



بسم الله الرحمن الرحيم

تلخيص مادة:

تفاضل وتكامل أ

Calculus A (Math A1401)
Ch1-ch4

مدرس المساق:

أ. اسماعيل الأسطل

تم الرفع بواسطة

فريق مدونة قسم الرياضيات

<http://iugmath.blogspot.com>

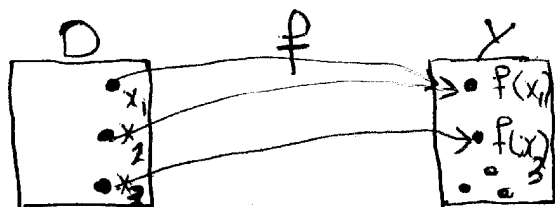
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ديسمبر 2013م

Functions and their graphs.

Def. A fn f from a set D to a set Y ($f: D \rightarrow Y$) is a rule that assigns each element x in D , a unique element $f(x)$ in Y .

The set D called ~~the~~ the domain of f , Y is called the codomain of f and the ~~case~~ collection of all $f(x)$ is called the range of f .



$$f(x_1) = f(x_2)$$

$$\text{Dom}(f) = \{x_1, x_2, x_3\}$$

$$\text{Rng}(f) = \{f(x_1), f(x_2)\}$$

$$\therefore \text{Rng} \subseteq Y$$

Example: Find the domain and range of the following fns

f	$\text{Dom}(f)$	$\text{Rng}(f)$
$f(x) = x^2 + 3$	R or $(-\infty, +\infty)$	$[3, +\infty)$
$f(x) = \sqrt{x} - 3$	$[0, +\infty)$	$[-3, +\infty)$
$f(x) = \sqrt{3-x}$	$(-\infty, 3]$	$[0, \infty)$
$*f(x) = \sqrt{16-x^2}$	$[-4, 4]$	$[0, 4]$
$*f(x) = \frac{100}{x-3}$	$R - \{3\}$	$R - \{0\}$

Following :-

f_n	Dom	Rng
[6] $f(x) = x + x $	$\mathbb{R} / (-\infty, \infty)$	$[0, \infty)$
[7] $f(x) = \sqrt{x^2 - 3x}$	$(-\infty, 0] \cup [3, \infty)$	$[0, \infty)$
[8] $f(x) = x $	$\mathbb{R} / (-\infty, \infty)$	$[0, \infty)$
[9] $f(x) = \sqrt{ x }$	$\mathbb{R} / (-\infty, \infty)$	$[0, \infty)$
[10] $f(x) = \frac{ x }{x} \text{ or } \frac{x}{ x }$	$\mathbb{R} - \{0\}$	$\{-1, 1\}$

$$[6] \quad x + |x| = \begin{cases} x + x & \text{if } x \geq 0 \\ x - x & \text{if } x < 0 \end{cases} \Rightarrow \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$[7] \quad x^2 - 3x \geq 0 \quad x(x-3) \geq 0 \quad \therefore x \geq 0$$

$$\text{Sign } x(x-3) = \begin{matrix} + & - & + \\ x & 0 & 3 & x \end{matrix}$$

$$\text{Sign } x(x-3) = \begin{matrix} - & + & - & + \\ x & 0 & 3 & x \end{matrix}$$

$$\text{Sign } f(x) = \begin{matrix} + & - & + \\ x & 0 & 3 & x \end{matrix}$$

$$[10] \quad \frac{|x|}{x} = \begin{cases} \frac{x}{x} & \text{if } x \geq 0 \\ \frac{-x}{x} & \text{if } x < 0 \end{cases} \Rightarrow \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Graphs of functions:-

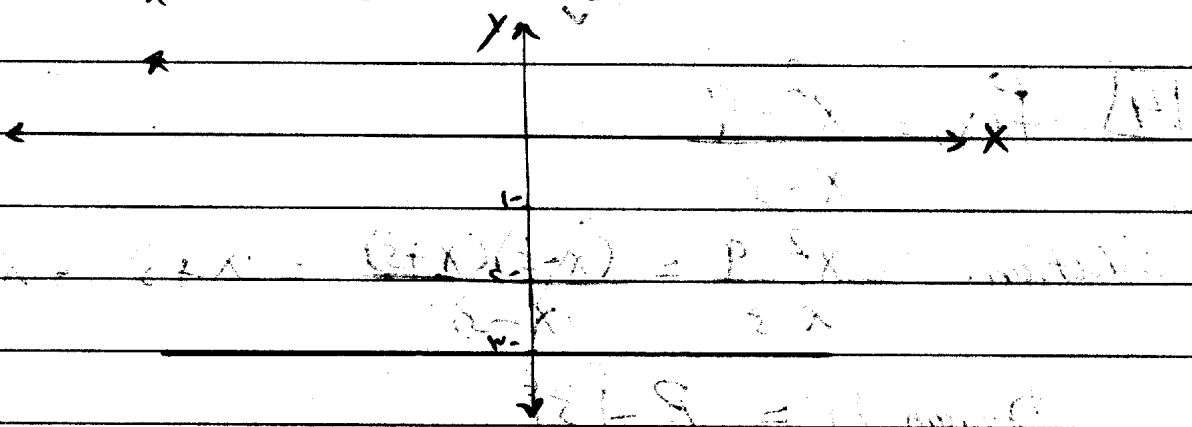
Def. The graph of $f: D \rightarrow \mathbb{R}$ with domain D is the set of points $\{(x, f(x)) \mid x \in D\}$.

Example: Find the domain and then graph the ~~func~~ followi

[1] $f(x) = -3$

Solution:-

$\text{Dom}(f) = (-\infty, \infty)$

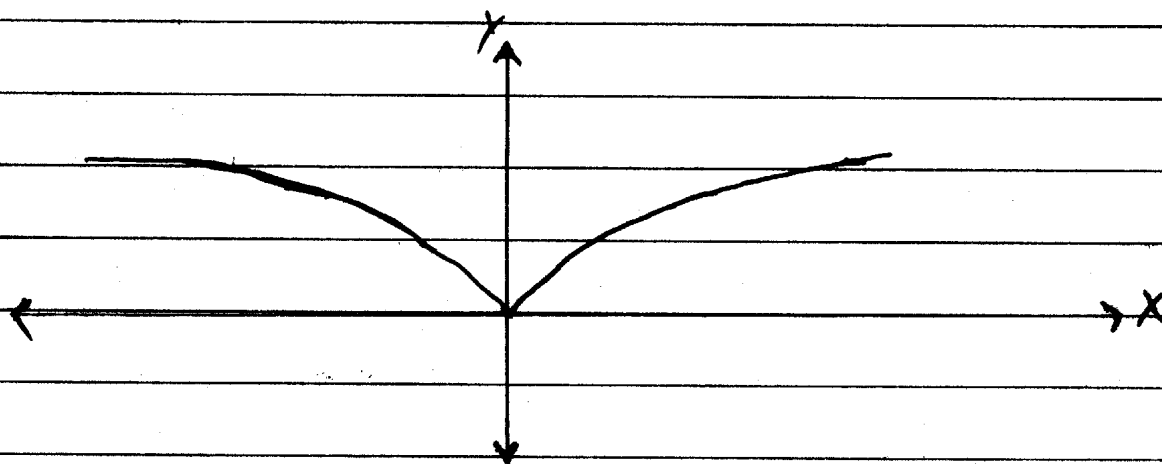


[2] $f(x) = \sqrt{|x|}$

Solution:-

$$\sqrt{|x|} = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$

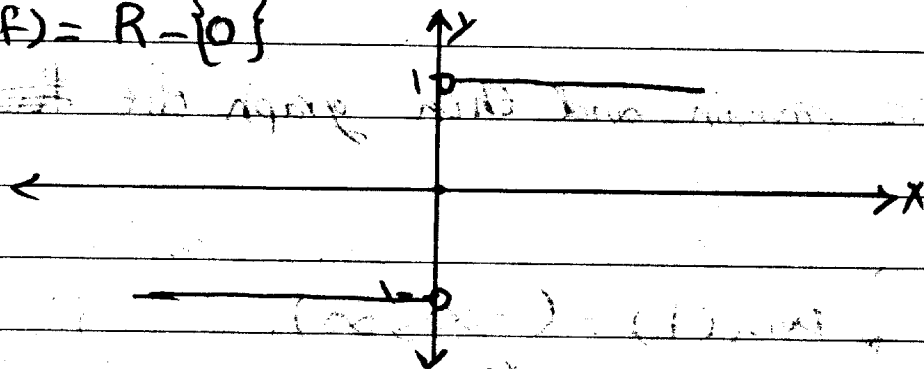
$\text{Dom}(f) = (-\infty, \infty)$



$$[3] f(x) = \frac{x}{|x|}$$

Solution: $\frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

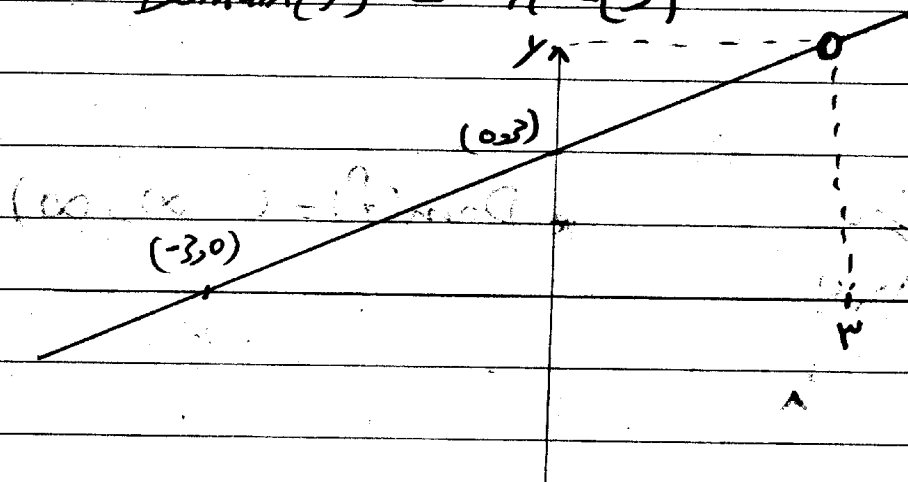
$$\text{Domain}(f) = \mathbb{R} - \{0\}$$



$$[4] f(x) = \frac{x^2 - 9}{x - 3}$$

Solution: $\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{(x-3)} = x+3, x \neq 3$

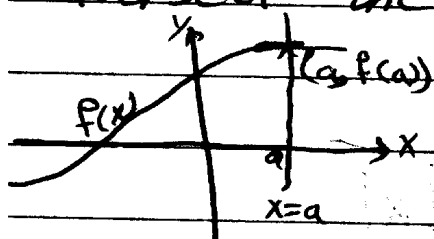
$$\text{Domain}(f) = \mathbb{R} - \{3\}$$



The Vertical Line Test For a Function.

If f is a function then no vertical line can intersect the graph of f more than once.

If $a \in \text{Domain}(f)$ then the vertical line $x=a$ will intersect the graph of f at the single point $(a, f(a))$.

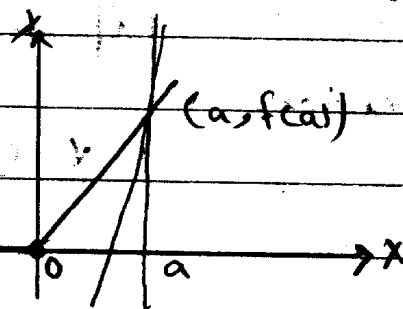


Example: Describe the graph of the following equations and determine whether the graph is that of a fn or not.

[1] $f(x) = x + |x|$

Solution: $x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

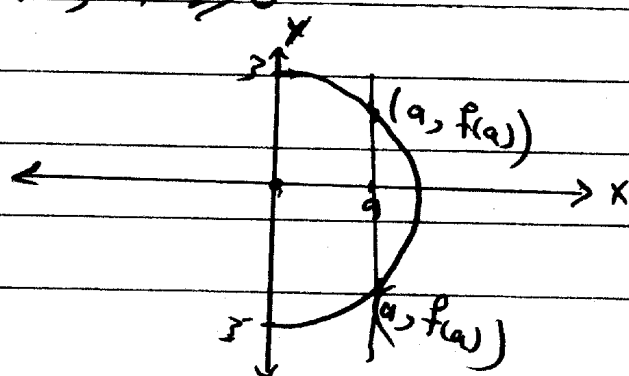
The equation is the graph of a fn. it satisfies the vertical line test.



[2] $f(x) = \sqrt{x^2 + y^2} = 9, x \geq 0$

Solution:

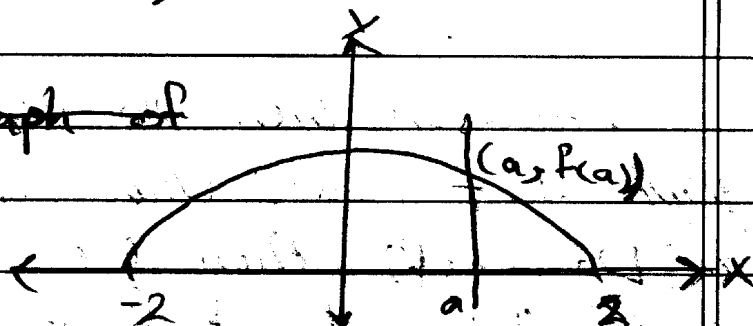
The circle isn't the graph of function. it fails the vertical line test.



[3] $f(x) = x^2 + y^2 = 4, y \geq 0$

Solution:-

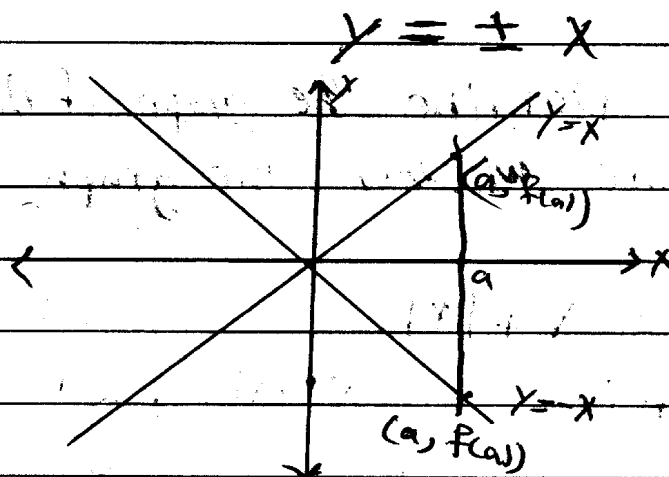
The equation is the graph of a function. it passes the vertical line test.



[4] $f(x) = y^2 = x^2$

Solution:- $\sqrt{y^2} = \sqrt{x^2} \Rightarrow |y| = |x|$

The equation isn't a function. it fails the vertical line test.



[5] $f(x) = |x| + |y| = 1$

Solution:-

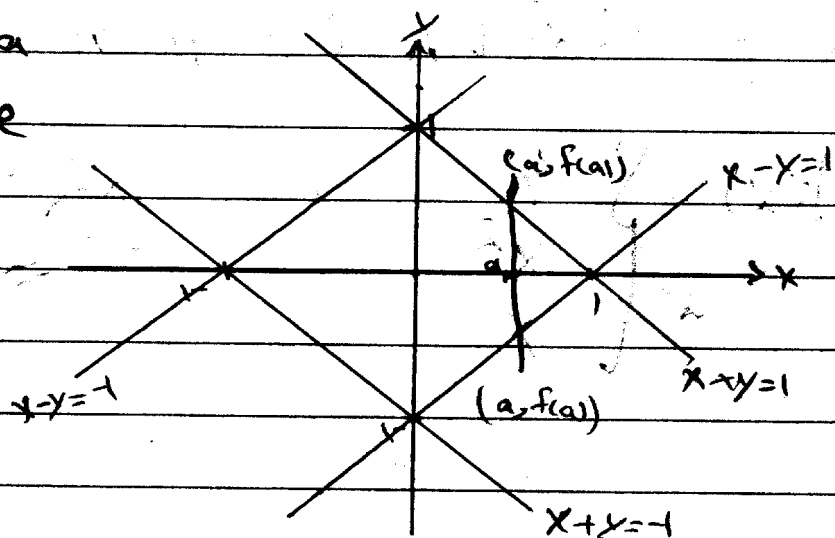
$$\pm x \pm y = 1 \Rightarrow x + y = 1$$

$$x - y = 1$$

$$-x + y = 1 \Rightarrow x - y = -1$$

$$-x - y = -1 \Rightarrow x + y = 1$$

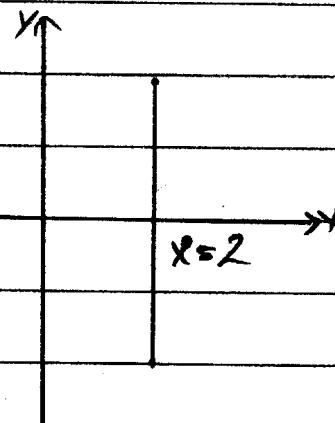
The equation is not a function. it fails the vertical line test.



[6] $f(x) = x = 2$

Solution.

The equation is not a function. It fails the vertical line test.



⑧ Piecewise-Defined Functions:-

Example:- Is the expression $f(x) = \begin{cases} 2-3x^3 & \text{if } x \leq 1 \\ 3x^4 & \text{if } x \geq 1 \end{cases}$ define a function? why?

Solution:-

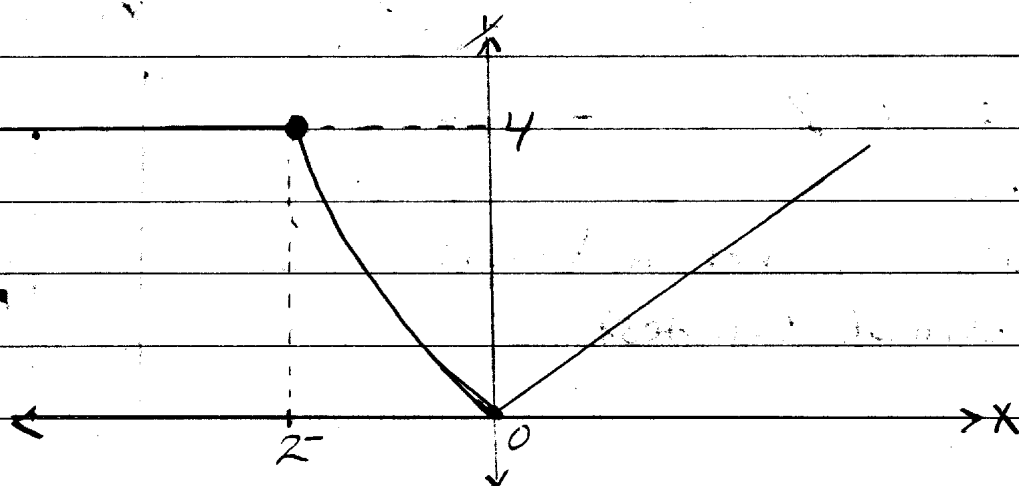
$$1. f(1) = \begin{cases} 2-3(1)^3 & = -1 \\ 3(1)^4 & = 3 \end{cases}$$

The given expression is not a fn because $-1 \neq 3$.

