



Chapter 4

Applications of Derivatives

First Semester 2020-2021
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4.1 Extreme values of functions

- This section shows how to find extreme (maximum or minimum) values of a function from its derivative.
- Once we can do this, we can solve a variety of problems in which we find the optimal (best) way to do something in a given situation.
- Finding extreme values is one of the most important applications of the derivative.

Chapter 4

Applications of Derivatives

In this chapter we use derivatives

- to find extreme values (قيم قصوى) of functions,
- to determine and analyze the shapes of graphs.

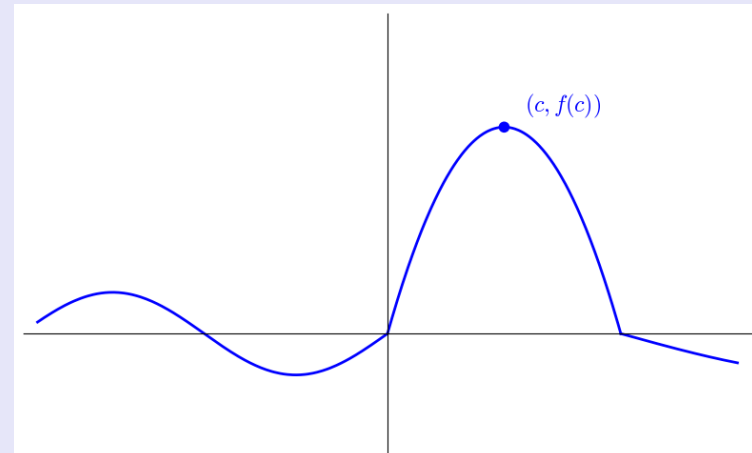
Absolute extreme

Definition

Let f be a function with domain D .

(1) f has an absolute maximum (عظمى) value on D at a point c if

$$f(x) \leq f(c) \quad \text{for all } x \in D.$$

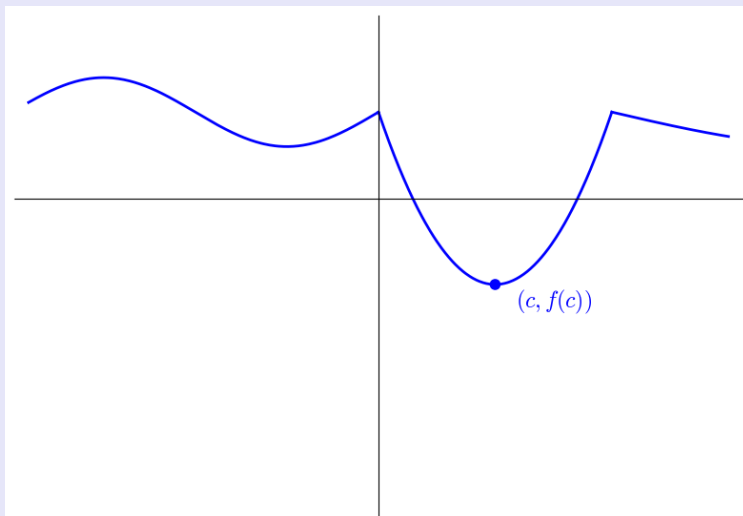


Absolute extreme

Definition

(2) f has an absolute minimum (صغرى) value on D at a point c if

$$f(x) \geq f(c) \quad \text{for all } x \in D.$$

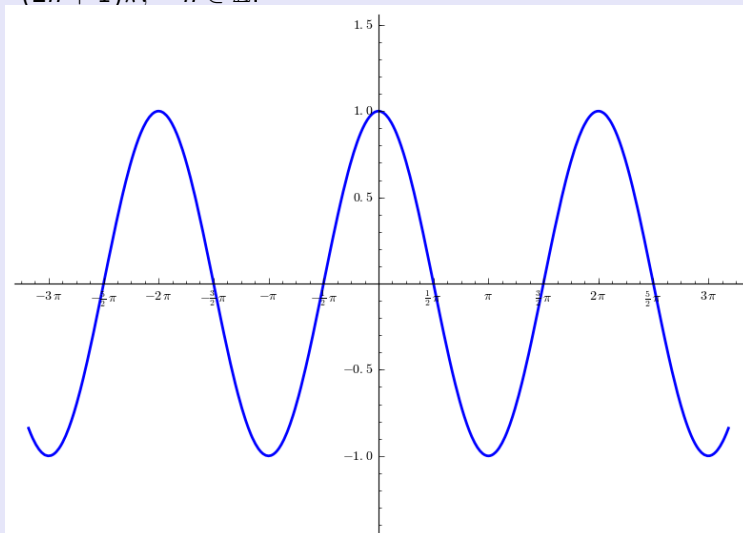


Remarks

- (1) Maximum and minimum values are called extreme values.
- (2) Absolute extreme values are also called global extreme values.

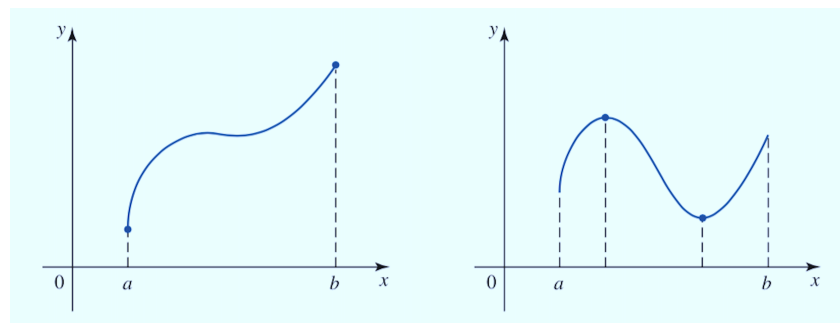
Example

The function $f(x) = \cos x$ has the absolute maximum value 1 that is assumed at $2n\pi$, and minimum value -1 , that is assumed at $x = (2n+1)\pi$, $n \in \mathbb{Z}$.



Theorem 1 (The Extreme Value Theorem)

If f is continuous at every point of a closed interval $[a, b]$, then f has absolute maximum and minimum values at some $x_1, x_2 \in [a, b]$.



Maximum and Minimum Values

Figures 8 and 9 show that a function need not possess extreme values if either hypothesis (continuity or closed interval) is omitted from the Extreme Value Theorem.

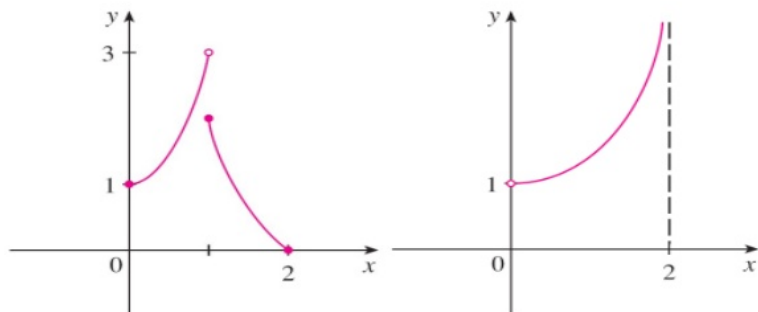


Figure 8 This function has minimum value $f(2) = 0$, but no maximum value.

Figure 9 This continuous function g has no maximum or minimum.

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Example 1

Show that $f(x) = \frac{1}{x}$ has absolute maximum and minimum values in $[2, 3]$ and find them.

Solution

- f is continuous on the closed interval $[2, 3]$.
- Thus f has both absolute maximum and absolute minimum in $[2, 3]$.
- $2 \leq x \leq 3 \implies \frac{1}{3} \leq \frac{1}{x} \leq \frac{1}{2} \implies \frac{1}{3} \leq f(x) \leq \frac{1}{2}$.
- Thus the absolute maximum value is $f(2) = \frac{1}{2}$ and the absolute minimum is $f(3) = \frac{1}{3}$.

Example 2

Show that $f(x) = x^3 - 3x$ has absolute maximum and minimum values in $[-2, 4]$. Can you find them?

Solution

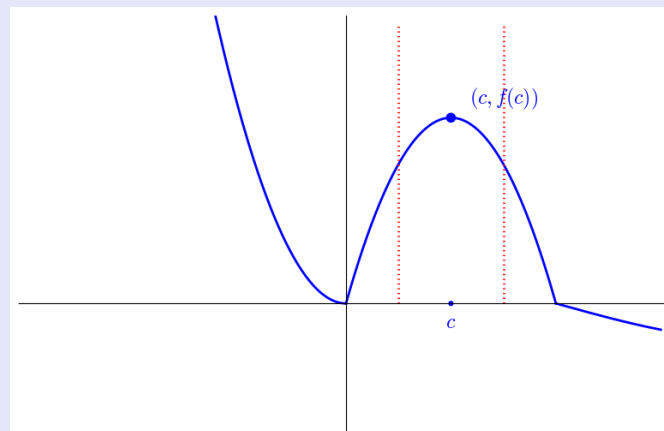
- f is continuous on the closed interval $[-2, 4]$.
- Thus, f has both absolute maximum and absolute minimum in $[-2, 4]$.
- If $-2 \leq x \leq 4$, then $\quad ? \leq f(x) = x^3 - 3x \leq ?$
- It is not easy to find the absolute extreme by the method used in Example 1.

Local Extreme Values

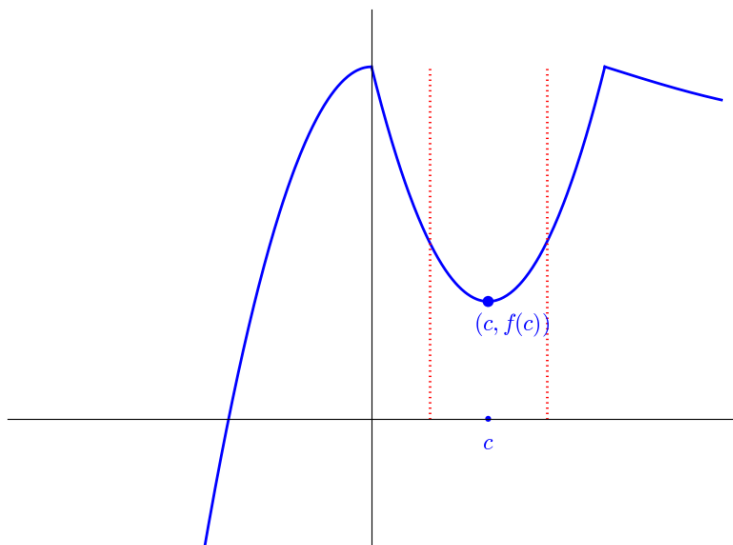
- In order to find the absolute extreme values, we should find the local extreme values.

Definition

- (1) A function f has a local maximum value at a point c within its domain D if $f(x) \leq f(c)$ for all x in some open interval $I \subseteq D$ containing c .



- (2) A function f has a local minimum value at a point c within its domain D if $f(x) \geq f(c)$ for all x in some open interval $I \subseteq D$ containing c .



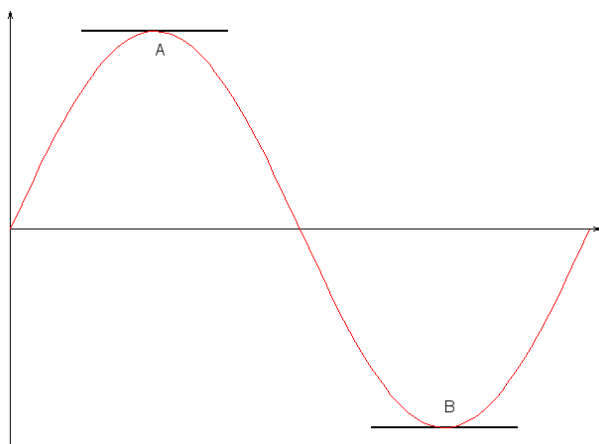
Remarks

- (1) We can extend the definition of local extreme values to the end points of intervals by replacing open intervals by half-open intervals.
- (2) Every absolute extreme value is also a local extreme value. Thus, if we find all local extreme values, we will find the absolute extreme values among these local extreme values.

Finding Extreme Values

Theorem 2 (The First Derivative Test for Local Extreme Values)

If f has a local extreme value at an interior point c of its domain, and if $f'(c)$ exists, then $f'(c) = 0$.



Remark

The solutions of $f'(x) = 0$ are candidates for local extreme values.

Example 1

$f(x) = x$ does not have any local extreme values since $f'(x) = 1 \neq 0$ exists for all $x \in \mathbb{R}$.

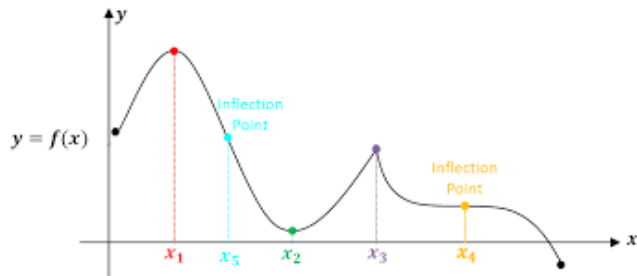
Example 2

$f(x) = x^2$ has a local minimum at $x = 0$ and $f'(0) = 0$.

Critical points

Definition (Critical points)

An interior point c of the domain of a function f is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.



Remark

The only points where a function can have extreme values are critical points and endpoints.

Finding the absolute extreme values of a continuous function f on a finite closed interval

- To find the absolute maximum and minimum values of a continuous function $f(x)$ on a closed interval $[a, b]$ use the following process:

- (1) Find the critical points of f .
- (2) Evaluate f at all critical points and endpoints.
- (3) Take the largest and smallest of these values.

Example 1

Find the absolute extreme values of $f(x) = \sqrt{x}$ on the interval $[1, 4]$. Then graph the function.

Solution

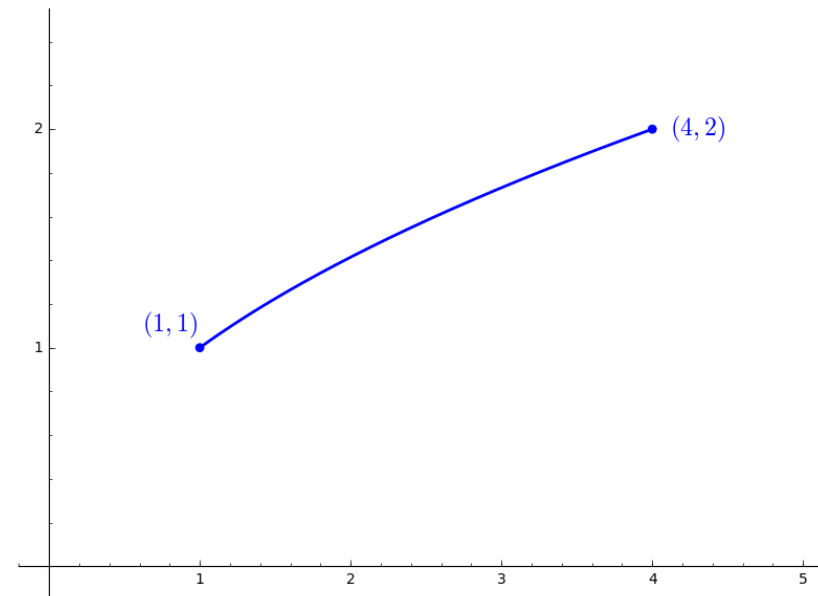
- Critical points: $f'(x) = \frac{1}{2\sqrt{x}}$
- $f'(x)$ is defined for all $x \in (1, 4)$.
- $f'(x) \neq 0$ for all $x \in (1, 4)$.
- Thus there is no critical points.
- Endpoints: $x = 1, 4$.

$$f(1) = 1$$

$$f(4) = 2$$

$$f(1) = 1 \quad \text{absolute minimum}$$

$$f(4) = 2 \quad \text{absolute maximum}$$



Example 2

Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x$ on the interval $[-2, 3]$.

Solution

- Critical points: $f'(x) = 3x^2 - 3$
- $f'(x)$ is defined for all $x \in (-2, 3)$.
- $f'(x) = 3x^2 - 3 = 0 \implies x = \pm 1 \in (-2, 3)$.
- Critical points: $x = 1, x = -1$
- Endpoints: $x = -2, x = 3$.

