not differentiable at $x = 0$ since it has discontinuity at $x = 0$.
- f is not differentiable at $x = 1$ since its graph has a corner at $x = 1$.
- f is not differentiable at $x = -2$ since its graph has a corner at $x = -2$.

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Differentiable functions are continuous

Theorem 1 (Differentiability implies continuity)

If f has a derivative at $x = x_0$, then f is continuous at $x = x_0$.

Remark

The converse of Theorem 1 is not true. That is, if f is continuous at x_0 , then f may or may not be differentiable at x_0 .

Examples

Example 1

The function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Example 2 (Exam)

$$\text{Let } f(x) = \begin{cases} x^2 - 1, & x < 3 \\ \frac{8x}{3}, & x \geq 3. \end{cases}$$

Show that f is continuous at $x = 3$ but not differentiable at $x = 3$.

Solution

- Continuity at $x = 3$:
 - 1 $f(3) = 8$ is defined.
 - 2 $\lim_{x \rightarrow 3} f(x) = L$.
 - 3 $L = f(3)$.
- $L = \lim_{x \rightarrow 3} f(x) = ?$
- $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 - 1) = 8$

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ \frac{8x}{3}, & x \geq 3. \end{cases}$$

- $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{8}{3}x = 8$
- $\implies \lim_{x \rightarrow 3} f(x) = 8 = f(3)$
- Therefore, f is continuous at $x = 3$.
- Differentiability at $x = 3$:
- $f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{f(3+h) - 8}{h}$
- $\lim_{h \rightarrow 0^+} \frac{f(3+h) - 8}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{8}{3}(3+h) - 8}{h}$
- $= \lim_{h \rightarrow 0^+} \frac{\frac{8}{3}h}{h} = \boxed{\frac{8}{3}}$

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ \frac{8x}{3}, & x \geq 3. \end{cases}$$

- $\lim_{h \rightarrow 0^-} \frac{f(3+h) - 8}{h} = \lim_{h \rightarrow 0^-} \frac{(3+h)^2 - 1 - 8}{h}$
- $= \lim_{h \rightarrow 0^-} \frac{9 + 6h + h^2 - 9}{h}$
- $= \lim_{h \rightarrow 0^-} \frac{6h + h^2}{h}$
- $= \lim_{h \rightarrow 0^-} (6 + h) = 6$
- Therefore, $f'(3)$ does not exist.

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3.3 Differentiation Rules

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أ.د. أيمن هاشم السقا

Differentiation Rules

- In this section we introduce several rules that allow us to differentiate many functions directly, without using the definition.

(1) If k is a constant, then $\frac{d}{dx}(k) = 0$.

(2) If n is a real number, then $\frac{d}{dx}x^n = nx^{n-1}$ for all x where x^n and x^{n-1} are defined.

(5) If f and g are differentiable functions of x , then

$$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x).$$

(6) If f and g are differentiable functions of x and $g(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Differentiation Rules

(3) If f is a differentiable function of x and k is a constant, then

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x).$$

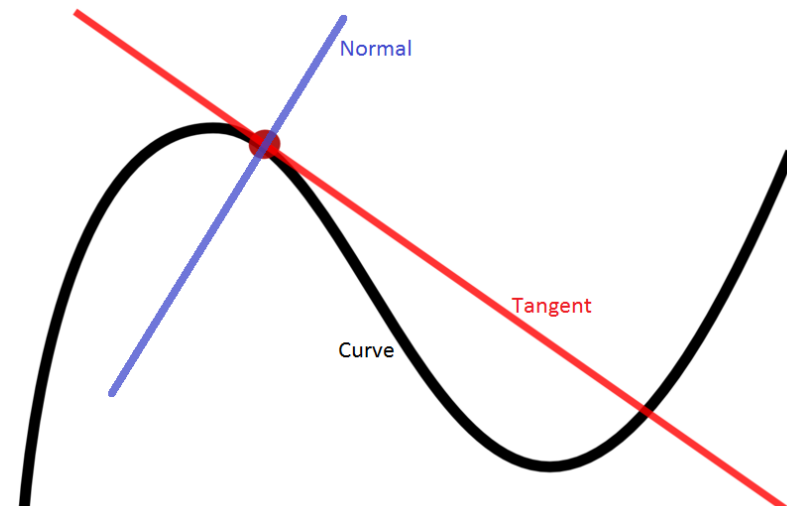
(4) If f and g are differentiable functions of x , then

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

Normal line

Definition

The normal line to a curve $y = f(x)$ at a point $P(x_0, f(x_0))$ is the line through P that is perpendicular to the tangent line of the curve at P .