Example: Sketch the graph of the fn.

$$f(x) = \begin{cases} 4 & \text{if } x < -2 \\ x^2 & \text{if } -2 \leq x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

Solution:-



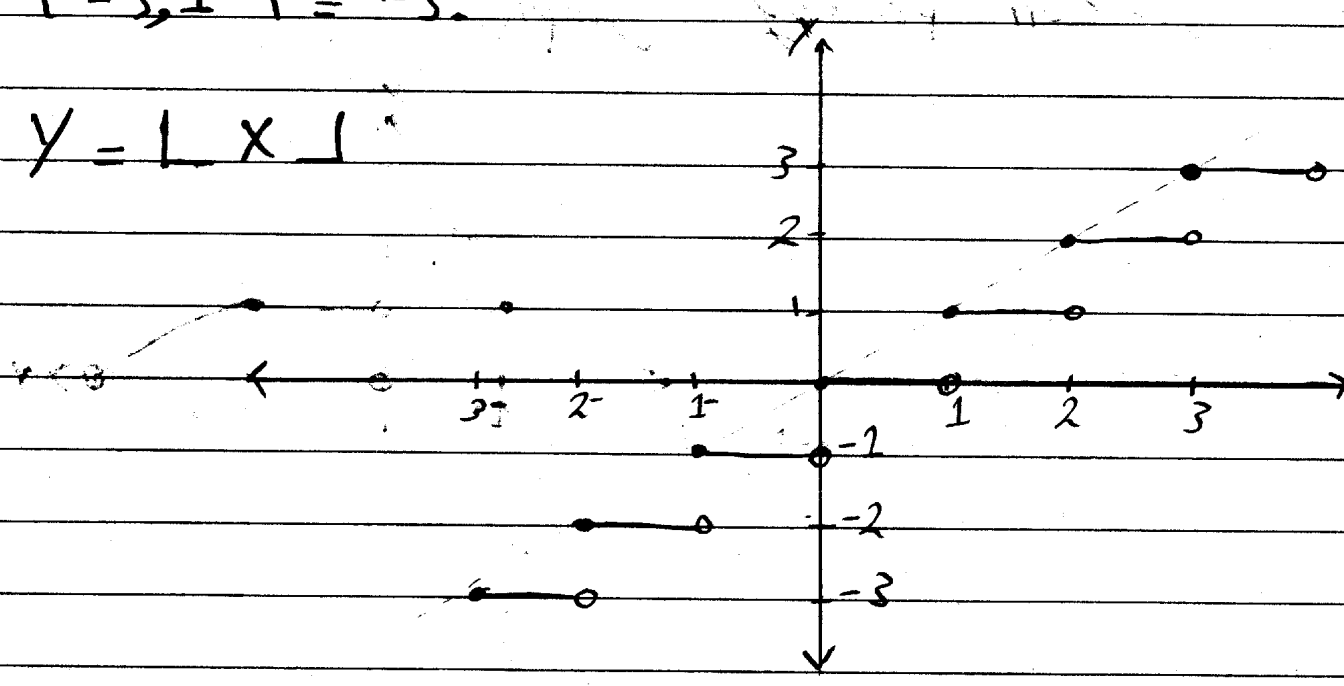
Example: (a) The integer floor fn whose value at any real number x is the greatest integer less than or equal to x . $f(x) = \lfloor x \rfloor$

e.g, $\lfloor 2.1 \rfloor = 2$, $\lfloor 2.99 \rfloor = 2$, $\lfloor 0 \rfloor = 0$
 $\lfloor -3.2 \rfloor = -4$, $\lfloor -0.3 \rfloor = -1$

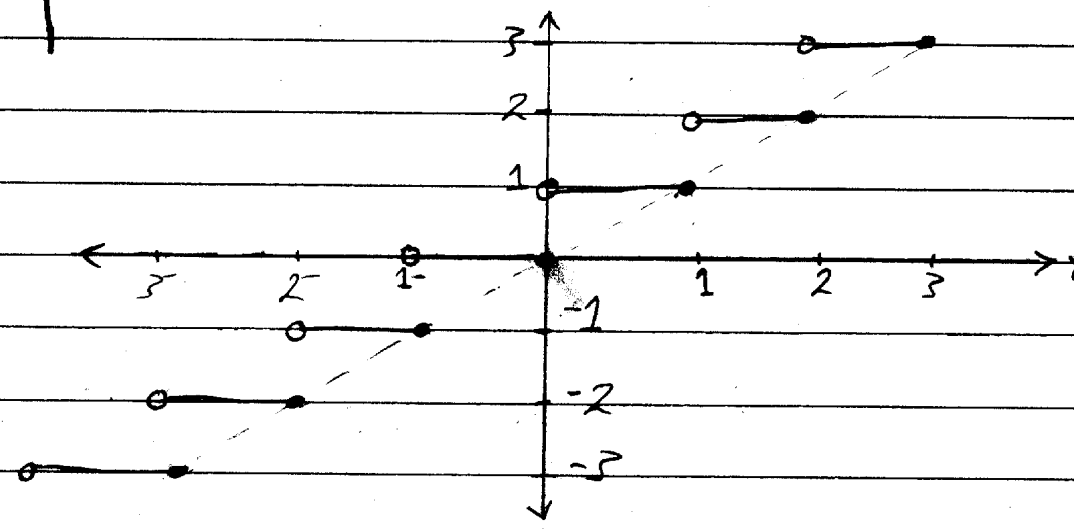
(b) The integer ceiling fn is the fn whose value at any real number x is the smallest integer greater than or equal to x . $f(x) = \lceil x \rceil$

e.g, $\lceil 2,1 \rceil = 3$, $\lceil 2,9 \rceil = 3$, $\lceil -0,3 \rceil = 0$.
 $\lceil -3,1 \rceil = -3$.

* $y = \lfloor x \rfloor$



* $y = \lceil x \rceil$



$$\text{Dom}(\lfloor x \rfloor) = \text{Dom}(\lceil x \rceil) = \mathbb{R}$$

$$\text{Rng} = \mathbb{Z} = \{ 0, +1, +2, \dots \}$$

Example: Find a formula for f with the corresponding graph.

$$-1 \leq x < 0$$

$$\Rightarrow f(x) = -x$$

$$0 \leq x \leq 1$$

$$\Rightarrow f(x) = 1$$

$$1 \leq x < 3$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{1 - 3} = -\frac{1}{2}$$

$$\frac{y - y_0}{x - x_0} = m \Rightarrow \frac{y - 0}{x - 3} = -\frac{1}{2}$$

$$\Rightarrow y = -\frac{1}{2}(3 - x)$$

$$\therefore f(x) = \begin{cases} -x & -1 \leq x < 0 \\ 1 & 0 \leq x \leq 1 \\ -\frac{1}{2}(3 - x) & 1 \leq x < 3 \end{cases}$$

$$\text{Dom}(f) = [-1, 3) - \{0\}$$

$$\text{Rng}(f) = (0, 1]$$

$$f = \{(x, y) \mid \text{such that } y = f(x)\}$$

$$f = \{(x, y) \mid \text{such that } y = f(x)\}$$

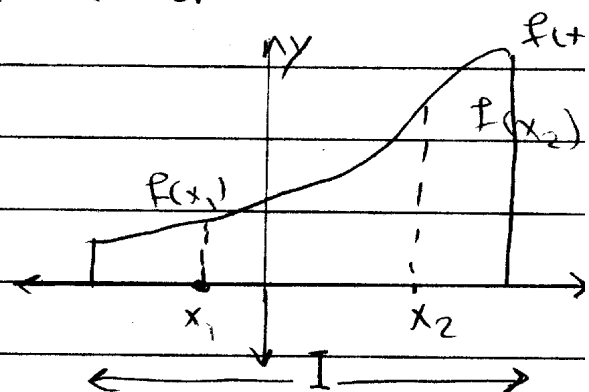
Increasing and decreasing fns.

Def: let f be a fn defined on an interval I and let $x_1, x_2 \in I$.

① If $f(x_2) \geq f(x_1)$, wherever $x_1 < x_2$, then f is said to be increasing fn on I .

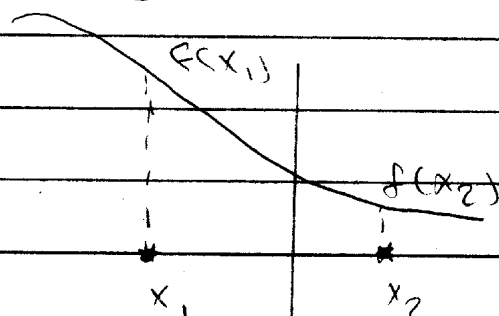
② If $f(x_2) \leq f(x_1)$, wherever $x_1 < x_2$, then f is said to be decreasing fn on I .

$x_1 < x_2$
 $f(x_2) > f(x_1)$
 $\Rightarrow$ increasing fn.



$x_1 < x_2$
 $f(x_2) < f(x_1)$

$\Rightarrow$ decreasing fn.



Example: Sketch the graph of the following fns and determine the intervals of increasing and decreasing.

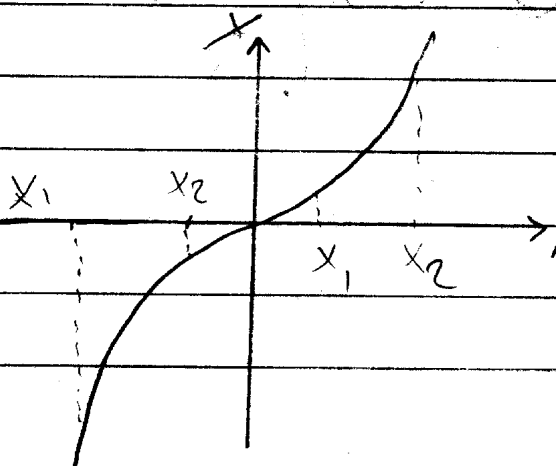
① $y = \frac{1}{2}x^3$

$(-\infty, 0]$

$x_1 < x_2$

$f(x_2) > f(x_1)$

$\therefore$ increasing fn



$$[0, \infty)$$

$$x_1 < x_2$$

$$f(x_2) > f(x_1)$$

$\therefore$ increasing fn

$\Rightarrow$ It is increasing on $(-\infty, \infty)$

$$\textcircled{2} y = \frac{-1}{|x|}$$

$$y = \begin{cases} +\frac{1}{+x} \\ -\frac{1}{+x} \end{cases}$$

$$x < 0, x \neq 0$$

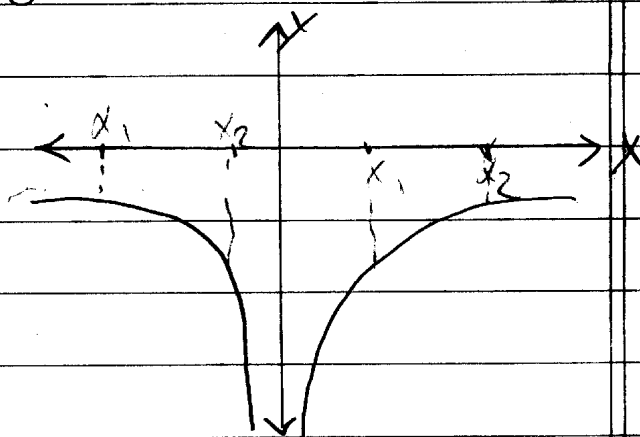
$$x > 0$$

$$(-\infty, 0)$$

$$x_1 < x_2$$

$$f(x_2) < f(x_1)$$

$\therefore$ decreasing fn.



$$(0, \infty)$$

$$x_1 < x_2$$

$$f(x_2) < f(x_1)$$

increasing

$$\text{Dom}(f) = (-\infty, \infty) - \{0\}$$

$$\text{Rng} = (-\infty, 0)$$

$$\textcircled{3} f(x) = \begin{cases} -x \\ x \end{cases}$$

$$x < 0$$

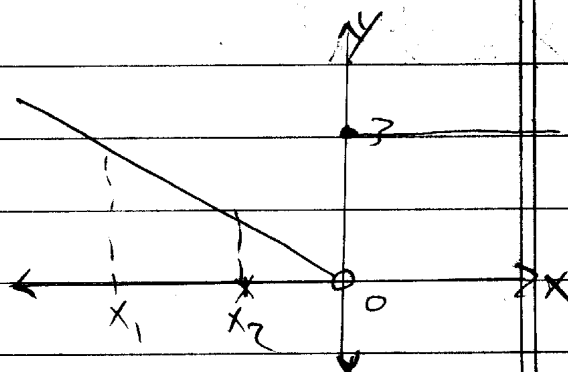
$$x \geq 0$$

$$(-\infty, 0)$$

$$x_1 < x_2$$

$$f(x_2) < f(x_1)$$

decreasing

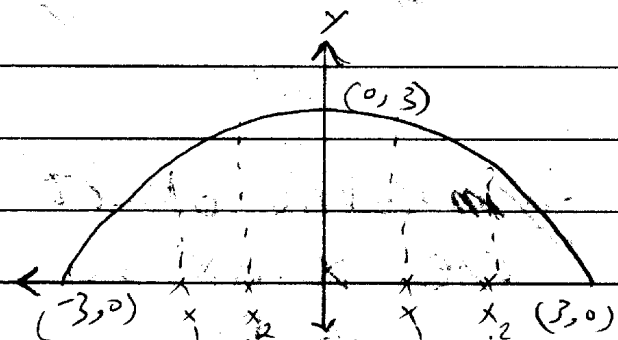


$(0, \infty)$

f is neither increasing nor decreasing or constant

④ $f(x) = \sqrt{9 - x^2}$

f is increasing on $[-3, 0]$
 & decreasing on $[0, 3]$



Even and odd fns, symmetry

Def: ① The fn f is called an even fn if $f(-x) = f(x), \forall x \in \text{Dom}(f)$.

② The fn f is called an odd fn if $f(-x) = -f(x), \forall x \in \text{Dom}(f)$.

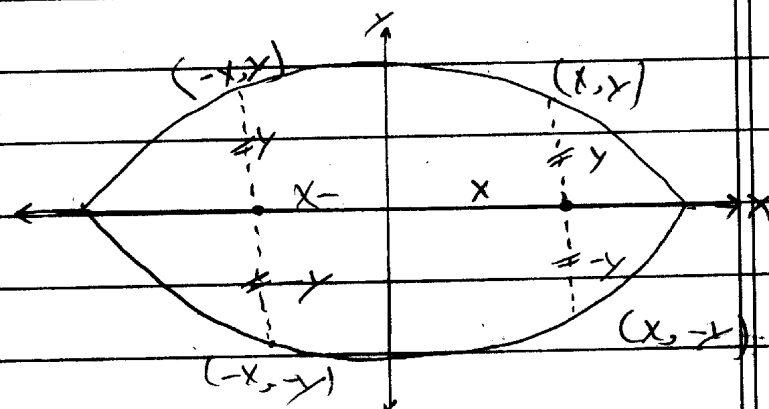
③ The graph of the fn is symmetric about the y -axis means that (x, y) lie on the graph iff $(-x, y)$ lie on the graph.

$(-x, y) \leftrightarrow (x, y)$: even fn,
 $(-x, y) \leftrightarrow (x, -y)$: odd fn

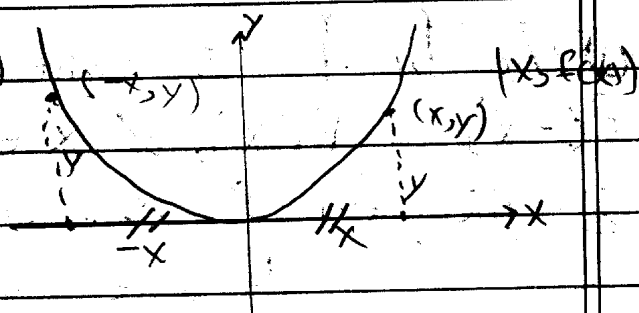
Remark/ ① The graph of an even fn is symmetric about the y -axis since $f(x) = f(-x)$ that is, a point (x, y) lie on the graph iff $(-x, y)$ lie on the graph.

② The graph of an odd fn is symmetric about the origin, since $f(-x) = -f(x)$, that is, a point (x, y) lies on the graph iff $(-x, -y)$ lies on the graph.

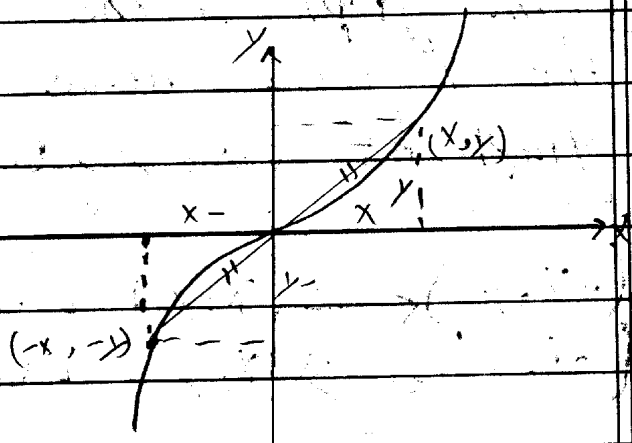
Symmetric about the x-axis.
"not fn"



Symmetric about the y-axis.
"Even fn"
 $f(-x) = f(x)$



Symmetric about origin.
"odd fn"



$$f(-x) = -f(x)$$

Example: Determine whether that the following fns are even, odd, or neither.

① $f(x) = x^4 + |x^3| + 5$ $(|-x| = |x|)$ note

Solution:-

$$f(-x) = (-x)^4 + |(-x)^3| + 5 = x^4 + |-x^3| + 5$$

$$= x^4 + |x^3| + 5$$

$$f(-x) = f(x) \Rightarrow \text{so } f \text{ is an even fn.}$$

② $f(x) = \frac{x}{x^2 - 3}$

Solution:- $f(-x) = \frac{-x}{(-x)^2 - 3} = \frac{-x}{x^2 - 3} = -f(x)$

$$f(-x) = -f(x) \Rightarrow \text{so } f \text{ is an odd fn.}$$

③ $f(x) = |x+1|$

Solution, $f(-x) = |-x+1| \neq f(x) \Rightarrow \text{not even}$

$$-f(x) = -|x+1| \neq f(-x) \Rightarrow \text{not odd}$$

$\Rightarrow$ So f neither even nor odd.

④ $f(x) = x^3 - 3x$

Solution: $f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x$

$$= -(x^3 - 3x) = -f(x)$$

$$f(-x) = -f(x)$$

So f is an odd fn.

