



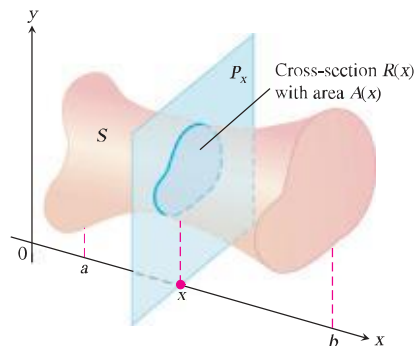
Chapter 6

Applications of Definite Integrals

First Semester 2020-2021
أ. د. أيمن هاشم السقا

6.1 Volumes Using Cross-Sections

- In this section we define volumes of solids using the area of their cross-sections.
- A cross-section of a solid S is the plane region formed by intersecting S with a plane.
- We present two different methods for obtaining the cross-sections appropriate to find the volume of a particular solid: the disk method and the washer method.

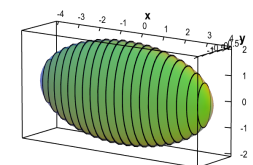


Chapter 6

Applications of Definite Integrals

- In this chapter we extend the applications of definite integrals to finding volumes, lengths of plane curves, and areas of surfaces of revolution.

- Suppose we want to find the volume of a solid S like the one in the figure.



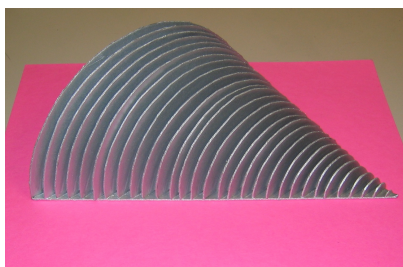
- We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases.
- If the cylindrical solid has a known base area A and height h , then the volume of the cylindrical solid is

$$V = Ah.$$

- This equation forms the basis for defining the volume of many solids that are not cylinders.

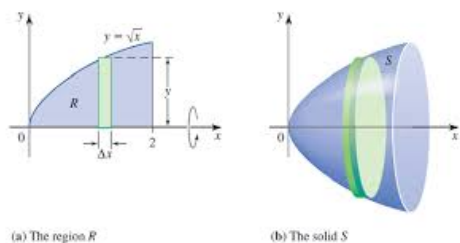


- If we partition the interval $[a, b]$ into subintervals of width Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x -axis at the partition points, slice S into slabs.
- Volume of the k th slab $\approx V_k = A(c_k)\Delta x_k$.



Solid of Revolution: The Disk Method

- The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a solid of revolution.



- To find the volume of a solid of revolution we have to observe that the cross-section area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution.

Volume

- Thus, the volume of the entire solid S is approximated by

$$V \approx \sum_{k=1}^n V_k = \sum_{k=1}^n A(c_k)\Delta x_k.$$

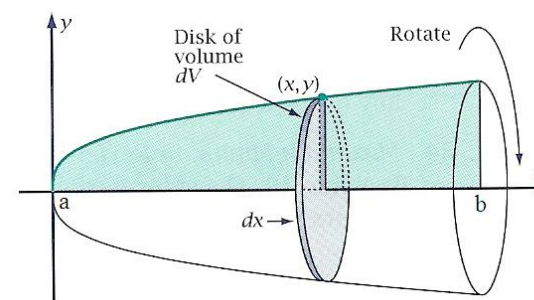
- We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero.

Definition

The volume of a solid of integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is given by

$$V = \int_a^b A(x) dx.$$

Rotation About the x -axis



- The area is then

$$A(x) = \pi R^2(x).$$

Rotation About the x-axis

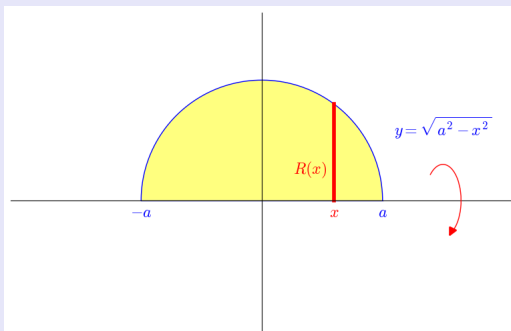
- The volume of the solid generated by revolving a region about the x-axis is

$$V = \pi \int_a^b R^2(x) dx.$$

Example 1

Find the volume of the solid generated by revolving the **region** between the curve $y = \sqrt{a^2 - x^2}$ and the x-axis **about** the x-axis.

Solution



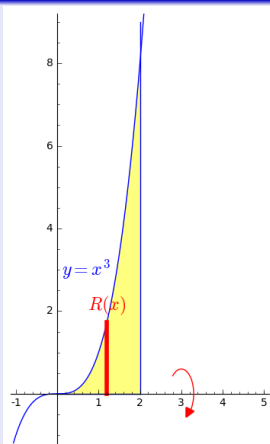
Solution

- The solid is a sphere of radius a .
- $R(x) = \sqrt{a^2 - x^2}$
- $V = \pi \int_{-a}^a R^2(x) dx = \pi \int_{-a}^a (\sqrt{a^2 - x^2})^2 dx$
- $V = \pi \int_{-a}^a (a^2 - x^2) dx = \pi \left[a^2 x - \frac{1}{3} x^3 \right]_{-a}^a = \frac{4\pi a^3}{3}$

Example 2

Find the volume of the solid generated by revolving the **region** bounded by the curves $y = x^3$, $y = 0$, $x = 2$ **about** the x-axis.