Example 1

Find equations for the tangent and normal lines to the curve $y = x^4 - 2x^3 + 2x - 5$ at the point $(2, -1)$.

Solution

- The equation of a line passing through the point $P(x_0, y_0)$ with slope m is

$$y - y_0 = m(x - x_0)$$

- Therefore, an equation of the tangent line is $y + 1 = 10(x - 2) \Rightarrow y = 10x - 13$
- and an equation of the normal line is

$$y + 1 = -\frac{1}{10}(x - 2) \Rightarrow y = -\frac{1}{10}x - \frac{4}{5}$$

- The slope of the tangent line to a curve $y = f(x)$ at a point $P(x_0, y_0)$ is

$$m_t = f'(x_0) = \left. \frac{dy}{dx} \right|_{x=x_0}$$

and the slope of the normal line is

$$m_n = \frac{-1}{m_t} = \frac{-1}{f'(x_0)}$$

- In this example, the point is $P = (x_0, y_0) = (2, -1)$ and $f(x) = x^4 - 2x^3 + 2x - 5$
- Thus $f'(x) = 4x^3 - 6x^2 + 2$
- and hence the slope of the tangent line is $m_t = f'(2) = 4(8) - 6(4) + 2 = 32 - 24 + 2 \Rightarrow m_t = 10$
- It follows that the slope of normal line is $m_n = \frac{-1}{10}$

Example 2

Find equations for the tangent and normal lines to the curve $y = x^3 - 3x^2 - x + 7$ at the points where the slope of the curve is 8.

Solution

- In this example, the slope of the tangent line is $m_t = 8$
- We have to find the points (x_0, y_0) .
- $f'(x) = 8 \Rightarrow 3x^2 - 6x - 1 = 8 \Rightarrow 3x^2 - 6x - 9 = 0$
- $\Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = 3, x = -1$
- Thus we have two points $P_1 = (x_1, y_1) = (3, f(3)) = (3, 4)$ and $P_2 = (x_2, y_2) = (-1, f(-1)) = (-1, 4)$
- Tangent and normal at $P_1 = (3, 4)$:
- Tangent line: $y - 4 = 8(x - 3) \Rightarrow y = 8x - 20$

- Normal line: $y - 4 = \frac{-1}{8}(x - 3) \Rightarrow y = -\frac{1}{8}x + \frac{35}{8}$
- Tangent an normal at $P_2 = (-1, 4)$:
- Tangent line: $y - 4 = 8(x + 1) \Rightarrow y = 8x + 12$
- Normal line: $y - 4 = \frac{-1}{8}(x + 1) \Rightarrow y = -\frac{1}{8}x + \frac{31}{8}$

Example 3

Find the value of a that makes

$$f(x) = \begin{cases} 2x^2 - 2, & x \leq -1 \\ a(x + 1), & x > -1. \end{cases}$$
differentiable for all x -values.

Solution

- We have $f'(x) = \begin{cases} 4x, & x < -1 \\ a, & x > -1. \end{cases}$
- Thus $f'(x)$ exists on $(-\infty, -1)$ and $(-1, \infty)$
- To check differentiability at $x = -1$, we use the definition of derivative.
- $f'(-1) = \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{f(-1+h)}{h}$

$$f(x) = \begin{cases} 2x^2 - 2, & x \leq -1 \\ a(x + 1), & x > -1 \end{cases}$$

- $\lim_{h \rightarrow 0^+} \frac{f(-1+h)}{h} = \lim_{h \rightarrow 0^+} \frac{a(-1+h+1)}{h} = a$
- $\lim_{h \rightarrow 0^-} \frac{f(-1+h)}{h} = \lim_{h \rightarrow 0^-} \frac{2(-1+h)^2 - 2}{h} = \lim_{h \rightarrow 0^-} \frac{2(1 - 2h + h^2) - 2}{h}$
- $\lim_{h \rightarrow 0^-} \frac{f(-1+h)}{h} = \lim_{h \rightarrow 0^-} \frac{2(-2h + h^2)}{h} = \lim_{h \rightarrow 0^-} 2(-2 + h) = -4$
- Thus $f'(-1)$ exists $\Rightarrow a = -4$

Second and Higher-Order Derivatives

- $y' = f'(x)$ is called the first (order) derivative of $y = f(x)$.
- Since $f'(x)$ is also a function, we can differentiate it to obtain the second derivative of $y = f(x)$,

$$y'' = \frac{d}{dx}y' = \frac{d^2y}{dx^2}.$$

- In general, the n -th derivative of $y = f(x)$ is

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n}.$$

Example 1

Find $f'''(x)$ if $f(x) = 2x^3 + 9x^2 + 5x^{-3}$.