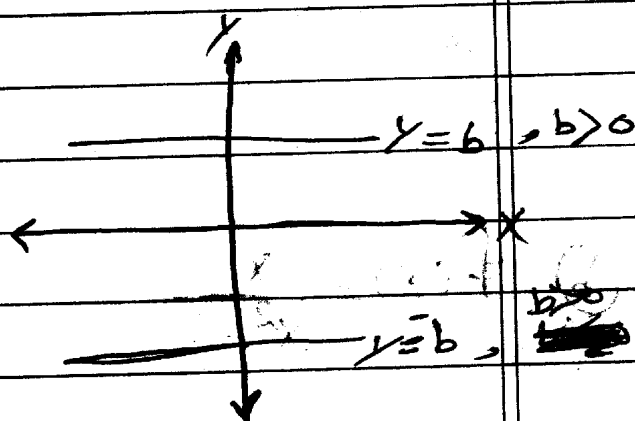
Common fns:

II Linear fns:

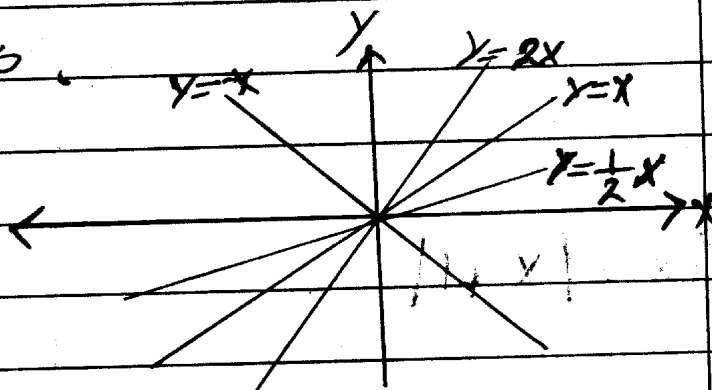
Def:- A fn of the form $f(x) = mx + b$ where m and b are constants is called a linear fn.

(A) If $m=0, b \neq 0$.

$$y = b$$



(B) If $b=0, m \neq 0$.

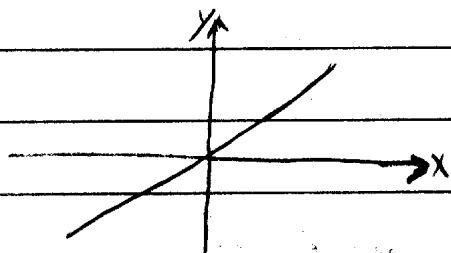


Def:- Two variable y and x are proportional if one is always a constant multiple of the other, that is, $y = kx$ for nonzero constant k ($k \neq 0$).
also $x = k(\frac{1}{y})$ means y is inversely proportional to x .

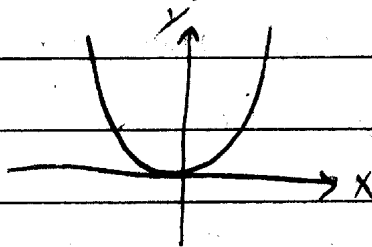
[2] The power fns:-

Def:- The fn $f(x) = x^a$, where a is constant, is called a power fn.

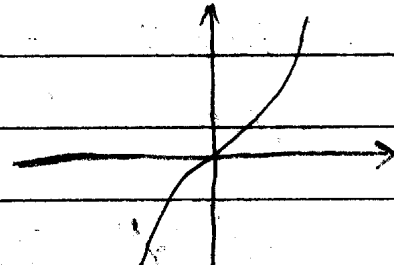
$$f(x) = x \rightarrow a = 1$$



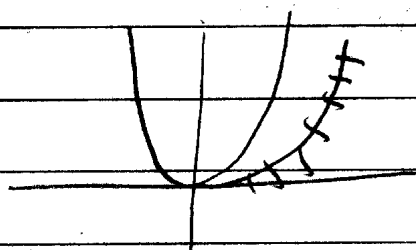
$$f(x) = x^2 \rightarrow a = 2$$



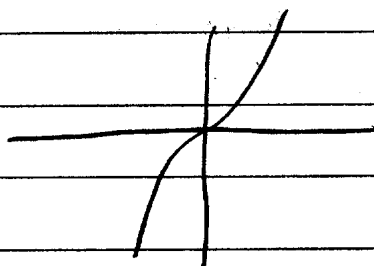
$$f(x) = x^3 \rightarrow a = 3$$



$$f(x) = x^4 \rightarrow a = 4$$

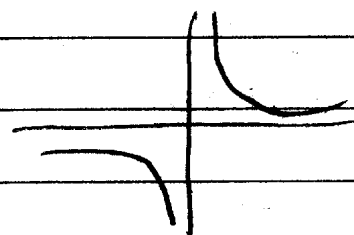


$$f(x) = x^5 \rightarrow a = 5$$

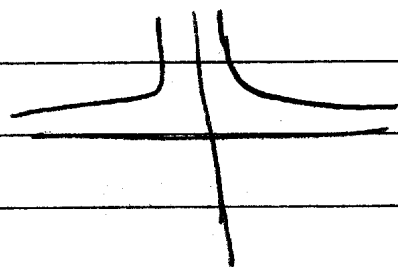


$$f(x) = x^{-1} \rightarrow a = -1$$

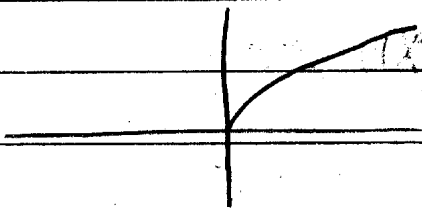
$$f(x) = \frac{1}{x}$$



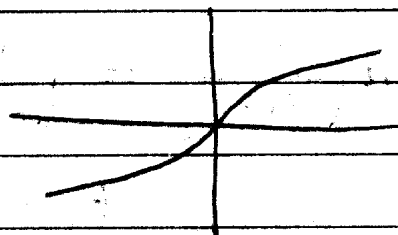
$$f(x) = x^{-2} = \frac{1}{x^2} \rightarrow a = -2$$



$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \rightarrow a = \frac{1}{2}$$

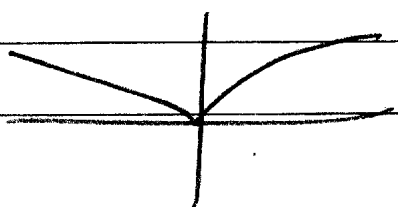


$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \rightarrow a = \frac{1}{3}$$

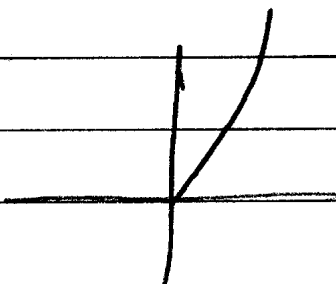


$$f(x) = \sqrt[3]{x^2} \rightarrow a = \frac{2}{3}$$

$$= x^{\frac{2}{3}}$$



$$f(x) = \sqrt{x^3} = x^{\frac{3}{2}} \rightarrow a = \frac{3}{2}$$



[3] Polynomials.

Def. A polynomial is an in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$
, where
 n is a nonnegative integer and the numbers $a_n, a_{n-1}, a_1, a_0, \dots$ are real numbers called the coefficients of the polynomials and n is called the degree of the polynomials.

For example / $2x^4 + 3x^2 + 10x - 1$ is a polynomial of degree 4.
~~ex: $2x^5$~~
 $P(x) = 2$ is a polynomial of degree zero.

[4] Rational Fns:-

Def:- The Rational fn is an expression in the form

$$f(x) = \frac{p(x)}{q(x)}$$
, where $p(x)$ and $q(x)$ are polynomials

$$\text{Dom}\left(\frac{p(x)}{q(x)}\right) = \{x : q(x) \neq 0\}$$

For example / $f(x) = \frac{x^4 + 3x^2 + 16}{x(x+5)}$. $\text{Dom}(f(x)) = \mathbb{R} - \{0, -5\}$

[5] Algebraic fns.

Def/ It's a fn that can be constructed using algebraic operations (addition, subtraction, multiplication, division, and taking roots).

~~any rational fn is rational fn.~~

any rational fn is algebraic fn. but not the opposite

= polynomial = = = =
= = = = rational = ,

[6] The transcendental fns.

Def/ It's the fns that aren't algebraic fns, as:-

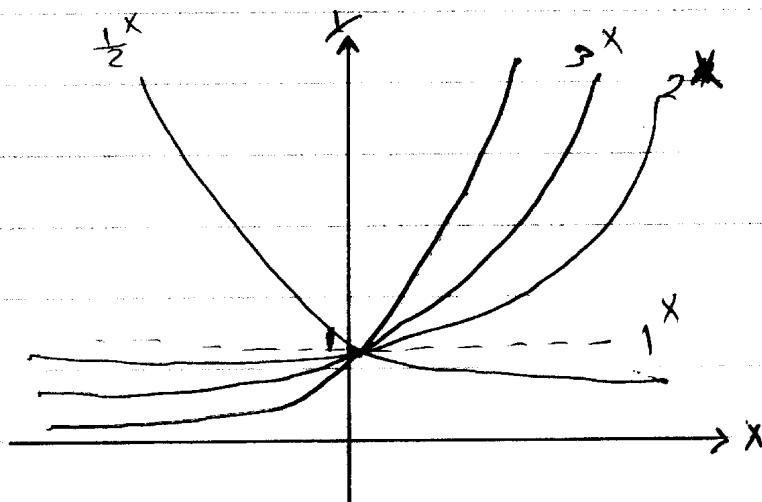
① Trigonometric fns.

② Exponential fns: $f(x) = a^x, a > 0, a \neq 1$.

$$a^x > 0 \forall x$$

$$\text{Dom}(a^x) = (-\infty, \infty)$$

$$\text{Rng}(a^x) = (0, \infty)$$

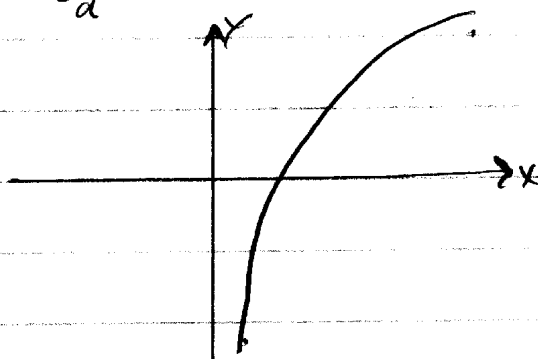


[3] Logarithmic fns. $f(x) = \log_a x$, $a > 0$, $a \neq 1$

$$x > 0$$

$$\text{Dom} = (0, \infty)$$

$$\text{Rng} = (-\infty, \infty)$$



④ Inverse fns

Exercises :- 1.

Functions. Find the Domain and Range.

$$1) f(x) = 1 + x^2. \quad \text{Dom}(f) = (-\infty, \infty) \quad \text{Rng}(f) = [1, \infty)$$

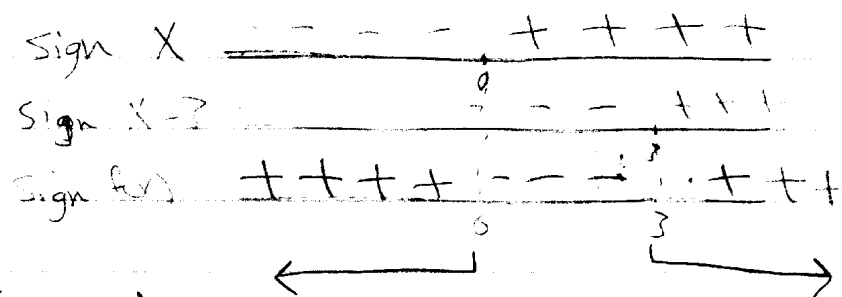
$$2) f(x) = 1 - \sqrt{x}. \quad \text{Dom}(f) = [0, \infty) \\ \text{Rng}(f) = (-\infty, 1]$$

$$3) f(x) = \sqrt{5x+10} \quad \text{Dom}(f) = \{x : 5x+10 \geq 0\} \\ = \{x : 5x \geq -10\} \\ = \{x : x \geq -2\}$$

$$\therefore \text{Dom}(f) = [-2, \infty)$$

$$\text{Rng}(f) = [0, \infty).$$

$$4) g(x) = \sqrt{x^2 - 3x}. \quad x^2 - 3x \geq 0 \\ x(x-3) \geq 0 \\ \Rightarrow x \geq 0, \quad x-3 \geq 0 \Rightarrow x \geq 3.$$



$$\text{Dom}(f) = (-\infty, 0] \cup [3, \infty)$$

$$\text{Rng}(f) = [0, \infty)$$

$$5) f(t) = \frac{4}{3-t} \quad \text{Dom}(f) = \mathbb{R} - \{3\}$$

$$\text{Rng}(f) = \mathbb{R} - \{0\}$$

$$6) G(t) = \frac{2}{t^2 - 16} \quad t^2 - 16 = 0$$

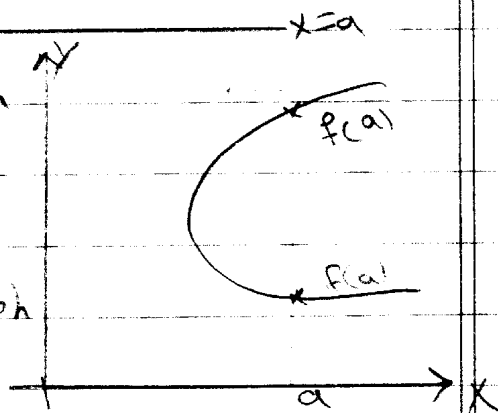
$$t^2 = 16 \Rightarrow t = \pm 4$$

$$\text{Dom}(G) = \mathbb{R} - \{\pm 4\}$$

$$\text{Rng}(G) = \mathbb{R} - \{0\}$$

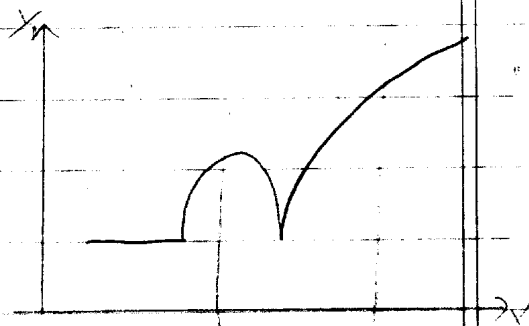
7) a) The graph is not a graph of function. Because the vertical line test fails.

the vertical line intersected the graph more than once.



b) The graph is a graph of function. Because the vertical line test succeeds.

the vertical line intersected the graph once.



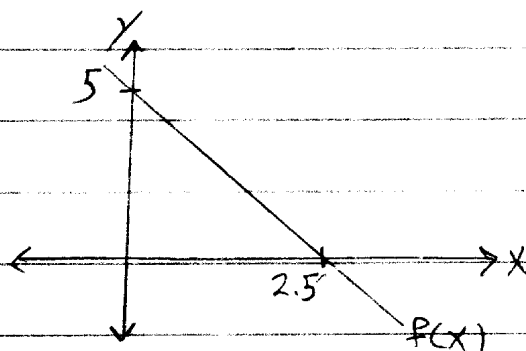
8) a) b) neither a nor b are graph of fns because the vertical line test fails. the vertical line intersected the graph more than once.

Functions and graphs.

Find the Domain and graph the fns.

15) $f(x) = 5 - 2x$.

* $\text{Dom}(f) = (-\infty, \infty)$.



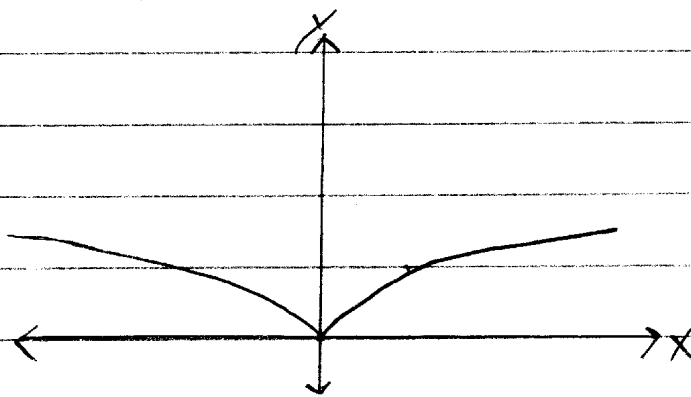
16) $f(x) = 1 - 2x - x^2$.

* $\text{Dom}(f) = (-\infty, \infty)$.

17) $g(x) = \sqrt{|x|}$

$$\sqrt{|x|} = \begin{cases} \sqrt{x} & , x \geq 0 \\ \sqrt{-x} & , x < 0 \end{cases}$$

* $\text{Dom}(g) = (-\infty, \infty)$

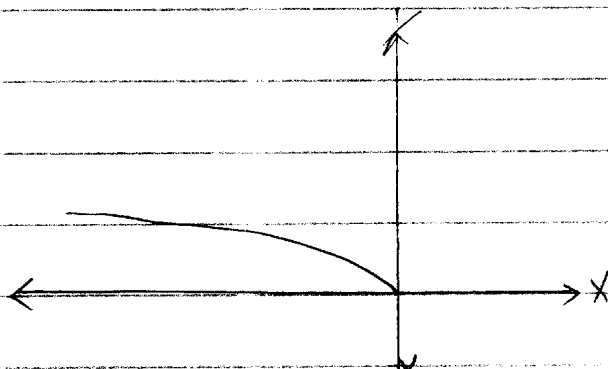


18) $g(x) = \sqrt{-x}$

$$-x \geq 0$$

$$\Rightarrow x \leq 0$$

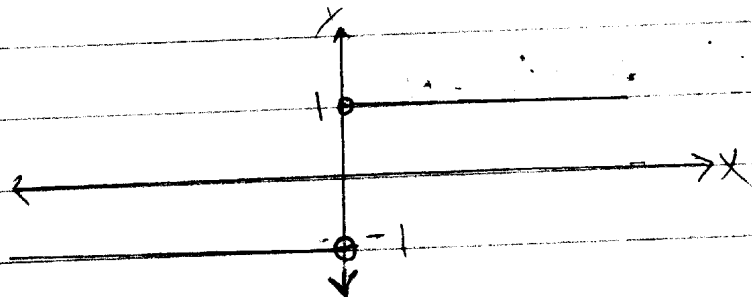
$\text{Dom}(g) = (-\infty, 0]$



19) $F(t) = \frac{t}{|t|}$

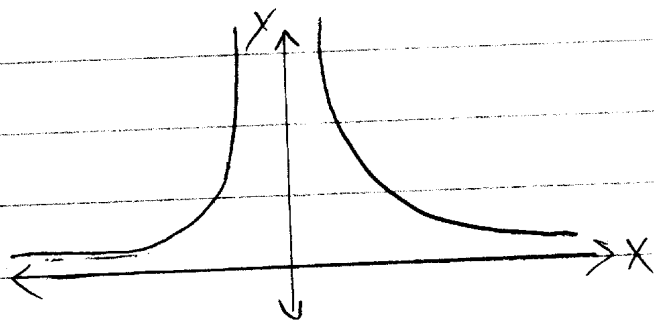
$$F(t) = \begin{cases} \frac{t}{t}, & t > 0, t \neq 0 \\ \frac{t}{-t}, & t < 0 \end{cases} \Rightarrow \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}, t \neq 0.$$

$\text{Dom}(f) = \mathbb{R} - \{0\}$



20) $G(t) = \frac{1}{|t|}$

$$G(t) = \begin{cases} \frac{1}{t}, & t > 0 \\ -\frac{1}{t}, & t < 0 \end{cases}, t \neq 0$$



