



Chapter 5 Integration

First Semester 2020-2021
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Antiderivatives

- We have studied how to find the derivative of a function.
- However, many problems require that we recover a function from its known derivatives.

Definition

(Antiderivative)

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example

Find an antiderivative of $f(x) = 3x^2$.

Solution

- Let $F(x) = x^3$.
- Then $F'(x) = 3x^2 = f(x)$.
- Thus, $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.

Chapter 5 Integration

- There are really two types of integrals that we will study in this chapter:
- Indefinite Integrals and
- Definite Integrals.

How to find all antiderivatives of a function

Theorem

If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C,$$

where C is an arbitrary constant.

Definition

(Indefinite integral)

The collection of all antiderivatives of f is called the indefinite integral of f with respect to x , and is denoted by

$$\int f(x) dx.$$

Remark

If F is an antiderivative of f , then

$$\int f(x) dx = F(x) + C,$$

where C is an arbitrary constant called the constant of integration.

Example 1

Find $\int x dx$.

Solution

- In this example $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$.
- Thus, $\int x dx = \frac{1}{2}x^2 + C$

Example 2

Find $\int \cos x dx$

Solution

- An antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$.
- It follows that $\int \cos x dx = \sin x + C$

Integral Formulas

$\int dx = x + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
$\int \cos(kx) dx = \frac{1}{k} \sin(kx) + C$	$\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \cot x \csc x dx = -\csc x + C$

Example 1

Find $\int \frac{1}{\sqrt{x}} dx$

Solution

- $\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1} + C = 2x^{1/2} + C$

Example 2

Find $\int \sin(3x) dx$

Solution

- $\int \sin(3x) dx = -\frac{1}{3} \cos(3x) + C$

Example 3

Find $\int \frac{\cos x}{\sin^2 x} dx$

Solution

- $\int \frac{\cos x}{\sin^2 x} dx = \int \cot x \csc x dx = -\csc x + C$

Antiderivative linearity rules

(1) If k is a constant, then $\int kf(x) dx = k \int f(x) dx$.

(2) $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$.

Example 1

Find $\int (x^3 - x^{1/4} + 6) dx$

Solution

- $\int (x^3 - x^{1/4} + 6) dx = \int x^3 dx - \int x^{1/4} dx + 6 \int dx$
- $\int (x^3 - x^{1/4} + 6) dx = \frac{1}{4}x^4 - \frac{1}{1/4+1}x^{1/4+1} + 6x + C$
- $\int (x^3 - x^{1/4} + 6) dx = \frac{1}{4}x^4 - \frac{4}{5}x^{5/4} + 6x + C$

Example 2

Find $\int [2 \sec^2 x + \cos(2x)] dx$

Solution

- $\int [2 \sec^2 x + \cos(2x)] dx = 2 \tan x + \frac{1}{2} \sin(2x) + C$



5.3 The Definite Integral

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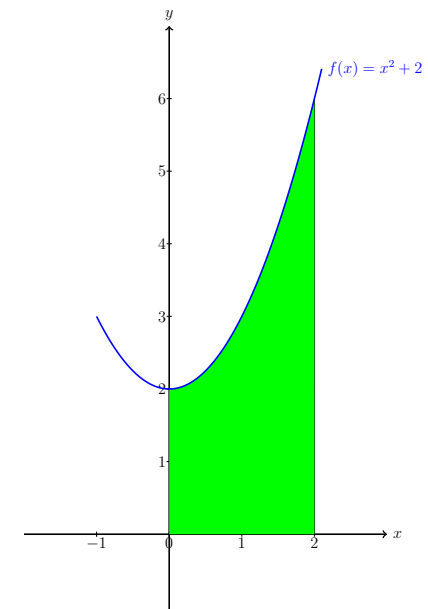
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Area and Estimating with Finite Sums

- The area problem is one of the interpretations of a definite integral and it will help us to understand the definition of the definite integral.
- Suppose we want to find the area of the region R that lies above the x -axis, below the graph of $y = x^2 + 2$, and between the vertical lines $x = 0$ and $x = 2$.
- Unfortunately, there is no simple formula for calculating the areas of general shapes having curved boundaries.

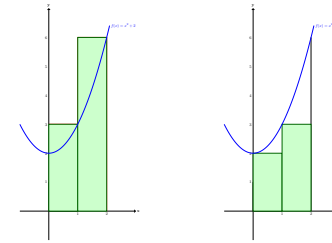
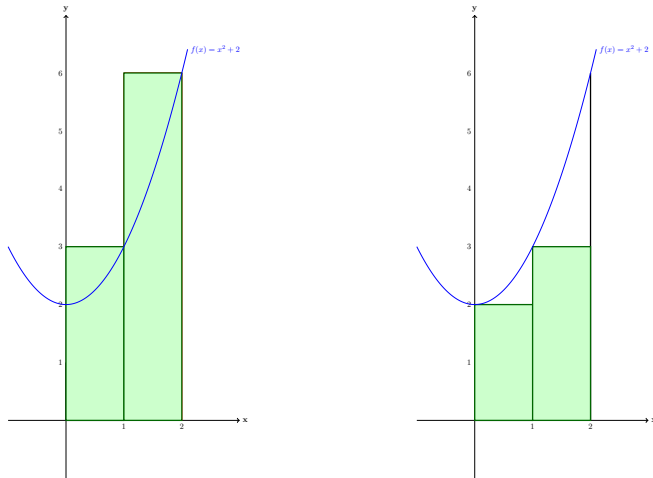
5.3 The Definite Integral

- This section defines the definite integral of a function as a limit of Riemann sum.
- In addition we will give many of the properties of definite integrals.
- The definite integral is the key tool in calculus for defining and calculating quantities important to mathematics and science.
- The idea behind the integral is that we can effectively compute such quantities by breaking them into small pieces and then summing the contribution from each piece.
- We then consider what happens when more and more, smaller and smaller pieces are taken in the summation process.



$$f(x) = x^2 + 2$$

While we do not yet have a method for determining the exact area of R , we can approximate it in simple way.

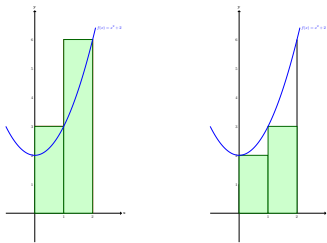


- We can divide the interval $[0, 2]$ into two intervals of equal length and we construct two rectangles with height equal the maximum value of the function f .
- The total area of the two rectangles approximate the area A of the R ,

$$A \approx (1)f(1) + (1)f(2) = 3 + 6 = 9.$$

- This estimate is larger than the true area A since the two rectangles contain R . We say that 9 is an **upper sum**.

$$f(x) = x^2 + 2$$



- If we use rectangles contained inside R , then the sum is called a **lower sum**.