$f(-1) = 2$

$f(1) = -2$

$f(-2) = -2$

$f(3) = 18$

$f(-1) = 2$	
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$f(1) = -2$	absolute minimum
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$f(-2) = -2$	absolute minimum
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$f(3) = 18$	absolute maximum
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Example 3 (Exam)

Find the absolute maximum and minimum values of the function $f(x) = 4 - x^2$ on the interval $[-2, 1]$.

Solution

- Critical points:
- $f'(x) = -2x = 0 \implies x = 0 \in (-2, 1)$
- Critical points: $x = 0$
- Endpoint: $x = -2, 1$.

$f(0) = 4$

$f(-2) = 0$

$f(1) = 3$

$f(0) = 4$	absolute maximum
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$f(-2) = 0$	absolute minimum
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$f(1) = 3$



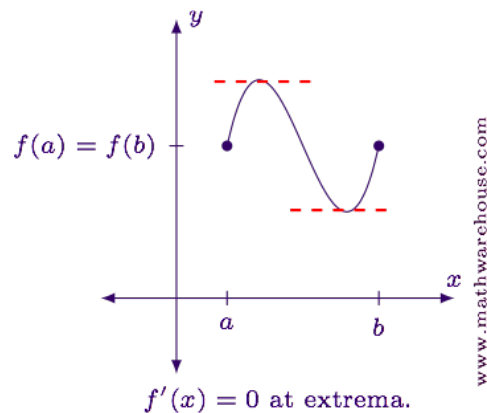
4.2 The Mean Value Theorem

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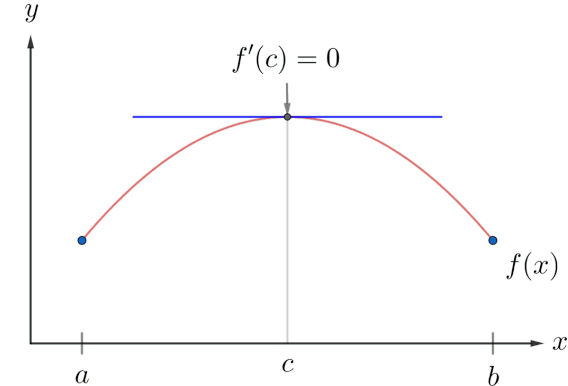
4.2 The Mean Value Theorem

- We know that the derivatives of constant functions are zeros, but could there be a more complicated function whose derivative is zero?
- If two functions have identical derivatives over an interval, how are the functions related?
- We answer these and other questions in this chapter by applying the Mean Value Theorem.
- First we introduce a special case, known as Rolle's Theorem.



Theorem 3 (Rolle's Theorem)

- Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$
- and differentiable at every point of its interior (a, b) .
- If $f(a) = f(b)$,
- then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$.



Examples

Example 1

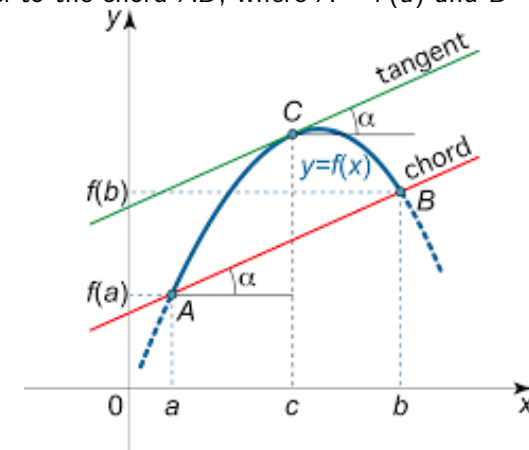
Let $f(x) = \frac{1}{3}x^3 - 3x$. Show that there is $c \in (-3, 3)$ such that $f'(c) = 0$.

Solution

- f is continuous on $[-3, 3]$.
- $f'(x) = x^2 - 3$ exists for all $x \in (-3, 3)$.
- $f(-3) = \frac{1}{3}(-3)^3 - 3(-3) = -9 + 9 = 0$
- $f(3) = \frac{1}{3}(3)^3 - 3(3) = 9 - 9 = 0$
- Thus $f(-3) = f(3)$
- By Rolle's theorem, there exists $c \in (-3, 3)$ such that $f'(c) = 0$.
- Note that $f'(c) = c^2 - 3 = 0 \implies c = \pm\sqrt{3} = \pm 1.732 \in (-3, 3)$

The Mean Value Theorem

- The mean value theorem was first stated by Joseph-Louis Lagrange.
- It guarantees that there is a point where the tangent line is parallel to the chord AB , where $A = f(a)$ and $B = f(b)$.



Example 2

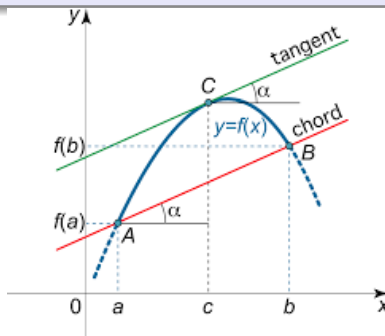
Let $f(x) = x^5 + x + 1$. Show that f has exactly one real zero.

Solution

- $f(x) = x^5 + x + 1$ is continuous on $[-1, 0]$
- $f(0) = 1$ and $f(-1) = -1$.
- Thus, by the Intermediate Value Theorem, f has a zero on $(-1, 0)$.
- $f'(x) = 5x^4 + 1$ exists for all $x \in (-1, 0)$
- If f has two zeros c_1, c_2 , then $f(c_1) = 0 = f(c_2)$
- Thus, by Rolle's theorem, there is $c \in (c_1, c_2)$ such that $f'(c) = 5c^4 + 1 = 0$.
- Impossible. ($5c^4 + 1 \geq 1$)

Theorem 4 (The Mean Value Theorem)

- Suppose that $y = f(x)$ is continuous on a closed interval $[a, b]$
- and differentiable on the interval's interior (a, b) .
- Then there is at least one number $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Example

Find the value or values of c that satisfy the equation
$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for } f(x) = \sqrt{x+1} \text{ and } [a, b] = [-1, 2].$$

Solution

- In this example, $f(x) = \sqrt{x+1}$ and $[a, b] = [-1, 2]$.
- $f(x)$ is continuous on $[-1, 2]$
- $f'(x) = \frac{1}{2\sqrt{x+1}}$ exists for all $x \in (-1, 2)$
- Thus by, the mean value theorem, there is at least one number $c \in (-1, 2)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
- $$\Rightarrow \frac{1}{2\sqrt{c+1}} = \frac{f(2) - f(-1)}{2 - (-1)}$$
- $$\Rightarrow \frac{1}{2\sqrt{c+1}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

- $\Rightarrow \frac{1}{2\sqrt{c+1}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$
- $\Rightarrow \frac{\sqrt{3}}{2} = \sqrt{c+1} \Rightarrow \frac{3}{4} = c+1$
- $\Rightarrow \boxed{c = -\frac{1}{4}}$

Corollary 1

If $f'(x) = 0$ at each point x of an open interval (a, b) , then $\boxed{f(x) = C}$ for all $x \in (a, b)$, where C is a constant.

Proof.

- Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.
- Then f is differentiable and hence continuous on $[x_1, x_2]$.
- Thus, by the Mean Value Theorem,

$$0 = f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$
- Hence $f(x_1) = f(x_2)$.
- Since x_1 and x_2 are any points in (a, b) , it follows that $f(x) = C$.

□

Corollary 2

If $f'(x) = g'(x)$ at each point $x \in (a, b)$, then there is a constant C such that $\boxed{f(x) = g(x) + C}$ for all $x \in (a, b)$.

Proof.

- We apply Corollary 1 to $h(x) = f(x) - g(x)$.
- $h'(x) = f'(x) - g'(x) = 0 \Rightarrow h(x) = C$
- $\Rightarrow f(x) - g(x) = C$

□

Example 1

Find all possible functions with derivative $y' = x^3$.

Solution

- $y' = x^3 \Rightarrow y = \frac{1}{4}x^4 + C$

Example 2

Find the function f with derivative $f'(x) = \cot x \csc x - 1$ and whose graph passes through the point $(\pi/2, 2)$.

Solution

- $f'(x) = \cot x \csc x - 1 \implies f(x) = -\csc x - x + C$
- The graph of f passes through the point $(\pi/2, 2) \implies f(\pi/2) = 2$
- $\implies -\csc(\pi/2) - \pi/2 + C = 2 \implies C = \pi/2 + 3$
- $f(x) = -\csc x - x + \frac{\pi}{2} + 3$



4.3 Monotonic Functions and the First Derivative Test (part 1)

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4.3 Monotonic Functions and the First Derivative Test

- In sketching the graph of a function it is useful to know where it is increasing and where it is decreasing.
- This section gives a test to determine where a differentiable function increases and where it decreases.
- We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions

Recall that

- (1) f is said to be increasing on I if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for any two points x_1 and x_2 in I .

- (2) f is said to be decreasing on I if

$$x_1 < x_2 \implies f(x_1) > f(x_2)$$

for any two points x_1 and x_2 in I .

Corollary 3 (First Derivative Test for Increasing and Decreasing)

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

- (1) If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
- (2) If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof.

- Assume that $x_1, x_2 \in (a, b)$ with $x_2 > x_1$.
- The mean value theorem says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$
- (1) If $f'(x) > 0$ for all $x \in (a, b)$, then $f'(c) > 0$
 - $x_2 > x_1 \implies (x_2 - x_1) > 0$
 - $\implies f(x_2) - f(x_1) > 0 \implies f(x_2) > f(x_1)$
- (2) If $f'(x) < 0$ for all $x \in (a, b)$, then $f'(c) < 0$
 - $x_2 > x_1 \implies (x_2 - x_1) > 0$
 - $\implies f(x_2) - f(x_1) < 0 \implies f(x_2) < f(x_1)$

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Example

Find the intervals on which the function $f(x) = x^2$ is increasing and decreasing.

Solution

- $f'(x) = 2x$
- **Find the intervals:**
- $f'(x)$ has one factor, namely $2x$.
- It is zero when $x = 0$
- This number divides the real line into the intervals

$$(-\infty, 0), \quad (0, \infty)$$
- **Make a table:**
- We make the following table using test points to determine the sign of each factor in each interval.

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Interval	$(-\infty, 0)$	$(0, \infty)$
sign of $2x$		
sign of $f'(x)$		
Behavior of $f(x)$		

Interval	$(-\infty, 0)$	$(0, \infty)$
sign of $2x$	$-$	$+$
sign of $f'(x)$	$-$	$+$
Behavior of $f(x)$		

Interval	$(-\infty, 0)$	$(0, \infty)$
sign of $2x$	$-$	$+$
sign of $f'(x)$	$-$	$+$
Behavior of $f(x)$	$\searrow$	$\nearrow$

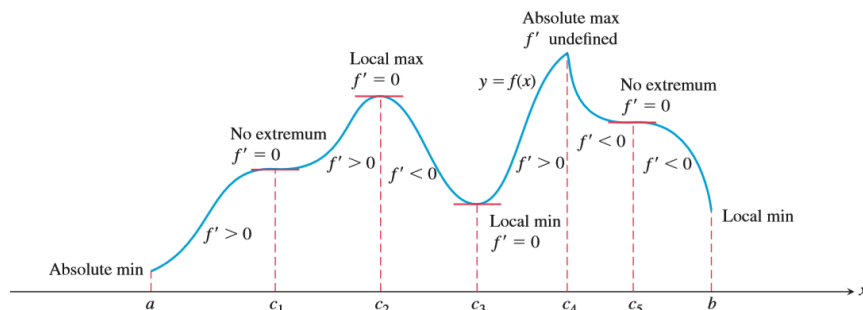
- Therefore, $f(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$

The First Derivative Test for Local Extrema

Theorem 5 (The First Derivative Test for Local Extrema)

Suppose that f is a continuous function and let c be a critical point of f . Assume that f is differentiable on an open interval containing c except possibly at c itself. Then

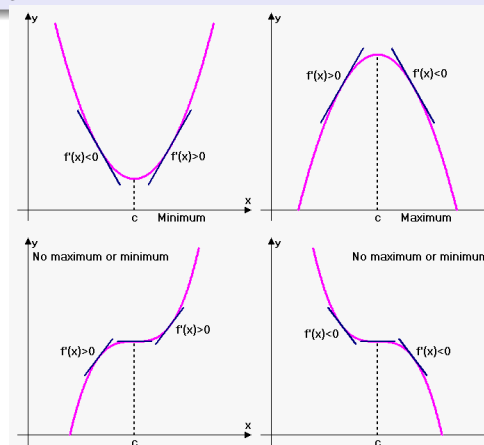
- (1) if f' changes from positive to negative at c , then f has a local maximum value at c ;



The First Derivative Test for Local Extrema

Theorem 5 (The First Derivative Test for Local Extrema)

- (2) if f' changes from negative to positive at c , then f has a local minimum value at c ;
- (3) if f' does not change sign at c , then f has no local extreme value at c .



Example 1

Let $f(x) = x^{1/5}(2x - 3)$.

- (a) Find the critical points of f .
- (b) Find the intervals on which f is increasing and decreasing.
- (c) Identify the function's local extreme values.
- (d) Which of the extreme values, if any, are absolute?

Solution

- (a) Find the critical points of f ?

- $f'(x) = \frac{1}{5}x^{-4/5}(2x - 3) + 2x^{1/5} = \frac{1}{5}x^{-4/5}(2x - 3 + 10x)$
- $\implies f'(x) = \frac{1}{5}x^{-4/5}(12x - 3) = \frac{12x - 3}{5x^{4/5}}$
- $f'(x)$ is not defined at $x = 0$
- $f'(x) = 0 \implies x = \frac{1}{4}$

- Therefore, the critical points are

$$x = 0 \text{ and } x = \frac{1}{4}$$

- (b) Find the intervals on which f is increasing and decreasing?

- $f'(x) = \frac{1}{5}x^{-4/5}(12x - 3) = \frac{12x - 3}{5x^{4/5}}$
- $f'(x)$ has the factors $12x - 3$ and $5x^{4/5}$.
- They are zero when $x = \frac{1}{4}$ and $x = 0$
- This number divides the real line into the intervals $(-\infty, 0)$, $(0, \frac{1}{4})$, $(\frac{1}{4}, \infty)$

- Make a table:

Interval	$(-\infty, 0)$	$(0, \frac{1}{4})$	$(\frac{1}{4}, \infty)$
sign of $12x - 3$			
sign of $5x^{4/5}$			
sign of $f'(x)$			
Behavior of $f(x)$			

Interval	$(-\infty, 0)$	$(0, \frac{1}{4})$	$(\frac{1}{4}, \infty)$
sign of $12x - 3$	-	-	+
sign of $5x^{4/5}$	+	+	+
• sign of $f'(x)$	-	-	+
Behavior of $f(x)$			

Interval	$(-\infty, 0)$	$(0, \frac{1}{4})$	$(\frac{1}{4}, \infty)$
sign of $12x - 3$	-	-	+
sign of $5x^{4/5}$	+	+	+
• sign of $f'(x)$	-	-	+
Behavior of $f(x)$	$\searrow$	$\searrow$	$\nearrow$

- Therefore, f is decreasing on $(-\infty, 1/4)$ and increasing on $(1/4, \infty)$

(c) Identify the function's local extreme values?

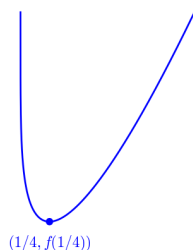
Interval	$(-\infty, 0)$	$(0, \frac{1}{4})$	$(\frac{1}{4}, \infty)$
sign of $12x - 3$	-	-	+
sign of $5x^{4/5}$	+	+	+
• sign of $f'(x)$	-	-	+
Behavior of $f(x)$	$\searrow$	$\searrow$	$\nearrow$

- $f(x)$ has a local minimum at $x = \frac{1}{4}$.
- $f(x) = x^{1/5}(2x - 3) \Rightarrow f(1/4) = (1/4)^{1/5}(\frac{1}{2} - 3) = -\frac{5}{2}(\frac{1}{2^{2/5}}) = -\frac{5}{2^{7/5}}$
- $\Rightarrow f\left(\frac{1}{4}\right) = -\frac{5}{2^{7/5}}$ is a local minimum.