$\text{Dom}(G) = \mathbb{R} - \{0\}$

21) Find Range of $Y = 2 + \frac{x^2}{x^2 + 4}$

$\text{Dom}(Y) = \mathbb{R}$

$$0 \leq \frac{x^2}{x^2 + 4} < 1$$

$\text{Rng}(Y) = [2, 3)$

$$2 + 0 \leq \frac{x^2}{x^2 + 4} < 1 \Rightarrow [2, 3)$$

21) Find the Domain of $Y = \frac{x+3}{4 - \sqrt{x^2 - 9}}$

$$4 - \sqrt{x^2 - 9} = 0 \Rightarrow \sqrt{x^2 - 9} = 4$$

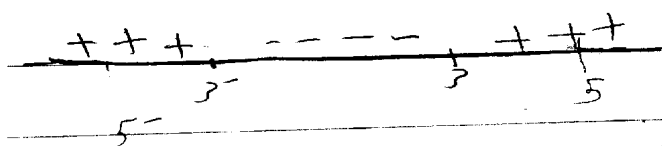
$$16 = x^2 - 9$$

$$x^2 = 25 \Rightarrow x = \pm 5 \Rightarrow \text{Dom}_1 = \mathbb{R} - \{\pm 5\}$$

$$\sqrt{x^2 - 9} > 0 \Rightarrow x^2 - 9 > 0 \Rightarrow x^2 > 9$$

$$|x| > 3$$

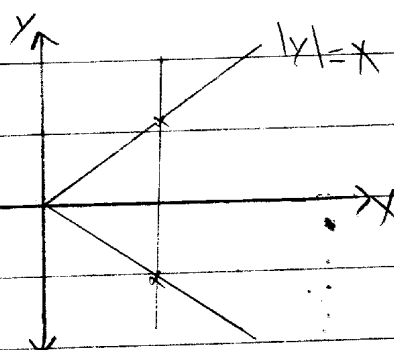
$$x > 3, x < -3$$



$$\text{Dom} = (-\infty, -3] \cup [3, \infty) - \{\pm 5\}$$

$$23) \text{ a) } |y| = x.$$

$$|y| = \begin{cases} y & , y \geq 0 \\ -y & , y < 0 \end{cases}$$



$$y = x, y \geq 0$$

$$-y = x \rightarrow y = -x, y < 0$$

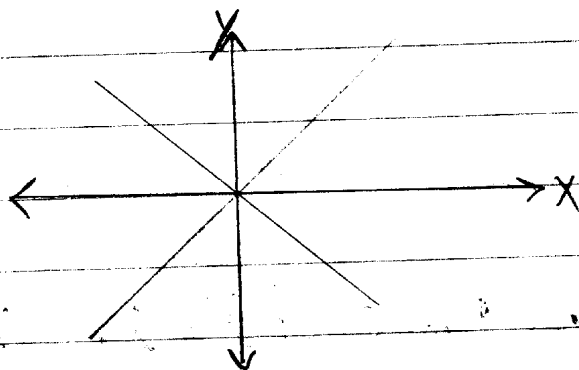
$$\text{b) } y^2 = x^2$$

$$|y| = |x|$$

$$y = \pm x$$

$$y = x$$

$$y = -x$$

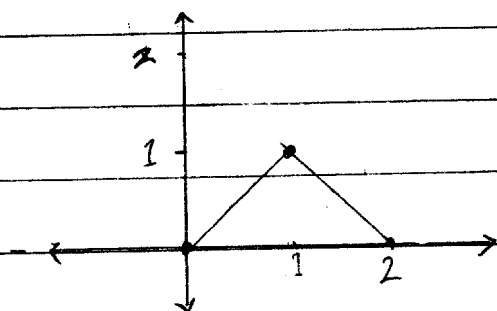


$$24) \text{ a) } |x| + |y| = 1$$

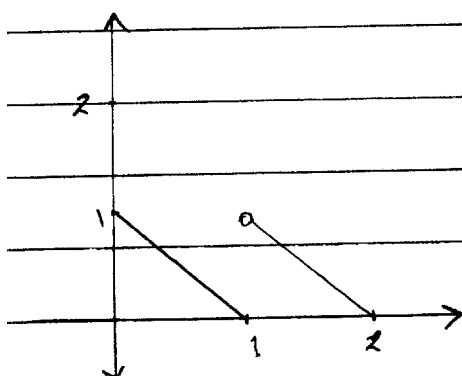
$$⑥ |x+y|=1$$

Piecewise-Defined Fns.

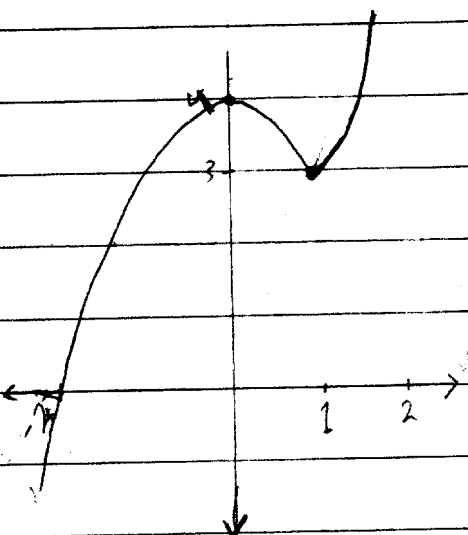
$$25) f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2-x, & 1 \leq x \leq 2 \end{cases}$$



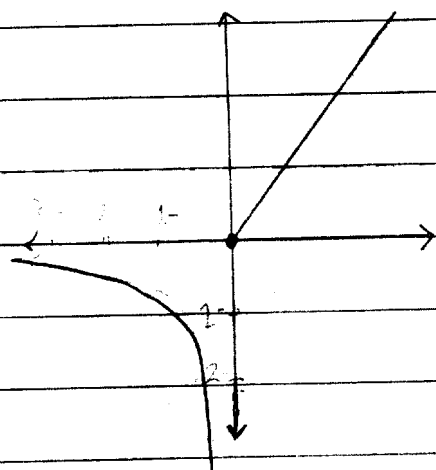
$$26) g(x) = \begin{cases} 1-x, & 0 \leq x < 1 \\ 2x, & 1 \leq x \leq 2 \end{cases}$$



$$27) f(x) = \begin{cases} 4-x^2, & x \leq 1 \\ x^2+2x, & x > 1 \end{cases}$$



$$28) G(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x, & 0 \leq x \end{cases}$$



Find a formula for each fns graphed :

$$29) \textcircled{a} \text{ slope}_1 = \frac{1-0}{1-0} = 1 \Rightarrow \frac{y-y_0}{x-x_0} = 1$$

$$\frac{y-0}{x-0} = 1 \Rightarrow \underline{y=x}$$

$$\text{slope}_2 = \frac{0-1}{2-1} = -1 \Rightarrow \frac{y-1}{x-1} = -1$$

$$y-1 = 1-x \Rightarrow y=2-x$$

$$F(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2-x, & 1 < x \leq 2 \end{cases}$$

$$\textcircled{b} F(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 0, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \\ 0, & 3 \leq x \leq 4 \end{cases}$$

$$\textcircled{30} \textcircled{a} \text{ slope}_1 = \frac{2-0}{0-2} = -1 \Rightarrow \frac{y-0}{x-2} = -1$$

$$y=2-x$$

$$\text{slope}_2 = \frac{1-0}{2-5} = -\frac{1}{3} \Rightarrow \frac{y-0}{x-5} = -\frac{1}{3}$$

$$y = \frac{5}{3} - \frac{x}{3}$$

$$F(x) = \begin{cases} 2-x, & 0 < x \leq 2 \\ \frac{5-x}{3}, & 2 < x \leq 5 \end{cases}$$

$$\textcircled{6} \text{ slope } 1 = \frac{-3-0}{0+1} = -3 \Rightarrow \frac{y-0}{x+1} = -3 \Rightarrow y = -x-3$$

$$\text{slope } 2 = \frac{-1-3}{2-0} = -2 \Rightarrow \frac{y-3}{x-0} = -2$$

$$\Rightarrow y-3 = -2x \Rightarrow y = -2x+3$$

$$F(x) = \begin{cases} -x-3, & -1 < x \leq 0 \\ -2x+3, & 0 < x \leq 2 \end{cases}$$

$$\begin{array}{r} 22) \quad \begin{array}{r} 1 \\ x^2+4 \overline{) x^2} \\ -x^2-4 \\ \hline -4 \end{array} \end{array}$$

$$1 - \frac{4}{x^2+4} \Rightarrow f(x) = 3 - \frac{4}{x^2+4}$$

$$-\infty < x < \infty$$

$$0 < x^2 < \infty$$

$$4 < x^2+4 < \infty$$

$$\frac{1}{4} > \frac{1}{x^2+4} > 0$$

$$1 > \frac{4}{x^2+4} > 0$$

$$\begin{array}{r} 4 \rightarrow 3 \rightarrow 4 \rightarrow 3 \\ -1 \leftarrow -4 \leftarrow 0 \\ \hline 1 \leftarrow \frac{-4}{x^2+4} \leftarrow 0 \end{array}$$

$$2 \leftarrow 3 - \frac{4}{x^2+4} \leftarrow 3$$

$$31) \textcircled{a} \text{ slope}_1 = \frac{+1-0}{-1-0} = -1 \Rightarrow$$

$$\frac{Y-0}{X-0} = -1 \rightarrow Y = -X$$

$$\text{slope}_2 = \frac{0-1}{3-1} = -\frac{1}{2} \Rightarrow$$

$$\frac{Y-1}{X-1} = -\frac{1}{2} \Rightarrow Y-1 = -\frac{1}{2}(X-1)$$

$$Y = \frac{3}{2} - \frac{1}{2}X$$

$$f(x) = \begin{cases} -X & , -1 \leq X \leq 0 \\ 1 & , 0 < X \leq 1 \\ \frac{3}{2} - \frac{1}{2}X & , 1 < X \leq 3 \end{cases}$$

$$\textcircled{b} \text{ slope}_1 = \frac{-1-0}{-2-0} = \frac{1}{2} \Rightarrow \frac{Y-0}{X-0} = \frac{1}{2} \rightarrow Y = \frac{1}{2}X$$

$$\text{slope}_2 = \frac{0-2}{1-0} = -2 \Rightarrow \frac{Y-2}{X-0} = -2$$

$$\Rightarrow Y-2 = -2X \rightarrow Y = -2X+2$$

$$f(x) = \begin{cases} \frac{1}{2}X & , -2 \leq X \leq 0 \\ -2X+2 & , 0 < X \leq 1 \\ -1 & , 1 < X \leq 3 \end{cases}$$

$$32) \text{ @ slope} = \frac{1-0}{T-\frac{T}{2}} = \frac{1}{\frac{T}{2}} = \frac{2}{T}$$

$$\frac{Y-0}{X-\frac{T}{2}} = \frac{2}{T} \Rightarrow Y = \frac{2}{T}X - \frac{2T}{2T} \Rightarrow Y = \frac{2X}{T} - 1$$

$$f(x) = \begin{cases} 0 & , 0 \leq x < \frac{T}{2} \\ \frac{2}{T}x - 1 & , \frac{T}{2} \leq x < T \end{cases}$$

$$\textcircled{b} \quad f(x) = \begin{cases} A & , 0 \leq x < \frac{T}{2} \\ -A & , \frac{T}{2} \leq x < T \\ A & , T \leq x < \frac{3T}{2} \\ -A & , \frac{3T}{2} \leq x < 2T \end{cases}$$

The Greatest and Least Integer Fns:-

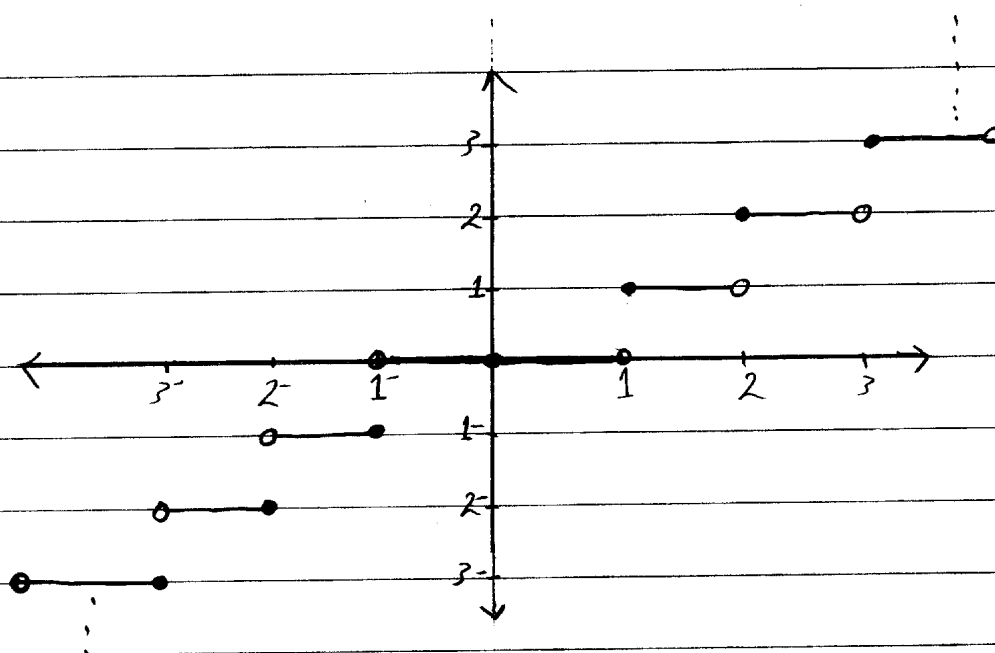
$$33) \text{ @ } \lfloor x \rfloor = 0, \quad 0 \leq x < 1$$

$$\textcircled{b} \quad \lceil x \rceil = 0, \quad -1 < x \leq 0$$

$$34) \quad 0$$

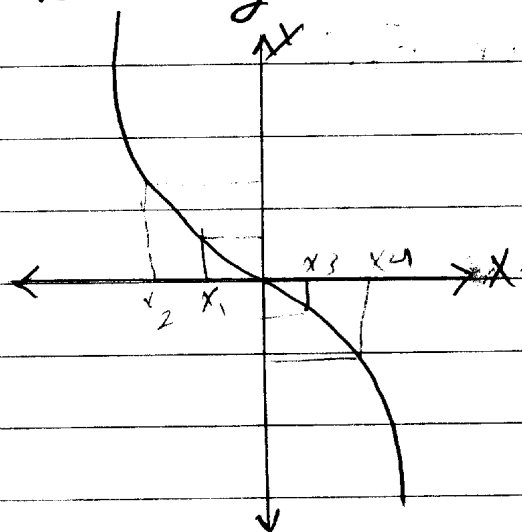
$$35)$$

26)



Increasing and Decreasing Functions.

27)



$$x_1 > x_2$$

$$\text{and } f(x_1) < f(x_2)$$

$\therefore$ the graph decreasing in the interval $(-\infty, 0)$.

$$x_3 < x_4$$

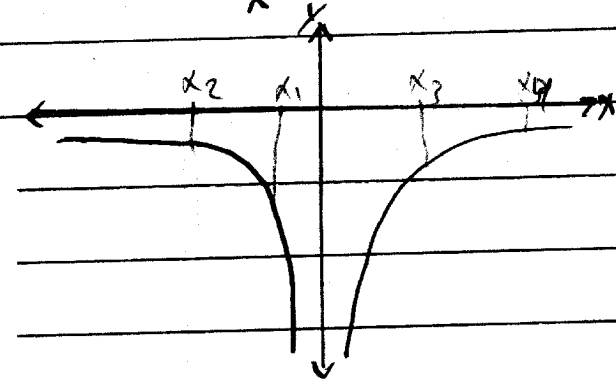
$$f(x_3) > f(x_4)$$

$$\therefore = = = =$$

$$\therefore = (0, \infty)$$

$\therefore$ The graph is decreasing in $(-\infty, \infty)$

$$38) y = -\frac{1}{x^2}$$



$$(-\infty, 0)$$

$$x_1 > x_2$$

$$f(x_1) < f(x_2)$$

$\therefore$ decreasing

$$(0, \infty)$$

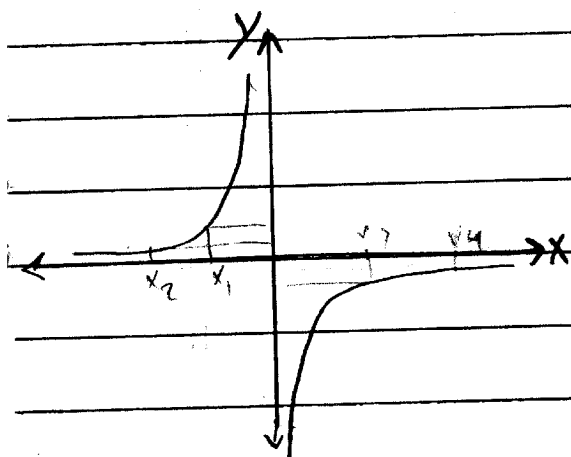
$$x_3 < x_4$$

$$f(x_3) < f(x_4)$$

$\therefore$ increasing

$\therefore$ the f_n increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$

$$39) y = -\frac{1}{x}$$



$$(-\infty, 0)$$

$$x_1 > x_2$$

$$f(x_1) > f(x_2)$$

$\therefore$ increasing

$$(0, \infty)$$

$$x_4 > x_3$$

$$f(x_4) > f(x_3)$$

$\therefore$ increasing

$\therefore$ the f_n increasing in $(-\infty, \infty)$

1.2 Combining Functions; Shifting and Scaling Graphs.

* Sums, Differences, Products, and Quotients.

Def/If f and g are two fns and $x \in \text{Dom}(f) \cap \text{Dom}(g)$, we define: ① $(f \pm g)(x) = f(x) \pm g(x)$.

$$\textcircled{2} (fg)(x) = f(x) \cdot g(x), \quad (cf)(x) = c f(x),$$

c is constant.

$$\textcircled{3} \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

$$\text{Dom}(f+g) = \text{Dom}(f-g) = \text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g).$$

$$\text{Dom}(cf) = \text{Dom}(f)$$

$$\text{Dom}\left(\frac{f}{g}\right) = (\text{Dom}(f) \cap \text{Dom}(g)) - \{x : g(x) = 0\}$$

Example:- Let $f(x) = \sqrt{x+4}$ and $g(x) = \sqrt{x^2-4}$. Find

① $f+g$, fg , $\frac{f}{g}$. ② $\text{Dom}(f+g)$, $(fg)\left(\frac{f}{g}\right)$

③ $\frac{f}{g}(1)$, $\frac{f}{g}(3)$.

Solution:- ① * $f+g(x) = f(x) + g(x) = \sqrt{x+4} + \sqrt{x^2-4}$.