Solution

- $f'(x) = \frac{d}{dx}(2x^3 + 9x^2 + 5x^{-3}) = 6x^2 + 18x - 15x^{-4}$
- $f''(x) = \frac{d}{dx}(6x^2 + 18x - 15x^{-4}) = 12x + 18 + 60x^{-5}$
- $f'''(x) = \frac{d}{dx}(12x + 18 + 60x^{-5}) = 12x - 300x^{-6}$

Example 2

Find $y''(x)$ if $y = \frac{x^2 - 3x}{2x^4 + 5}$.

Solution

- $y' = \frac{d}{dx} \left(\frac{x^2 - 3x}{2x^4 + 5} \right) = \frac{(2x - 3)(2x^4 + 5) - (8x^3)(x^2 - 3x)}{(2x^4 + 5)^2}$
- $y' = \frac{4x^5 + 10x - 6x^4 - 15 - 8x^5 + 24x^4}{(2x^4 + 5)^2}$
- $y' = \frac{-4x^5 + 18x^4 + 10x - 15}{4x^8 + 20x^4 + 25}$

$$(2x^4 + 5)^2 = 4x^8 + 20x^4 + 25$$

- $y'' = \frac{d}{dx} \left(\frac{-4x^5 + 18x^4 + 10x - 15}{4x^8 + 20x^4 + 25} \right)$
- $y'' = \frac{1}{(4x^8 + 20x^4 + 25)^2} \left[(-20x^4 + 72x^3 + 10)(4x^8 + 20x^4 + 25) - (-4x^5 + 18x^4 + 10x - 15)(32x^7 + 80x^3) \right]$



3.5 Derivatives of Trigonometric Functions

3.6 The Chain Rule

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أ.د. أيمن هاشم السقا

3.5 Derivatives of Trigonometric Functions

- $\frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x$

Proof.

- $\frac{d}{dx} \sin x = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$
- $\frac{d}{dx} \sin x = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$
- $\frac{d}{dx} \sin x = \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} = \cos x.$

□

Example 1

Prove the following rules:

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \tan x \sec x$$

$$\frac{d}{dx} \csc x = -\cot x \csc x$$

Solution

- $\frac{d}{dx} \tan x = \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right)$
- $\frac{d}{dx} \tan x = \frac{\cos x \cos x - (-\sin x)(\sin x)}{\cos^2 x}$
- $\frac{d}{dx} \tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$
- $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} = \sec^2 x$

Example 2

Find y' if $y = x^2 \cot x - \frac{1}{x^4}$.

Solution

- $y' = \frac{d}{dx} \left(x^2 \cot x - \frac{1}{x^4} \right)$
- $y' = 2x \cot x - x^2 \csc^2 x + 4x^{-5}$

Example 3

Find y' if $y = \frac{\sqrt{x} + \cos x}{\tan x - \cot x}$.

Solution

$$\begin{aligned} y' &= \frac{d}{dx} \left(\frac{\sqrt{x} + \cos x}{\tan x - \cot x} \right) \\ y' &= \frac{\left(\frac{1}{2\sqrt{x}} - \sin x \right) (\tan x - \cot x) - (\sec^2 x + \csc^2 x)(\sqrt{x} + \cos x)}{(\tan x - \cot x)^2} \end{aligned}$$

Example 4

Find y'' if $y = \csc x$.

Solution

$$\begin{aligned} y' &= \frac{d}{dx} (\csc x) \\ y' &= -\csc x \cot x \\ y'' &= \frac{d}{dx} (-\csc x \cot x) \\ y'' &= - \left[(-\csc x \cot x) \cot x + \csc x (-\csc^2 x) \right] \\ y'' &= - \left[-\csc x \cot^2 x - \csc^3 x \right] = \csc x \cot^2 x + \csc^3 x \end{aligned}$$

3.6 The Chain Rule

- If $h(x) = \sin(x^2 + 5)$, then $h'(x) = ?$
- Let $g(x) = x^2 + 5$ and $f(x) = \sin x$
- Then $h(x) = (f \circ g)(x)$.
- We know $f'(x) = \cos x$ and $g'(x) = 2x$
- Thus, if we can write the derivative of h in terms of the derivatives of g and f , then we can find $h'(x)$.
- In this section we will develop a rule to differentiate composite function $(f \circ g)(x)$.

Theorem 2 (The Chain Rule)

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x)$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

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Example 1

Let $f \circ g$ be a composite of the differentiable functions $y = f(u)$ and $u = g(x)$. If $f'(4) = -3$, $g(1) = 4$, $g'(1) = 7$, then find $(f \circ g)'(1)$.