(d) Which of the extreme values, if any, are absolute?

Interval	$(-\infty, 0)$	$(0, \frac{1}{4})$	$(\frac{1}{4}, \infty)$
sign of $12x - 3$	-	-	+
sign of $5x^{4/5}$	+	+	+
• sign of $f'(x)$	-	-	+
Behavior of $f(x)$	$\searrow$	$\searrow$	$\nearrow$

- $f(1/4) = -5/2^{7/5}$ is an absolute minimum since f is decreasing on $(-\infty, 1/4)$ and increasing on $(1/4, \infty)$.
- No absolute maximum



4.3 Monotonic Functions and the First Derivative Test (part 2)

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$$\Rightarrow f'(x) = \frac{8(1-x^2)}{3x^{1/3}}$$

- $f'(x)$ is not defined at $x = 0$
- $f'(x) = 0 \Rightarrow x = \pm 1$
- Therefore, the critical points are $x = 0, x = -1, \text{ and } x = 1$

(b) Find the intervals on which f is increasing and decreasing?

- $f'(x)$ has the factors $1-x$, $1+x$, and $x^{1/3}$.
- They are zero when $x = \pm 1$ and $x = 0$
- These numbers divide the real line into the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$

• Make a table:

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of $1-x$				
sign of $1+x$				
sign of $x^{1/3}$				
sign of $f'(x)$				
Behavior of $f(x)$				

•

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of $1-x$	+	+	+	-
sign of $1+x$	-	+	+	+
sign of $x^{1/3}$	-	-	+	+
sign of $f'(x)$	+	-	+	-
Behavior of $f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

- Thus, f is increasing on $(-\infty, -1) \cup (0, 1)$ and decreasing on $(-1, 0) \cup (1, \infty)$

Example 2

Let $f(x) = x^{2/3}(4-x^2)$.

- Find the critical points of f .
- Find the intervals on which f is increasing and decreasing.
- At what points, if any, does f assume local extreme values.
- Which of the extreme values, if any, are absolute?

Solution

(a) Find the critical points of f ?

- $f'(x) = \frac{2}{3}x^{-1/3}(4-x^2) - 2xx^{2/3} = \frac{2}{3}x^{-1/3}(4-x^2) - 2x^{5/3}$
- $f'(x) = \frac{2}{3}x^{-1/3}(4-x^2-3x^2)$
- $\Rightarrow f'(x) = \frac{8(1-x^2)}{3x^{1/3}}$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of $1-x$	+	+	+	-
sign of $1+x$	-	+	+	+
sign of $x^{1/3}$	-	-	+	+
sign of $f'(x)$	+	-	+	-
Behavior of $f(x)$				

•

$$f(x) = x^{2/3}(4 - x^2)$$

(c) At what points, if any, does f assume local extreme values?

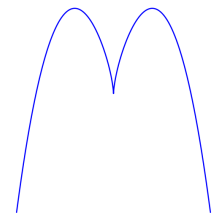
Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of $1 - x$	+	+	+	-
sign of $1 + x$	-	+	+	+
sign of $x^{1/3}$	-	-	+	+
sign of $f'(x)$	+	-	+	-
Behavior of $f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

- Thus, $f(1) = f(-1) = 3$ are local maximum values and $f(0) = 0$ is a local minimum value.

(d) Which of the extreme values, if any, are absolute?

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of $1 - x$	+	+	+	-
sign of $1 + x$	-	+	+	+
sign of $x^{1/3}$	-	-	+	+
sign of $f'(x)$	+	-	+	-
Behavior of $f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

- $f(-1) = f(1) = 3$ is absolute maximum.
- NO absolute minimum.



Example 3

Let $f'(x) = (2 - x)(x + 1)^2$.

- Find the critical points of f .
- Find the intervals on which f is increasing and decreasing.
- Find the local extreme values, if any, of f .

Solution

(a) Find the critical points of f ?

- $f'(x) = (2 - x)(x + 1)^2$ is defined for all x
- $f'(x) = (2 - x)(x + 1)^2 = 0 \implies x = 2, x = -1$ are the critical points.

(b) Find the intervals on which f is increasing and decreasing?

- $f'(x)$ has the factors $2 - x$ and $(x + 1)^2$.
- They are zero at the critical points.

- These numbers divide the real line into the intervals $(-\infty, -1)$, $(-1, 2)$, $(2, \infty)$

• Make a table:

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
sign of $2 - x$			
sign of $(x + 1)^2$			
sign of $f'(x)$			
Behavior of $f(x)$			

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
sign of $2 - x$	+	+	-
sign of $(x + 1)^2$	+	+	+
sign of $f'(x)$	+	+	-
Behavior of $f(x)$			

•

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
sign of $2 - x$	+	+	-
sign of $(x + 1)^2$	+	+	+
sign of $f'(x)$	+	+	-
Behavior of $f(x)$	$\nearrow$	$\nearrow$	$\searrow$

•

- It follows that f is increasing on $(-\infty, 2)$ and decreasing on $(2, \infty)$

(c) Find the local extreme values, if any, of f ?

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
sign of $2 - x$	+	+	-
sign of $(x + 1)^2$	+	+	+
sign of $f'(x)$	+	+	-
Behavior of $f(x)$	$\nearrow$	$\nearrow$	$\searrow$

•

- From the table, we see that f has a local maximum value at $x = 2$.

Example 4

Find the local extrema of $f(x) = \cos x$ on the interval $0 \leq x \leq 2\pi$.

Solution

- Critical points:**
- $f'(x) = -\sin x$
- $f'(x) = -\sin x = 0 \implies x = 0, \pm\pi, \pm2\pi, \dots$
- $x \in (0, 2\pi) \implies x = \pi$ is the only critical point.
- Make a table:**

Interval	$(0, \pi)$	$(\pi, 2\pi)$
sign of $-\sin x$	-	+
sign of $f'(x)$	-	+
Behavior of $f(x)$	$\searrow$	$\nearrow$

Interval	$(0, \pi)$	$(\pi, 2\pi)$
sign of $-\sin x$	$-$	$+$
sign of $f'(x)$	$-$	$+$
Behavior of $f(x)$	$\searrow$	$\nearrow$

- It follows that f has local minimum at $x = \pi$ and local maximum at $x = 0, 2\pi$.

Example 5

Find the extreme values of the function

$$f(x) = \begin{cases} x^2 + 2x + 3, & x \leq 0 \\ 2x^2 - 8x + 3, & x > 0. \end{cases}$$

Solution

- Critical points:

$$f'(x) = \begin{cases} 2x + 2, & x < 0 \\ 4x - 8, & x > 0 \end{cases}$$

$$f'(0) = \lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{f(h) - 3}{h}$$

$$\lim_{h \rightarrow 0^-} \frac{h^2 + 2h}{h} = 2$$

$$\lim_{h \rightarrow 0^+} \frac{2h^2 - 8h}{h} = -8$$

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$$f'(x) = \begin{cases} 2x + 2, & x < 0 \\ 4x - 8, & x > 0 \end{cases}$$

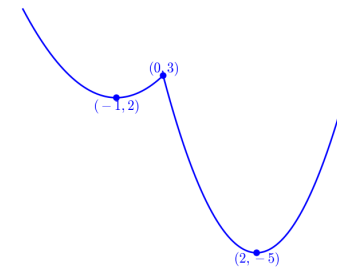
- Thus $f'(0)$ does not exist.
- Therefore, the critical points are $0, -1, 2$

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

- By the table we see that f has local minimum at $x = -1$ and $x = 2$ and f has a local maximum at $x = 0$

$$f(x) = \begin{cases} x^2 + 2x + 3, & x \leq 0 \\ 2x^2 - 8x + 3, & x > 0. \end{cases} \Rightarrow$$

- $f(0) = 3$ is a local maximum value,
 $f(-1) = 2$ and $f(2) = -5$ are local minimum values.



- $f(2) = -5$ is the absolute minimum.
- No absolute maximum.



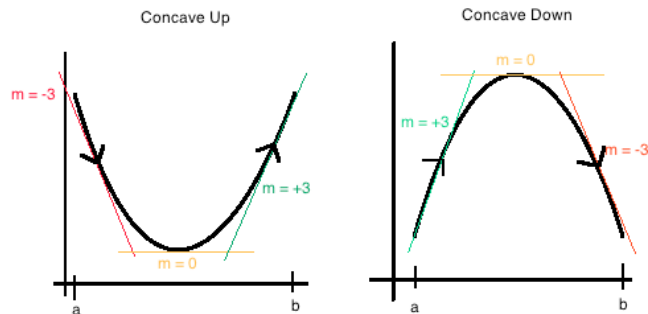
4.4 Concavity and Curve Sketching (part1)

First Semester 2020-2021
أ.د. أيمن هاشم السقا

Concavity (تقعر)

Definition

The graph of a differentiable function $y = f(x)$ is concave up on an open interval I if f' is increasing on I and it is concave down on I if f' is decreasing on I .



4.4 Concavity and Curve Sketching

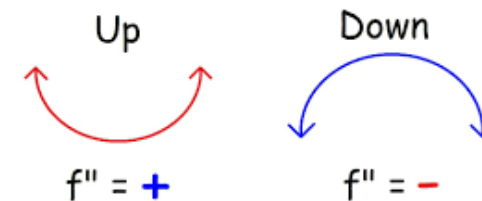
- In this section, we will use the second derivative to find how the graph of a function bends or turns.
- Then we will use this knowledge and the previous knowledge about increasing and decreasing and asymptotes to draw accurate graph of a function.

The Second Derivative test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I .

- (1) If $f''(x) > 0$ on I , then the graph of f is concave up.
- (2) If $f''(x) < 0$ on I , then the graph of f is concave down.

Concavity

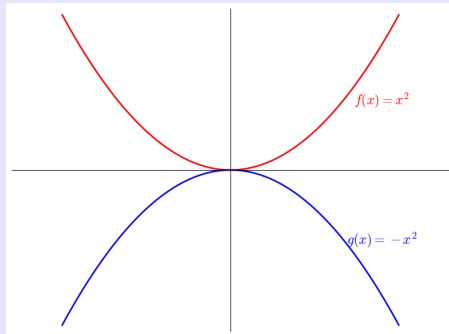


Example 1

Show that $f(x) = x^2$ is concave up on $\mathbb{R}$ and $g(x) = -x^2$ is concave down on $\mathbb{R}$.

Solution

- $f'(x) = 2x$ and $g'(x) = -2x$
- $f''(x) = 2 > 0$ for all $x \in \mathbb{R} \implies f$ is concave up on $\mathbb{R}$
- $g''(x) = -2 < 0$ for all $x \in \mathbb{R} \implies g$ is concave down on $\mathbb{R}$



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Example 2

Show that $f(x) = \cos x$ is concave down on $(0, \pi/2)$ and concave up on $(\pi/2, 3\pi/2)$.

Solution

- $f'(x) = -\sin x$, $f''(x) = -\cos x$

Interval	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{3\pi}{2})$
sign of $-\cos x$	-	+
sign of $f''(x)$	-	+
Behavior of $f(x)$	$\cap$	$\cup$

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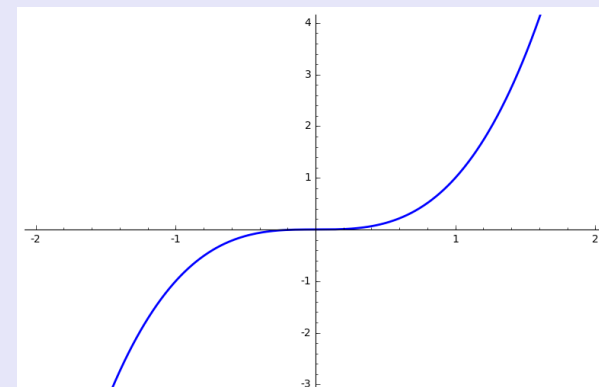
Points of inflection

Definition

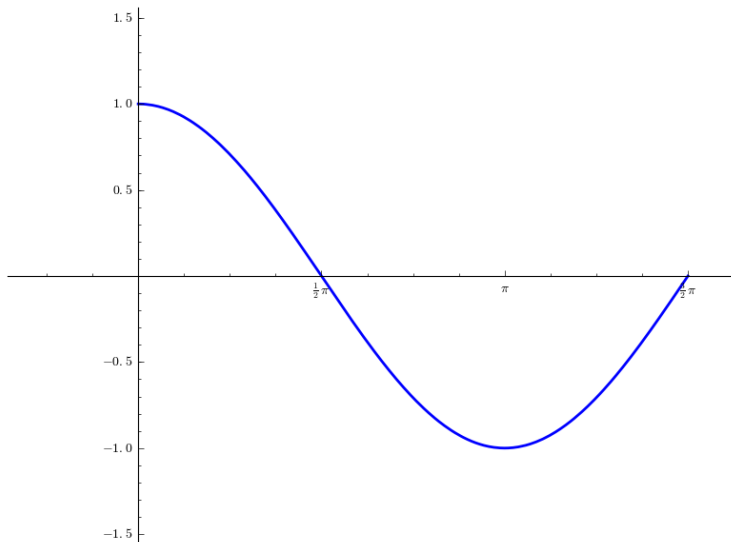
Let f be a continuous function on an open interval containing c . If f changes concavity at the point $(c, f(c))$, then $(c, f(c))$ is called a point of inflection.

Example

The function $f(x) = x^3$ has an inflection point $(0, 0)$.



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Second derivative test for inflection points

If $(c, f(c))$ is a point of inflection of the graph of f , then either $f''(c) = 0$ or $f''(c)$ is undefined.

Example 1

Show that $x = 0$ is a point of inflection of $f(x) = x^3$.

Solution

- $f'(x) = 3x^2, \quad f''(x) = 6x = 0 \implies x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
sign of $6x$	-	+
sign of $f''(x)$	-	+
Behavior of $f(x)$	$\cap$	$\cup$

- Thus $x = 0$ is a point of inflection of $f(x) = x^3$

Example 2

Show that $x = 0$ is not a point of inflection of $f(x) = x^4$

Solution

- $f'(x) = 4x^3, \quad f''(x) = 12x^2 = 0 \implies x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
sign of $12x^2$	+	+
sign of $f''(x)$	+	+
Behavior of $f(x)$	$\cup$	$\cup$

- Thus $x = 0$ is NOT a point of inflection of $f(x) = x^4$

Second derivative test for local extrema

Theorem 5 (Second derivative test for local extrema)

Suppose f'' is continuous on an open interval that contains $x = c$.

- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
- If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

Example

Show that $f(x) = x^2 + 3$ has a local minimum at $x = 0$.

Solution

- $f'(x) = 2x = 0$ implies $x = 0$ is a critical point.
- $f''(x) = 2 \implies f''(0) = 2 > 0$.
- Thus $f(0) = 3$ is a local minimum.

Curve Sketching

- To graph $y = f(x)$ we follow the following steps:

- Identify the domain of f .
- Find the asymptotes that may exist.
- Find the critical points of f , if any, and classify them. Find where the function is increasing and where it is decreasing.
- Find the points of inflection, if any, and determine the concavity of the curve.
- Plot key points, such as the intercepts and the points found in previous steps.
- Sketch the graph of f together with any asymptotes that exist.

Example 1

Sketch the graph of $f(x) = 2x^2 - x^4$.

Solution

(1) Domain of f

- Since f is a polynomial, $\text{Dom}(f) = \mathbb{R}$

(2) Asymptotes

- $\text{Dom}(f) = \mathbb{R} \implies$ the graph of f has no vertical asymptotes.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} (2x^2 - x^4) = \lim_{x \rightarrow \infty} x^4 \left(\frac{2}{x^2} - 1 \right) = -\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} (2x^2 - x^4) = \lim_{x \rightarrow -\infty} x^4 \left(\frac{2}{x^2} - 1 \right) = -\infty$$

- It follows that the graph of f has no horizontal asymptotes.

- Since f is a polynomial, the graph of f has no oblique asymptotes.