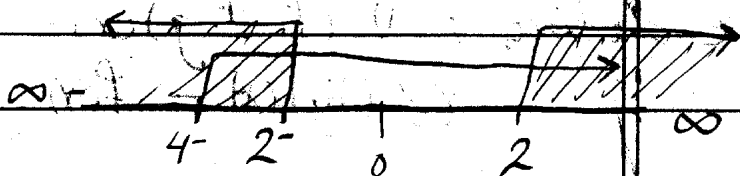
$$\begin{aligned} * fg(x) &= f(x) \cdot g(x) = \sqrt{x+4} \cdot \sqrt{x^2-4} \\ &= \sqrt{(x+4)(x^2-4)}. \end{aligned}$$

$$\begin{aligned} * \frac{f}{g}(x) &= \frac{f(x)}{g(x)} = \frac{\sqrt{x+4}}{\sqrt{x^2-4}} = \sqrt{\frac{x+4}{x^2-4}} \end{aligned}$$

⑥ * $\text{Dom}(f+g) :=$

$$\text{Dom}(f) = \{x : x+4 \geq 0\} \Rightarrow \{x : x \geq -4\} = [-4, \infty).$$

$$\begin{aligned} \text{Dom}(g) &= \{x : x^2 - 4 \geq 0\} \Rightarrow \{x : x^2 \geq 4\} \\ &= \{x : |x| \geq 2\} = (-\infty, -2] \cup [2, \infty) \end{aligned}$$



$$\text{Dom}(f+g) = \text{Dom}(f) \cap \text{Dom}(g) = [-4, -2] \cup [2, \infty).$$

$$* \text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g) = [-4, -2] \cup [2, \infty).$$

$$* \text{Dom}\left(\frac{f}{g}\right) = (\text{Dom}(f) \cap \text{Dom}(g)) - \{x : g(x) = 0\}$$

$$= [-4, -2] \cup [2, \infty) - \{-2, 2\} = [-4, -2) \cup (2, \infty).$$

⑦ $\frac{f}{g}(1)$ does not exist because $1 \notin [-4, -2) \cup (2, \infty)$

$$\frac{f}{g}(3) = \sqrt{\frac{3+4}{3^2-4}} = \sqrt{\frac{7}{5}}$$

$$\text{Dom}(f \circ g) = \{x : x \in \text{Dom}(g) \text{ and } g(x) \in \text{Dom}(f)\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } \sqrt{5-x} \in (-\infty, \infty)\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } \sqrt{5-x} \in [0, \infty)\}$$

Dom(f) = $\mathbb{R}$
Rng(g) = $[0, \infty)$

$$= \{x : x \in (-\infty, 5] \text{ and } \sqrt{5-x} \geq 0\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } 5-x \geq 0\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } 5 \geq x\}$$

$$= (-\infty, 5]$$

$$\star \text{Dom}(g \circ g) = \{x : x \in \text{Dom}(g) \text{ and } g(x) \in \text{Dom}(g)\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } \sqrt{5-x} \in (-\infty, 5]\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } \sqrt{5-x} \in [0, 5]\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } 0 \leq \sqrt{5-x} \leq 5\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } 0 \leq 5-x \leq 25\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } -5 \leq -x \leq 20\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } 5 \geq x \geq -20\}$$

$$= \{x : x \in (-\infty, 5] \text{ and } x \in [-20, 5]\}$$

$$= [-20, 5]$$

Example/ If $f(x) = \frac{x+1}{x}$ and $f \circ g(x) = x$. Find $g(x)$.

$$(f \circ g)(x) = f(g(x)) = x \Rightarrow \frac{g(x)+1}{g(x)} = x$$

$$g(x)+1 = x g(x) \Rightarrow g(x) - x g(x) = -1$$

$$x g(x) - g(x) = 1 \Rightarrow g(x)(x-1) = 1$$

$$g(x) = \frac{1}{x-1}, x \neq 1$$

Example/ If $g(x) = x+3$ and $f \circ g(x) = x^2 + 1$. Find a formula for $f(x)$.

$$(f \circ g)(x) = f(g(x)) = x^2 + 1 \Rightarrow f(x+3) = x^2 + 1$$

$$\text{put } y = x+3 \Rightarrow x = y-3$$

$$\text{Now: } f(y) = (y-3)^2 + 1 = y^2 - 6y + 10$$

$$\therefore f(x) = x^2 - 6x + 10$$

Example/ If f is an even fn and g is an odd fn. what can you say about?

$$\textcircled{a} gf \quad \textcircled{b} gof \quad \textcircled{c} gog \quad \textcircled{d} X^3 \left(\frac{f}{g} \right) X$$

$$f \text{ is an even } \rightarrow f(x) = f(-x), x \in \text{Dom}(f)$$

$$g \text{ is an odd } \rightarrow g(-x) = -g(x), x \in \text{Dom}(g)$$

$$\textcircled{a} gf(-x) = g(-x) \cdot f(-x) = -g(x) \cdot f(x) = -gf(x)$$

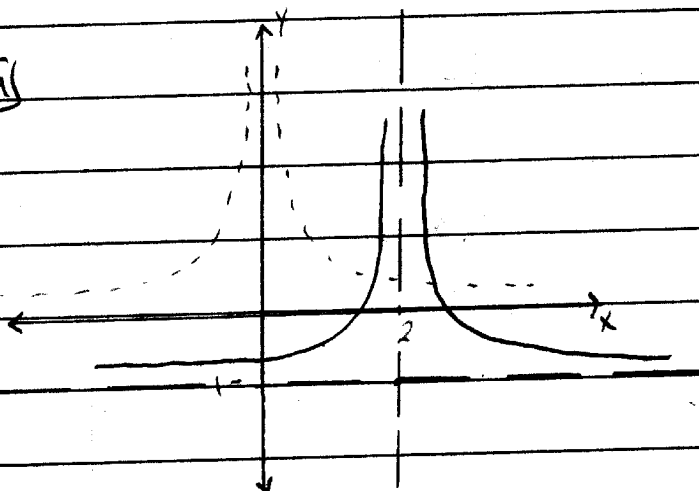
$$\therefore gf(x) \text{ is an odd fn.}$$

Example sketch and write the following equations.

(a) $y = \frac{1}{x^2}$ after shifting right 2 and down 1.

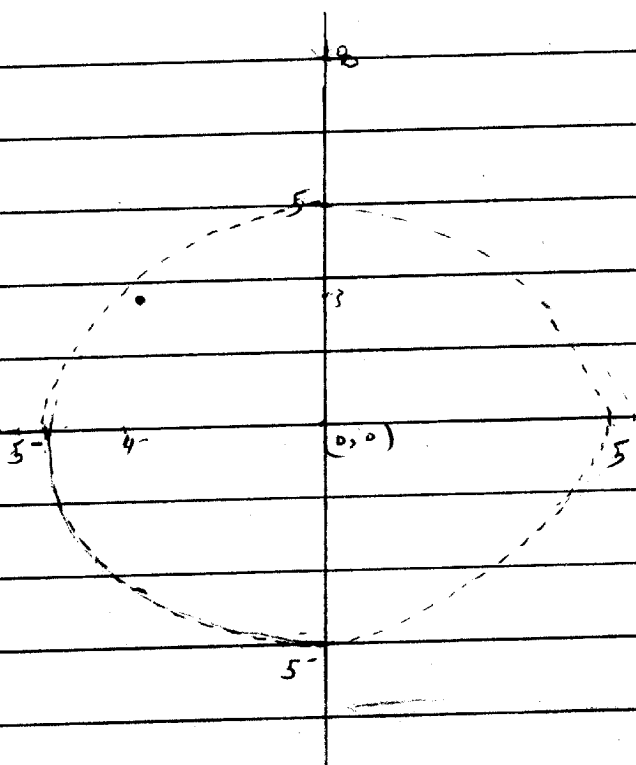
(b) $x^2 + y^2 = 25$ after shifting up 3 and left 4.

(a)



$$y = \frac{1}{(x-2)^2} - 1$$

(b)



$$y^2 = 25 - x^2$$

$$y = \sqrt{25 - x^2}$$

$$y = \sqrt{25 - (x+4)^2} + 3$$

$$y - 3 = \sqrt{25 - (x+4)^2}$$

$$(y-3)^2 = 25 - (x+4)^2$$

$$(y-3)^2 + (x+4)^2 = 25$$

center of the circle
 $(-4, 3)$

$$\textcircled{a} (g \circ f)(-x) = g(f(-x)) = g(f(x)) = g \circ f(x).$$

$\therefore g \circ f$ is an even fn.

$$\textcircled{c} g \circ g(-x) = g(g(-x)) = g(-g(x)) = -g(g(x)) \\ = -g \circ g(x)$$

$\therefore g \circ g$ is an odd fn.

$$\textcircled{d} x^3 \left(\frac{f}{g} \right)(x) := (-x)^3 \left(\frac{f}{g} \right)(-x) =$$

$$-x^3 \frac{f(-x)}{g(-x)} = -x^3 \frac{f(x)}{-g(x)} = x^3 \frac{f(x)}{g(x)}$$

$$= x^3 \frac{f}{g}(x) \quad \therefore x^3 \left(\frac{f}{g} \right) \text{ is an even fn.}$$

Shifting a graph of a fn.

1] Vertical Shifting: If k is a positive real number, then:
 ① $y = f(x) + k$, shift the graph of $y = f(x)$ up k units.
 ② $y = f(x) - k$, $= = = =$ down $= =$

2] Horizontal Shifting: If h is positive real number, then:
 ① $y = f(x - h)$, shift the graph of $y = f(x)$ to the right h units.
 ② $y = f(x + h)$, shift the graph of $y = f(x)$ to the left h units.

Sketch the graph of the following fns.

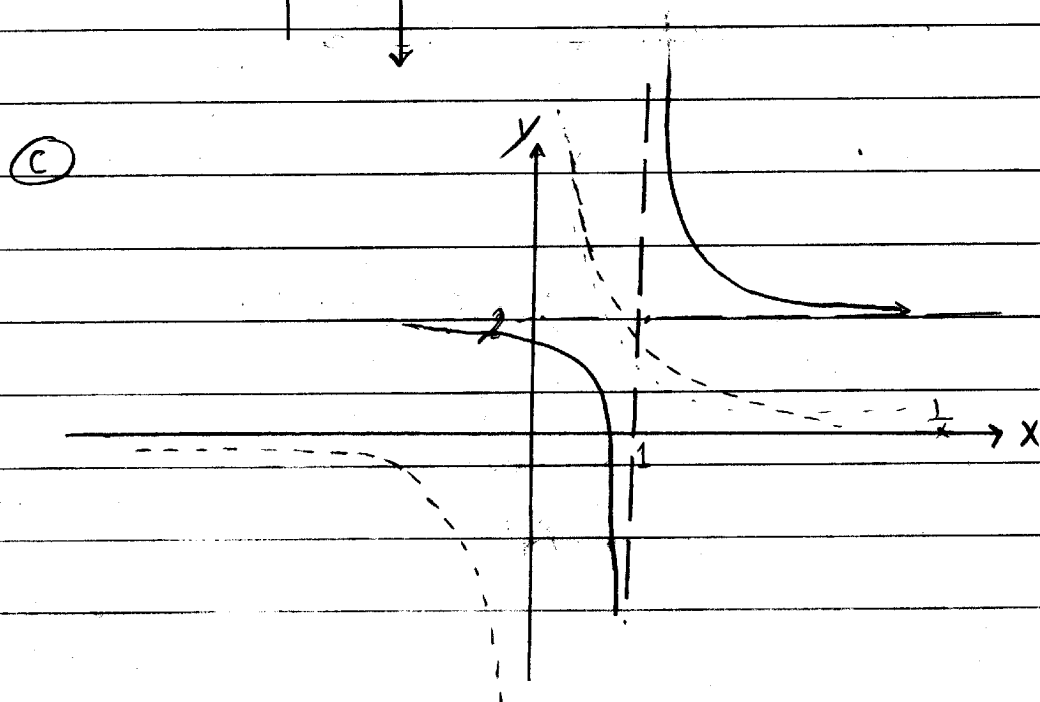
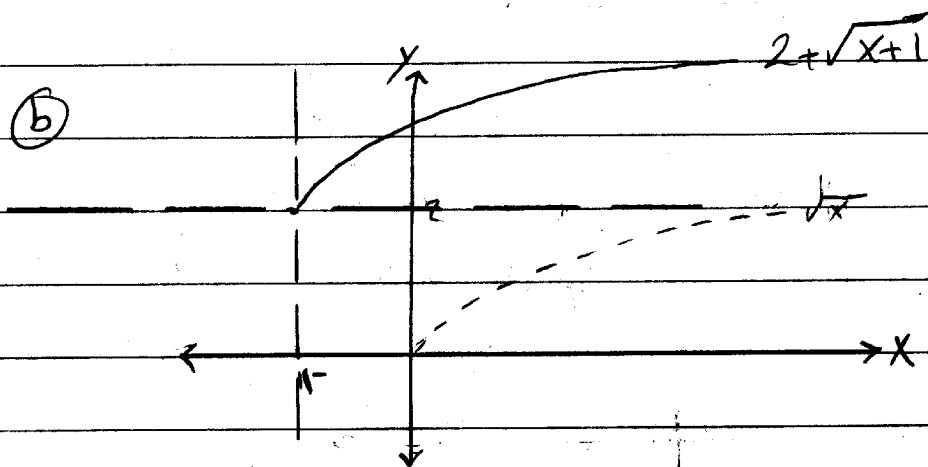
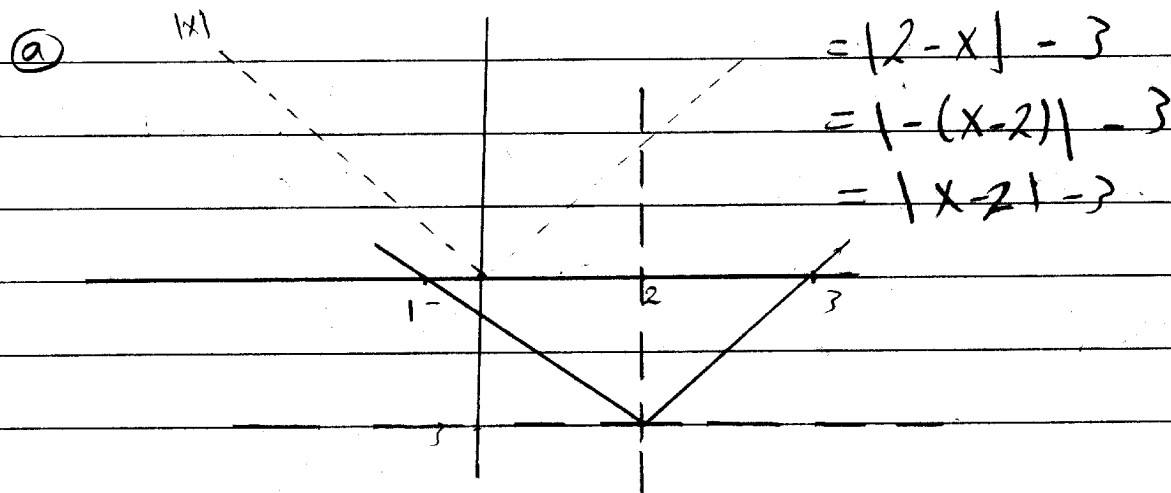
① $y = |2-x| - 3$

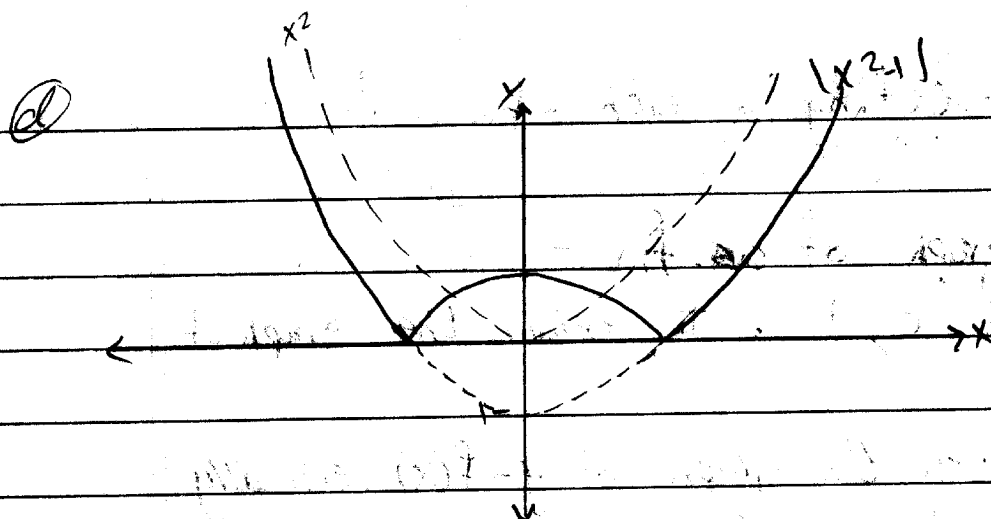
② $y = 2 + \sqrt{x+1}$

③ $y = \frac{1}{x-1} + 2$

④ $y = |x^2 - 1|$

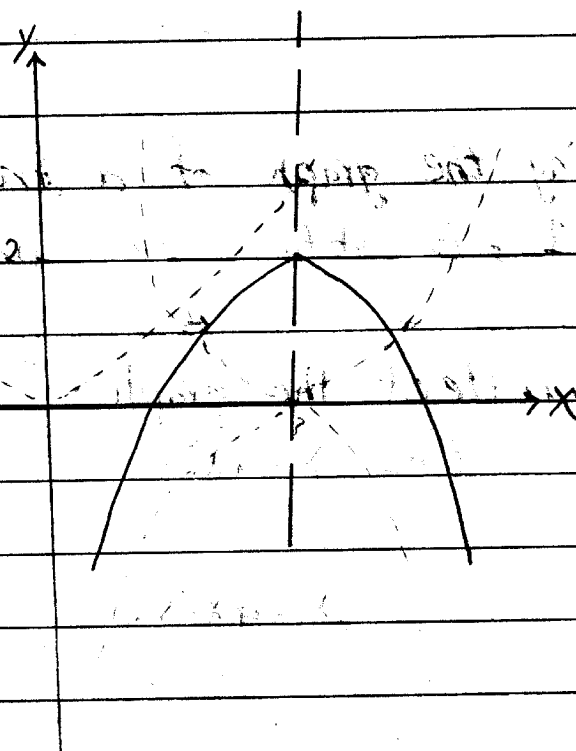
⑤ $y = -x^2 + 6x - 7$





③ $y = -(x^2 - 6x) - 7$
 $= -(x^2 - 6x + 9) + 9 - 7$
 $y = -(x - 3)^2 + 2$

Parabola open down with vertex $(h, k) = (3, 2)$.



Scaling and Reflecting a graph of a fn.

[A] Scaling a graph of a fn:-

Suppose that $c > 1$, To obtain the graph of:

① $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c .