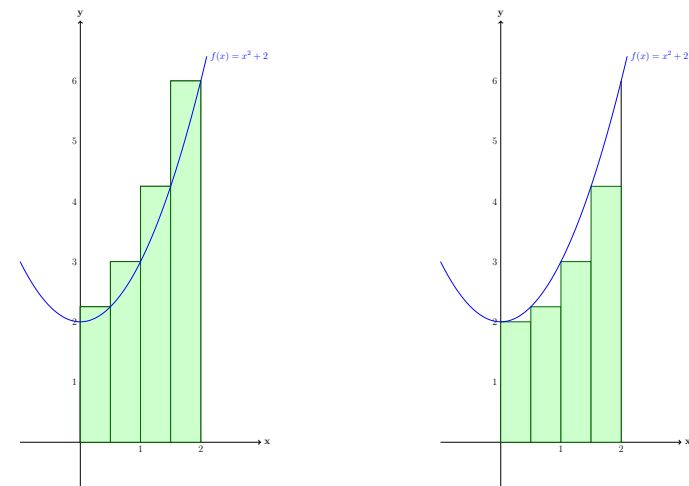
$$A \approx (1)f(0) + (1)f(1) = 2 + 3 = 5$$

- Thus,

$$5 < A < 9$$

and the error is less than $9 - 5 = 4$.

Let us divide the interval $[0, 2]$ into four intervals of equal length



Riemann Sums

- An upper sum of the area A of the R is

$$A \approx \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) = \frac{1}{2}\left(\frac{9}{4} + 3 + \frac{17}{4} + 6\right) = \frac{31}{4}.$$

- A lower sum of the area A of the R is

$$A \approx \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) = \frac{1}{2}\left(2 + \frac{9}{4} + 3 + \frac{17}{4}\right) = \frac{23}{4}.$$

- Thus,

$$\frac{23}{4} < A < \frac{31}{4}$$

and the error is less than 2.

- Therefore, when more and more intervals are taken, the estimation of the area becomes more and more close to the exact value.

- Let $f(x)$ be a continuous function on a closed interval $[a, b]$.
- We subdivide the interval $[a, b]$ into n subintervals, not necessarily of equal width, by choosing $n - 1$ points $x_1, \dots, x_{n-1}$ between a and b such that

$$x_0 = a < x_1 < x_2 < \dots < x_{n-1} < b = x_n.$$

- The set $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ is called a partition of $[a, b]$.
- The width (length) of the k th subinterval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
- A Riemann sum for a function f on $[a, b]$ is

$$S_P = \sum_{k=1}^n f(c_k) \Delta x_k.$$

Definite Integrals

- The finite sum approximations we considered above became more accurate as the number of terms increased and the subinterval widths narrowed.

Definition

(Definite integral)

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We define the definite integral of f over $[a, b]$ to be

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k,$$

provided that the limit exists.

Integrable and Nonintegrable Functions

Definition

If $\int_a^b f(x) dx$ exists, then we say that f is integrable over $[a, b]$.
Otherwise we say that f is nonintegrable over $[a, b]$.

Theorem

(Continuous functions are integrable)

If a function is continuous over $[a, b]$, then f is integrable over $[a, b]$.

Properties of Definite Integrals

Theorem

If f and g are integrable over the interval $[a, b]$, then

- (1) we define $\int_a^a f(x)dx = 0$;
- (2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
- (3) for any constant k , $\int_a^b kf(x)dx = k \int_a^b f(x)dx$;
- (4) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
- (5) for any $c \in (a, b)$, $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- (6) if $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$;

$$m \leq f(x) \leq M$$

- (7) if f has maximum value M and minimum value m on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Example 1

If $\int_1^2 f(x)dx = 2$ and $\int_2^3 f(x)dx = 3$, then find $\int_1^3 2f(x)dx$.

Solution

$$\begin{aligned} \bullet \int_1^3 2f(x)dx &= 2 \int_1^3 f(x)dx = 2 \left[\int_1^2 f(x)dx + \int_2^3 f(x)dx \right] \\ \bullet &= 2[2 + 3] = 10 \end{aligned}$$

Example 2

Find upper and lower bounds for $\int_0^2 \frac{1}{x^4 + 1} dx$.

Solution

- $? = L \leq \int_0^2 \frac{1}{x^4 + 1} dx \leq U = ?$
- Theorem 2, part (7)

$$\Rightarrow m(2-0) \leq \int_0^2 \frac{1}{x^4 + 1} dx \leq M(2-0)$$

- $f(x) = \frac{1}{x^4 + 1} \Rightarrow f'(x) = \frac{-4x^3}{(x^4 + 1)^2}$
- $f'(x)$ is defined for all $x \in (0, 2)$
and $f'(x) = 0 \Rightarrow x = 0 \notin (0, 2)$

- Thus, there is no critical points.

$$\bullet f(0) = 1 \quad f(2) = \frac{1}{17}$$

$$\bullet \Rightarrow m = \frac{1}{17}, \quad M = 1$$

- Therefore,

$$\frac{2}{17} \leq \int_0^2 \frac{1}{x^4 + 1} dx \leq 2$$

Remark

The variable of integration is a dummy variable; that is

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = \dots$$



5.4 The Fundamental Theorem of Calculus

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Fundamental Theorem, Part 1

- Let f be an integrable function over an interval $[a, b]$.
- Then for each $x \in [a, b]$,

$$F(x) = \int_a^x f(t) dt$$

is a function from $[a, b]$ to $\mathbb{R}$.

- For example, $F(a) = 0$, $F(b) = \int_a^b f(t) dt, \dots$

Theorem 3

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t) dt$ is continuous on $[a, b]$ and differentiable on (a, b) . Moreover

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

5.4 The Fundamental Theorem of Calculus

- This section presents the Fundamental Theorem of Calculus, which is the central theorem of integral calculus.
- It connects integration and differentiation, enabling us to compute integrals using antiderivatives.
- Leibniz and Newton exploited this relationship and started mathematical developments that fuelled the scientific revolution for the next 200 years.

Remark

The conclusion of The Fundamental Theorem of Calculus, Part 1, is beautiful, powerful, deep, and surprising. The equation

$$F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

may be the most important equation in mathematics for the following reasons:

- 1 It says that the processes of integration and differentiation are inverses of one another.
- 2 It says that the differential equation $F'(x) = f(x)$ has a solution for every continuous functions f .
- 3 It says that every continuous function is the derivative of some other function.