Solution

- By the chain rule, $(f \circ g)'(1) = f'(g(1)) g'(1)$
- $(f \circ g)'(1) = f'(4) (7) = -3(7) = -21$

Example 2

Given $y = \tan u$ and $u = x^2 + \sin x$, find dy/dx .

Solution

- By the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$
- We have $\frac{dy}{du} = \frac{d}{du}(\tan u) = \sec^2 u$
- and $\frac{du}{dx} = \frac{d}{dx}(x^2 + \sin x) = 2x + \cos x$
- $\Rightarrow \frac{dy}{dx} = (\sec^2 u)(2x + \cos x) = \sec^2(x^2 + \sin x) \cdot (2x + \cos x)$

Example 3

Find y' at $x = 1$ if $y = \cos(5x^3 - 3x + 6)$.

Solution

- "Outside-Inside" Rule
- $y' = \frac{d}{dx} \cos(5x^3 - 3x + 6) = -\sin(5x^3 - 3x + 6) \cdot (15x^2 - 3)$
- $\left. \frac{dy}{dx} \right|_{x=1} = -\sin(5 - 3 + 6) \cdot (15 - 3) = -12 \sin(8)$

Example 4

Find y' if $y = \sec^3(\sin(3x))$.

Solution

- $y' = \frac{d}{dx} \sec^3(\sin(3x)) = \frac{d}{dx} \left(\sec(\sin(3x)) \right)^3$
- $y' = 3 \left(\sec(\sin(3x)) \right)^2 \cdot \sec(\sin(3x)) \tan(\sin(3x)) \cdot \cos(3x) \cdot 3$

Example 5

Find y' if $y = f\left(\frac{1}{x}\right)$.

Solution

- $y' = \frac{d}{dx} f\left(\frac{1}{x}\right)$
- $y' = f'\left(\frac{1}{x}\right) \cdot \left(\frac{-1}{x^2}\right)$

Example 6 (Exam)

Find $\frac{dy}{dx}$ if $y = \sec(\sqrt{x} + x) \tan\left(\frac{1}{x^2}\right)$.

Solution

- $y' = \frac{d}{dx} \left[\sec(\sqrt{x} + x) \tan\left(\frac{1}{x^2}\right) \right]$
- $y' = \tan\left(\frac{1}{x^2}\right) \cdot \sec(\sqrt{x} + x) \cdot \tan(\sqrt{x} + x) \cdot \left(\frac{1}{2\sqrt{x}} + 1\right)$
 $+ \sec(\sqrt{x} + x) \cdot \sec^2\left(\frac{1}{x^2}\right) \cdot (-2x^{-3})$

Example 7 (Exam)

Find $\frac{dy}{dx}$ if $y = [x^3 + \cos(2x)]^{-3}$.

Solution

- $y' = \frac{d}{dx} [x^3 + \cos(2x)]^{-3}$
- $y' = -3 [x^3 + \cos(2x)]^{-4} \cdot [3x^2 - \sin(2x) \cdot 2]$

Example 8

Find y'' if $y = (3x^5 - 5) \cot(3x^2 - 1)$.

Solution

- $y' = \frac{d}{dx} [(3x^5 - 5) \cot(3x^2 - 1)]$
- $y' = 15x^4 \cdot \cot(3x^2 - 1) + (3x^5 - 5) \cdot [-\csc^2(3x^2 - 1) \cdot (6x)]$
- $y' = 15x^4 \cot(3x^2 - 1) - 6x(3x^5 - 5) \csc^2(3x^2 - 1)$
- $y'' = \frac{d}{dx} [15x^4 \cot(3x^2 - 1) - 6x(3x^5 - 5) \csc^2(3x^2 - 1)]$
- $y'' = 60x^3 \cdot \cot(3x^2 - 1) + 15x^4 \cdot [-\csc^2(3x^2 - 1) \cdot (6x)]$
 $- 6(18x^5 - 5) \cdot \csc^2(3x^2 - 1)$
 $- 6(3x^6 - 5x) \cdot 2 \csc(3x^2 - 1) \cdot [-\csc(3x^2 - 1) \cdot \cot(3x^2 - 1) \cdot (6x)]$

Example 9

Let $y = f(x)g^4(x)$ and $f(3) = 5$, $g(3) = 2$, $f'(3) = -3$, $g'(3) = 4$. Find dy/dx at $x = 3$.