② $y = \frac{1}{c}f(x)$, compress = = = = =

③ $y = f(cx)$, compress = = = = horizontally by = = = =

④ $y = f\left(\frac{x}{c}\right)$, stretch = = = = =

[B] Reflecting the graph of a fn:-

If $c = -1$, to obtain the graph of:

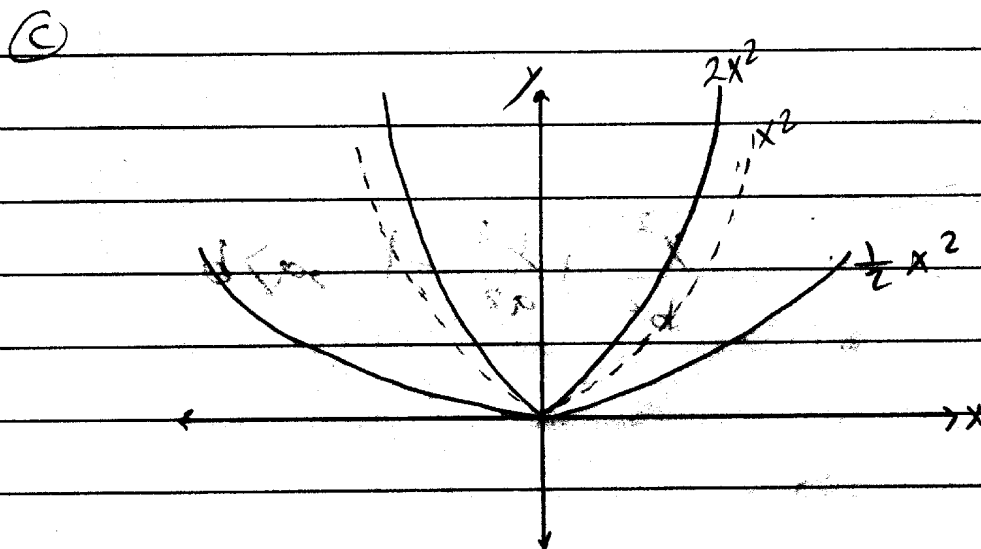
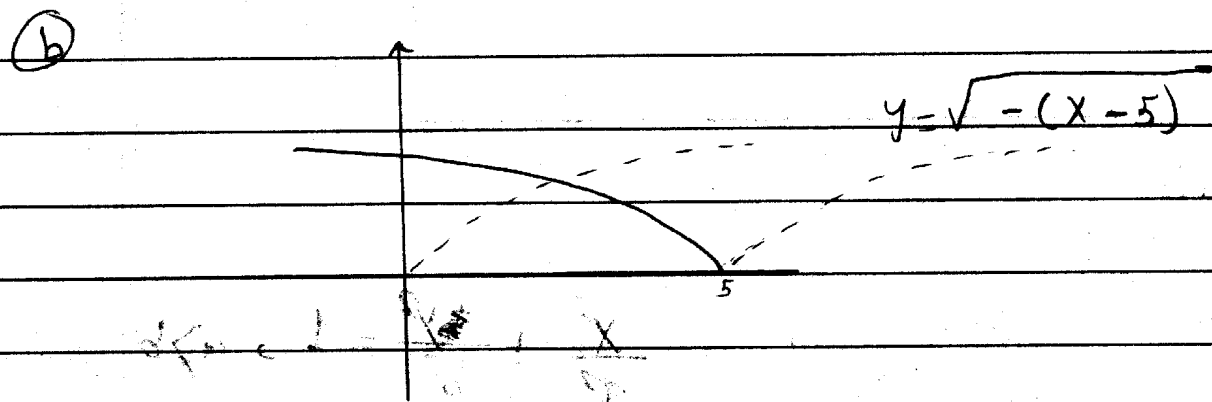
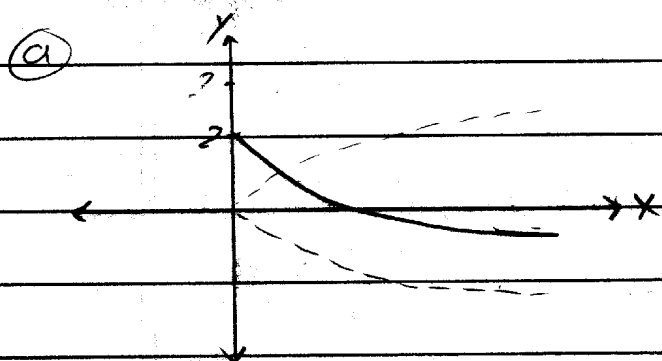
① $y = -f(x)$, reflect the graph of $y = f(x)$ across the x -axis.

② $y = f(-x)$, = = = = =
= y -axis.

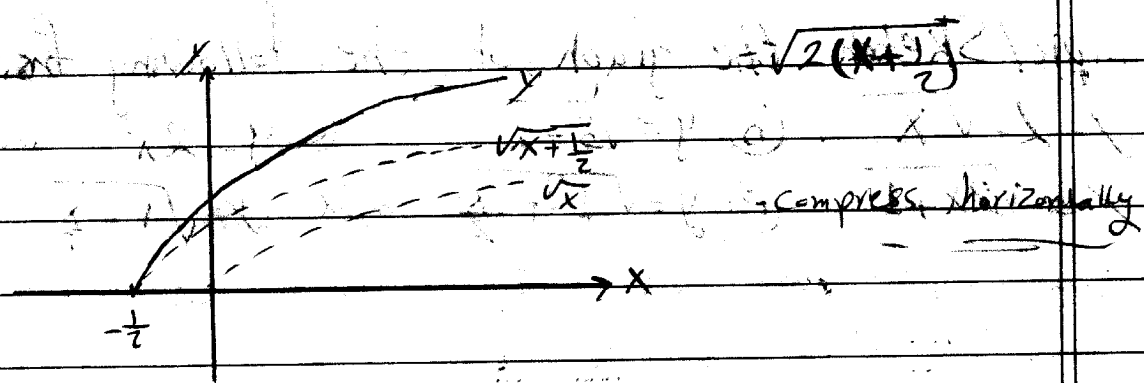
Example/Sketch the graph of the following functions:

(a) $y = 2 - \sqrt{x}$ (b) $y = \sqrt{5-x}$ (c) $y = 2x^2$ and $\frac{1}{2}x^2$.

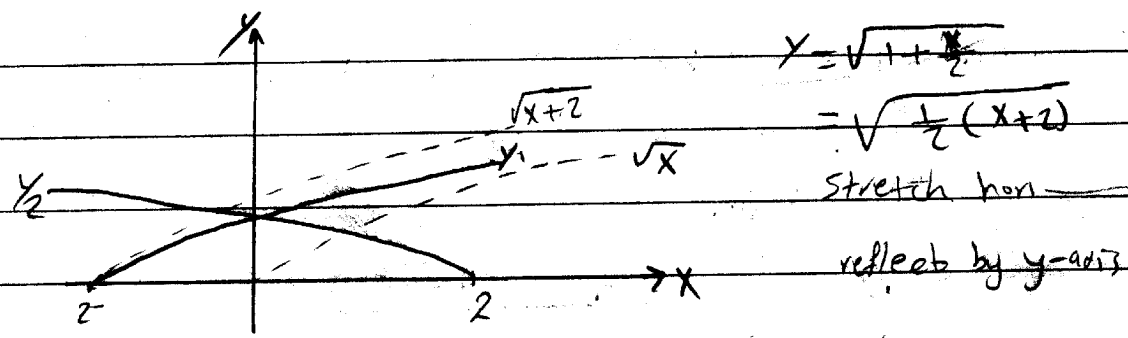
(d) $y = \sqrt{2x+1}$ (e) $y = \sqrt{1+\frac{x}{2}}$ and $y = \sqrt{1-\frac{x}{2}}$.



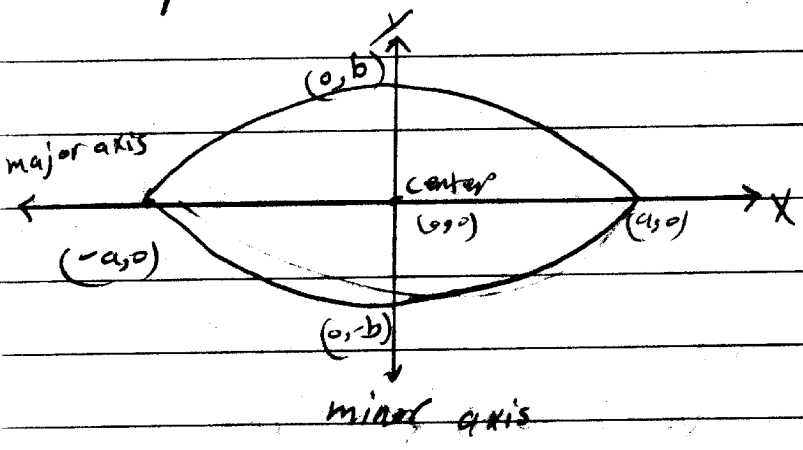
(d)



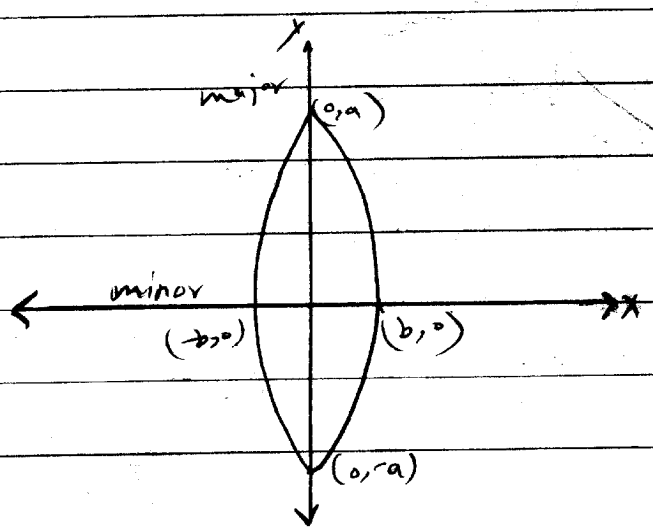
(e)



Ellipses:-



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$



$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$$

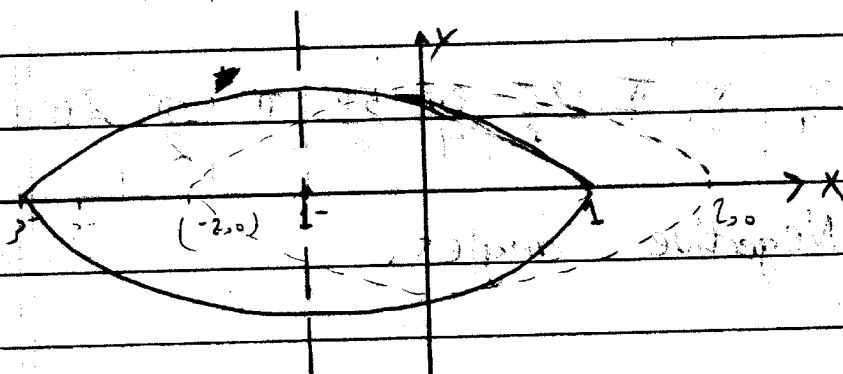
If the center of the ellipse is (h, k) then ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Example/Sketch the ellipse $(x+1)^2 + 2y^2 = 4$

$$\frac{(x+1)^2}{4} + \frac{y^2}{2} = 1$$

ellipse with center $(h, k) = (-1, 0)$

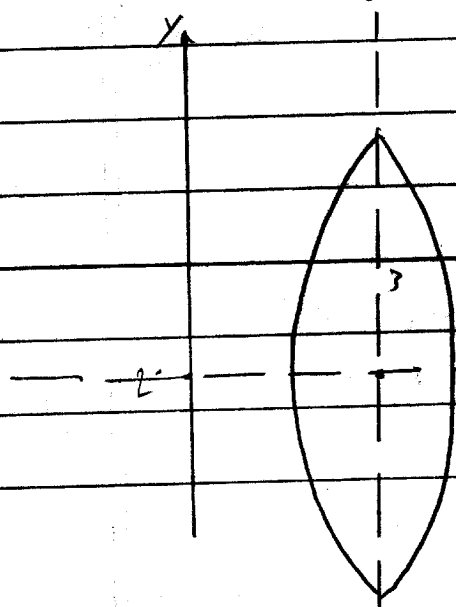


Example/Write an equation for the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$

that shifted 3 units to the right and 2 units down. Sketch the ellipse and identify its center and major axis.

For the given ellipse, the major axis is y-axis

$$a = 5, b = 2$$



The center is $(3, -2)$ and the major axis is $x = 3$

the equation

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{25} = 1$$

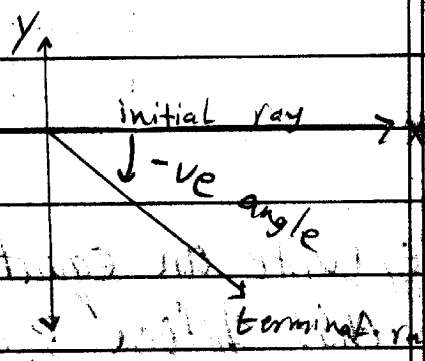
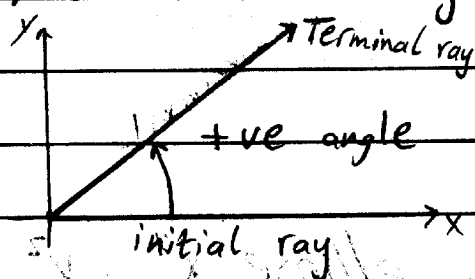
1.3 Trigonometric Functions

Radian measure and degree:

To change Radian measure to degree, multiply by $\frac{180^\circ}{\pi}$
 $\text{degree} = \text{radian} \times \frac{180^\circ}{\pi}$

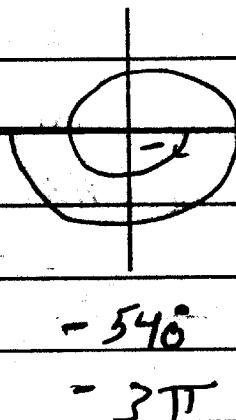
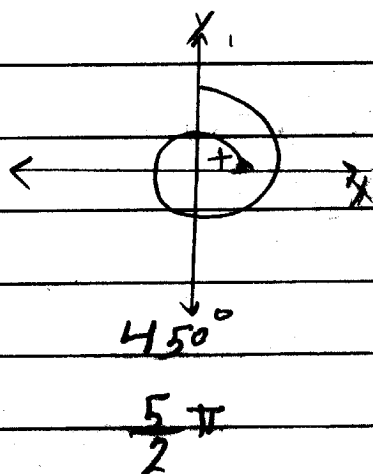
Degree	-45	0	30	45	60	90	120	135	150	180	270	360
Radian	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Positive and Negative angles.



Counter clock wise
Clockwise

Angles $> 360^\circ$.

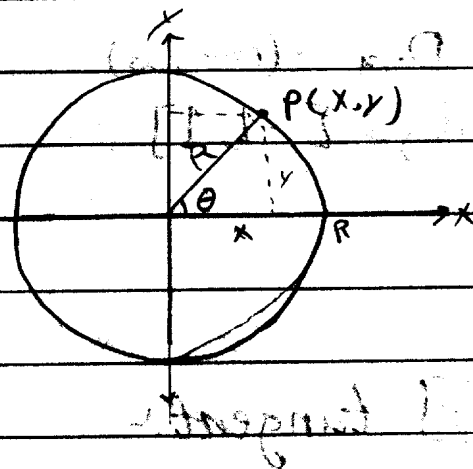


The Six Basic Trigonometric Functions.

sine: $\sin \theta = \frac{y}{R}$ cosecant: $\csc \theta = \frac{R}{y}$

cosine: $\cos \theta = \frac{x}{R}$ secant: $\sec \theta = \frac{R}{x}$

tangent: $\tan \theta = \frac{y}{x}$ cotangent: $\cot \theta = \frac{x}{y}$



$$\Rightarrow \tan \theta = \frac{\sin \theta}{\cos \theta} \rightarrow \cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta} \quad \cdot \quad \csc \theta = \frac{1}{\sin \theta}$$

sin, csc

+

All

+

four quadrants.

All students take

tan, cot

+

cos, sec

+

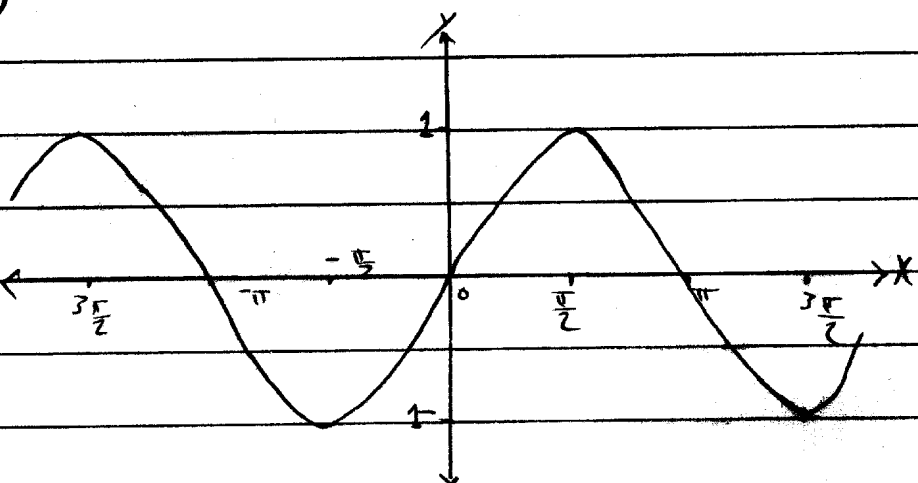
calculus.