Examples

Example 1

Find $F'(x)$ if $F(x) = \int_1^x \cot t \, dt$.

Solution

- $F'(x) = \frac{d}{dx} \int_1^x \cot t \, dt$
- $F'(x) = \cot x$

Example 2 (Exam)

Find $\frac{dy}{dx}$ if $y = \int_4^{\sec x} (t - t^{-1}) dt$.

Solution

- Note that if $F(x) = \int_4^x (t - t^{-1}) dt$, then
- $F(\sec x) = \int_4^{\sec x} (t - t^{-1}) dt$
- $\Rightarrow y' = F'(\sec x) \sec x \tan x$
- But $F'(x) = x - \frac{1}{x}$
- $\Rightarrow y' = F'(\sec x) \sec x \tan x = \left[\sec x - \frac{1}{\sec x} \right] \sec x \tan x$
- $y' = \sec^2 x \tan x - \tan x$

Example 3 (Exam)

Let $x \sin(\pi x) = \int_0^{x^2} f(t) dt$, where f is a continuous function.

Use differentiation to find the value of $f(4)$.

Solution

- $\frac{d}{dx} [x \sin(\pi x)] = \frac{d}{dx} \int_0^{x^2} f(t) dt$
- $\sin(\pi x) + \pi x \cos(\pi x) = f(x^2) 2x$
- $x = 2 \Rightarrow \sin(2\pi) + 2\pi \cos(2\pi) = 4f(4)$
- $\Rightarrow f(4) = \frac{\pi}{2}$

Example 4

Find $\frac{dy}{dx}$ if $y = \int_{\sqrt{x}}^{2\sqrt{x}} \frac{\cos t}{t} dt$.

Solution

- $y' = \frac{d}{dx} \int_{\sqrt{x}}^{2\sqrt{x}} \frac{\cos t}{t} dt = \frac{d}{dx} \left[\int_{\sqrt{x}}^a \frac{\cos t}{t} dt + \int_a^{2\sqrt{x}} \frac{\cos t}{t} dt \right]$
- $y' = \frac{d}{dx} \left[- \int_a^{\sqrt{x}} \frac{\cos t}{t} dt + \int_a^{2\sqrt{x}} \frac{\cos t}{t} dt \right]$
- $y' = -\frac{\cos \sqrt{x}}{\sqrt{x}} \frac{1}{2\sqrt{x}} + \frac{\cos(2\sqrt{x})}{2\sqrt{x}} \frac{1}{\sqrt{x}}$
- $y' = \frac{\cos(2\sqrt{x}) - \cos(\sqrt{x})}{2x}$

Fundamental Theorem, Part 2 (The Evaluation Theorem)

Theorem 4

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t) dt = F(b) - F(a).$$

Notation

We denote $F(b) - F(a)$ by $F(x) \Big|_a^b$ or $\left[F(x) \right]_a^b$.

Example

Evaluate the following integrals:

- (1) $\int_0^1 (x^2 + 1) dx.$
- (2) $\int_{-2}^{-1} \frac{1}{x^2} dx.$
- (3) $\int_0^{\pi/2} \sin x dx.$
- (4) $\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4) dt.$
- (5) $\int_{-\pi}^{\pi/2} |\cos x| dx.$

Solution

$$(1) \int_0^1 (x^2 + 1) dx = \left. \frac{1}{3}x^3 + x \right|_0^1 = \frac{1}{3} + 1 - [0 + 0] = \frac{4}{3}$$

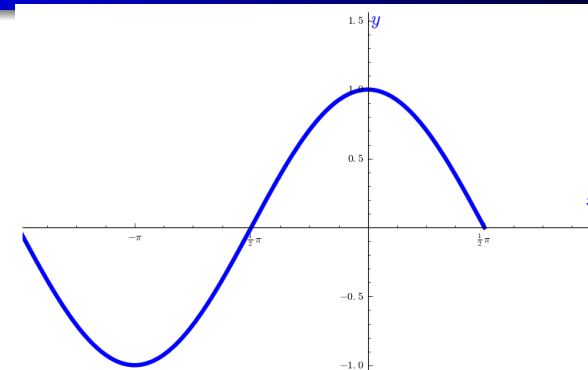
$$(2) \int_{-2}^{-1} \frac{1}{x^2} dx = \left. \frac{-1}{x} \right|_{-2}^{-1} = \frac{-1}{-1} + \frac{1}{-2} = \frac{1}{2}$$

$$(3) \int_0^{\pi/2} \sin x dx = \left. -\cos x \right|_0^{\pi/2} = -\cos(\pi/2) + \cos 0 = 1$$

$$(4) \int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4) dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^3 + t^2 + 4t + 4) dt$$

$$\bullet = \left[\frac{1}{4}t^4 + \frac{1}{3}t^3 + 2t^2 + 4t \right]_{-\sqrt{3}}^{\sqrt{3}}$$

$$\bullet = \frac{1}{4}(9 - 9) + \frac{1}{3}(3\sqrt{3} + 3\sqrt{3}) + 2(3 - 3) + 4(\sqrt{3} + \sqrt{3}) = 10\sqrt{3}$$

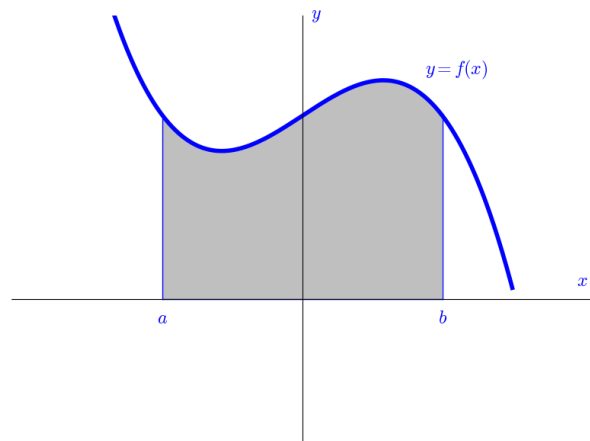


$$(5) \int_{-\pi}^{\pi/2} |\cos x| dx = \int_{-\pi}^{-\pi/2} -\cos x dx + \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$\bullet \int_{-\pi}^{\pi/2} |\cos x| dx = \left. -\sin x \right|_{-\pi}^{-\pi/2} + \left. \sin x \right|_{-\pi/2}^{\pi/2}$$

$$\bullet = -[\sin(-\pi/2) - \sin(-\pi)] + [\sin(\pi/2) - \sin(-\pi/2)] = 1 + 2 = 3$$