Solution

- $y' = \frac{d}{dx} [f(x)g^4(x)] = f'(x)g^4(x) + f(x) \cdot 4g^3(x) \cdot g'(x)$
- $\left. \frac{dy}{dx} \right|_{x=3} = f'(3)g^4(3) + f(3) \cdot 4g^3(3) \cdot g'(3)$
- $\left. \frac{dy}{dx} \right|_{x=3} = (-3)(2)^4 + 5 \cdot 4(2)^3 \cdot (4)$
- $\left. \frac{dy}{dx} \right|_{x=3} = (-3)(16) + 80(8) = -48 + 640 = 592$

Example 10 (Exam)

Let f and g be two functions such that $g(x) = f(\sqrt{x}) + \sqrt{f(x)}$. If $f(1) = 4$ and $f'(1) = 8$, then find $g'(1)$.

Solution

- $g'(x) = \frac{d}{dx} [f(\sqrt{x}) + \sqrt{f(x)}]$
- $g'(x) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{f(x)}} \cdot f'(x)$
- $\Rightarrow g'(1) = f'(1) \cdot \frac{1}{2} + \frac{1}{2\sqrt{f(1)}} \cdot f'(1)$
- $\Rightarrow g'(1) = (8)\left(\frac{1}{2}\right) + \frac{1}{2\sqrt{4}}(8) = 4 + 2 = 6$

Example 11

Let f be a differentiable function. Show that if f is even, then f' is odd.

Solution

- $f \text{ is even} \Rightarrow f(-x) = f(x)$
- $\frac{d}{dx} f(-x) = \frac{d}{dx} f(x)$
- $\Rightarrow f'(-x) \cdot (-1) = f'(x)$
- $\Rightarrow f'(-x) = -f'(x)$
- Therefore, f' is odd.



3.7 Implicit Differentiation

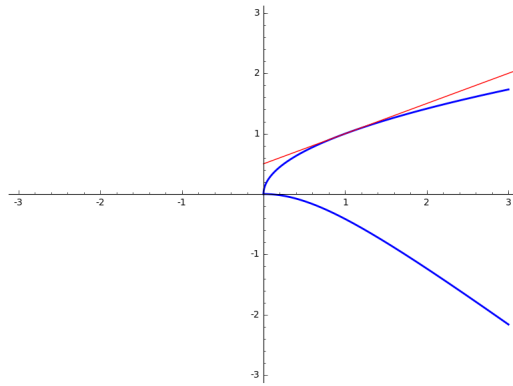
3.9 Linearization and Differentials

First Semester 2020-2021

أ.د. أيمن هاشم السقا

3.7 Implicit Differentiation

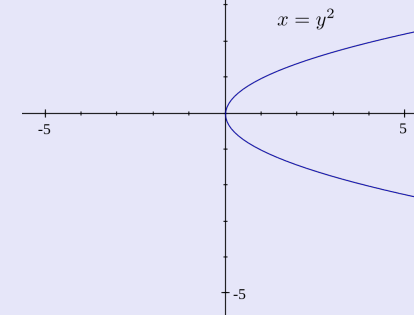
- A plane curve can often be represented in Cartesian coordinates by an implicit equation of the form $F(x, y) = 0$.



- When we can not put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find y' by implicit differentiation.

Implicit defined functions

- The graph of the equation $x = y^2$ has a well-defined slope at nearly every point because it is the union of the graphs of the two differentiable functions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$.
- In order to find the slope we treat y as a differentiable function of x and differentiate both sides of the equation $y^2 = x$ with respect to x , using the differentiation rules.



Example 1

Find $\frac{dy}{dx}$ if $x^4 - 4xy + y^2 = 8$.

Solution

- (1) We differentiate both sides with respect to x , treating

$y = f(x)$ as a differentiable function of x :

$$\begin{aligned} \bullet \frac{d}{dx} [x^4 - 4xy + y^2] &= \frac{d}{dx} [8] \\ \bullet \implies 4x^3 - 4[y + xy'] + 2yy' &= 0 \end{aligned}$$

- (2) Collect the terms with $y' = \frac{dy}{dx}$ on one side of the equation and solve for $y' = \frac{dy}{dx}$.

$$\bullet -4xy' + 2yy' = -4x^3 + 4y \implies 2(y - 2x)y' = 4(y - x^3)$$

$$\bullet \implies y' = \frac{4(y - x^3)}{2(y - 2x)}$$

Example 2

Find $\frac{dy}{dx}$ if $y = 3x^3 + 2xy^{3/2} + \cos(x^2y)$.