The graphs of trigonometric fns:-

[I] Sine:-

$$\text{Dom}(\sin(x)) = (-\infty, \infty)$$

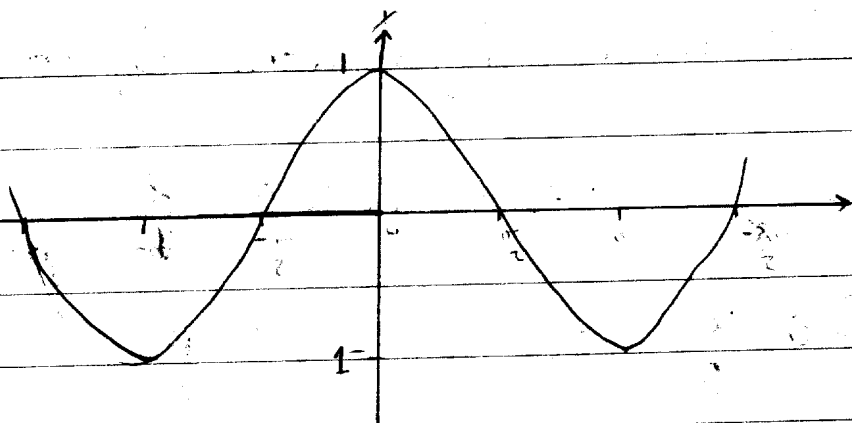
$$\text{Rng}(\sin(x)) = [-1, 1]$$



[2] Cosine:-

$$\text{Dom} = (-\infty, \infty)$$

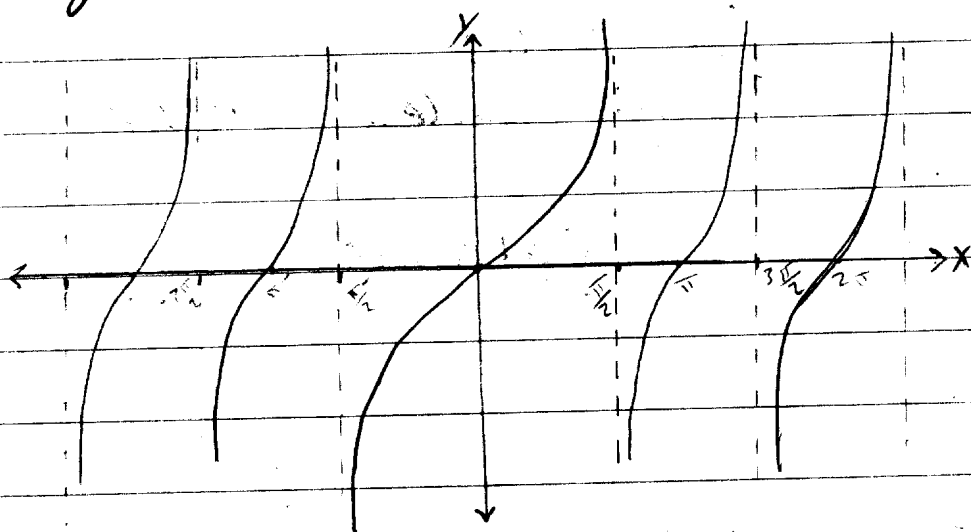
$$\text{Rng} = [-1, 1]$$



[3] tangent:-

$$\text{Dom} = \mathbb{R} - \left\{ n\frac{\pi}{2}, n \text{ is odd} \right\}$$

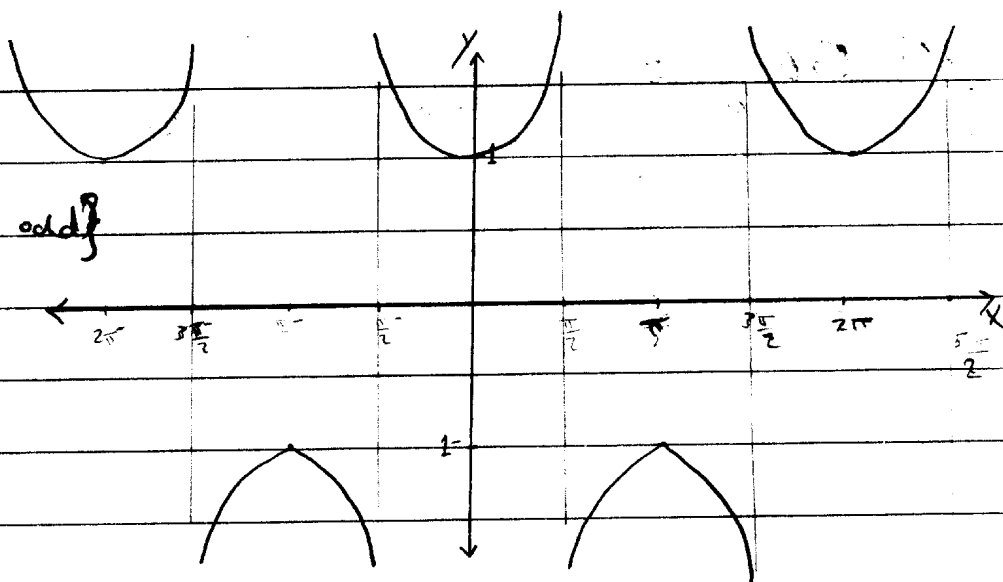
$$\text{Rng} = (-\infty, \infty)$$



[4] Secant:

$$\text{Dom} = \mathbb{R} - \left\{ n\frac{\pi}{2}, n \text{ is odd} \right\}$$

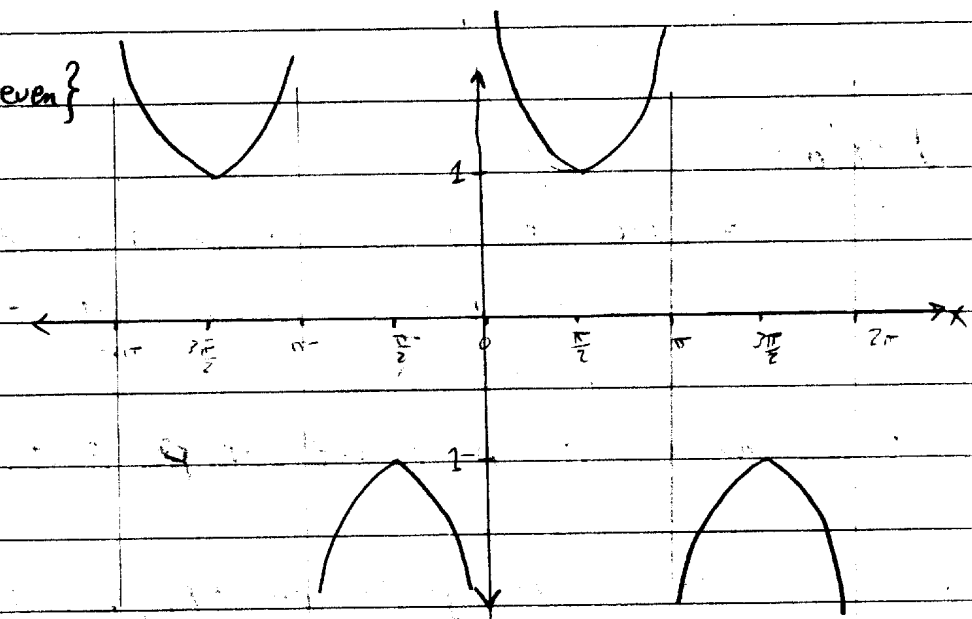
$$\text{Rng} = \mathbb{R} - (-1, 1)$$



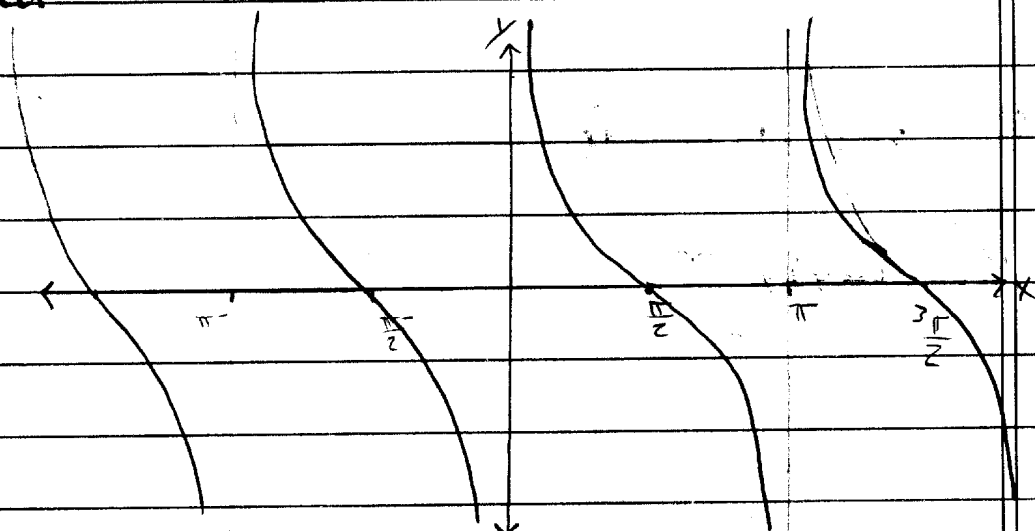
[5] Cosecant:

$$\text{Dom} = \mathbb{R} - \left\{ n\frac{\pi}{2}, n \text{ is even} \right\}$$

$$\text{Rng} = \mathbb{R} - (-1, 1)$$



[6] Cotangent:-



$$\text{Dom} = \mathbb{R} - \{n\pi, n \text{ is an even}\}$$

$$\text{Rng} = (-\infty, \infty)$$

Period:-

Def:- a fn $f(x)$ is called periodic if there is a positive number p such that $f(x+p) = f(x)$ for all x .

* the smallest such value of p is the period of f .

Example/ From the graph of trigonometric fns we have

① The period of $\tan(x)$ and $\cot(x)$ is π , that is,
 $\tan(x+\pi) = \tan(x)$ and $\cot(x+\pi) = \cot(x)$.

② The period of $\cos(x)$, $\sin(x)$, $\sec(x)$ and $\csc(x)$ is 2π , that is, $\sin(x+2\pi) = \sin(x)$.

$$\sin(x-2\pi) = \sin(x), \quad \cos(x-2\pi) = \cos(x)$$

$$\sec(x+2\pi) = \sec(x)$$

Example/ Find the period of the following fns:-

① $f(x) = \cot(\pi x)$

② $f(x) = \sin(5x)$

③ $f(x) = \cos(2x + \frac{\pi}{2})$

④ $f(x) = \tan(\frac{3x}{2}) + 3$

⑤ $\sin(\frac{x}{3})$

① Put $\pi x = \pi \Rightarrow \frac{x}{3} = 1 \Rightarrow x = 3$
the period $P = 3$

② the period of $\sin(x)$ is 2π

put $5x = 2\pi \rightarrow x = \frac{2\pi}{5}$
the period $P = \frac{2\pi}{5}$

③ the period of $\cos(x)$ is 2π (period (π) & shifting)

put $2\pi = 2x \rightarrow x = \pi$
the period $P = \pi$

④ the period of $\tan(x) = \pi$

put $\frac{3x}{2} = \pi \Rightarrow x = \frac{2\pi}{3}$

the period $P = \frac{2\pi}{3}$

⑤ the period of $\sin(x) = 2\pi$

put $\frac{x}{3} = 2\pi \rightarrow x = 6\pi$

$P = 6\pi$

Exercises 1.2...

Algebraic Combinations

$$1) f(x) = x, \quad g(x) = \sqrt{x-1}$$

$$\star \text{Dom}(f) = (-\infty, \infty) \Rightarrow \text{Rng}(f) = (-\infty, \infty)$$

$$\star \text{Dom}(g) = [1, \infty) \Rightarrow \text{Rng}(g) = [0, \infty)$$

$$\star \text{Dom}(f+g) = \text{Dom}(f) \cap \text{Dom}(g) = [1, \infty)$$

$$\text{Rng}(f+g) = [1, \infty)$$

$$\star \text{Dom}(f \cdot g) = [1, \infty) \Rightarrow \text{Rng}(f \cdot g) = [0, \infty)$$

$$3) f(x) = 2, \quad g(x) = x^2 + 1$$

$$\star \text{Dom}(f) = (-\infty, \infty) \Rightarrow \text{Rng}(f) = 2$$

$$\star \text{Dom}(g) = (-\infty, \infty) \Rightarrow \text{Rng}(g) = [1, \infty)$$

$$\star \text{Dom}\left(\frac{f}{g}\right) = \{ \text{Dom}(f) \cap \text{Dom}(g) \} - \{x : g(x) = 0\}$$

$$= (-\infty, \infty)$$

$$\text{Rng}\left(\frac{f}{g}\right) = (0, 2] \quad \text{--- ub. is } \infty$$

$$\star \text{Dom}\left(\frac{g}{f}\right) = \{x : x \in \text{Dom}(g) \cap \text{Dom}(f)\} - \{x : f(x) = 0\}$$

$$= (-\infty, \infty)$$

$$\text{Rng}\left(\frac{g}{f}\right) = \left[\frac{1}{2}, \infty\right)$$

Composites of Functions.

$$[5] f(x) = x + 5, \quad g(x) = x^2 - 3$$

$$a) f(g(0)) := f(0^2 - 3) = f(-3) = -3 + 5 = \underline{\underline{2}}$$

$$b) g(f(0)) := g(0 + 5) = g(5) = (5)^2 - 3 = \underline{\underline{22}}$$

$$\textcircled{c} f(g(x)) := f(x^2 - 3) = x^2 - 3 + 5 = x^2 + 2.$$

$$\textcircled{d} g(f(x)) := g(x+5) := (x+5)^2 - 3 = x^2 + 10x + 22.$$

$$\textcircled{e} f(h(-5)) := f(-5+5) = f(0) = 5.$$

$$\textcircled{f} g(g(2)) := g(2^2 - 3) = g(1) = -2.$$

$$\textcircled{g} f(f(x)) := f(x+5) = (x+5) + 5 = x + 10.$$

$$\textcircled{h} g(g(x)) := g(x^2 - 3) = (x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6.$$

$$\boxed{7} \quad f(x) = x+1, \quad g(x) = 3x, \quad h(x) = 4-x.$$

$$f \circ g \circ h = f \circ g(h(x)) = f(g(h(x))).$$

$$f(g(h(x))) = f(g(4-x)) = f(3(4-x)) = f(12-3x) = 12-3x+1 = 13-3x.$$

$$\boxed{9} \quad f(x) = \sqrt{x+1}, \quad g(x) = \frac{1}{x+4}, \quad h(x) = \frac{1}{x}.$$

$$f \circ g \circ h = f(g(h(x))) = f(g(\frac{1}{x})) = f(\frac{1}{\frac{1}{x}+4})$$

$$= \sqrt{\frac{1}{\frac{1}{x}+4} + 1} =$$

III

a) $f \circ g(x)$.

b) $g \circ f(x)$.

c) $g \circ g(x)$

d) $f \circ f(x)$.

e) $g \circ h \circ f(x)$

f) $h \circ j \circ f(x)$.

[13] a) $f \circ g(x) = f(g(x)) = f(x-7) = \sqrt{x-7}$.

b) $f \circ g(x) = f(g(x)) = \sqrt{g(x)-5} = \sqrt{x^2-5}$

$g(x)-5 = x^2-5 \rightarrow g(x) = x^2$.

[14] $f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = x$

put $y = \frac{1}{x} \rightarrow f(y) = \frac{1}{y}$

$\therefore f(x) = \frac{1}{x}$

[15] a) $f(1) = 1$.

b) $g(-2) = 2$.

$\frac{1}{x} = (x) \cdot \frac{1}{1+x} = 1+x \therefore (x) \cdot \frac{1}{1+x} = 1+x$

$$[19] f \circ g(x) = f(g(x)) = \frac{g(x)}{g(x)-2} = x$$

$$g(x) = x g(x) - 2x$$

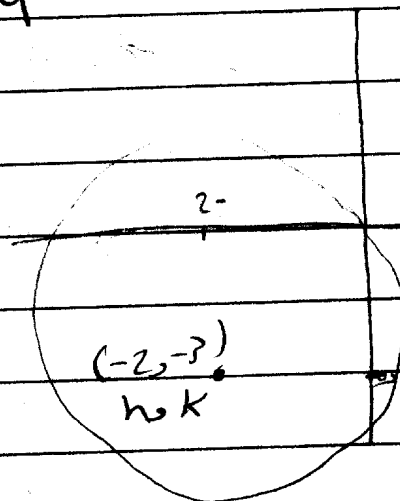
$$g(x)(1-x) = -2x$$

$$\Rightarrow g(x) = \frac{-2x}{1-x} = \frac{2x}{x-1} \quad \#$$

$$[21] \quad -(x-4)^2 \quad \text{and} \quad -(x+7)^2$$

$$[25] \quad (x-h)^2 + (y-k)^2 = R^2$$

$$(x+2)^2 + (y+3)^2 = 49$$



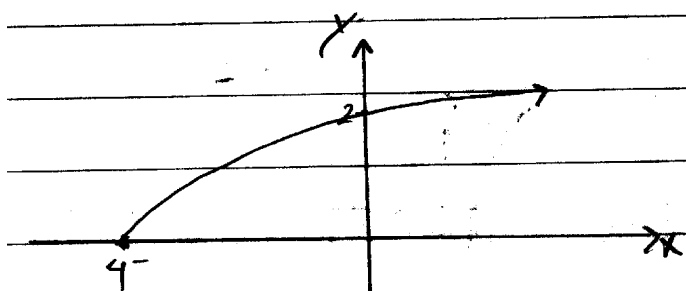
$$[27] \quad y = (x+1)^3 - 1$$

$$[29] \quad y = \sqrt{x+0.81}$$

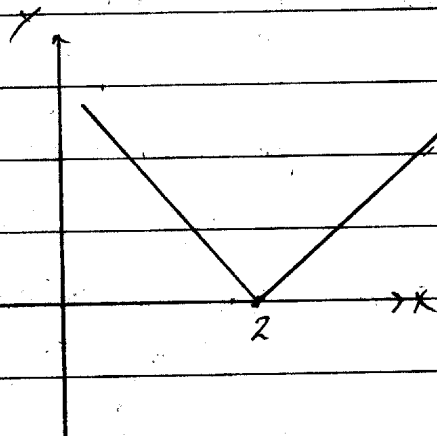
$$[31] \quad y = 2x - 7 + 7 = 2x$$

$$[39] \quad y = \frac{1}{(x+2)^2} - 1$$

25)

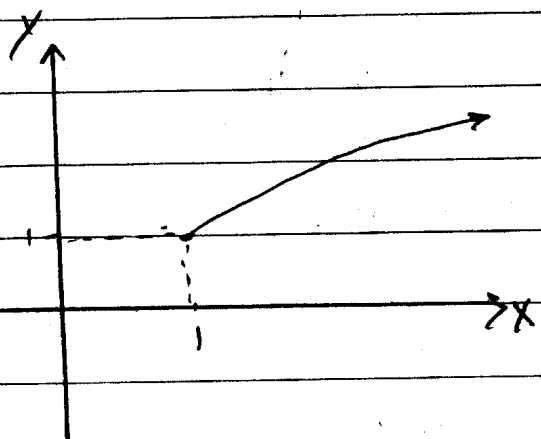


37)

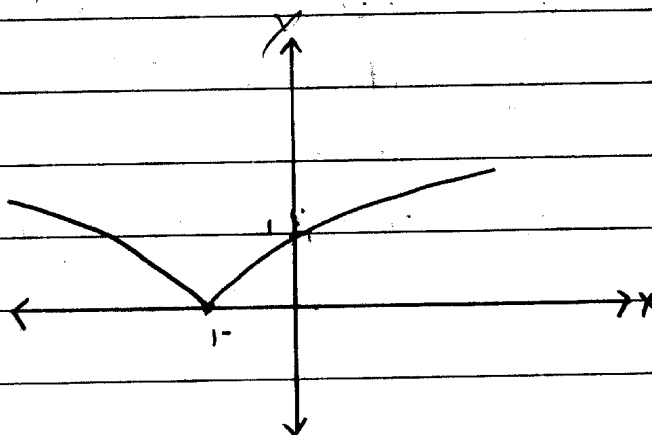


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29)

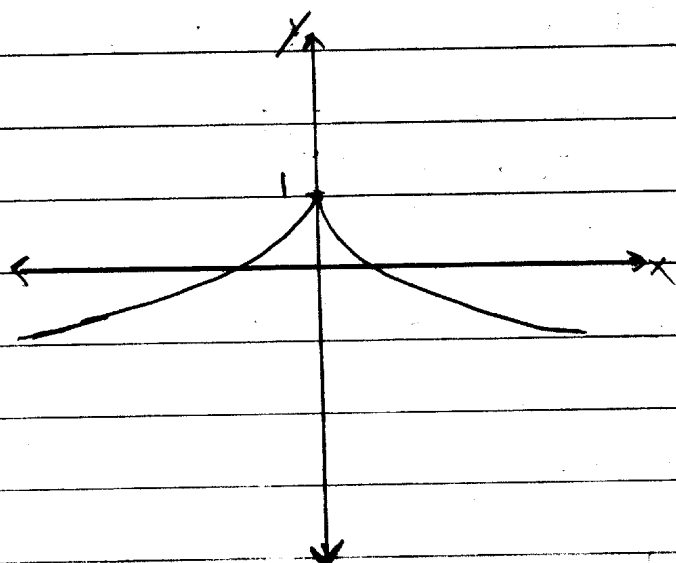


41)