Solution

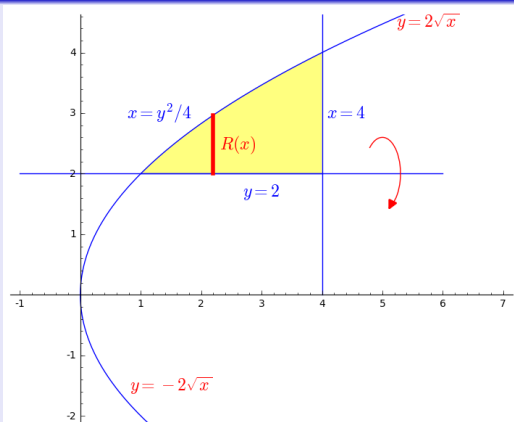


- $R(x) = x^3$
- $V = \pi \int_0^2 R^2(x) dx = \pi \int_0^2 (x^3)^2 dx$
- $V = \pi \int_0^2 x^6 dx = \pi \left[\frac{x^7}{7} \right]_0^2 = \frac{128\pi}{7}$

Example 3

Find the volume of the solid generated by revolving the **region** in the first quadrant bounded by the curve $x = y^2/4$, $y = 2$, $x = 4$ **about** the line $y = 2$.

Solution

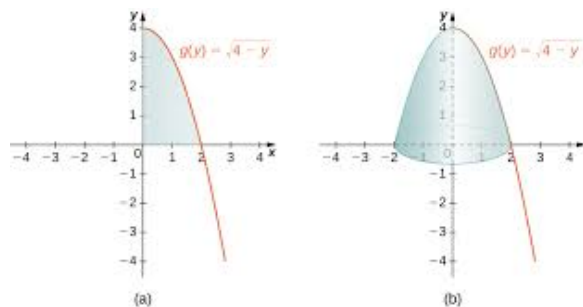


- $R(x) = 2\sqrt{x} - 2$
- $y = 2$ and $x = y^2/4 \implies x = 1$
- $V = \pi \int_1^4 R^2(x) dx = \pi \int_1^4 (2\sqrt{x} - 2)^2 dx$
- $V = 4\pi \int_1^4 (x - 2\sqrt{x} + 1) dx = 4\pi \left[\frac{x^2}{2} - \frac{4}{3}x^{3/2} + x \right]_1^4$
- $V = 4\pi \left[\frac{1}{2}(16 - 1) - \frac{4}{3}(8 - 1) + (4 - 1) \right] = \frac{14\pi}{3}$

Rotation About the y -axis

- The volume of the solid generated by revolving a region about the y -axis is

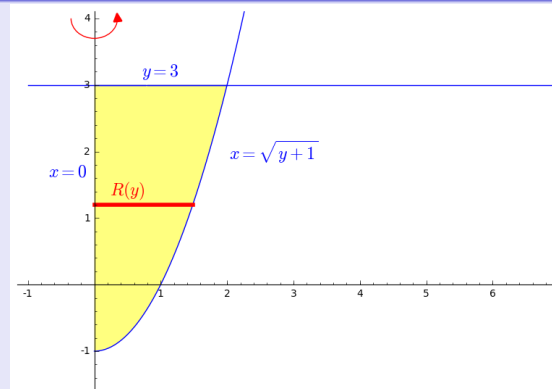
$$V = \pi \int_a^b R^2(y) dy.$$



Example 1

Find the volume of the solid generated by revolving the **region** bounded by $x = \sqrt{y+1}$, $x = 0$, and $y = 3$ **about** the y -axis.

Solution

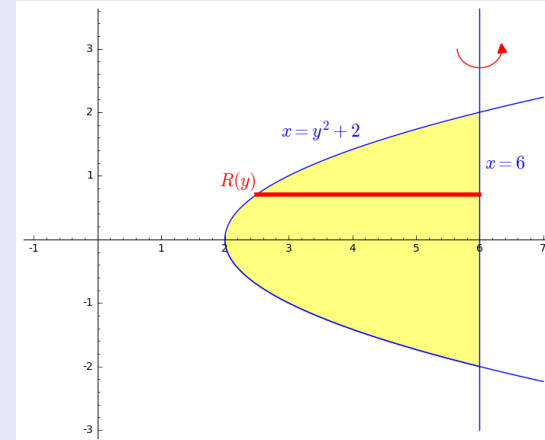


- $R(y) = \sqrt{y+1}$
- From $y = -1$ to $y = 3$
- $V = \pi \int_{-1}^3 R^2(y) dy$
- $V = \pi \int_{-1}^3 (\sqrt{y+1})^2 dy = \pi \int_{-1}^3 (y+1) dy$
- $V = \pi \left[\frac{1}{2}y^2 + y \right]_{-1}^3$
- $V = \pi \left[\frac{1}{2}(9-1) + (3+1) \right] = 8\pi$

Example 2

Find the volume of the solid generated by revolving the **region** between the curve $x = y^2 + 2$ and the line $x = 6$ **about** the line $x = 6$.

Solution



- $R(y) = 6 - (y^2 + 2) = 4 - y^2$
- $x = 6$ and $x = y^2 + 2 \implies y^2 = 4 \implies y = \pm 2$
- $V = \pi \int_{-2}^2 R^2(y) dy$
- $V = \pi \int_{-2}^2 (4 - y^2)^2 dy = \pi \int_{-2}^2 (16 - 8y^2 + y^4) dy$
- $V = \pi \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5 \right]_{-2}^2$
- $V = \pi \left[16(2+2) - \frac{8}{3}(8+8) + \frac{1}{5}(32+32) \right]$
- $V = \pi \left[64 - \frac{(16)(8)}{3} + \frac{64}{5} \right] = 64\pi \left[1 - \frac{2}{3} + \frac{1}{5} \right]$
- $V = 64\pi \left[\frac{5+3}{15} \right] = \frac{512\pi}{15}$

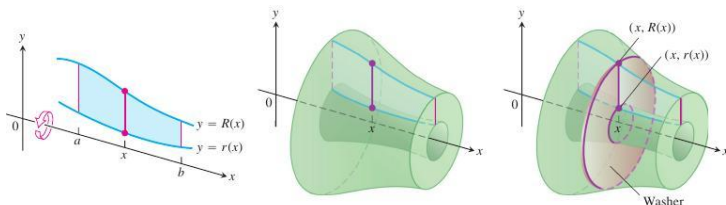


6.1 Volumes Using Cross-Sections (part 2)

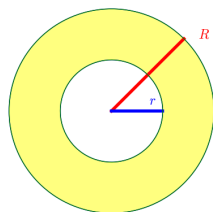
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Solid of Revolution: The Washer Method

- If the region we revolved does not border on or cross the axis of revolution, then the solid has a hole in it.