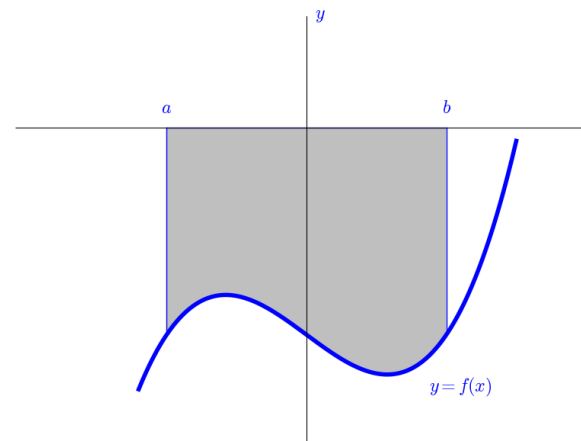
Total area between the curve $y = f(x)$ and the x -axis



- (1) If $y = f(x) \geq 0$ on $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is

$$A = \int_a^b f(x) dx$$

Total area between the curve $y = f(x)$ and the x -axis



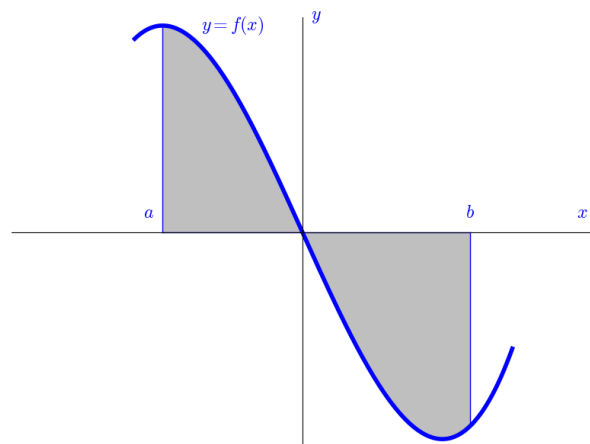
- (2) If $f(x) < 0$ on $[a, b]$, then the area between the curve $y = f(x)$ and the x -axis over $[a, b]$ is

$$A = - \int_a^b f(x) dx$$

Total area between the curve $y = f(x)$ and the x -axis

- (3) If $f(x)$ change signs on $[a, b]$, then the area between the curve $y = f(x)$ and the x -axis over $[a, b]$ is

$$A = \int_a^b |f(x)| dx$$

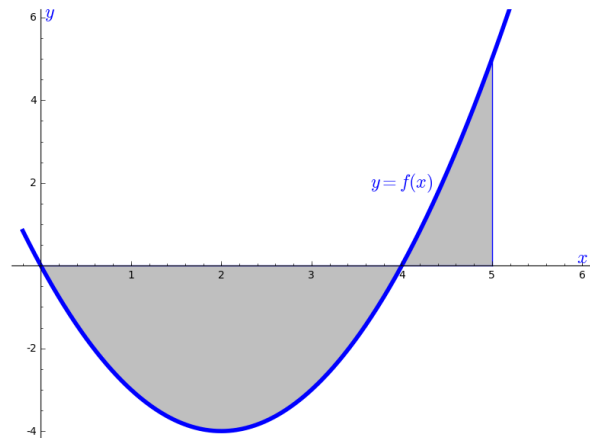


Example

Find the area between the curve $y = x^2 - 4x$ and the x -axis over $[0, 5]$.

Solution

- $A = \int_a^b |f(x)| dx$
- $f(x) = x^2 - 4x = 0 \Rightarrow x = 0, x = 4.$
- $\int_0^4 f(x) dx = \int_0^4 (x^2 - 4x) dx = \left. \frac{x^3}{3} - 2x^2 \right|_0^4 = \frac{64}{3} - 32 = -\frac{32}{3}$
- $\int_4^5 f(x) dx = \left. \frac{x^3}{3} - 2x^2 \right|_4^5 = \frac{125}{3} - 50 - \frac{64}{3} + 32$
- $= \frac{61}{3} - 18 = \frac{7}{3}$



- Total area = $|-32/3| + 7/3 = 13$



5.5 Indefinite Integrals and the Substitution Method

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5.5 Indefinite Integrals and the Substitution Method

- Antiderivatives generally turn out to be more difficult to find than derivatives.
- However, there are many techniques for computing them.
- In this section we will study the substitution method for computing antiderivatives.

Integral Formulas

$\int dx = x + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
$\int \cos(kx) dx = \frac{1}{k} \sin(kx) + C$	$\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \cot x \csc x dx = -\csc x + C$

Theorem 5 (The Substitution Rule)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Proof.

- Let $F(u)$ be an antiderivative of $f(u)$.
- Then $F'(u) = f(u)$.
- Since $u = g(x)$, the chain rule implies that

$$\frac{d}{dx}F(g(x)) = F'(g(x))\frac{d}{dx}g(x) = f(g(x))g'(x).$$

□

- Thus

$$\int f(g(x))g'(x)dx = \int \frac{d}{dx}F(g(x))dx$$

-

$$= F(g(x)) + C = F(u) + C = \int f(u)du.$$

Example 1

Evaluate the integral $\int (4x - 7)^3 dx$.

Solution

- Let $u = 4x - 7$
- $\Rightarrow \frac{du}{dx} = 4$
- $\Rightarrow du = 4dx$
- Thus $\int (4x - 7)^3 dx = \int u^3 \frac{du}{4} = \frac{1}{4} \int u^3 du$
- $\Rightarrow \int (4x - 7)^3 dx = \frac{1}{4} \frac{u^4}{4} + C$
- $\Rightarrow \boxed{\int (4x - 7)^3 dx = \frac{1}{16} (4x - 7)^4 + C}$

Example 2

Evaluate the integral $\int x (5x^2 + 4)^{15} dx$.

Solution

- Let $u = 5x^2 + 4$
- $\Rightarrow du = 10x dx$
- Thus $\int x (5x^2 + 4)^{15} dx = \int x u^{15} \frac{du}{10x} = \frac{1}{10} \int u^{15} du$
- $\Rightarrow \int x (5x^2 + 4)^{15} dx = \frac{1}{10} \frac{u^{16}}{16} + C$
- $\Rightarrow \boxed{\int x (5x^2 + 4)^{15} dx = \frac{1}{160} (5x^2 + 4)^{16} + C}$

Example 3

Evaluate the integral $\int x^2 \sqrt{x^3 - 4} \, dx$.