(3) Critical points, interval on which f is increasing and decreasing

$$f'(x) = 4x - 4x^3 = 4x(1 - x^2)$$

- The critical points are $x = 0, x = -1, x = 1$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
x	-	-	+	+
$1 - x^2$	-	+	+	-
$f'(x)$	+	-	+	-
f	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

- From the table we see that
- f is increasing on $(-\infty, -1]$ and on $[0, 1]$
- f is decreasing on $[-1, 0]$ and on $[1, \infty)$
- f has local minimum at $x = 0$
- f has local maximum at $x = -1$ and $x = 1$

$$f'(x) = 4x - 4x^3$$

(4) Points of inflection and concavity

$$f''(x) = 4 - 12x^2 = 4(1 - 3x^2) = 0 \implies x = \pm \frac{1}{\sqrt{3}}$$

Table for f''

Interval	$(-\infty, -1/\sqrt{3})$	$(-1/\sqrt{3}, 1/\sqrt{3})$	$(1/\sqrt{3}, \infty)$
$1 - 3x^2$	-	+	-
$f''(x)$	-	+	-
f	$\cap$	$\cup$	$\cap$

- f is concave up on the interval $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and concave down on the intervals $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$

- $x = \pm \frac{1}{\sqrt{3}}$ are inflection points

$$f(x) = 2x^2 - x^4$$

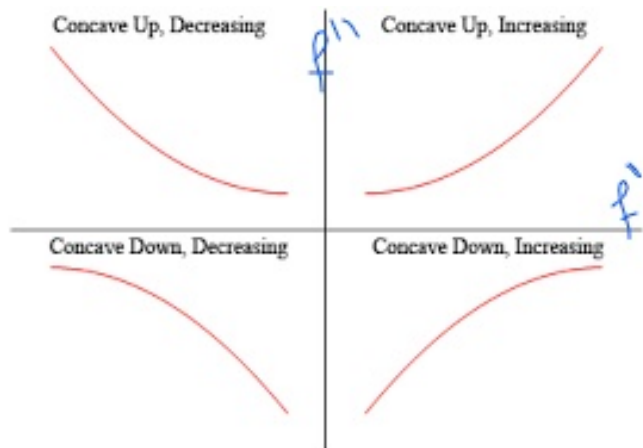
Plot key points

- x -intercept: $y = 0 \implies x^2(2 - x^2) = 0 \implies x = 0, \pm\sqrt{2}$
- y -intercept: $x = 0 \implies y = 0$
- Local minimum: $f(0) = 0$
- Local maximum: $f(1) = 1$ and $f(-1) = 1$
- Inflection points: $f\left(\pm \frac{1}{\sqrt{3}}\right) = \frac{2}{3} - \frac{1}{9} = \frac{5}{9}$

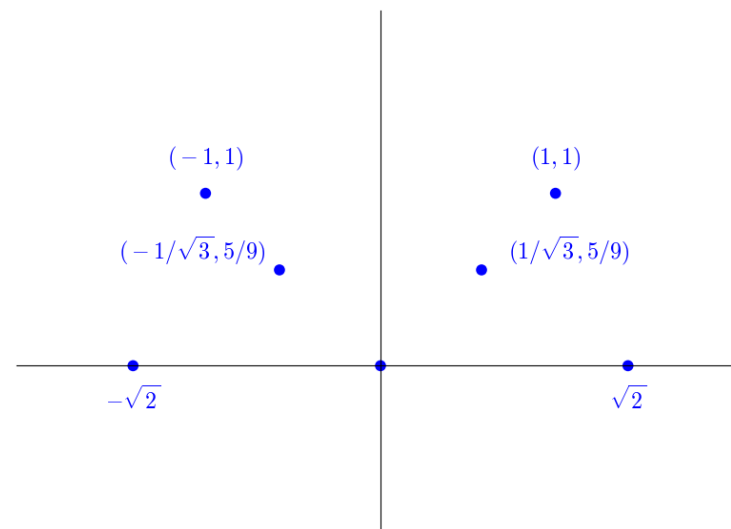
Graphing

Table for f'' and f'

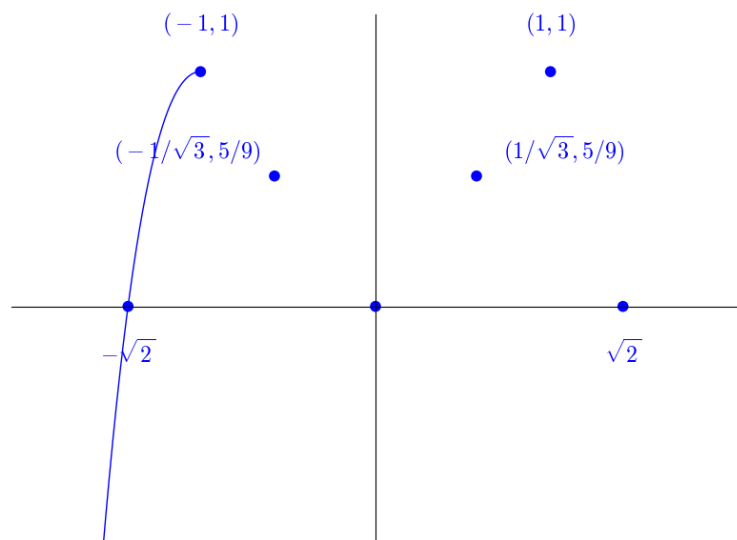
Interval	$(-\infty, -1)$	$(-1, \frac{-1}{\sqrt{3}})$	$(\frac{-1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f'(x)$	+	-	-	+	+	-
$f''(x)$	-	-	+	+	-	-
$f(x)$						



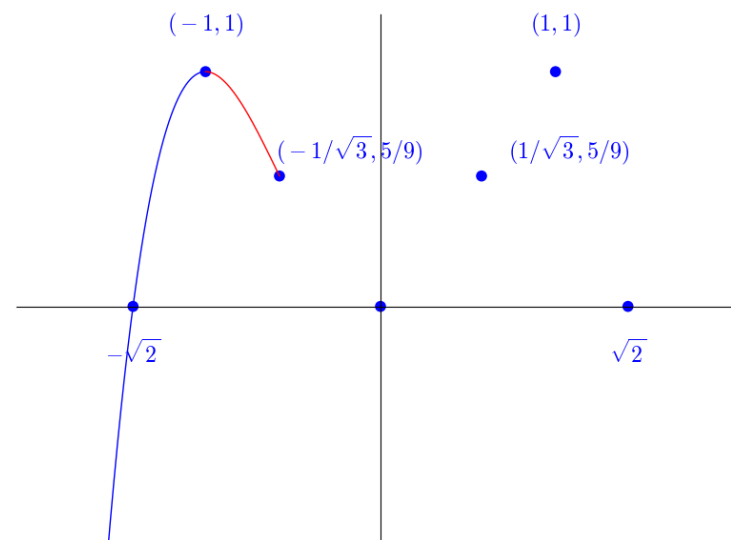
Interval	$(-\infty, -1)$	$(-1, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						

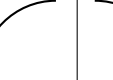
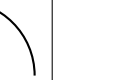


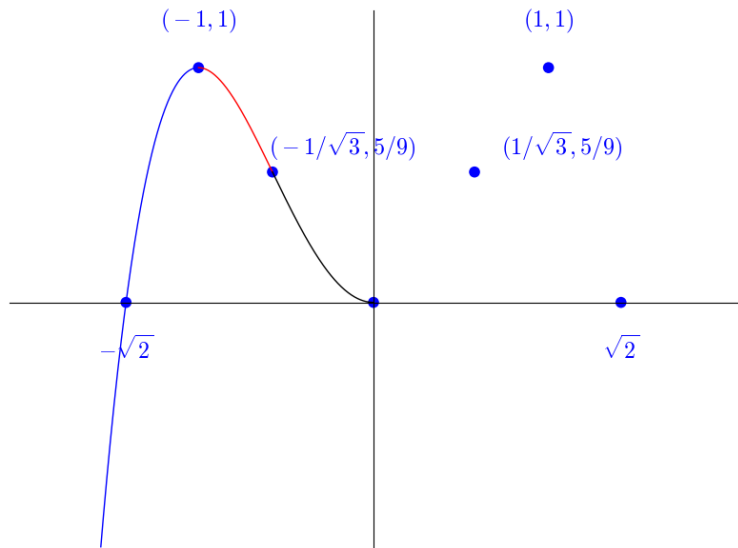
Interval	$(-\infty, -1)$	$(-1, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						

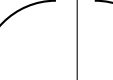
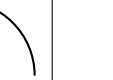


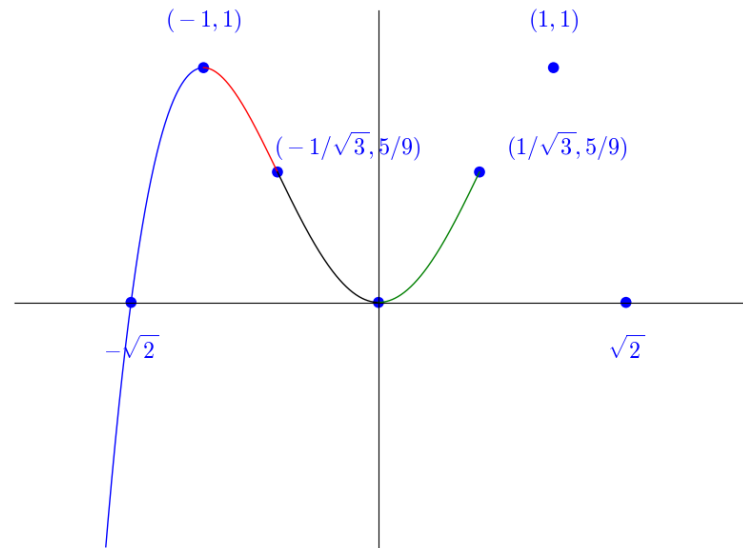
Interval	$(-\infty, -1)$	$(-1, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						

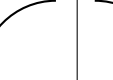
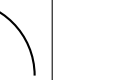


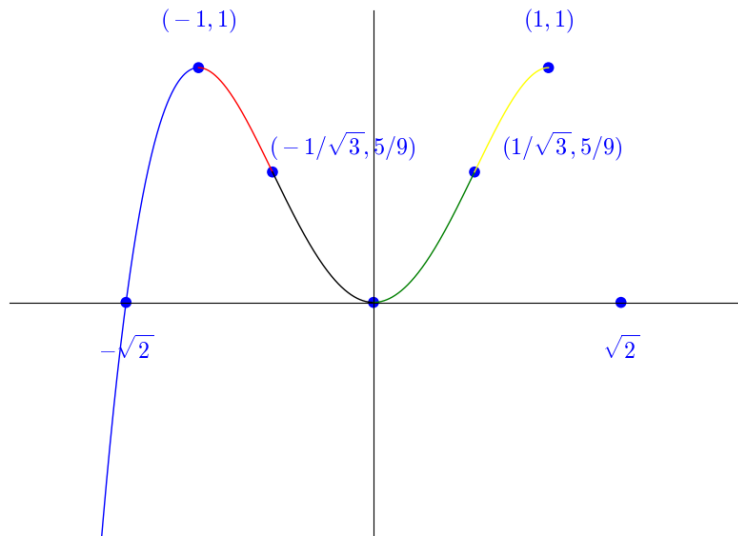
Interval	$(-\infty, -1)$	$(-1, \frac{-1}{\sqrt{3}})$	$(\frac{-1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						

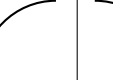
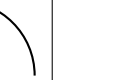


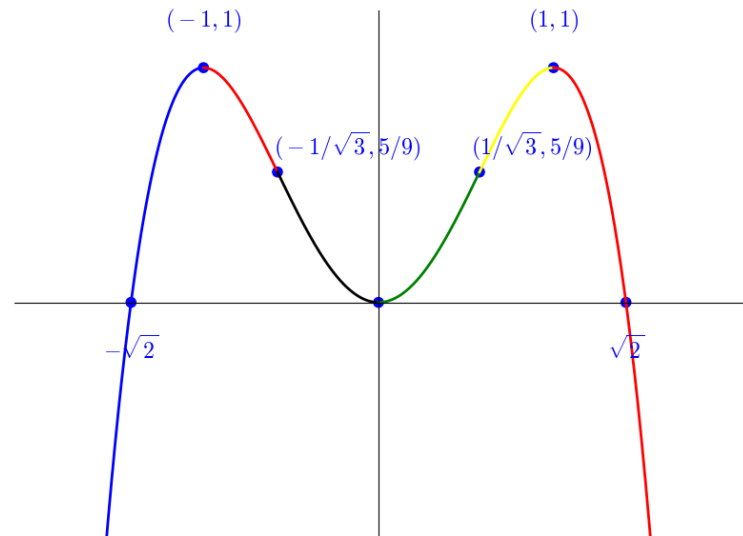
Interval	$(-\infty, -1)$	$(-1, \frac{-1}{\sqrt{3}})$	$(\frac{-1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						



Interval	$(-\infty, -1)$	$(-1, \frac{-1}{\sqrt{3}})$	$(\frac{-1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						



Interval	$(-\infty, -1)$	$(-1, \frac{-1}{\sqrt{3}})$	$(\frac{-1}{\sqrt{3}}, 0)$	$(0, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, 1)$	$(1, \infty)$
$f(x)$						





4.4 Concavity and Curve Sketching (part 2)

First Semester 2020-2021
أ.د. أيمن هاشم السقا

$$f'(x) = \frac{3}{5x^{2/5}}$$

- $f'(x) \neq 0$ for all $x \in \mathbb{R}$
- So $x = 0$ is the only critical point.

Table for f'

Interval	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	+
f	↗	↗

- Thus, $f(x)$ is increasing on $\mathbb{R}$ and $f(x)$ has no local extreme values
- (c) Find the inflection points of f and intervals on which f is concave down and up?
 - $f''(x) = \frac{d}{dx} \left(\frac{3}{5} x^{-2/5} \right) = \frac{-6}{25x^{7/5}}$
 - $f''(x)$ is not defined at $x = 0$
 - $f''(x) \neq 0$ for all $x \in \mathbb{R}$

Example 2

Let $f(x) = x^{3/5}$.

- Find the critical points of f and identify the function's local extreme values.
- Find the intervals on which f is increasing and decreasing.
- Find the inflection points of f and intervals on which f is concave down and up.
- Sketch the graph of f .

Solution

- In this example, $\text{Dom}(f) = \mathbb{R}$
- (a) Find the critical points of f and identify the function's local extreme values?
 - $f'(x) = \frac{3}{5} x^{-2/5} = \frac{3}{5x^{2/5}}$
 - $f'(x)$ is not defined at $x = 0$.