Solution

- (1) We differentiate both sides with respect to x , treating

$y = f(x)$ as a differentiable function of x :

$$\begin{aligned} \bullet \frac{d}{dx} [y] &= \frac{d}{dx} [3x^3 + 2xy^{3/2} + \cos(x^2y)] \\ \bullet \implies y' &= 9x^2 + 2[y^{3/2} + x \cdot \frac{3}{2}y^{1/2}y'] - \sin(x^2y) \cdot [2xy + x^2y'] \\ \bullet \implies y' &= 9x^2 + 2y^{3/2} + 3xy^{1/2}y' - 2xy \sin(x^2y) - x^2 \sin(x^2y)y' \end{aligned}$$

- $\Rightarrow y' = 9x^2 + 2y^{3/2} + 3xy^{1/2}y' - 2xy \sin(x^2y) - x^2 \sin(x^2y)y'$
- (2) Collect the terms with y' on one side of the equation and solve for y' .
 - $\Rightarrow y' - 3xy^{1/2}y' + x^2 \sin(x^2y)y' = 9x^2 + 2y^{3/2} - 2xy \sin(x^2y)$
 - $\Rightarrow [1 - 3xy^{1/2} + x^2 \sin(x^2y)] y' = 9x^2 + 2y^{3/2} - 2xy \sin(x^2y)$
 - $\Rightarrow y' = \frac{9x^2 + 2y^{3/2} - 2xy \sin(x^2y)}{1 - 3xy^{1/2} + x^2 \sin(x^2y)}$

- $\Rightarrow 2yy'' + 2(y')^2 + 3y'' = 0$
- (2) Collect the terms with y'' on one side of the equation and solve for y'' .
 - $\Rightarrow (2y + 3)y'' = -2(y')^2$
 - $\Rightarrow y'' = \frac{-2(y')^2}{2y + 3}$
 - $\Rightarrow y''|_{(3,2)} = \frac{-2(y')^2}{2(2)+3}$
 - $2yy' + 3y' - 2 = 0 \Rightarrow y' = \frac{2}{2y+3}$
 - $y'|_{(3,2)} = \frac{2}{7}$
 - $\Rightarrow y''|_{(3,2)} = \frac{-2(2/7)^2}{7}$
 - $\Rightarrow y''|_{(3,2)} = \frac{-8}{343}$

Example 3

If $y^2 + 3y - 2x = 4$, find $\frac{d^2y}{dx^2}$ at the point $(3, 2)$

Solution

- (1) We differentiate both sides with respect to x , treating $y = f(x)$ as a differentiable function of x :
 - $\frac{d}{dx} [y^2 + 3y - 2x] = \frac{d}{dx} [4]$
 - $\Rightarrow 2yy' + 3y' - 2 = 0$
 - We differentiate both sides with respect to x again:
 - $\frac{d}{dx} [2yy' + 3y' - 2] = \frac{d}{dx} [0]$
 - $\Rightarrow 2yy'' + 2(y')^2 + 3y'' = 0$

Example 4

Find $\frac{d^2y}{dx^2}$ if $x^{5/2} + 4y^{5/2} = y$.

Solution

- (1) We differentiate both sides with respect to x , treating $y = f(x)$ as a differentiable function of x :
 - $\frac{d}{dx} [x^{5/2} + 4y^{5/2}] = \frac{d}{dx} [y]$
 - $\Rightarrow \frac{5}{2}x^{3/2} + 4 \cdot \frac{5}{2}y^{3/2}y' = y'$
 - $\Rightarrow \frac{5}{2}x^{3/2} + 10y^{3/2}y' = y'$
 - We differentiate both sides with respect to x again:
 - $\frac{d}{dx} [\frac{5}{2}x^{3/2} + 10y^{3/2}y'] = \frac{d}{dx} [y']$

- $\frac{d}{dx} \left[\frac{5}{2}x^{3/2} + 10y^{3/2}y' \right] = \frac{d}{dx} [y']$
- $\implies \frac{15}{4}x^{1/2} + 10y^{3/2}y'' + 15y^{1/2}(y')^2 = y''$

(2) Collect the terms with y'' on one side of the equation and solve for y'' .

- $\implies 10y^{3/2}y'' - y'' = -\frac{15}{4}x^{1/2} - 15y^{1/2}(y')^2$

- $\implies y'' = \frac{-\frac{15}{4}x^{1/2} - 15y^{1/2}(y')^2}{10y^{3/2} - 1}$

- $\frac{5}{2}x^{3/2} + 10y^{3/2}y' = y' \implies y' = \frac{\frac{5}{2}x^{3/2}}{1 - 10y^{3/2}}$

$$2(-1) - 1 - (-1)m_t + 6(1)^2m_t = 0$$

- $\implies 7m_t = 3 \implies m_t = \frac{3}{7}$

- $\implies m_n = -\frac{7}{3}$

- Tangent line: $y - 1 = \frac{3}{7}(x + 1) \implies y = \frac{3}{7}x + \frac{10}{7}$

- Normal line: $y - 1 = -\frac{7}{3}(x + 1) \implies y = -\frac{7}{3}x - \frac{4}{3}$

Example 5

Verify that the point $(-1, 1)$ is on the curve $x^2 - xy + 2y^3 = 4$ and find the tangent and normal lines to the curve at this point.