Solution

- Let $u = x^3 - 4$
- $\Rightarrow du = 3x^2 \, dx$
- Thus $\int x^2 \sqrt{x^3 - 4} \, dx = \int x^2 \sqrt{u} \frac{du}{3x^2} = \frac{1}{3} \int \sqrt{u} \, du$
- $\Rightarrow \int x^2 \sqrt{x^3 - 4} \, dx = \frac{1}{3} u^{3/2} \frac{2}{3} + C$
- $\Rightarrow \boxed{\int x^2 \sqrt{x^3 - 4} \, dx = \frac{2}{9} (x^3 - 4)^{3/2} + C}$

Example 4

Evaluate the integral $\int \sin(5x - 2) \, dx$.

Solution

- Let $u = 5x - 2$
- $\Rightarrow du = 5 \, dx$
- Thus $\int \sin(5x - 2) \, dx = \int \sin u \frac{du}{5}$
- $\Rightarrow \int \sin(5x - 2) \, dx = -\frac{1}{5} \cos u + C$
- $\Rightarrow \boxed{\int \sin(5x - 2) \, dx = -\frac{1}{5} \cos(5x - 2) + C}$

Example 5

Evaluate the integral $\int x \cos(x^2 + 5) \, dx$.

Solution

- Let $u = x^2 + 5$
- $\Rightarrow du = 2x \, dx$
- Thus $\int x \cos(x^2 + 5) \, dx = \int x \cos u \frac{du}{2x} = \frac{1}{2} \int \cos u \, du$
- $\Rightarrow \int x \cos(x^2 + 5) \, dx = \frac{1}{2} \sin u + C$
- $\Rightarrow \boxed{\int x \cos(x^2 + 5) \, dx = \frac{1}{2} \sin(x^2 + 5) + C}$

Example 6

Evaluate the integral $\int \frac{1}{\sqrt{3x - 2}} \, dx$.

Solution

- Let $u = 3x - 2$
- $\Rightarrow du = 3 \, dx$
- Thus $\int \frac{1}{\sqrt{3x - 2}} \, dx = \int \frac{1}{\sqrt{u}} \frac{du}{3} = \frac{1}{3} \int u^{-1/2} \, du$
- $\Rightarrow \int \frac{1}{\sqrt{3x - 2}} \, dx = \frac{1}{3} u^{1/2} 2 + C$
- $\Rightarrow \boxed{\int \frac{1}{\sqrt{3x - 2}} \, dx = \frac{2}{3} \sqrt{3x - 2} + C}$

Example 7

Evaluate the integral $\int \frac{x}{(x+2)^4} dx$.

Solution

- Let $u = x + 2 \Rightarrow du = dx$
- Thus $\int \frac{x}{(x+2)^4} dx = \int \frac{x}{u^4} du$
- $u = x + 2 \Rightarrow x = u - 2$
- $\Rightarrow \int \frac{x}{(x+2)^4} dx = \int \frac{u-2}{u^4} du$
- $\Rightarrow \int \frac{x}{(x+2)^4} dx = \int (u^{-3} - 2u^{-4}) du$
- $\Rightarrow \int \frac{x}{(x+2)^4} dx = \frac{u^{-2}}{-2} - 2 \frac{u^{-3}}{-3} + C = \frac{-1}{2u^2} + \frac{2}{3u^3} + C$

- $\Rightarrow \int \frac{x}{(x+2)^4} dx = \frac{-1}{2u^2} + \frac{2}{3u^3} + C$
- $\Rightarrow \int \frac{x}{(x+2)^4} dx = \frac{-1}{2(x+2)^2} + \frac{2}{3(x+2)^3} + C$

Example 8

Evaluate the integral $\int \cos x \sin^3 x dx$.

Solution

- Let $u = \sin x \Rightarrow du = \cos x dx$
- Thus $\int \cos x \sin^3 x dx = \int u^3 du$
- $\Rightarrow \int \cos x \sin^3 x dx = \frac{1}{4} u^4 + C$
- $\Rightarrow \int \cos x \sin^3 x dx = \frac{1}{4} \sin^4 x + C$

Example 9

Evaluate the integral $\int \sqrt{\frac{x^2-1}{x^8}} dx, x > 0$.

Solution

- First of all let us write $\int \sqrt{\frac{x^2-1}{x^8}} dx = \int \frac{1}{x^3} \sqrt{\frac{x^2-1}{x^2}} dx$
- Let $u = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2} \Rightarrow du = \frac{2}{x^3} dx$
- Thus $\int \sqrt{\frac{x^2-1}{x^8}} dx = \frac{1}{2} \int \sqrt{u} du$
- $\Rightarrow \int \sqrt{\frac{x^2-1}{x^8}} dx = \frac{1}{2} u^{3/2} \frac{2}{3} + C$
- $\Rightarrow \int \sqrt{\frac{x^2-1}{x^8}} dx = \frac{1}{3} \left(\frac{x^2-1}{x^2} \right)^{3/2} + C$

Example 10 (Exam 2019)

Evaluate the integral $\int x^7(x^4 + 1)^{-1/4} dx$.

Solution

- Let $u = x^4 + 1 \implies du = 4x^3 dx$
- Thus $\int x^7(x^4 + 1)^{-1/4} dx = \frac{1}{4} \int x^4 u^{-1/4} du$
- $u = x^4 + 1 \implies x^4 = u - 1$
- $\int x^7(x^4 + 1)^{-1/4} dx = \frac{1}{4} \int (u - 1)u^{-1/4} du$
- $\int x^7(x^4 + 1)^{-1/4} dx = \frac{1}{4} \int (u^{3/4} - u^{-1/4}) du$
- $$= \frac{1}{4} \left(\frac{4}{7} u^{7/4} - \frac{4}{3} u^{3/4} \right) + C = \frac{1}{7} (x^4 + 1)^{7/4} - \frac{1}{3} (x^4 + 1)^{3/4} + C$$

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5.6 Substitution and Area between Curves (part 1)

First Semester 2020-2021

أ.د. أيمن هاشم السقا

5.6 Substitution and Area between Curves

- In this section we use substitution to evaluate definite integrals and to find the area between two curves.

Substitution in definite integrals

- There are two methods for evaluating a definite integral by substitution.
- The first method is to find an antiderivative using substitution, and then to evaluate the definite integral by applying the Fundamental Theorem.
- The second method extends the process of substitution directly to definite integrals so the limits of integration change when the variable of integration is changed by substitution.