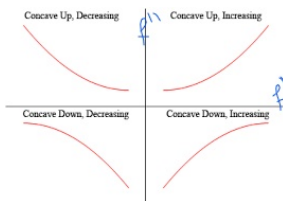
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$$f''(x) = \frac{-6}{25x^{7/5}}$$

Table for f''

Interval	$(-\infty, 0)$	$(0, \infty)$
-6	−	−
$25x^{7/5}$	−	+
$f''(x)$	+	−
f	∪	∩

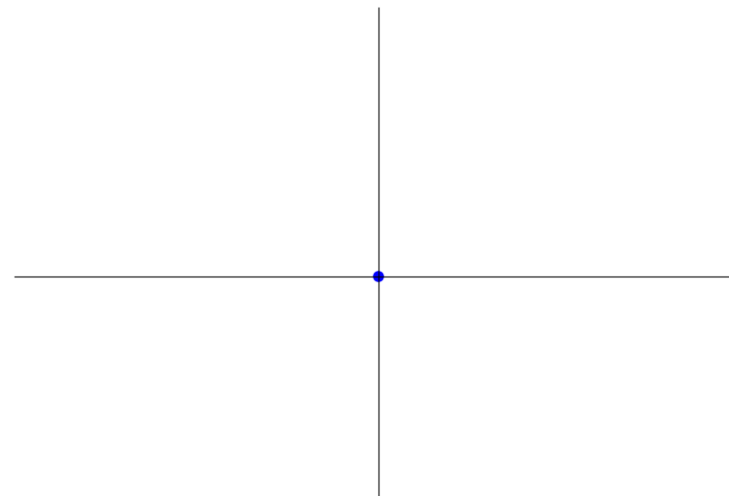
- f is concave up on the interval $(-\infty, 0)$, and concave down on the interval $(0, \infty)$
- f has an inflection point at $x = 0$
- (d) Sketch the graph of f ?
 - x-intercept: $y = 0 \implies x = 0$
 - y-intercept: $x = 0 \implies y = 0$
 - Inflection point: $f(0) = 0$



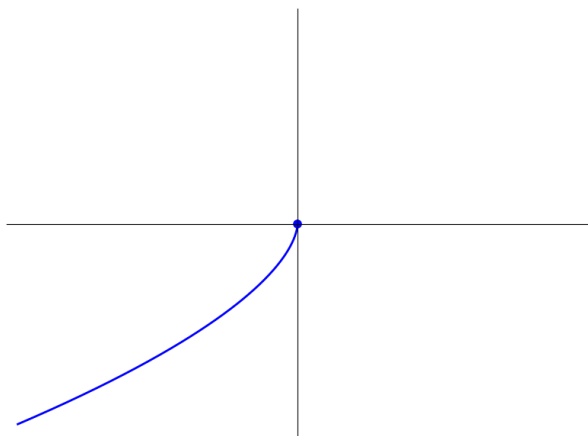
• Table for f' and f''

Interval	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
f		

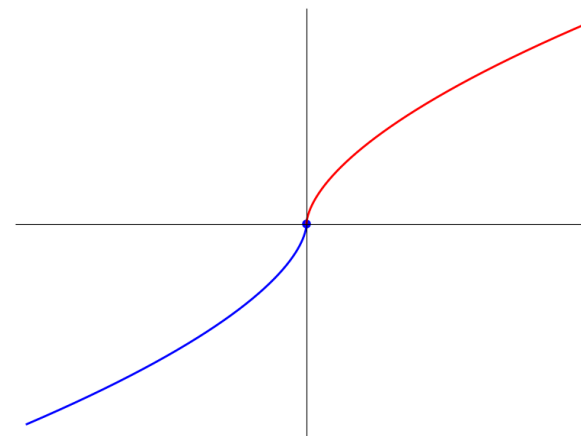
Interval	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
f		



Interval	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
f		



Interval	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
f		



$$f(x) = \frac{x-4}{2x+4}$$

Example 3

Let $f(x) = \frac{x-4}{2x+4}$.

- Find the critical points of f and identify the function's local extreme values.
- Find the intervals on which f is increasing and decreasing.
- Find the inflection points of f and intervals on which f is concave down and up.
- Find the asymptotes.
- Sketch the graph of f .

Solution

- $Dom(f) = \mathbb{R} - \{-2\}$

$$f'(x) = \frac{3}{(x+2)^2}$$

- Find the inflection points of f and intervals on which f is concave down and up?

- $f''(x) = \frac{-6}{(x+2)^3} \neq 0 \Rightarrow$ No inflection points.

Interval	$(-\infty, -2)$	$(-2, \infty)$
-6	$-$	$-$
$(x+2)^3$	$-$	$+$
$f''(x)$	$+$	$-$
f	$\cup$	$\cap$

- f is concave up on the interval $(-\infty, -2)$, and concave down on the interval $(-2, \infty)$

- Find the critical points of f and identify the function's local extreme values?

- $f'(x) = \frac{x+2-(x-4)}{2(x+2)^2} = \frac{3}{(x+2)^2}$

- $f'(x) \neq 0$ and it is defined for all $x \in Dom(f)$

- $\Rightarrow$ No critical points $\Rightarrow$ No local extreme values

Interval	$(-\infty, -2)$	$(-2, \infty)$
$f'(x)$	$+$	$+$
f	$\nearrow$	$\nearrow$

- f is increasing on $(-\infty, -2)$ and on $(-2, \infty)$

- Find the asymptotes?

- Horizontal asymptote:

- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x-4}{2x+4} = \frac{1}{2}$

- $\Rightarrow y = \frac{1}{2}$ is a horizontal asymptote

- $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x-4}{2x+4} = \frac{1}{2}$

- Vertical asymptote: $Dom(f) = \mathbb{R} - \{-2\}$

- $\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \frac{x-4}{2x+4} = -\infty$

- $\Rightarrow x = -2$ is a vertical asymptote

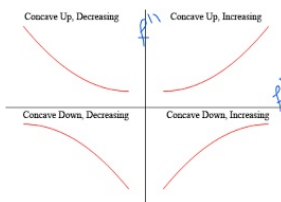
- $\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{x-4}{2x+4} = \infty$

$$f(x) = \frac{x-4}{2x+4}$$

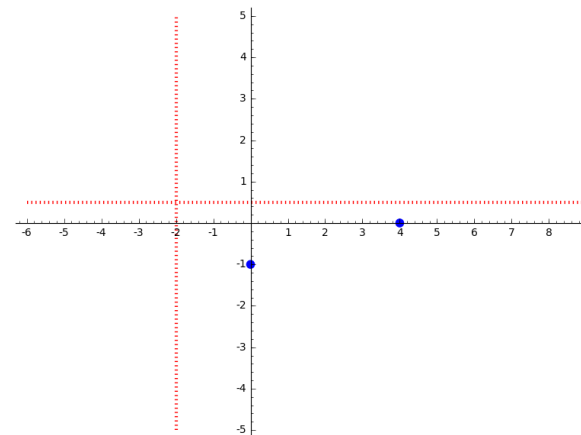
(e) Sketch the graph of f ?

- x -intercept: $y = 0 \implies x = 4$.
- y -intercept: $x = 0 \implies y = -1$.

Interval	$(-\infty, -2)$	$(-2, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
$f(x)$		

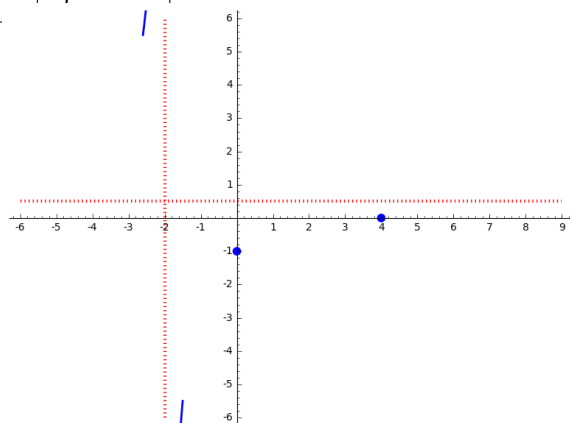


Interval	$(-\infty, -2)$	$(-2, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
$f(x)$		

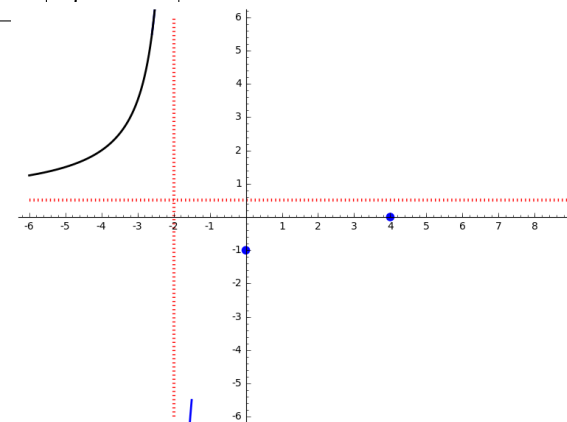


$$\lim_{x \rightarrow -2^+} f(x) = -\infty \quad \lim_{x \rightarrow -2^-} f(x) = \infty$$

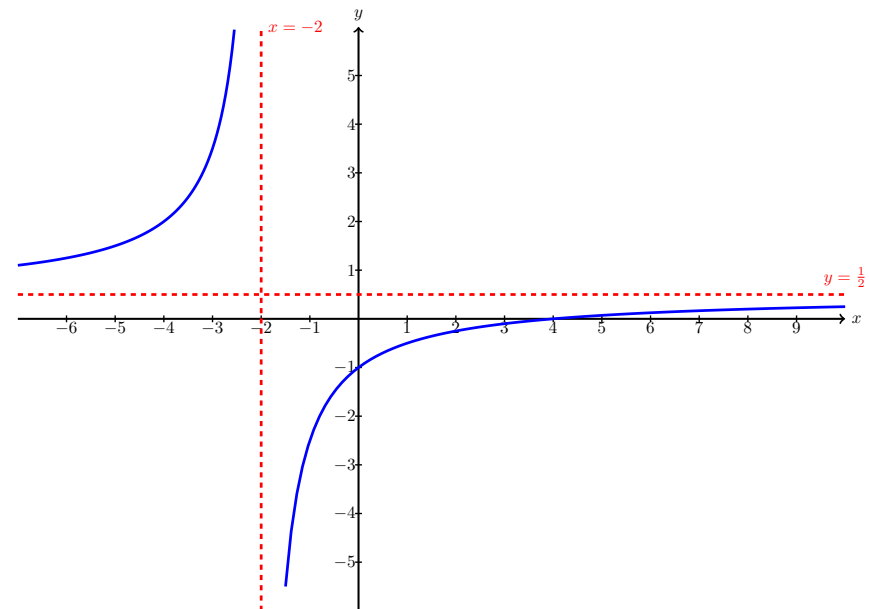
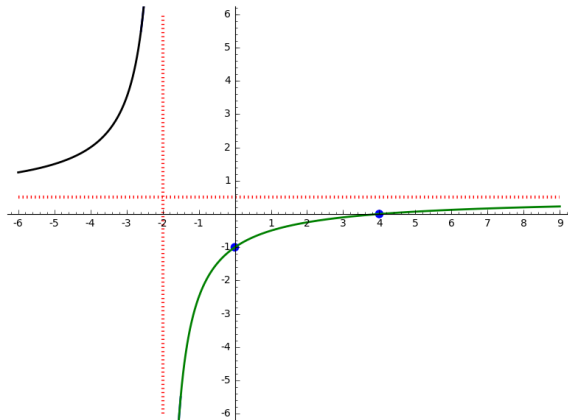
Interval	$(-\infty, -2)$	$(-2, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
$f(x)$		



Interval	$(-\infty, -2)$	$(-2, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
$f(x)$		



Interval	$(-\infty, -2)$	$(-2, \infty)$
$f'(x)$	+	+
$f''(x)$	+	-
$f(x)$		



$$f(x) = \frac{x^2 - 8}{x - 3}$$

Solution

- $Dom(f) = \mathbb{R} - \{3\}$

(a) Find the critical points of f and identify the function's local extreme values?

- $f'(x) = \frac{2x(x-3) - (x^2-8)}{(x-3)^2} = \frac{x^2 - 6x + 8}{(x-3)^2}$

- $f'(x)$ is defined for all $x \in Dom(f)$

- $f'(x) = 0 \implies x^2 - 6x + 8 = 0 \implies (x-4)(x-2) = 0$

- The critical points of f are $x = 2$ and $x = 4$

- Table for f' :**

Real line	2	3	4
$(x-2)$	-	+	+
$(x-4)$	-	-	+
$(x-3)^2$	+	+	+
$f'(x)$	+	-	+
f	$\nearrow$	$\searrow$	$\nearrow$

Example 4 (Exam)

Let $f(x) = \frac{x^2 - 8}{x - 3}$.

- Find the critical points of f and identify the function's local extreme values.
- Find the intervals on which f is increasing and decreasing.
- Find the inflection points of f and intervals on which f is concave down and up.
- Find the asymptotes.
- Sketch the graph of f .

$$f(x) = \frac{x^2-8}{x-3}$$

- $f(2) = 4$ is a local maximum and $f(4) = 8$ is a local minimum.
- (b) Find the intervals on which f is increasing and decreasing?
 - f is increasing on $(-\infty, 2]$ and on $[4, \infty)$.
 - f is decreasing on $[2, 3)$ and on $(3, 4]$.
- (c) Find the inflection points of f and intervals on which f is concave down and up?

$$\bullet f'(x) = \frac{x^2 - 6x + 8}{(x-3)^2}$$

$$\implies f''(x) = \frac{(2x-6)(x-3)^2 - 2(x-3)(x^2-6x+8)}{(x-3)^4}$$

$$\bullet f''(x) = \frac{(2x-6)(x-3) - 2(x^2-6x+8)}{(x-3)^3}$$

$$\bullet f''(x) = \frac{2x^2 - 6x - 6x + 18 - 2(x^2 - 6x + 8)}{(x-3)^3}$$

$$\bullet f''(x) = \frac{2}{(x-3)^3}$$

$$f''(x) = \frac{2}{(x-3)^3}$$

- $f''(x)$ is defined for all $x \in \text{Dom}(f)$
- $f''(x) \neq 0$
- No inflection point
- Table for f''

Real line	3	
$(x-3)^3$	-	+
$f''(x)$	-	+
f	$\cap$	$\cup$

- f is concave up on $(3, \infty)$ and concave down on $(-\infty, 3)$

- (d) Find the asymptotes?

$$\bullet \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2-8}{x-3} = \lim_{x \rightarrow \infty} \frac{x^2(1-\frac{8}{x^2})}{x(1-\frac{3}{x})}$$

$$\bullet \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x(1-\frac{8}{x^2})}{(1-\frac{3}{x})} = \infty$$

$$\bullet \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x(1-\frac{8}{x^2})}{(1-\frac{3}{x})} = -\infty$$

- There is no horizontal asymptotes.

$$f(x) = \frac{x^2-8}{x-3}$$

$$\bullet \begin{array}{r} x+3 \\ x-3 \overline{) x^2-8} \\ \underline{x^2-3x} \\ 3x-8 \\ \underline{3x-9} \\ 1 \end{array}$$

$$\bullet \implies f(x) = x+3 + \frac{1}{x-3}$$

- Thus $y = x+3$ is an oblique asymptote.

$$\bullet \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \frac{x^2-8}{x-3} = \infty$$

- Thus $x = 3$ is a vertical asymptote

$$\bullet \lim_{x \rightarrow 3^-} f(x) = -\infty$$

- (e) Sketch the graph of f :

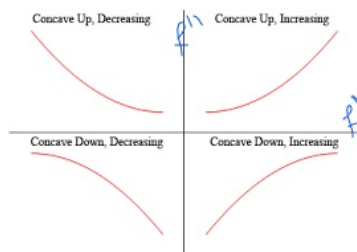
$$\bullet x\text{-intercept: } y = 0 \implies x^2 - 8 = 0 \implies x = \pm 2\sqrt{2} = \pm 2.8$$

$$\bullet y\text{-intercept: } x = 0 \implies y = \frac{8}{3} = 2.6$$

$$\bullet \begin{array}{r} x \\ x-3 \overline{) x^2-8} \\ \underline{x^2-3x} \\ 3x-8 \end{array}$$

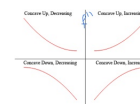
• Table for f' and f'' :

Real line	2		3	4	
$f'(x)$	+	−		−	+
$f''(x)$	−	−		+	+
$f(x)$					



• Table for f' and f'' :

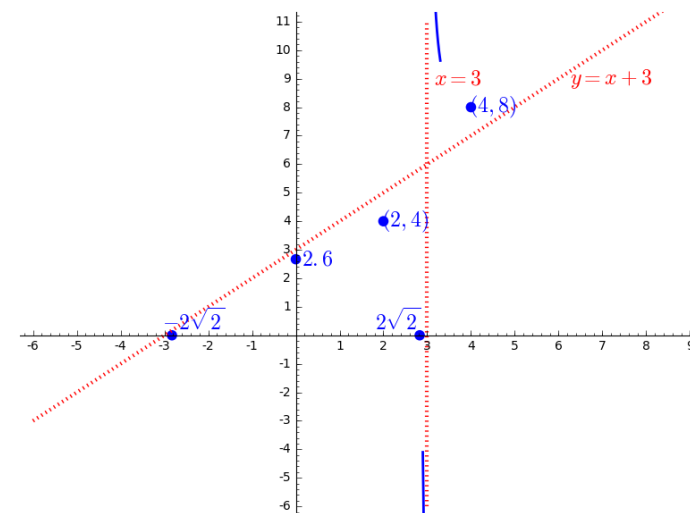
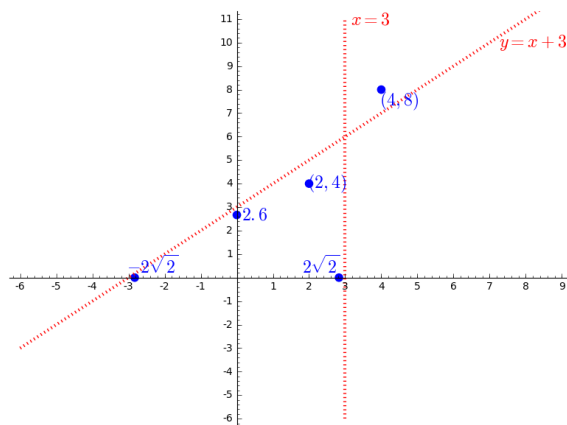
Real line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