- In this case the cross-sections are washer instead of disks.

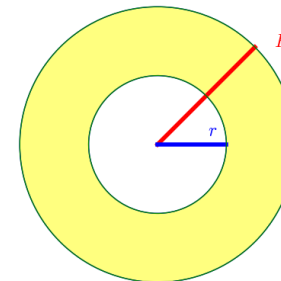


Volume by Washer for Rotation About the x-axis

- The volume of the solid obtained by revolving the region between the curves $y = f(x)$ and $y = g(x)$ over $[a, b]$ about the x-axis or a line parallel to it is given by

$$V = \int_a^b \pi [R^2(x) - r^2(x)] dx.$$

Solid of Revolution: The Washer Method



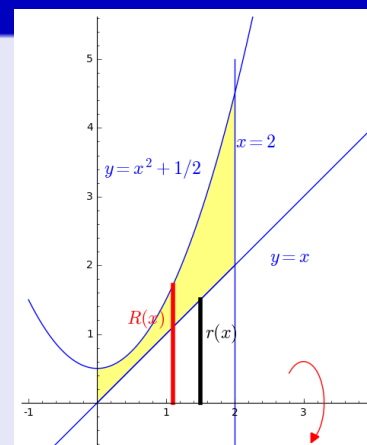
- The dimensions of a typical washer are
- Outer radius: $R(x)$
- Inner radius: $r(x)$
- The washer's area is

$$A(x) = \pi [R^2(x) - r^2(x)].$$

Example 1

Find the **volume** of the solid generated by revolving the **region** bounded by $y = x^2 + \frac{1}{2}$, $y = x$, $x = 0$ and $x = 2$ **about** the x-axis.

Solution



- $V = \pi \int_a^b [R^2(x) - r^2(x)] dx$
- $R(x) = x^2 + \frac{1}{2}$, $r(x) = x$

- $a = 0$ and $b = 2$

- $V = \pi \int_0^2 \left[\left(x^2 + \frac{1}{2} \right)^2 - x^2 \right] dx = \pi \int_0^2 \left[x^4 + \frac{1}{4} \right] dx$

- $V = \pi \left[\frac{1}{5}x^5 + \frac{1}{4}x \right]_0^2$

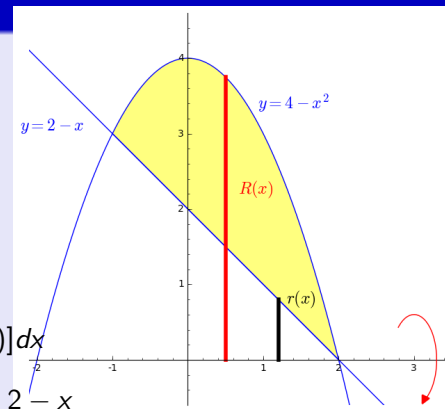
- $V = \pi \left[\frac{32}{5} + \frac{1}{2} \right]$

- $V = \frac{69\pi}{10}$

Example 2

Find the volume of the solid generated by revolving the **region** enclosed by $y = 4 - x^2$ and $y = 2 - x$ **about** the x -axis.

Solution



- $V = \pi \int_a^b [R^2(x) - r^2(x)] dx$

- $R(x) = 4 - x^2, \quad r(x) = 2 - x$

- $y = 4 - x^2$ and $y = 2 - x \implies 4 - x^2 = 2 - x \implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0$

- $a = -1$ and $b = 2$

- $V = \pi \int_{-1}^2 [(4 - x^2)^2 - (2 - x)^2] dx$
 $= \pi \int_{-1}^2 [16 - 8x^2 + x^4 - (4 - 4x + x^2)] dx$

- $V = \pi \int_{-1}^2 [12 + 4x - 9x^2 + x^4] dx$

- $V = \pi \left[12x + 2x^2 - 3x^3 + \frac{1}{5}x^5 \right]_{-1}^2$

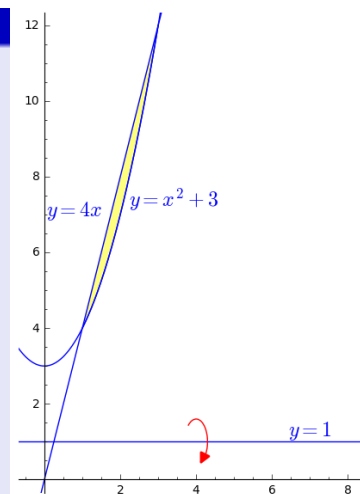
- $V = \pi \left[12(2 + 1) + 2(4 - 1) - 3(8 + 1) + \frac{1}{5}(32 + 1) \right]$

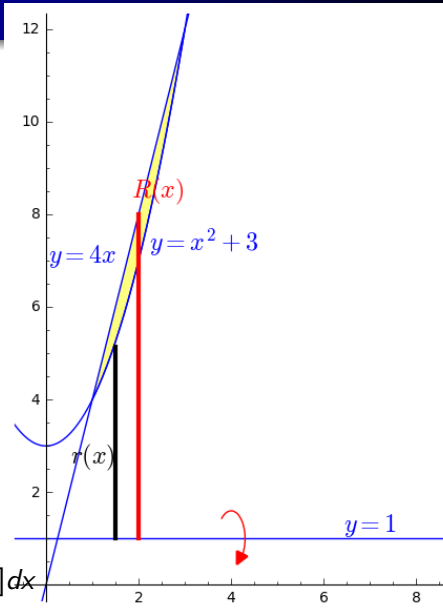
- $V = \pi \left[15 + \frac{33}{5} \right] \implies V = \frac{108\pi}{5}$

Example 3

Find the volume of the solid generated by revolving the **region** bounded by $y = x^2 + 3$ and $y = 4x$ **about** the line $y = 1$.