Solution

- $(x, y) = (-1, 1)$ is on the curve?
- $(-1)^2 - (-1)(1) + 2(1)^3 = 1 + 1 + 2 = 4$ is true
- Thus $(x, y) = (-1, 1)$ is on the curve.

- Find $m_t = \frac{dy}{dx} \Big|_{(-1,1)}$

(1) We differentiate both sides with respect to x :

- $\frac{d}{dx} [x^2 - xy + 2y^3] = \frac{d}{dx} [4]$

- $\implies 2x - y - xy' + 6y^2y' = 0$

- $(x, y) = (-1, 1) \implies 2(-1) - 1 - (-1)m_t + 6(1)^2m_t = 0$

Example 6

Find the tangent and normal lines to the curve $x \sin(2y) = y \cos(2x)$ at the point $(\pi/4, \pi/2)$.

Solution

- Find $m_t = \frac{dy}{dx} \Big|_{(\pi/4, \pi/2)}$

(1) We differentiate both sides with respect to x :

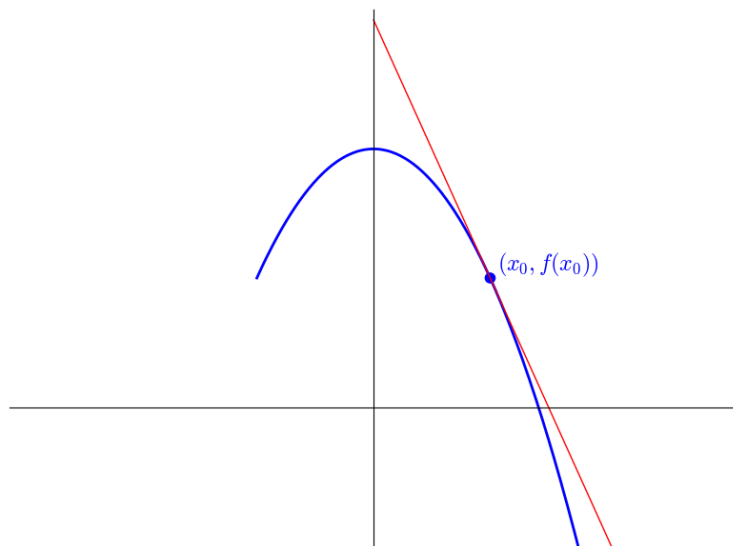
- $\frac{d}{dx} [x \sin(2y)] = \frac{d}{dx} [y \cos(2x)]$

- $\implies \sin(2y) + x \cos(2y) \cdot 2y' = y' \cdot \cos(2x) - y \cdot \sin(2x) \cdot 2$

- $(x, y) = (\pi/4, \pi/2) \implies \sin(\pi) + \frac{\pi}{4} \cos(\pi) \cdot 2m_t = m_t \cdot \cos(\pi/2) - \frac{\pi}{2} \cdot \sin(\pi/2) \cdot 2$

- $\implies -\frac{\pi}{2}m_t = -\pi \implies m_t = 2$

- $m_t = 2$
- $\Rightarrow m_n = -\frac{1}{2}$
- Tangent line: $y - \frac{\pi}{2} = 2(x - \frac{\pi}{4}) \Rightarrow y = 2x$
- Normal line: $y - \frac{\pi}{2} = -\frac{1}{2}(x - \frac{\pi}{4}) \Rightarrow y = -\frac{1}{2}x + \frac{5\pi}{8}$



3.9 Linearization and Differentials (الفوارق)

Linearization is a method to approximate complicated functions with simpler ones.

Definition

Let f be differentiable function at $x = x_0$.

- (1) The linearization of f at x_0 is the approximating function

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

- (2) The approximation $f(x) \approx L(x)$ of f by L is the standard linear approximation of f at x_0 .

Remark

If L is the linear approximation of f at x_0 , then x_0 is the center of the approximation.

Examples

Example 1

Find the linearization of $f(x) = \sqrt{x^2 + 9}$ at $x = -4$ and use it to approximate $f(-4.5)$.