Theorem 6 (Substitution in definite integrals)

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g , then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 1

Evaluate $\int_0^{\pi/4} [1 - \cos(3x)] \sin(3x) dx$.

Solution

- Let $u = \cos(3x) \Rightarrow du = -3 \sin(3x) dx$
- $\int_0^{\pi/4} [1 - \cos(3x)] \sin(3x) dx = \int_{x=0}^{x=\pi/4} [1 - u] \left[\frac{du}{-3} \right]$
- $x = 0 \Rightarrow u = \cos(0) = 1$
- $x = \pi/4 \Rightarrow u = \cos(3\pi/4) = -\frac{1}{\sqrt{2}}$
- $= \frac{-1}{3} \int_{u=1}^{u=-1/\sqrt{2}} [1 - u] du = \frac{-1}{3} \left[u - \frac{u^2}{2} \right]_{u=1}^{u=-1/\sqrt{2}}$
- $= \frac{-1}{3} \left[-\frac{1}{\sqrt{2}} - \frac{1}{4} - \left(1 - \frac{1}{2} \right) \right] = \frac{1}{3} \left[\frac{1}{\sqrt{2}} + \frac{3}{4} \right] = \frac{(2\sqrt{2} + 3)}{12}$

Example 2

Evaluate $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt$.

Solution

- Let $u = t^5 + 2t \Rightarrow du = (5t^4 + 2) dt$
- $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt = \int_{t=0}^{t=1} \sqrt{u} du$
- $t = 0 \Rightarrow u = 0$
- $t = 1 \Rightarrow u = 3$
- $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt = \int_{u=0}^{u=3} \sqrt{u} du$
- $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt = u^{3/2} \frac{2}{3} \Big|_{u=0}^{u=3}$
- $\int_0^1 \sqrt{t^5 + 2t} (5t^4 + 2) dt = \frac{2}{3} (3\sqrt{3} - 0) = 2\sqrt{3}$

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Example 3

Evaluate $\int_0^{\pi/2} \frac{\sin x}{(3 + \cos x)^2} dx$.

Solution

- Let $u = 3 + \cos x \Rightarrow du = -\sin x dx$
- $\int_0^{\pi/2} \frac{\sin x}{(3 + \cos x)^2} dx = \int_{u=4}^{u=3} \frac{-du}{u^2}$
- $= \frac{1}{u} \Big|_{u=4}^{u=3}$
- $= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

Example 4

Evaluate $\int_0^1 x \sqrt{1 - x^2} dx$.

Solution

- Let $u = 1 - x^2 \Rightarrow du = -2x dx$
- $\int_0^1 x \sqrt{1 - x^2} dx = \int_{u=1}^{u=0} \sqrt{u} \frac{du}{-2}$
- $= -\frac{1}{2} \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=0}$
- $= -\frac{1}{3} (0 - 1) = \frac{1}{3}$

Example 5

Evaluate $\int_{-\pi/4}^0 \tan x \sec^2 x \, dx$.

Solution

- Let $u = \tan x \implies du = \sec^2 x \, dx$
- $\int_{-\pi/4}^0 \tan x \sec^2 x \, dx = \int_{u=-1}^{u=0} u \, du$
- $= \frac{1}{2} u^2 \Big|_{u=-1}^{u=0}$
- $= \frac{1}{2} (0 - 1) = -\frac{1}{2}$

Example 6 (Exam)

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$.

Solution

- Let $u = \sin x \implies du = \cos x \, dx$
- $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx = \int_{u=-1}^{u=1} u^2 \, du$
- $= \frac{1}{3} u^3 \Big|_{u=-1}^{u=1}$
- $= \frac{1}{3} (1 + 1) = \frac{2}{3}$

Example 7

Evaluate $\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}} \, dt$.

Solution

- Let $u = \sqrt{t} \implies du = \frac{dt}{2\sqrt{t}}$
- $\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}} \, dt = \int_{u=\pi/6}^{u=\pi/2} \frac{\cos u}{\sqrt{t} \sin u} 2\sqrt{t} \, du$
- $= 2 \int_{u=\pi/6}^{u=\pi/2} \frac{\cos u}{\sin u} \, du$
- Let $w = \sin u \implies dw = \cos u \, du$
- $\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}} \, dt = 2 \int_{w=1/2}^{w=1} \frac{dw}{\sqrt{w}}$
- $= 2 \left(2\sqrt{w} \right) \Big|_{w=1/2}^{w=1}$

- $= 2 \left(2\sqrt{w} \right) \Big|_{w=1/2}^{w=1}$
- $\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}} \, dt = 4 \left(1 - \frac{1}{\sqrt{2}} \right) = \frac{4(\sqrt{2} - 1)}{\sqrt{2}}$

Example 8 (Exam)

Use the integral $\int_1^2 f(z) dz = 3$ to find the value of the integral $\int_{1/2}^1 \frac{1}{x^2} f\left(\frac{1}{x}\right) dx$.

Solution

- Let $z = \frac{1}{x} \implies dz = \frac{-dx}{x^2}$
- $\int_{1/2}^1 \frac{1}{x^2} f\left(\frac{1}{x}\right) dx = \int_{z=2}^{z=1} -f(z) dz$
- $= \int_{z=1}^{z=2} f(z) dz = 3$

Example 9 (Exam)

Evaluate $\int_0^{\pi/3} \frac{\tan x}{\sqrt{2 \sec x}} dx$.

Solution

- Let $u = \sec x \implies du = \sec x \tan x dx$
- $\int_0^{\pi/3} \frac{\tan x}{\sqrt{2 \sec x}} dx = \int_{u=1}^{u=2} \frac{1}{\sqrt{2u}} \frac{du}{u}$
- $= \frac{1}{\sqrt{2}} \int_{u=1}^{u=2} u^{-3/2} du$
- $= \frac{1}{\sqrt{2}} u^{-1/2} (-2) \Big|_{u=1}^{u=2} = \frac{-2}{\sqrt{2}} \frac{1}{\sqrt{u}} \Big|_{u=1}^{u=2}$
- $= -\sqrt{2} \left(\frac{1}{\sqrt{2}} - 1 \right) = -1 + \sqrt{2}$

Definite Integrals of Symmetric Functions

Theorem 8

Let f be continuous on the symmetric interval $[-a, a]$.

- (1) If f is even, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
- (2) If f is odd, then $\int_{-a}^a f(x) dx = 0$.

Proof.