Solution





- $V = \pi \int_a^b [R^2(x) - r^2(x)] dx$
- $R(x) = 4x - 1, \quad r(x) = x^2 + 3 - 1 = x^2 + 2$

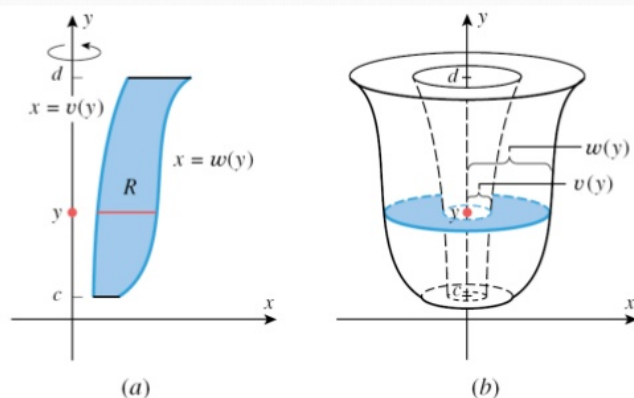
- $y = 4x$ and $y = x^2 + 3 \implies x^2 + 3 = 4x \implies x^2 - 4x + 3 = 0 \implies (x - 3)(x - 1) = 0$
- $a = 1$ and $b = 3$
- $V = \pi \int_1^3 [(4x - 1)^2 - (x^2 + 2)^2] dx$
 $= \pi \int_0^2 [16x^2 - 8x + 1 - (x^4 + 4x^2 + 4)] dx$
- $V = \pi \int_1^3 [-x^4 + 12x^2 - 8x - 3] dx$
- $V = \pi \left[-\frac{1}{5}x^5 + 4x^3 - 4x^2 - 3x \right]_1^3$
- $V = \pi \left[-\frac{1}{5}(243 - 1) + 4(27 - 1) - 4(9 - 1) - 3(3 - 1) \right]$
- $V = \pi \left[-\frac{242}{5} + 66 \right] = 2\pi \left[\frac{165 - 121}{5} \right] \implies V = \frac{88\pi}{5}$

Volume by Washer for Rotation About the y-axis

Volume by Washer for Rotation About the y-axis

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2) dy$$

Washers



Washers

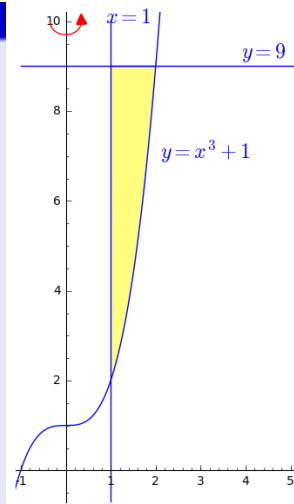
- The volume of the solid obtained by revolving the region between the curves $x = f(y)$ and $x = g(y)$ over $[a, b]$ about the y-axis or a line parallel to it is given by

$$V = \int_a^b \pi[R^2(y) - r^2(y)] dy.$$

Example 1

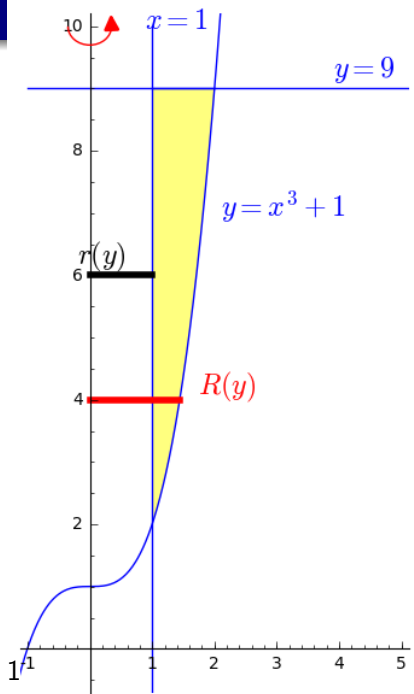
Find the volume of the solid generated by revolving the **region** bounded by $y = x^3 + 1$, $x = 1$ and $y = 9$ **about** the y -axis.