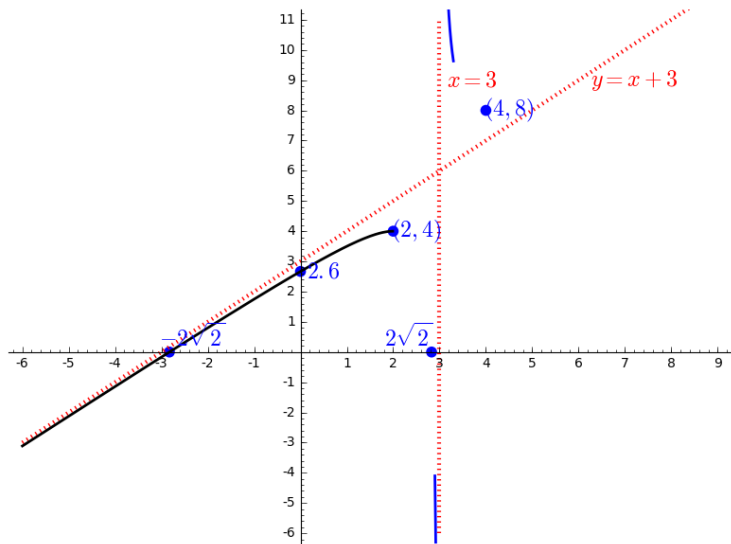
- $f(2) = 4$ is a local maximum and $f(4) = 8$ is a local minimum.
- x -intercept: $x = \pm 2\sqrt{2} = \pm 2.8$
 y -intercept: $y = 8/3 = 2.6$
- $y = x + 3$ is an oblique asymptote
 $x = 3$ is a vertical asymptote
 $\lim_{x \rightarrow 3^+} f(x) = \infty$ and $\lim_{x \rightarrow 3^-} f(x) = -\infty$

$$\lim_{x \rightarrow 3^+} f(x) = \infty \text{ and } \lim_{x \rightarrow 3^-} f(x) = -\infty$$

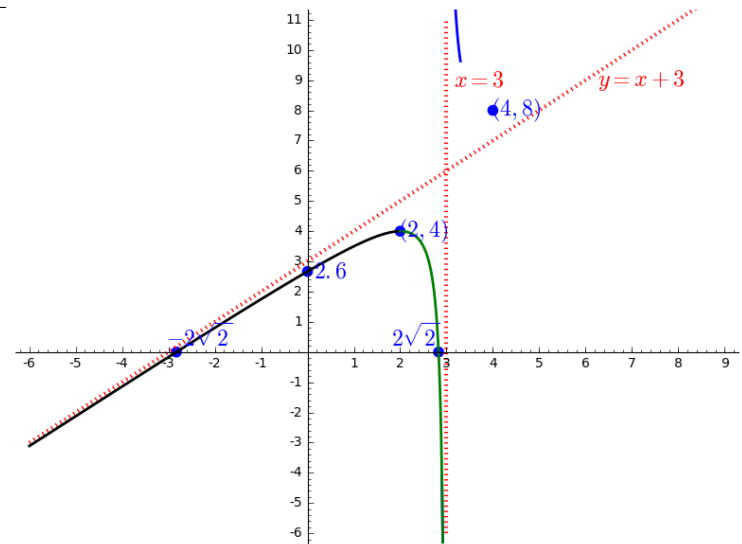
$f(2) = 4$ is a local maximum and $f(4) = 8$ is a local minimum
 x -intercept: $x = \pm 2\sqrt{2} = \pm 2.8$
 y -intercept: $y = 8/3 = 2.6$
 $y = x + 3$ is an oblique asymptote
 $x = 3$ is a vertical asymptote
 $\lim_{x \rightarrow 3^+} f(x) = \infty$ and $\lim_{x \rightarrow 3^-} f(x) = -\infty$



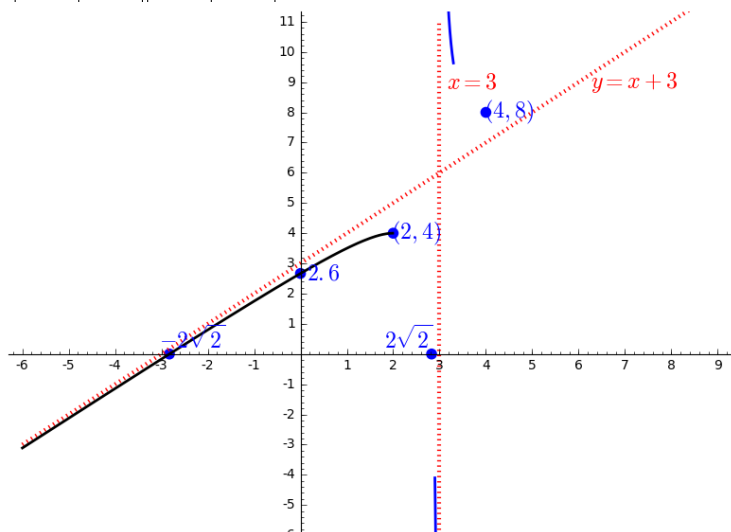
line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



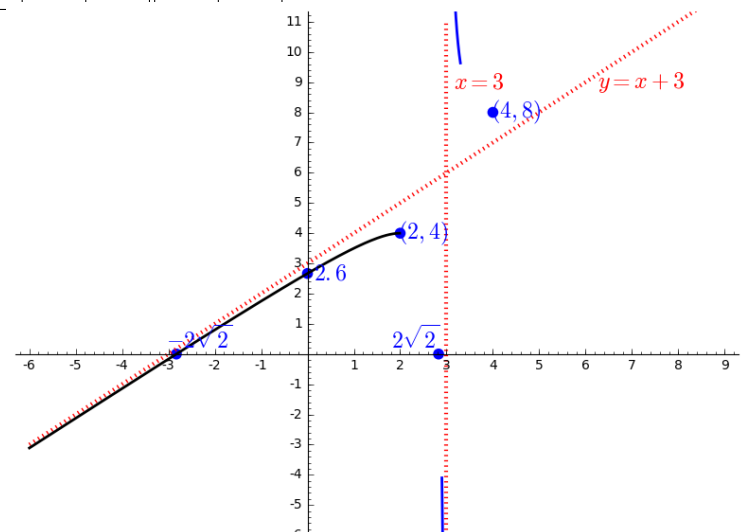
line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



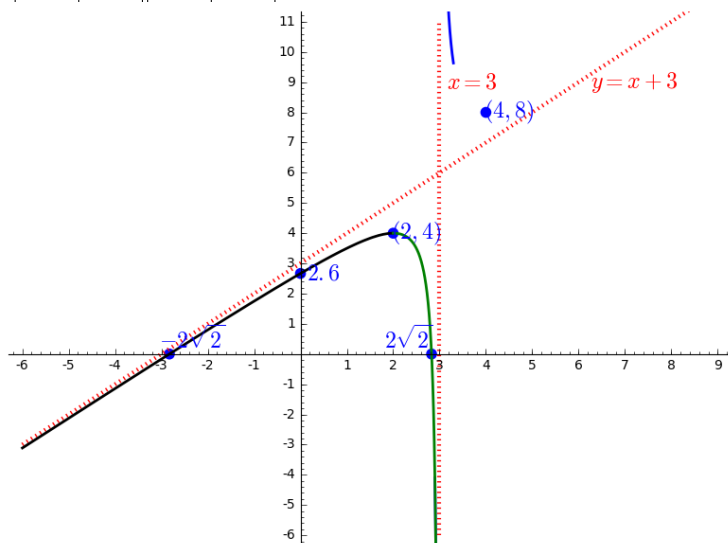
line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



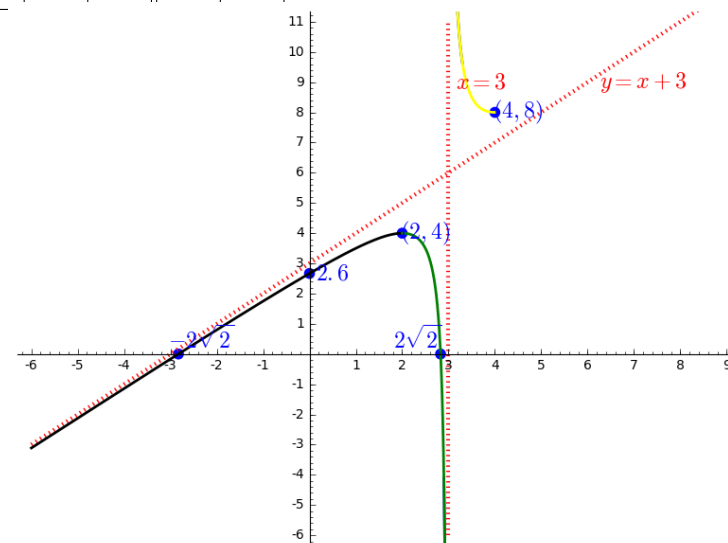
line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



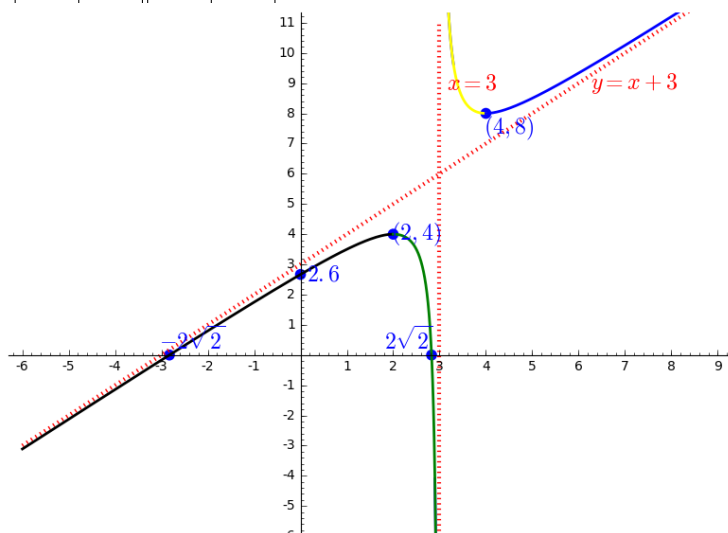
line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$	∩	∪	∩	∪



line	2	3	4	
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
$f(x)$				



4.4 Concavity and Curve Sketching (part 3)

First Semester 2020-2021
أ.د. أيمن هاشم السقا

$$f(x) = \frac{x-7}{x-2}$$

Example 5

Let $f(x) = \frac{x^2 - 49}{x^2 + 5x - 14}$.

- Find the critical points of f and identify the function's local extreme values.
- Find the intervals on which f is increasing and decreasing.
- Find the inflection points of f and intervals on which f is concave down and up.
- Find the asymptotes.
- Sketch the graph of f .

Solution

- $x^2 + 5x - 14 = 0 \implies (x-2)(x+7) = 0$
- $\implies \text{Dom}(f) = \mathbb{R} - \{-7, 2\}$
- $f(x)$ can be written as $f(x) = \frac{(x-7)(x+7)}{(x-2)(x+7)} = \frac{x-7}{x-2}$

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$$f(x) = \frac{x-7}{x-2}$$

- Find the inflection points of f and intervals on which f is concave down and up?

- $f'(x) = \frac{5}{(x-2)^2}$
 $\implies f''(x) = \frac{-10}{(x-2)^3}$
- $f''(x)$ is defined for all $x \in \text{Dom}(f)$
- $f''(x) \neq 0$
- No inflection point

Table for f''

Interval	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$
-10	-	-	-
$(x-2)^3$	-	-	+
$f''(x)$	+	+	-
$f(x)$	U	U	∩

- f is concave up on $(-\infty, -7)$ and $(-7, 2)$, and concave down on $(2, \infty)$

- Find the critical points of f and identify the function's local extreme values?

- $f'(x) = \frac{(x-2) - (x-7)}{(x-2)^2} = \frac{5}{(x-2)^2}$
- $f'(x)$ is defined for all $x \in \text{Dom}(f)$
- $f'(x) \neq 0$ for all $x \in \text{Dom}(f)$
- No critical points and hence no local extreme values.

- Find the intervals on which f is increasing and decreasing?

Table for f' :

Interval	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$
$f'(x)$	+	+	+
$f(x)$	↗	↗	↗

- Thus f is increasing on $\text{Dom}(f) = \mathbb{R} - \{-7, 2\}$

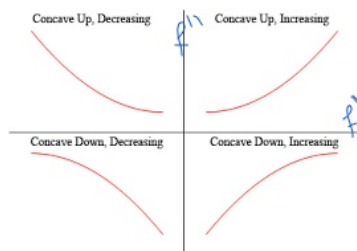
- Find the asymptotes?

- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x-7}{x-2} = 1$
- Thus $y = 1$ is a horizontal asymptote
- $\lim_{x \rightarrow -\infty} f(x) = 1$
- $\lim_{x \rightarrow -7^\pm} f(x) = \lim_{x \rightarrow -7^\pm} \frac{x-7}{x-2} = \frac{14}{9}$
- $\implies x = -7$ is not a vertical asymptote
- $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x-7}{x-2} = -\infty$
- $\implies x = 2$ is a vertical asymptote
- $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x-7}{x-2} = \infty$

(e) Sketch the graph of f :

• Table for f' and f''

x	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$
$f'(x)$	+	+	+
$f''(x)$	+	+	-
$f(x)$			



$$f(x) = \frac{x-7}{x-2}$$

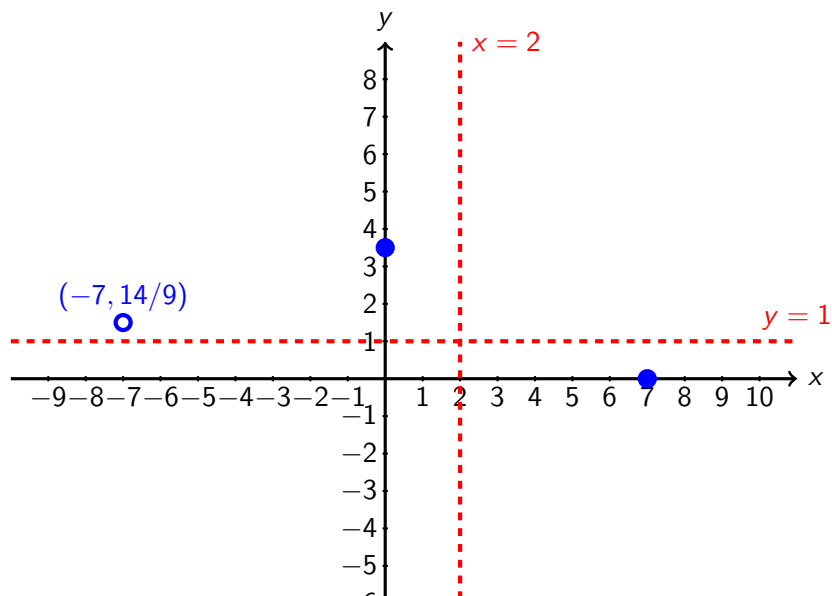
(e) Sketch the graph of f :

• Table for f' and f''