- $\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$
- Let $u = -x \implies du = -dx$
- $\int_{-a}^0 f(x) dx = - \int_a^0 f(-u) du + \int_0^a f(x) dx$
- $= \int_0^a f(x) dx + \int_0^a f(-x) dx$

□

Example 1

Evaluate $\int_{-\pi/2}^{\pi/2} \cos(x) dx$.

Solution

- $f(x) = \cos x$ is even
- $\int_{-\pi/2}^{\pi/2} \cos(x) dx = 2 \int_0^{\pi/2} \cos(x) dx$
- $= 2 \sin(x) \Big|_0^{\pi/2} = 2$



5.6 Substitution and Area between Curves (part 2)

First Semester 2020-2021
أ.د. أيمن هاشم السقا

Example 2

Evaluate $\int_{-1}^1 \frac{x}{(x^4 + 1)^2} dx$.

Solution

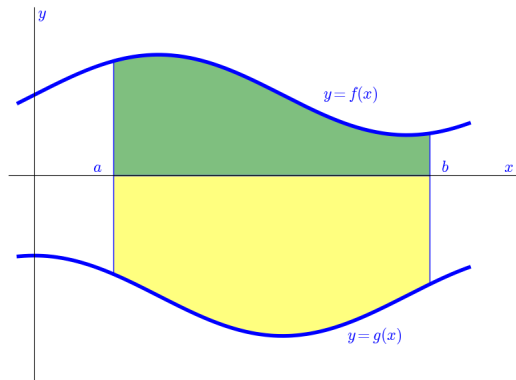
- $f(x) = \frac{x}{(x^4 + 1)^2}$ is odd since $f(-x) = \frac{-x}{(x^4 + 1)^2} = -f(x)$
- $\Rightarrow \int_{-1}^1 \frac{x}{(x^4 + 1)^2} dx = 0$

Area Between Curves

Definition

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is

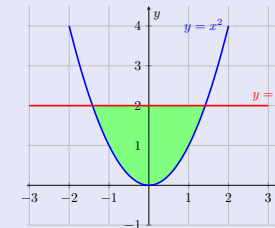
$$A = \int_a^b [f(x) - g(x)] dx.$$



Example 1

Find the area of the region enclosed by $y = x^2$, $y = 2$.

Solution



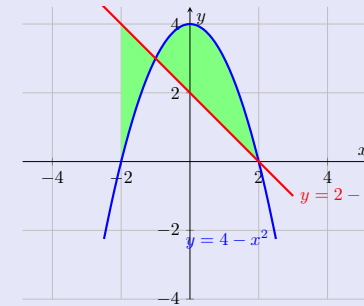
- $\text{Area} = \int_a^b [2 - x^2] dx$
- $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

- $\text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}} [2 - x^2] dx = 2x - \frac{1}{3}x^3 \Big|_{-\sqrt{2}}^{\sqrt{2}}$
- $= 2[\sqrt{2} + \sqrt{2}] - \frac{1}{3}[2\sqrt{2} + 2\sqrt{2}] = \frac{8\sqrt{2}}{3}$

Example 2 (Exam)

Find the area of the region enclosed by $y = 4 - x^2$, $y = 2 - x$, $x = -2$, and $x = 2$.

Solution



- $4 - x^2 = 2 - x \implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0$
- $\implies x = 2, -1$

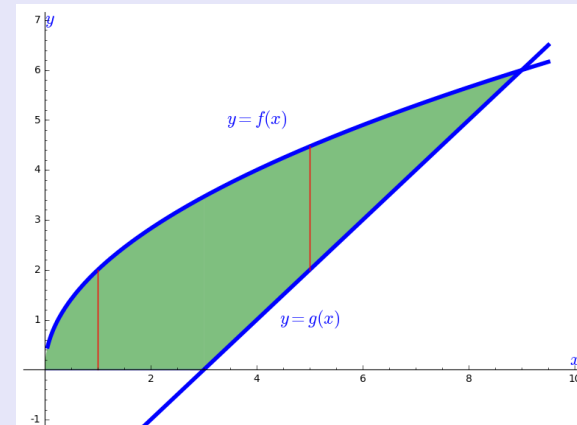
82

- $\text{Area} = \int_{-2}^{-1} [2 - x - (4 - x^2)] dx + \int_{-1}^2 [4 - x^2 - (2 - x)] dx$
- $= \int_{-2}^{-1} [x^2 - x - 2] dx - \int_{-1}^2 [x^2 - x - 2] dx$
- $= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \right]_{-2}^{-1} - \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x \right]_{-1}^2$
- $= \frac{1}{3}[-1 + 8] - \frac{1}{2}[1 - 4] - 2[-1 + 2] - \frac{1}{3}[8 + 1] + \frac{1}{2}[4 - 1] + 2[2 + 1]$
- $= \frac{7}{3} + \frac{3}{2} - 2 - 3 + \frac{3}{2} + 6 = \frac{19}{3}$

Example 3

Find the area of the region in the first quadrant that is bounded above by $y = 2\sqrt{x}$ and below by the x -axis and the line $y = x - 3$.

Solution



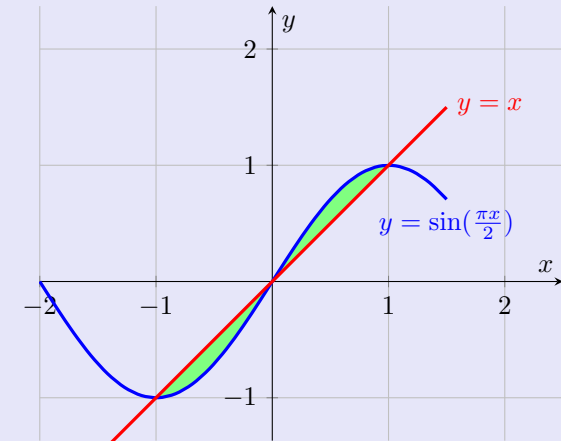
84

- $2\sqrt{x} = x - 3 \implies 4x = (x - 3)^2 = x^2 - 6x + 9 \implies x^2 - 10x + 9 = 0$
- $\implies (x - 9)(x - 1) = 0 \implies x = 9, 1$
- $\text{Area} = \int_0^3 2\sqrt{x} dx + \int_3^9 [2\sqrt{x} - (x - 3)] dx$
- $= \left[\frac{4}{3} x^{3/2} \right]_0^3 + \left[\frac{4}{3} x^{3/2} - \frac{1}{2} x^2 + 3x \right]_3^9$
- $= \frac{4}{3} [3\sqrt{3}] + \frac{4}{3} [27 - 3\sqrt{3}] - \frac{1}{2} [81 - 9] + 3[9 - 3]$
- $= 36 - 36 + 18 = 18$

Example 4

Find the area of the region enclosed by $y = \sin\left(\frac{\pi x}{2}\right)$ and $y = x$.