Solution

- We want to find $L(x) = f(x_0) + f'(x_0)(x - x_0)$
- In this example, $f(x) = \sqrt{x^2 + 9}$ and $x_0 = -4$
- $\Rightarrow f(x_0) = f(-4) = \sqrt{16 + 9} = 5$
- $\Rightarrow f'(x) = \frac{2x}{2\sqrt{x^2 + 9}} = \frac{x}{\sqrt{x^2 + 9}}$
- $\Rightarrow f'(-4) = \frac{-4}{\sqrt{16 + 9}} = -\frac{4}{5}$
- Thus $L(x) = f(x_0) + f'(x_0)(x - x_0) =$
 $f(-4) + f'(-4)(x + 4) \Rightarrow L(x) = 5 - \frac{4}{5}(x + 4)$

- $f(x) \approx L(x) = 5 - \frac{4}{5}(x + 4)$
- $\implies f(-4.5) \approx L(-4.5) = 5 - \frac{4}{5}(-4.5 + 4) = 5 + \frac{2}{5} = \boxed{5.4}$
- True value $f(-4.5) = \sqrt{(-4.5)^2 + 9} = \sqrt{29.25} = \boxed{5.40832}$

Example 2

Find the linearization of $f(x) = 2x^2 + x^{-2}$ at $x = 1$.

Solution

- We want to find $L(x) = f(x_0) + f'(x_0)(x - x_0)$
- In this example, $f(x) = 2x^2 + x^{-2}$ and $x_0 = 1$
- $\implies f(x_0) = f(1) = 2 + 1 = 3$
- We calculate $f'(x) = 4x - 2x^{-3} \implies f'(1) = 4 - 2 = 2$
- Thus $L(x) = f(x_0) + f'(x_0)(x - x_0) =$
 $f(1) + f'(1)(x - 1) \implies \boxed{L(x) = 3 + 2(x - 1)}$

Example 3

Find a linearization of $f(x) = x + \sqrt{x}$ at suitably chosen integer near $x = 2.1$ at which $f(x)$ and $f'(x)$ are easy to evaluate. Then use the linearization to approximate $f(2.1)$.

Solution

- We want to find $L(x) = f(x_0) + f'(x_0)(x - x_0)$
- In this example, $f(x) = x + \sqrt{x}$ and $x_0 = ?$
- We want x_0 to be near $x = 2.1$ and $f(x_0)$ and $f'(x_0)$ are easy to evaluate.
- $f(x_0)$ and $f'(x_0)$ are easy to evaluate when $x_0 = 0, 1, 4, 9, \dots$
- We choose $\boxed{x_0 = 1}$ to be near $x = 2.1$

- $f(x) = x + \sqrt{x}$
- $\implies f(x_0) = f(1) = 2$
- We calculate $f'(x) = 1 + \frac{1}{2\sqrt{x}} \implies f'(1) = 1 + \frac{1}{2} = \frac{3}{2}$
- Thus $L(x) = f(x_0) + f'(x_0)(x - x_0) =$
 $f(1) + f'(1)(x - 1) \implies \boxed{L(x) = 2 + \frac{3}{2}(x - 1) = \frac{1}{2}(3x + 1)}$
- $f(x) \approx L(x) = \frac{1}{2}(3x + 1)$
- $\implies f(2.1) \approx L(2.1) = \frac{1}{2}(6.3 + 1) = \frac{7.3}{2} = \boxed{3.65}$
- True value $f(2.1) = 2.1 + \sqrt{2.1} = \boxed{3.5491}$

Differentials (الفوارق)

- We sometimes use the Leibniz notation $\frac{dy}{dx}$ to represent the derivative of y with respect to x .
- Contrary to its appearance, it is not a ratio.
- We now introduce two new variables dx and dy with the property that when their ratio exists, it is equal to the derivative.

Definition

Let $y = f(x)$ be differentiable function. The differential dx is an independent variable. The differential dy is

$$dy = f'(x)dx.$$

Remark

The variable dy is always a dependent variable and it depends on both x and dx .

Example

Let $y = x\sqrt{4-x^2}$. Find dy and the value of dy when $x = 0$ and $dx = 0.1$

Solution

- $dy = f'(x)dx = \left[\sqrt{4-x^2} + x \cdot \frac{(-2x)}{2\sqrt{4-x^2}} \right] dx$
- $\Rightarrow dy = \left[\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \right] dx$
- When $x = 0$ and $dx = 0.1$, $dy = \sqrt{4}(0.1) = 0.2$

تم



مَا اصعب جزء في الموضوع؟

What is the most difficult part of this section?