Solution



- $V = \pi \int_a^b [R^2(y) - r^2(y)] dy$

- $R(y) = (y - 1)^{1/3}$, $r(y) = 1$

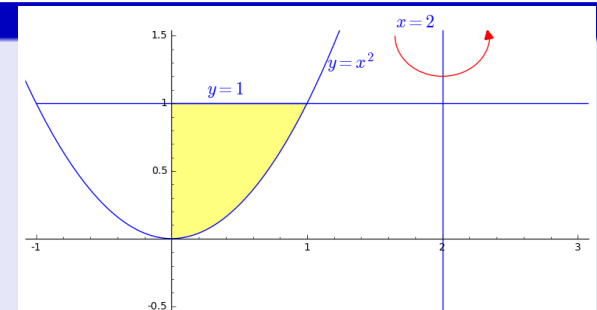


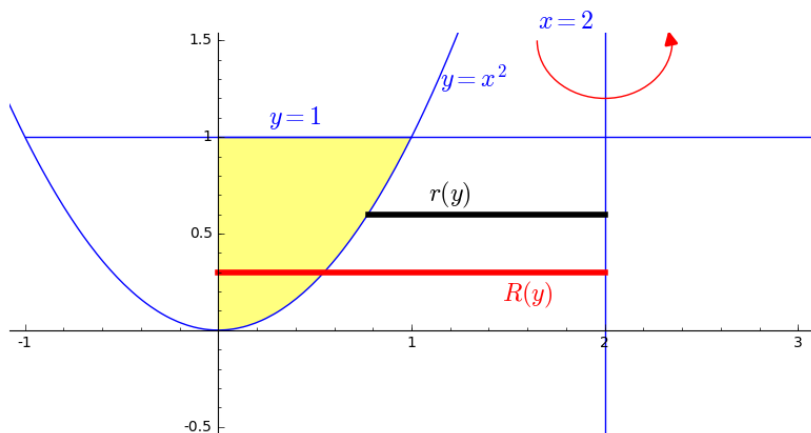
- $x = 1 \implies y = 2$
- $a = 2$ and $b = 9$
- $V = \pi \int_2^9 [(y - 1)^{2/3} - 1] dy$
 $= \pi \left[\frac{3}{5}(y - 1)^{5/3} - y \right]_2^9$
- $V = \pi \left[\frac{3}{5}(32 - 1) - (9 - 2) \right]$
- $V = \pi \left[\frac{93}{5} - 7 \right]$
- $\implies V = \frac{58\pi}{5}$

Example 2

The **region** in the first quadrant enclosed by the parabola $y = x^2$, the y -axis, and the line $y = 1$ is revolved **about** the line $x = 2$ to generate a solid. Find the volume of the solid.

Solution





- $V = \pi \int_a^b [R^2(y) - r^2(y)] dy$
- $R(y) = 2, \quad r(y) = 2 - \sqrt{y}$

- $a = 0$ and $b = 1$

- $V = \pi \int_0^1 [4 - (2 - \sqrt{y})^2] dy$

- $V = \pi \int_0^1 [4 - (4 - 4\sqrt{y} + y)] dy = \pi \int_0^1 [4\sqrt{y} - y] dy$

- $= \pi \left[\frac{8}{3} y^{3/2} - \frac{1}{2} y^2 \right]_0^1$

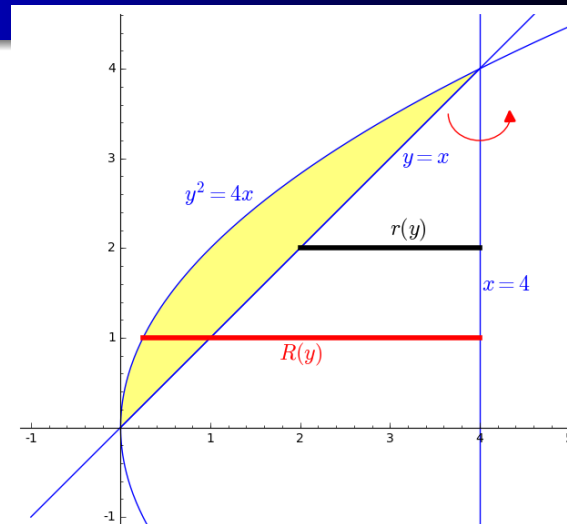
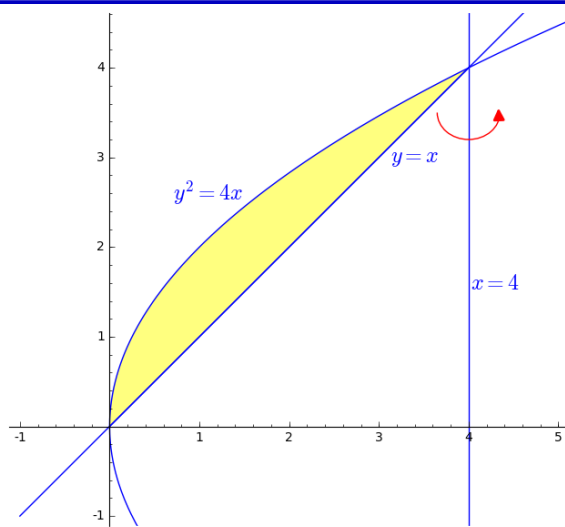
- $V = \pi \left[\frac{8}{3} - \frac{1}{2} \right]$

- $\Rightarrow V = \frac{13\pi}{6}$

Example 3 (Exam)

Find the volume of the solid generated by revolving the region bounded by the curves $y^2 = 4x$ and $y = x$ about the line $x = 4$.

Solution



- $V = \pi \int_a^b [R^2(y) - r^2(y)] dy$
- $R(y) = 4 - \frac{y^2}{4}, \quad r(y) = 4 - y$

- $x = y^2/4$ and
 $x = y \implies y = y^2/4 \implies y^2 - 4y = 0 \implies y(y - 4) = 0$
- $a = 0$ and $b = 4$
- $V = \pi \int_0^4 \left[\left(4 - \frac{y^2}{4}\right)^2 - (4 - y)^2 \right] dy$
- $V = \pi \int_0^4 \left[16 - 2y^2 + \frac{y^4}{16} - (16 - 8y + y^2) \right] dy$
 $= \pi \int_0^4 \left[8y - 3y^2 + \frac{y^4}{16} \right] dy$
- $= \pi \left[4y^2 - y^3 + \frac{1}{80}y^5 \right]_0^4$
- $V = \pi \left[4(16) - 64 + \frac{(4)^5}{80} \right]$
- $\implies V = \frac{64\pi}{5}$