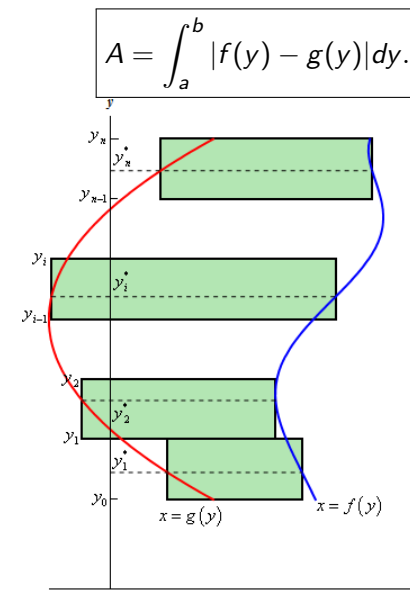
Solution



- $\text{Area} = \int_{-1}^0 \left[x - \sin\left(\frac{\pi x}{2}\right) \right] dx + \int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x \right] dx$
- $= \left[\frac{1}{2} x^2 + \frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) \right]_{-1}^0 - \left[\frac{1}{2} x^2 + \frac{2}{\pi} \cos\left(\frac{\pi x}{2}\right) \right]_0^1$
- $= \frac{1}{2} [0 - 1] + \frac{2}{\pi} [1 - 0] - \frac{1}{2} [1 - 0] - \frac{2}{\pi} [0 - 1]$
- $= \frac{4}{\pi} - 1$

Integration with Respect to y

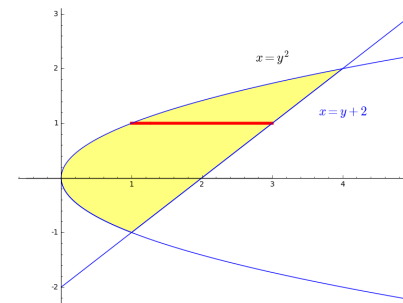
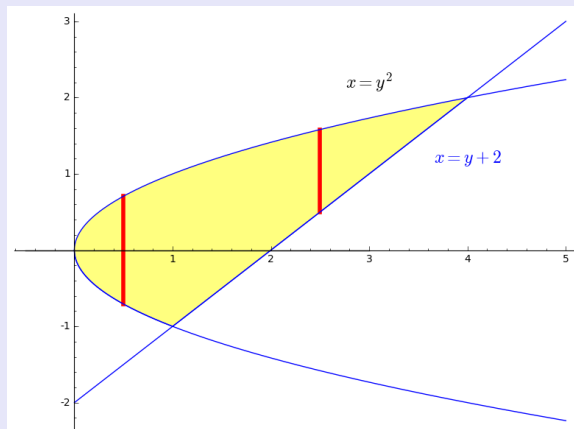
- The area of the region between the curves $x = f(y)$ and $x = g(y)$ from $y = a$ to $y = b$ is



Example 1

Find the area of the region enclosed by $x = y^2$, $x = y + 2$.

Solution

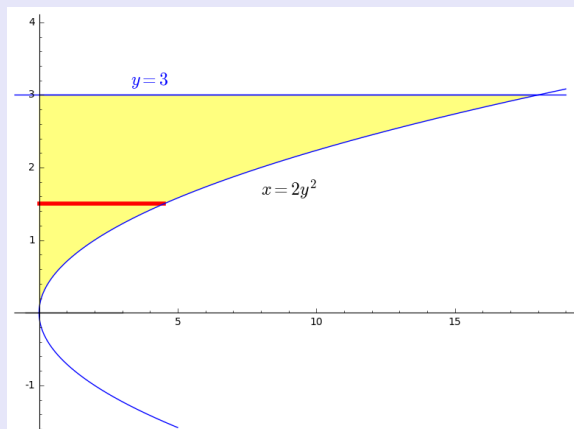


- $\text{Area} = \int_{-1}^2 [y + 2 - y^2] dy$
- $y + 2 = y^2 \implies y^2 - y - 2 = 0 \implies (y + 1)(y - 2) = 0$
- $= \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right]_{-1}^2$
- $= \frac{1}{2}[4 - 1] + 2[2 + 1] - \frac{1}{3}[8 + 1] = \frac{3}{2} + 3 = \frac{9}{2}$

Example 2

Find the area of the region enclosed by the lines $x = 0$, $y = 3$, and the curve $x = 2y^2$.

Solution

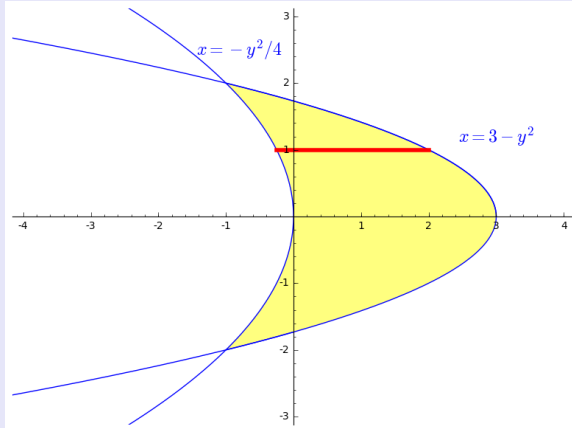


- $\text{Area} = \int_0^3 2y^2 dy$
- $= \left[\frac{2}{3}y^3 \right]_0^3$
- $= \frac{2}{3}[27 - 0] = 18$

Example 3

Find the area of the region enclosed by the curves $x = 3 - y^2$ and $x = -\frac{1}{4}y^2$.

Solution



- $\text{Area} = \int_a^b \left[3 - y^2 + \frac{1}{4}y^2 \right] dy$
- $3 - y^2 = -\frac{1}{4}y^2 \implies 3 = \frac{3}{4}y^2 \implies y^2 = 4 \implies y = \pm 2$
- $\text{Area} = \int_{-2}^2 \left[3 - \frac{3}{4}y^2 \right] dy$
- $= \left[3y - \frac{1}{4}y^3 \right]_{-2}^2$
- $= 3[2 + 2] - \frac{1}{4}[8 + 8] = 8$

تم



مَا اصعب جزء في الموضوع؟

What is the most difficult part of this section?