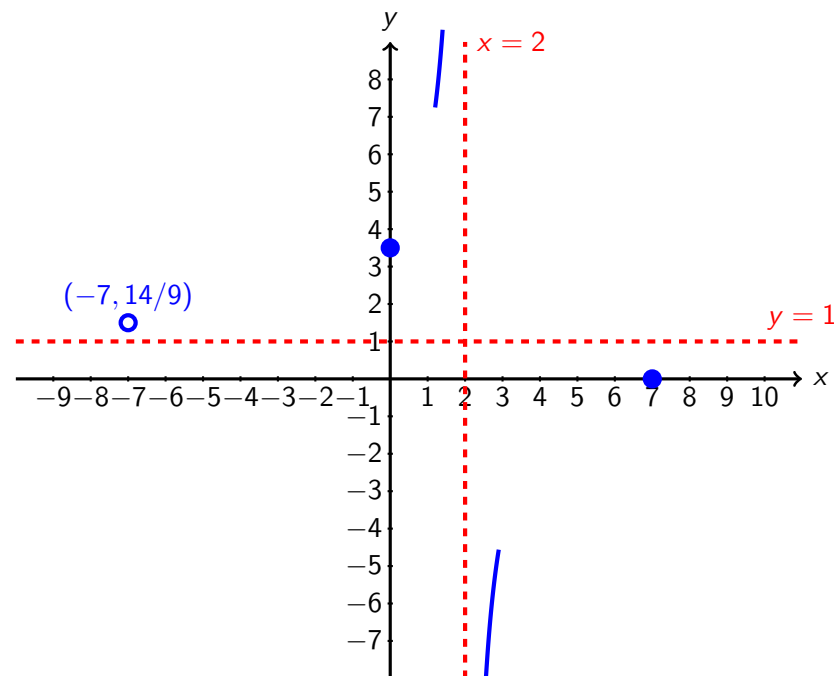
x	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$
$f'(x)$	+	+	+
$f''(x)$	+	+	-
$f(x)$			

- x - intercept: $x = 7$, y - intercept: $y = 3.5$
- $\text{Dom}(f) = \mathbb{R} - \{-7, 2\}$
- $y = 1$ is a horizontal asymptote
- $\lim_{x \rightarrow -7^\pm} f(x) = \frac{14}{9}$
- $x = 2$ is a vertical asymptote
- $\lim_{x \rightarrow 2^\pm} f(x) = \mp\infty$

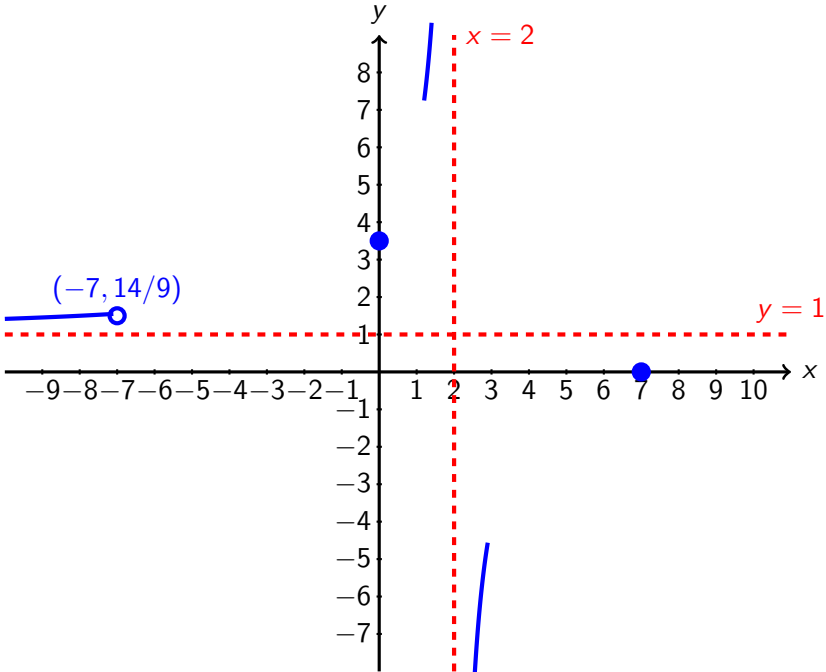
x - intercept: $x = 7$, y - intercept: $y = 3.5$
 $y = 1$ is a horizontal asymptote
 $x = 2$ is a vertical asymptote



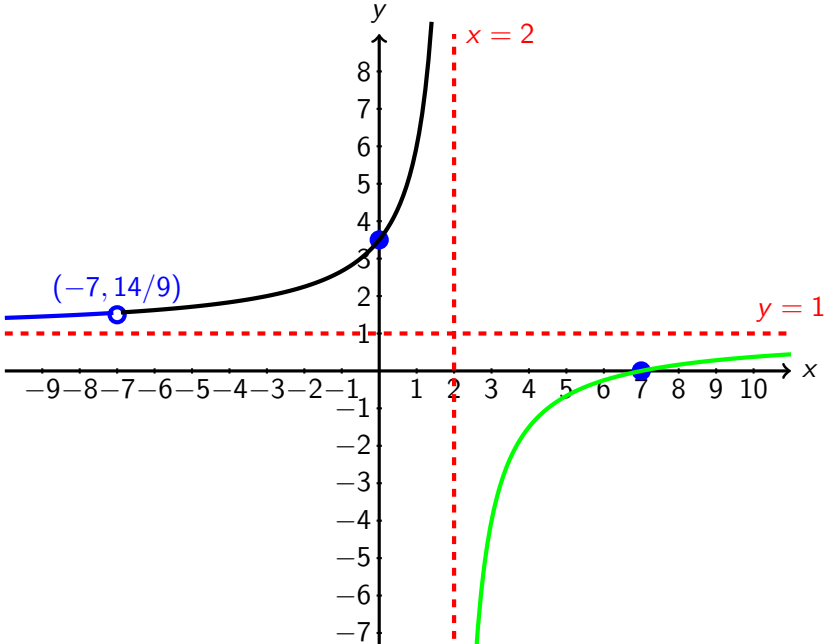
$$\lim_{x \rightarrow 2^\pm} f(x) = \mp\infty$$



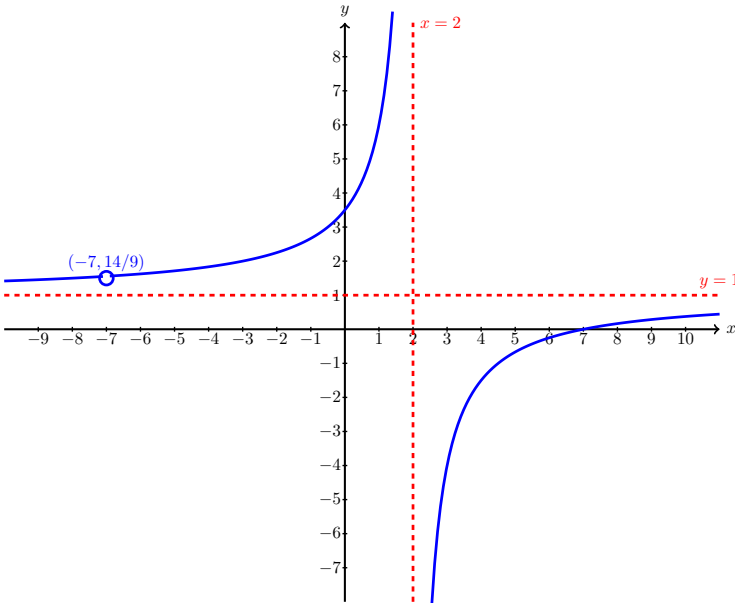
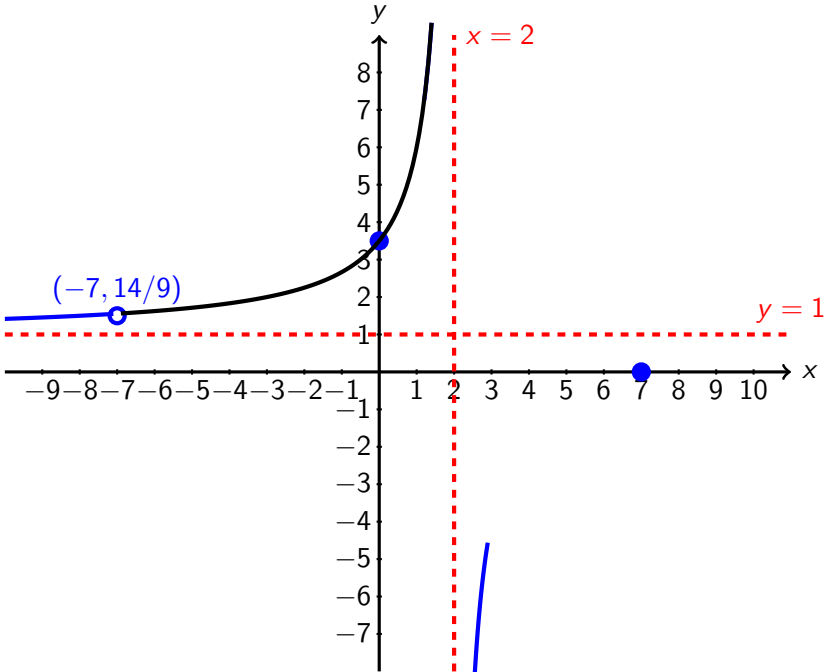
x	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$	
$f(x)$	$\nearrow$	$\nearrow$	$\nearrow$	



x	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$	
$f(x)$	$\nearrow$	$\nearrow$	$\nearrow$	



x	$(-\infty, -7)$	$(-7, 2)$	$(2, \infty)$	
$f(x)$	$\nearrow$	$\nearrow$	$\nearrow$	



$$f(x) = \frac{9x}{(x-1)^2}$$

Example 6 (Exam)

Let $f(x) = \frac{9x}{(x-1)^2}$.

- Find the critical points of f and identify the function's local extreme values.
- Find the intervals on which f is increasing and decreasing.
- Find the inflection points of f and intervals on which f is concave down and up.
- Find the asymptotes.
- Sketch the graph of f .

Solution

- $Dom(f) = \mathbb{R} - \{1\}$

- Find the critical points of f and identify the function's local extreme values?

- $f'(x) = \frac{9(x-1)^2 - 2(x-1)(9x)}{(x-1)^4} = \frac{9(x-1) - 18x}{(x-1)^3}$

- $\Rightarrow f'(x) = \frac{-9(x+1)}{(x-1)^3}$

- $f'(x)$ is defined for all $x \in Dom(f)$

- $f'(x) = 0 \Rightarrow x = -1$

- The critical point of f is $x = -1$

- Table for f' :**

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
-9	—	—	—
$x+1$	—	+	+
$(x-1)^3$	—	—	+
$f'(x)$	—	+	—
f	$\searrow$	$\nearrow$	$\searrow$

$$f(x) = \frac{9x}{(x-1)^2}$$

- $f(-1) = -\frac{9}{4}$ is a local minimum
- Find the intervals on which f is increasing and decreasing?
 - f is increasing on $[-1, 1)$
 - f is decreasing on $(-\infty, -1]$ and on $(1, \infty)$.
 - Find the inflection points of f and intervals on which f is concave down and up?

- $f'(x) = \frac{-9(x+1)}{(x-1)^3}$
 $\Rightarrow f''(x) = \frac{-9(x-1)^3 - 3(x-1)^2(-9(x+1))}{(x-1)^6}$

- $f''(x) = \frac{-9(x-1) + 27(x+1)}{(x-1)^4}$

- $f''(x) = \frac{18x + 36}{(x-1)^4}$

- $f''(x) = \frac{18(x+2)}{(x-1)^4}$

$$f''(x) = \frac{18(x+2)}{(x-1)^4}$$

- $f''(x)$ is defined for all $x \in Dom(f)$

- $f''(x) = 0 \Rightarrow x = -2$

- Table for f''**

x	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
$18(x+2)$	—	+	+
$(x-1)^4$	+	+	+
$f''(x)$	—	+	+
f	$\cap$	$\cup$	$\cup$

- $f(-2) = -2$ is an inflection point.

- f is concave down on $(-\infty, -2)$ and concave up on $(-2, 1)$ and on $(1, \infty)$

- Find the asymptotes?

- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{9x}{(x-1)^2} = \lim_{x \rightarrow \infty} \frac{9x}{x^2(1 - \frac{1}{x})^2}$

- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{9}{x(1 - \frac{1}{x})^2} = 0$

$$f(x) = \frac{9x}{(x-1)^2}$$

- $\Rightarrow y = 0$ is a horizontal asymptote
- $\lim_{x \rightarrow -\infty} f(x) = 0$
- $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{9x}{(x-1)^2} = \infty$
- Thus $x = 1$ is a vertical asymptote
- $\lim_{x \rightarrow 1^-} f(x) = \infty$

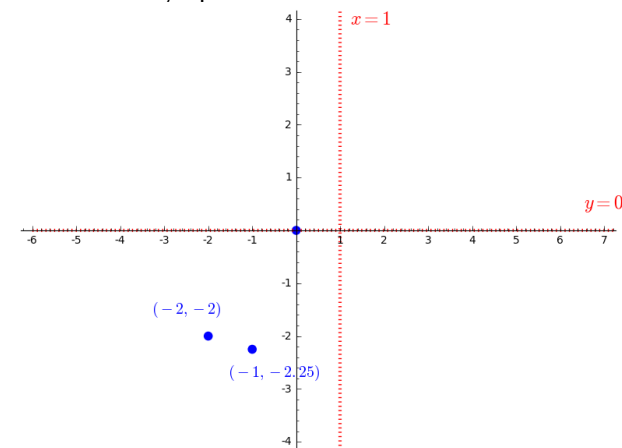
(e) Sketch the graph of f :

- x -intercept: $y = 0 \Rightarrow x = 0$
- y -intercept: $x = 0 \Rightarrow y = 0$

• Table for f' and f'' :

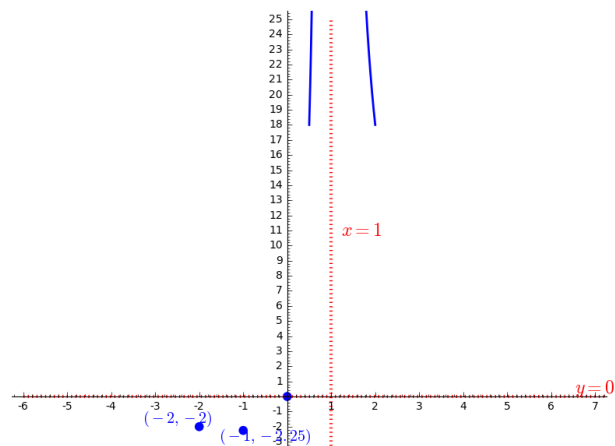
Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, \infty)$
$f'(x)$	-	-	+	-
$f''(x)$	-	+	+	+
f				

- x -intercept and y -intercept: $(0, 0)$
- $(-1, -\frac{9}{4})$ is a local minimum and $(-2, -2)$ is an inflection point.
- $\Rightarrow y = 0$ is a horizontal asymptote
- $x = 1$ is a vertical asymptote