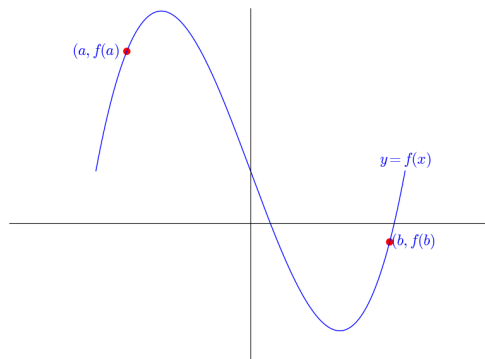
6.3 Arc Length

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6.3 Arc Length

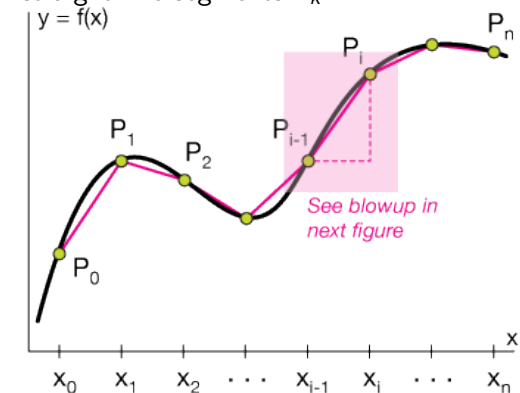
- Suppose that we want to find the length of a curve
 $y = f(x)$ from $x = a$
to $x = b$.



- In order to derive an integral formula for the length of the curve, we assume that f has a continuous derivative at every point of $[a, b]$.
- We partition the interval into n subintervals with

$$a = x_1 < x_2 < \dots < x_n = b.$$

- Next we connect successive points $P_{k-1} = (x_{k-1}, f(x_{k-1}))$ and $P_k = (x_k, f(x_k))$ with straight line segments L_k .



- Each line segment L_k has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

- By mean value theorem, $\exists c_k \in (x_{k-1}, x_k)$ such that

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}.$$

- Thus

$$L_k = \sqrt{1 + [f'(c_k)]^2} \Delta x_k$$

and hence the length of the curve L is approximated by

$$L \approx \sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x_k.$$

Arc length

- We expect the approximation to improve as the partition become finer. As a result we obtain the length of the curve is given by

$$L = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{1 + [f'(c_k)]^2} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Definition

If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx.$$

Examples

Example 1

Find the length of the curve $y = (x+1)^{3/2}$ from $x = -1$ to $x = 3$.

Solution

- We have $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$
- In this example $f(x) = (x+1)^{3/2} \Rightarrow f'(x) = \frac{3}{2}(x+1)^{1/2}$
- Thus $L = \int_{-1}^3 \sqrt{1 + \left[\frac{3}{2}(x+1)^{1/2}\right]^2} dx$
- $L = \int_{-1}^3 \sqrt{1 + \frac{9}{4}(x+1)} dx = \int_{-1}^3 \sqrt{\frac{9}{4}x + \frac{13}{4}} dx$
- Let $u = \frac{9}{4}x + \frac{13}{4} \Rightarrow du = \frac{9}{4}dx$

- $L = \int_{u=1}^{u=10} \sqrt{u} \frac{4}{9} du = \frac{4}{9} \left(\frac{2}{3} u^{3/2} \right) \Big|_{u=1}^{u=10}$
- $\Rightarrow L = \frac{8}{27} (10\sqrt{10} - 1)$

Example 2

Find the length of the curve $y = \frac{1}{3}x^3 + \frac{1}{4x}$ from $x = 1$ to $x = 2$.

Solution

- We have $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$
- In this example $f(x) = \frac{1}{3}x^3 + \frac{1}{4x} \Rightarrow f'(x) = x^2 - \frac{1}{4x^2}$
- Thus $L = \int_1^2 \sqrt{1 + \left[x^2 - \frac{1}{4x^2}\right]^2} dx$
- $$L = \int_1^2 \sqrt{1 + \left[x^4 - \frac{1}{2} + \frac{1}{16x^4}\right]} dx = \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16x^4}} dx$$
- $$L = \int_1^2 \sqrt{\left[x^2 + \frac{1}{4x^2}\right]^2} dx = \int_1^2 \left[x^2 + \frac{1}{4x^2}\right] dx$$

- $$L = \int_1^2 \left[x^2 + \frac{1}{4x^2}\right] dx$$
- $$L = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^2$$
- $$L = \frac{1}{3}(8 - 1) - \frac{1}{4}\left(\frac{1}{2} - 1\right) = \frac{7}{3} + \frac{1}{8}$$
- $$\Rightarrow L = \frac{59}{24}$$

Dealing with discontinuities in dy/dx

- At a point on a curve where $\frac{dy}{dx}$ fails to exist, $\frac{dx}{dy}$ may exist and we may be able to find the length of the curve by expressing x as a function of y .

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If $g'(y)$ is continuous on $[c, d]$, then the length of the curve $x = g(y)$ from the point $(g(c), c)$ to the point $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2} dy.$$

Example 1

Find the length of the curve $y = (x/4)^{2/3}$ from $x = 0$ to $x = 4$.

Solution

- $y = (x/4)^{2/3} = f(x) \Rightarrow f'(x) = \frac{dy}{dx} = \frac{1}{6}(x/4)^{-1/3}$
- $\frac{dy}{dx}$ is not defined at $x = 0$, so we can not use the formula
$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$
- Rewrite the equation as $x = 4y^{3/2} = g(y)$.
- Thus $g'(y) = \frac{dx}{dy} = 6y^{1/2}$ which is continuous on $[0, 1]$
- Therefore, we can use the formula $L = \int_c^d \sqrt{1 + [g'(y)]^2} dy$
- $x = 0 \Rightarrow y = 0$, $x = 4 \Rightarrow y = 1$

- It follows that

$$L = \int_0^1 \sqrt{1 + [g'(y)]^2} dy = \int_0^1 \sqrt{1 + [6y^{1/2}]^2} dy$$

- $L = \int_0^1 \sqrt{1 + 36y} dy$
- Let $u = 1 + 36y \implies du = 36dy$

$$\implies L = \frac{1}{36} \int_{u=1}^{u=37} \sqrt{u} du$$

$$L = \frac{1}{36} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=37}$$

$$\implies L = \frac{1}{54} (37\sqrt{37} - 1)$$

Example 2

Find the length of the curve $x = \frac{y^3}{6} + \frac{1}{2y}$ from $y = 2$ to $y = 3$.

Solution

- In this example $x = g(y) = \frac{y^3}{6} + \frac{1}{2y}$.
- Thus $g'(y) = \frac{y^2}{2} - \frac{1}{2y^2}$ which is continuous on $[2, 3]$
- Therefore, we can use the formula $L = \int_c^d \sqrt{1 + [g'(y)]^2} dy$
- It follows that $L = \int_2^3 \sqrt{1 + \left[\frac{y^2}{2} - \frac{1}{2y^2} \right]^2} dy$
- $L = \int_2^3 \sqrt{1 + \left[\frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4} \right]} dy$

$$L = \int_2^3 \sqrt{1 + \left[\frac{y^4}{4} - \frac{1}{2} + \frac{1}{4y^4} \right]} dy = \int_2^3 \sqrt{\left[\frac{y^4}{4} + \frac{1}{2} + \frac{1}{4y^4} \right]} dy$$

$$L = \int_2^3 \sqrt{\left[\frac{y^2}{2} + \frac{1}{2y^2} \right]^2} dy$$

$$L = \int_2^3 \left[\frac{y^2}{2} + \frac{1}{2y^2} \right] dy$$

$$L = \left[\frac{y^3}{6} - \frac{1}{2y} \right]_2^3$$

$$L = \frac{1}{6} (27 - 8) - \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} \right) = \frac{19}{6} + \frac{1}{12} = \frac{39}{12}$$

$$\implies L = \frac{13}{4}$$



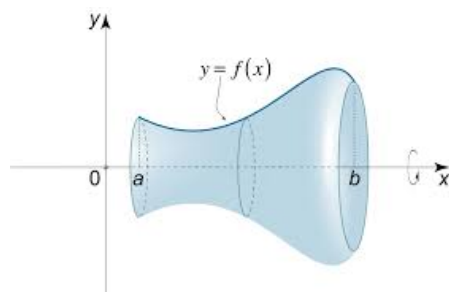
6.4 Areas of Surfaces of Revolution

First Semester 2020-2021

أ.د. أيمن هاشم السقا

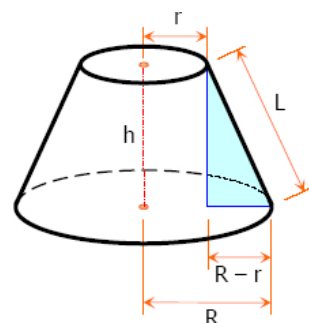
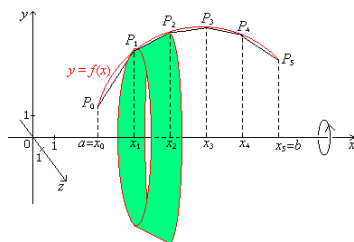
6.4 Areas of Surfaces of Revolution

- In this section we study how to find the surface area of a solid of revolution.



- Each line segment L_k sweeps out a frustum of a cone whose surface area is given by

$$\begin{aligned}\Delta S_k &= \pi[f(x_k) + f(x_{k-1})]\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi[f(x_k) + f(x_{k-1})]\sqrt{1 + f'(c_k)^2}\Delta x_k.\end{aligned}$$



Frustum of a Right Circular Cone

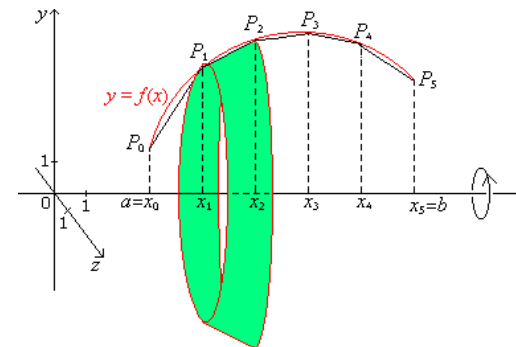
Figure: $A = \pi(r + R)L$

Revolution About the x -Axis

- Suppose that we want to find the area of the surface of a solid of revolution.
- We partition the interval into n subintervals with

$$a = x_1 < x_2 < \cdots < x_n = b$$

and we connect successive points $P_{k-1} = (x_{k-1}, f(x_{k-1}))$ and $P_k = (x_k, f(x_k))$ with straight line segments L_k .