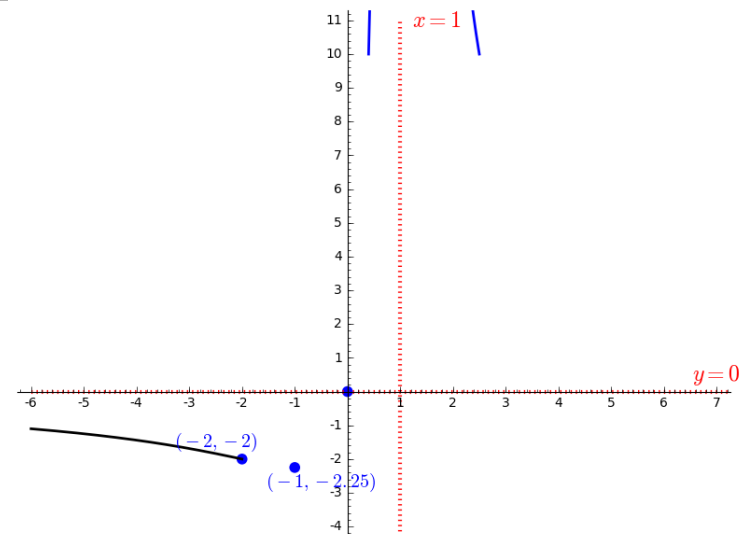
$$f(x) = \frac{9x}{(x-1)^2}$$

- $\lim_{x \rightarrow 1^\pm} f(x) = \infty$



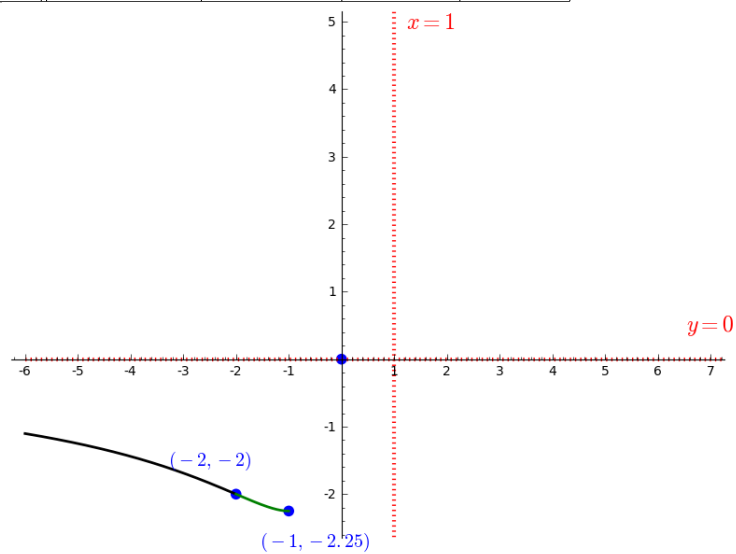
$$f(x) = \frac{9x}{(x-1)^2}$$

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, \infty)$
f				



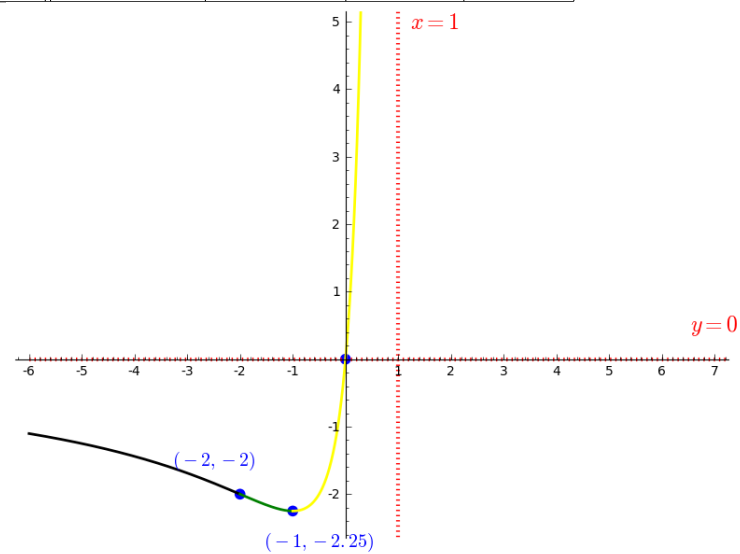
$$f(x) = \frac{9x}{(x-1)^2}$$

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, \infty)$
f				



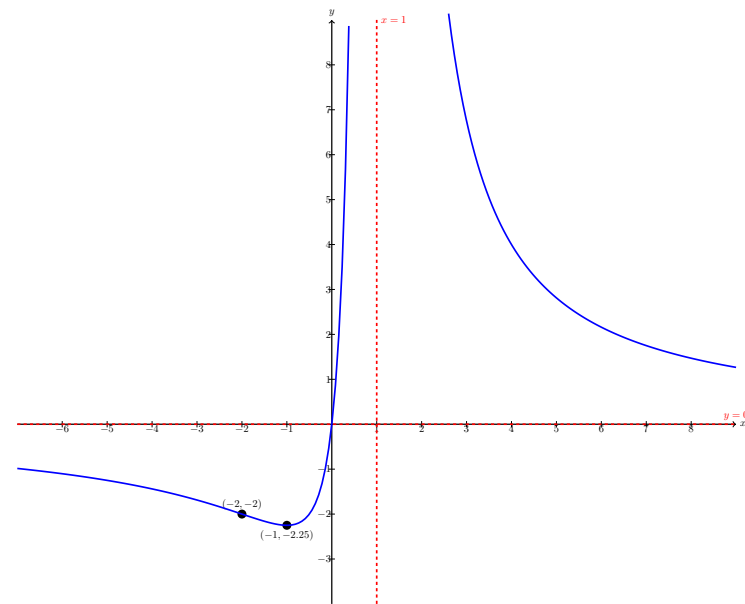
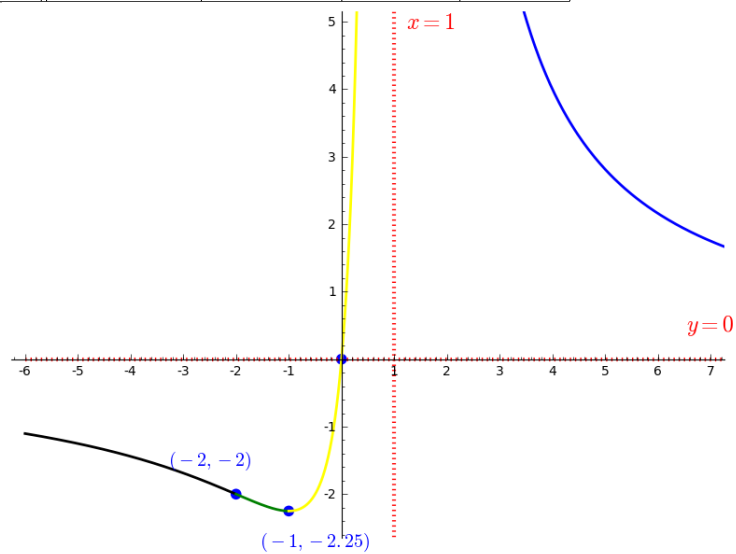
$$f(x) = \frac{9x}{(x-1)^2}$$

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, \infty)$
f				



$$f(x) = \frac{9x}{(x-1)^2}$$

Interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 1)$	$(1, \infty)$
f				



Example 7 (Exam)

Let $f(x) = \frac{x^2 + 4}{x}$.

- (i) Find the asymptotes of f .
- (ii) Find the intervals on which f is increasing and decreasing.
- (iii) Find the local extreme values.
- (iv) Find intervals on which f is concave down and up.
- (v) Find the inflection points if it exists.
- (vi) Sketch the graph of f .

Solution

- $Dom(f) = \mathbb{R} - \{0\}$

$$f'(x) = \frac{x^2 - 4}{x^2} \quad f(x) = \frac{x^2 + 4}{x}$$

- The critical points of f are $x = 2$ and $x = -2$

Table for f' :

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$x^2 - 4$	+	-	-	+
x^2	+	+	+	+
$f'(x)$	+	-	-	+
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

- f is increasing on $(-\infty, -2]$ and on $[2, \infty)$.
- f is decreasing on $[-2, 0)$ and on $(0, 2]$.

(iii) Find the local extreme values?

- $f(-2) = -4$ is a local maximum
- $f(2) = 4$ is a local minimum

(i) Find the asymptotes?

- $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 + 4}{x} = \lim_{x \rightarrow \infty} \left(x + \frac{4}{x}\right) = \infty$

- $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \left(x + \frac{4}{x}\right) = -\infty$

- There is no horizontal asymptotes.

- $f(x) = x + \frac{4}{x}$

- Thus $y = x$ is an oblique asymptote.

- $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x^2 + 4}{x} = \infty$

- Thus $x = 0$ is a vertical asymptote

- $\lim_{x \rightarrow 0^-} f(x) = -\infty$

(ii) Find the intervals on which f is increasing and decreasing?

- $f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$

- $f'(x)$ is defined for all $x \in Dom(f)$

- $f'(x) = 0 \implies x^2 - 4 = 0 \implies x = \pm 2$

$$f'(x) = \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

(iv) Find intervals on which f is concave down and up?

- $f''(x) = \frac{8}{x^3} \neq 0 \implies$ No inflection points.

Table for f''

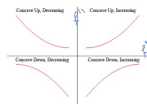
Interval	$(-\infty, 0)$	$(0, \infty)$
8	+	+
x^3	-	+
$f''(x)$	-	+
f	$\cap$	$\cup$

- f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$

(v) Find the inflection points if it exists?

- No inflection points.

$$f(x) = \frac{x^2+4}{x}$$

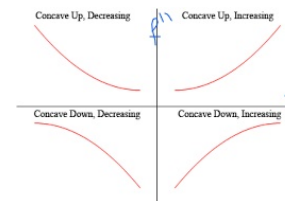


(vi) Sketch the graph of f ?

- $\text{Dom}(f) = \mathbb{R} - \{0\}$
- $y = x$ is an oblique asymptote.
- $x = 0$ is a vertical asymptote.
- No x -intercept and no y -intercept.
- $\lim_{x \rightarrow 0^\pm} f(x) = \pm\infty$
- $f(-2) = -4$ is a local maximum and $f(2) = 4$ is a local minimum

• Table for f' and f'' :

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
f				

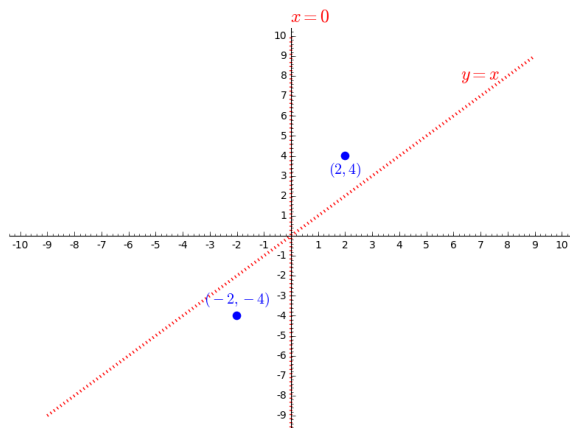


• Table for f' and f'' :

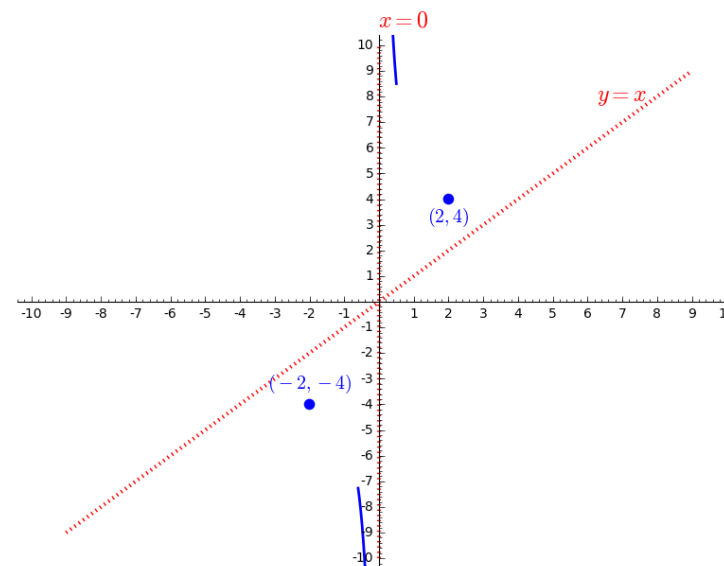
Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	+	-	-	+
$f''(x)$	-	-	+	+
f				

$$f(x) = \frac{x^2+4}{x}$$

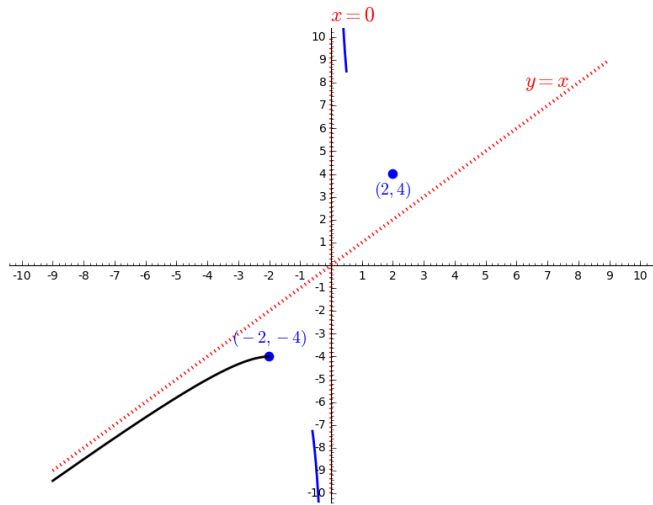
- $\text{Dom}(f) = \mathbb{R} - \{0\}$
- $y = x$ is an oblique asymptote.
- $x = 0$ is a vertical asymptote.
- No x -intercept.
- No y -intercept.
- $f(-2) = -4$ is a local maximum
- $f(2) = 4$ is a local minimum



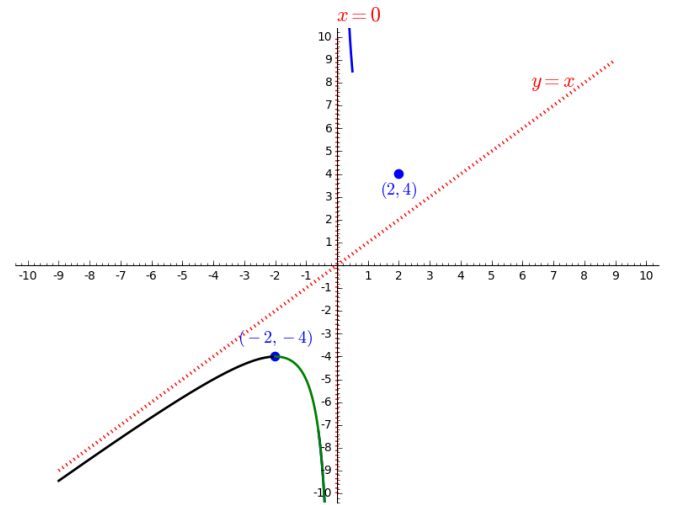
- $\lim_{x \rightarrow 0^\pm} f(x) = \pm\infty$



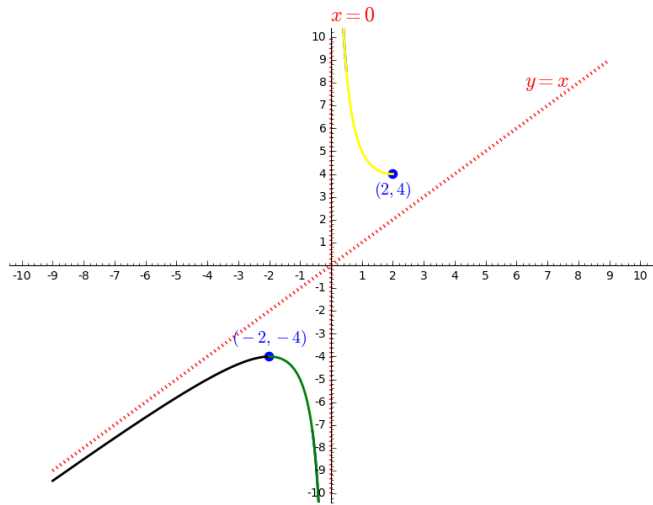
Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
f				



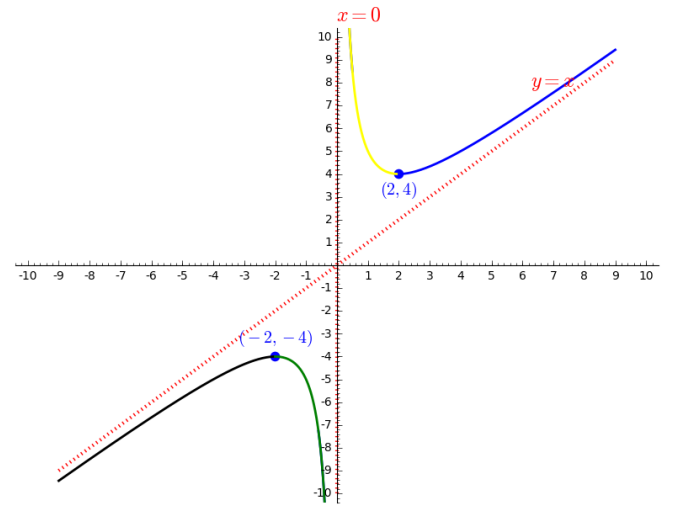
Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
f				



Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
f				



Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
f				



تم



مَا أصعب جزء في الموضوع؟

What is the most difficult part of this section?