- The area of the original surface is approximated by the sum of these areas:

$$S \approx \sum_{k=1}^n \pi[f(x_k) + f(x_{k-1})]\sqrt{1 + f'(c_k)^2}\Delta x_k.$$

- These sums are not the Riemann sums of any function since the points x_{k-1} , x_k , and c_k are not the same.
- However, it can be proved that

$$\begin{aligned}S &= \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \pi[f(x_k) + f(x_{k-1})]\sqrt{1 + f'(c_k)^2}\Delta x_k \\ &= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} dx.\end{aligned}$$

Surface area

Definition

If $f(x) \geq 0$ is continuously differentiable on $[a, b]$, then the area of the surface generated by revolving the graph of $y = f(x)$ about the x -axis is

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx = 2\pi \int_a^b y \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx.$$

- $S = 4\pi \int_2^7 \sqrt{x+2} dx$
- Let $u = x + 2 \implies du = dx$
- $S = 4\pi \int_{u=4}^{u=9} \sqrt{u} du = 4\pi \left[\frac{2}{3} u^{3/2} \right]_{u=4}^{u=9}$
- $S = \frac{8\pi}{3} [27 - 8]$
- $S = \frac{152\pi}{3}$

Examples

Example 1

Find the area of surface generated by revolving the curve $y = 2\sqrt{x+1}$, $2 \leq x \leq 7$ about the x -axis.

Solution

- $S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx = 2\pi \int_a^b y \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx.$
- $f(x) = 2\sqrt{x+1} \implies f'(x) = \frac{1}{\sqrt{x+1}}$
- $S = 2\pi \int_2^7 2\sqrt{x+1} \sqrt{1 + \left[\frac{1}{\sqrt{x+1}}\right]^2} dx$
- $S = 4\pi \int_2^7 \sqrt{x+1} \sqrt{1 + \frac{1}{x+1}} dx$
- $S = 4\pi \int_2^7 \sqrt{x+1} \sqrt{\frac{x+2}{x+1}} dx = 4\pi \int_2^7 \sqrt{x+2} dx$

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Examples

Example 2

Find the area of surface generated by revolving the curve $y = x^3/9$, $0 \leq x \leq 2$ about the x -axis.

Solution

- $S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx.$
- $f(x) = x^3/9 \implies f'(x) = \frac{x^2}{3}$
- $S = 2\pi \int_0^2 \frac{x^3}{9} \sqrt{1 + \left[\frac{x^2}{3}\right]^2} dx = \frac{2\pi}{9} \int_0^2 x^3 \sqrt{1 + \left[\frac{x^4}{9}\right]} dx$
- $S = \frac{2\pi}{27} \int_0^2 x^3 \sqrt{x^4 + 9} dx$
- Let $u = x^4 + 9 \implies du = 4x^3 dx$

$$u = x^4 + 9 \implies du = 4x^3 dx$$

- $S = \frac{2\pi}{27} \int_0^2 x^3 \sqrt{x^4 + 9} dx$
- $S = \frac{2\pi}{27} \int_{u=9}^{u=25} \frac{1}{4} \sqrt{u} du = \frac{\pi}{54} \left[\frac{2}{3} u^{3/2} \right]_{u=9}^{u=25}$
- $S = \frac{\pi}{81} [125 - 27]$
- $S = \frac{98\pi}{81}$

Revolution About the y -Axis

- If $f(y) \geq 0$ is continuously differentiable on $[a, b]$, then the area of the surface generated by revolving the graph of $x = f(y)$ about the y -axis is

$$S = 2\pi \int_a^b f(y) \sqrt{1 + [f'(y)]^2} dy$$

$$= 2\pi \int_a^b x \sqrt{1 + \left[\frac{dx}{dy} \right]^2} dy.$$

Examples

Example 1

Find the area of surface generated by revolving the curve $x = 1 - y$, $0 \leq y \leq 1$ about the y -axis.

Solution

- $S = 2\pi \int_a^b f(y) \sqrt{1 + [f'(y)]^2} dy.$
- $f(y) = 1 - y \implies f'(y) = -1$
- $S = 2\pi \int_0^1 (1 - y) \sqrt{1 + 1} dy = 2\sqrt{2} \pi \int_0^1 (1 - y) dy$
- $S = 2\sqrt{2} \pi \left(y - \frac{y^2}{2} \right) \Big|_0^1$
- $S = 2\sqrt{2} \pi \left[1 - \frac{1}{2} \right] \implies S = \sqrt{2} \pi$

Examples

Example 2

Find the area of surface generated by revolving the curve $x = \frac{1}{3}y^{3/2} - y^{1/2}$, $3 \leq y \leq 4$ about the y -axis.

Solution

- $S = 2\pi \int_a^b f(y) \sqrt{1 + [f'(y)]^2} dy.$
- $f(y) = \frac{1}{3}y^{3/2} - y^{1/2} \implies f'(y) = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$
- $S = 2\pi \int_3^4 \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) \sqrt{1 + \left[\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \right]^2} dy$
- $S = 2\pi \int_3^4 \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) \sqrt{1 + \left[\frac{1}{4}y - \frac{1}{2} + \frac{1}{4y} \right]} dy$
- $S = 2\pi \int_3^4 \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) \sqrt{\left[\frac{1}{4}y + \frac{1}{2} + \frac{1}{4y} \right]} dy$

- $S = 2\pi \int_3^4 \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) \sqrt{\left[\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \right]^2} dy$
- $S = 2\pi \int_3^4 \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \right) dy$
- $S = \pi \int_3^4 \left(\frac{1}{3}y^{3/2} - y^{1/2} \right) (y^{1/2} + y^{-1/2}) dy$
- $S = \pi \int_3^4 \left(\frac{1}{3}y^2 + \frac{1}{3}y - y - 1 \right) dy = \pi \int_3^4 \left(\frac{1}{3}y^2 - \frac{2}{3}y - 1 \right) dy$
- $S = \pi \left(\frac{1}{9}y^3 - \frac{1}{3}y^2 - y \right) \Big|_3^4$
- $S = \pi \left(\frac{1}{9}[64 - 27] - \frac{1}{3}[16 - 9] - [4 - 3] \right) = \frac{\pi(37 - 21 - 9)}{9}$
- $S = \frac{7\pi}{9}$

تم



مَا أَصْعَبُ جُزْءٍ فِي الْمَوْضُوعِ؟
What is the most difficult part of this section?