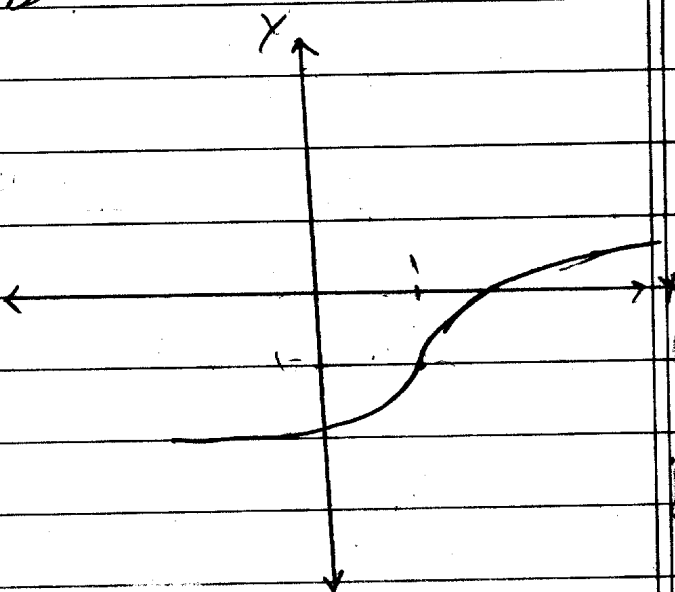


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43)



45)



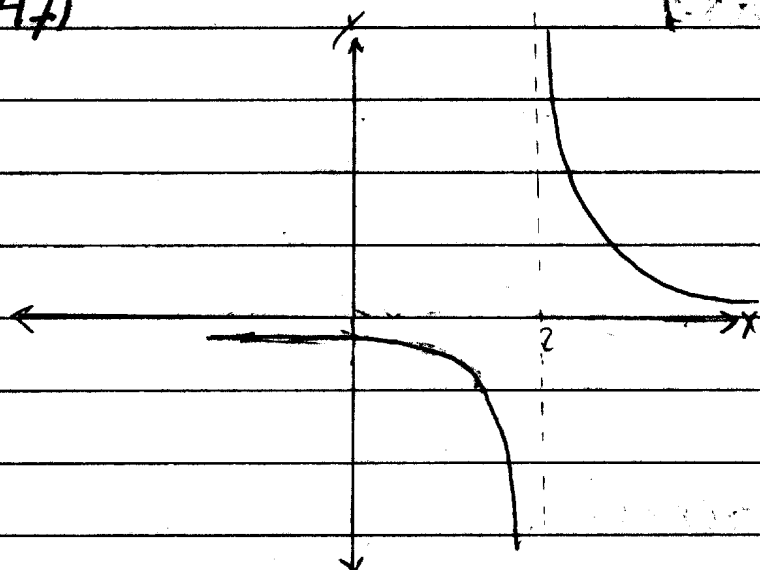
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118

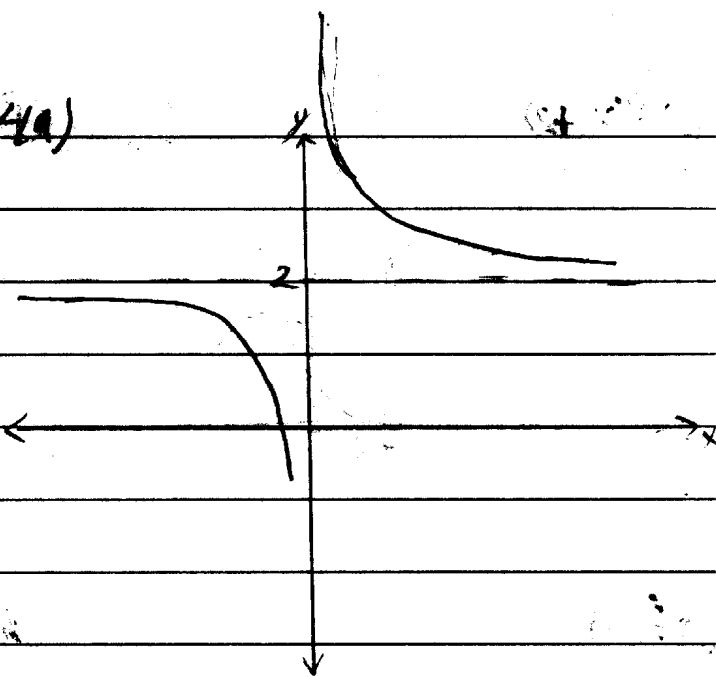
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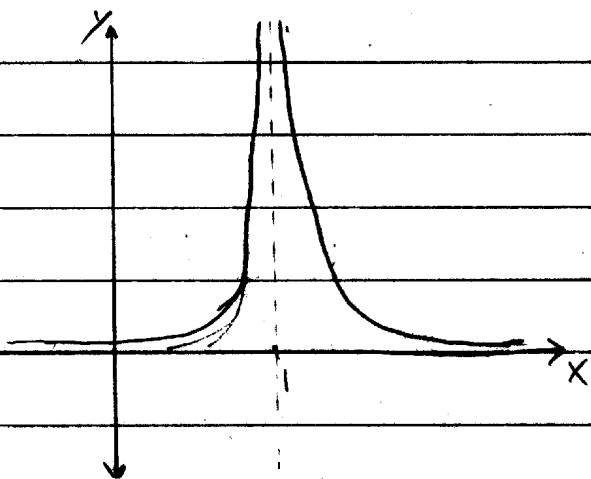
47)



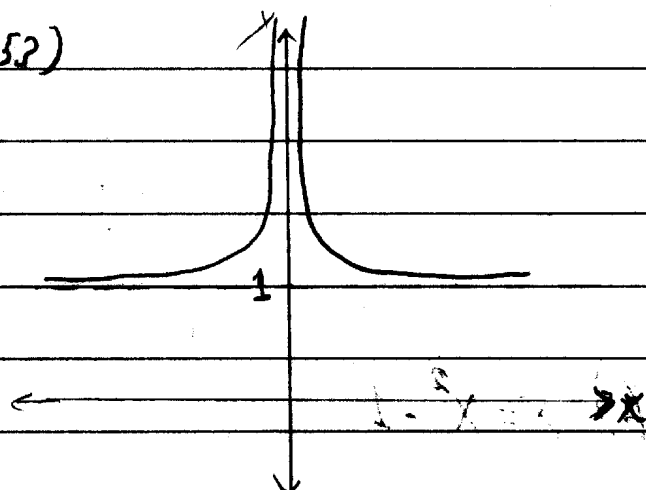
48)



51)



52)

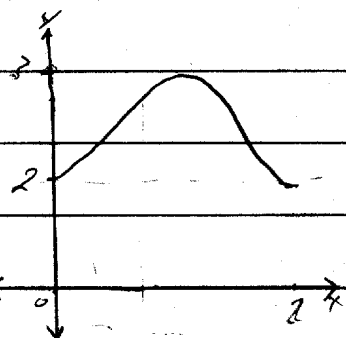


55)

a) $f(x) + 2$

Dom = $[0, 2]$

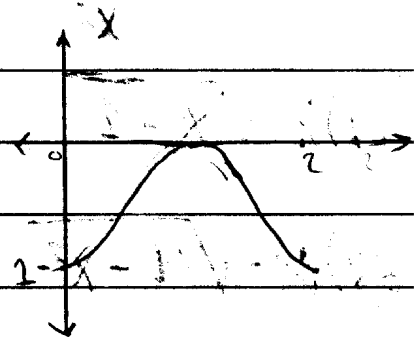
Rng = $[2, 3]$



b) $f(x) - 1$

Dom = $[0, 2]$

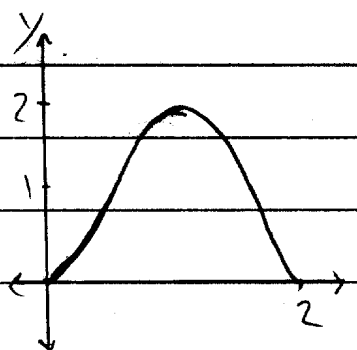
Rng = $[-1, 0]$



c) $2f(x)$

Dom = $[0, 2]$

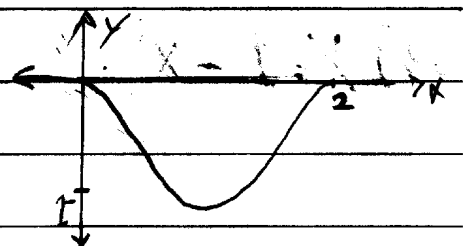
Rng = $[0, 2]$



d) $-f(x)$

Dom = $[0, 2]$

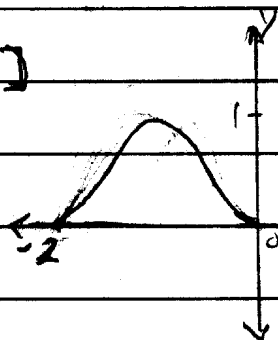
Rng = $[-1, 0]$



⑥ $f(x+2)$

Dom = $[-2, 0]$

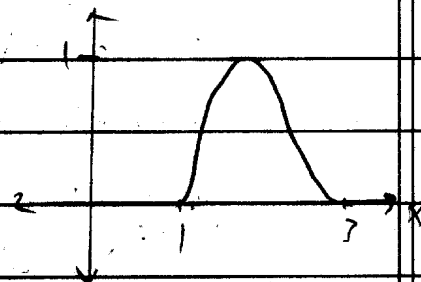
Rng = $[0, 1]$



⑦ $f(x-1)$

Dom = $[1, 3]$

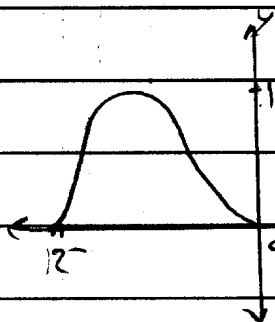
Rng = $[0, 1]$



⑧ $f(-x)$

Dom = $[-2, 0]$

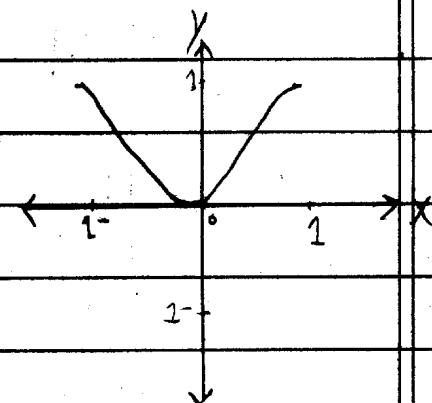
Rng = $[0, 1]$



⑨ $-f(x+1)+1$

Dom = $[-1, 1]$

Rng = $[0, 1]$



57) $y = x^2 - 1 \Rightarrow y = 3(x^2 - 1) = 3x^2 - 3$

59) $y = 1 + \frac{1}{x^2} \Rightarrow y = \frac{1}{2} \left(1 + \frac{1}{x^2} \right) = \frac{1}{2} + \frac{1}{2x^2}$

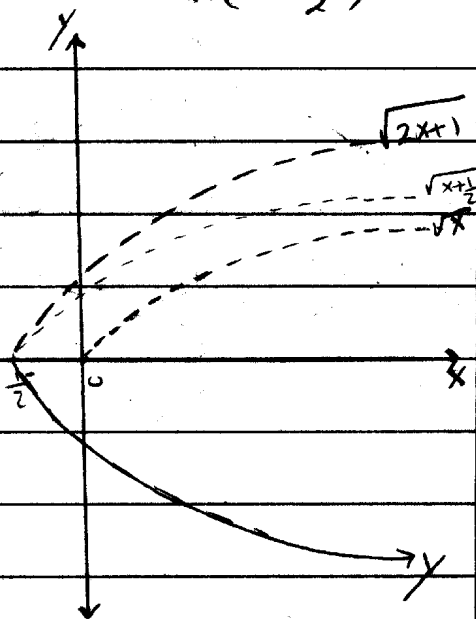
61) $y = \sqrt{x+1} \Rightarrow y = \sqrt{4x+1}$

63) $y = \sqrt{4-x^2} \Rightarrow y = \sqrt{4 - \left(\frac{x}{2}\right)^2} = \sqrt{4 - \frac{x^2}{4}}$

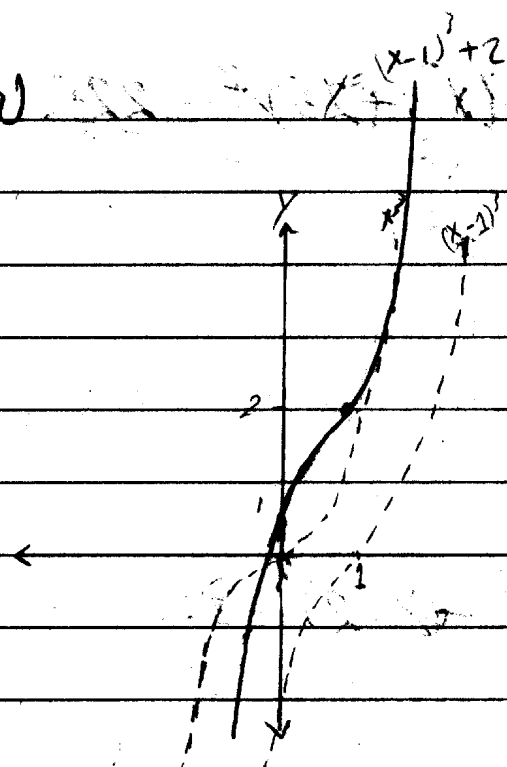
65) $y = 1 - x^3 \Rightarrow y = 1 - (3x)^3 = 1 - 27x^3$

67)

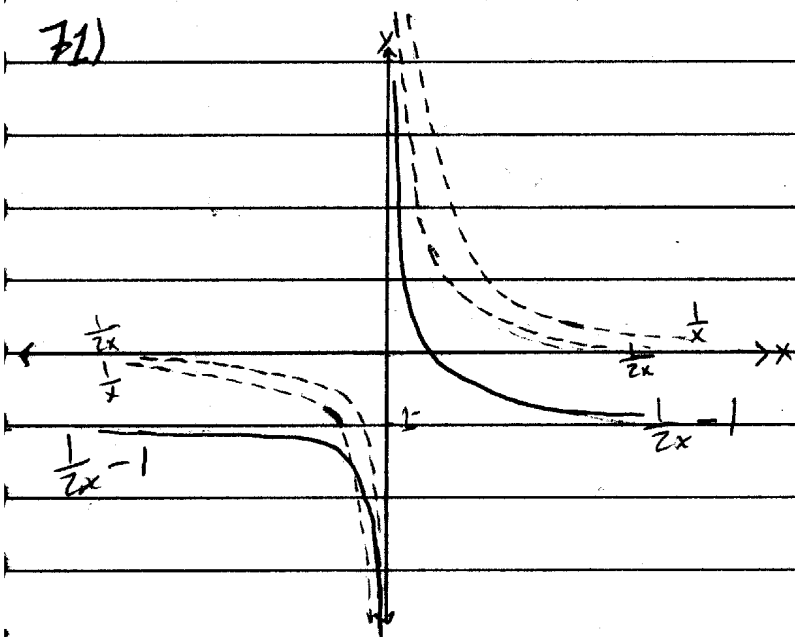
$$\sqrt{2(x + \frac{1}{2})}$$



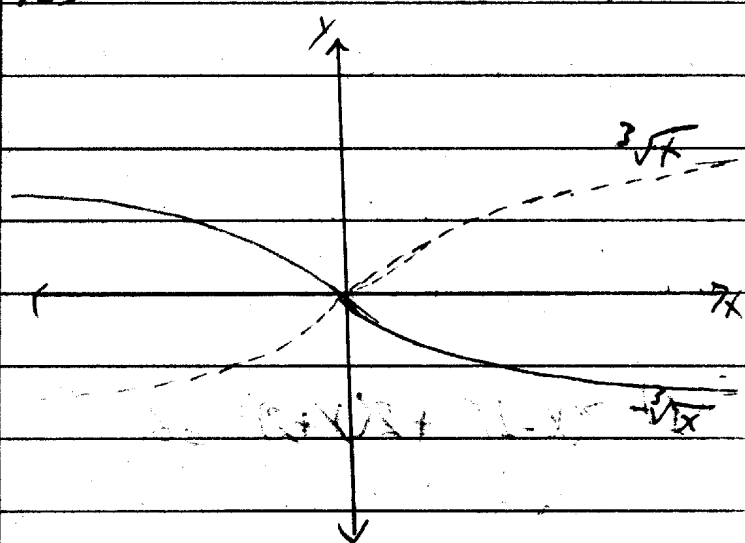
69)



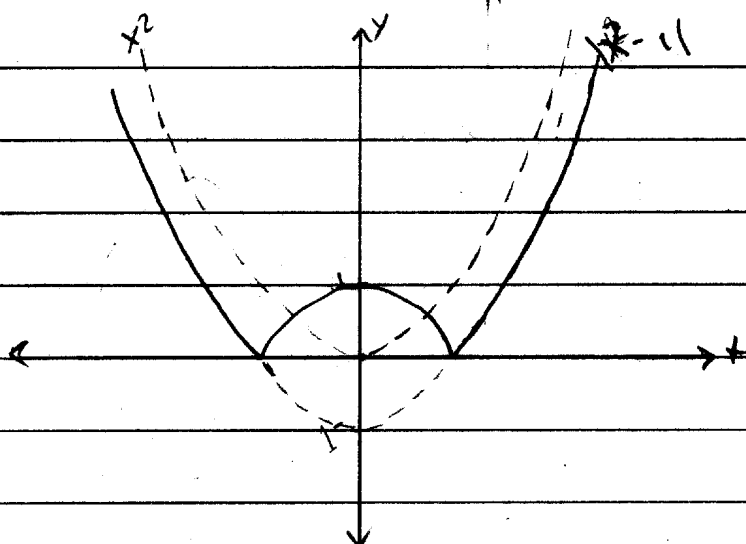
71)



73)



75)



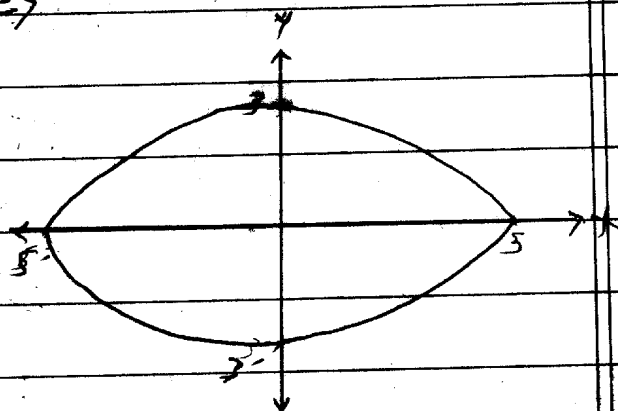
$$77) 9x^2 + 25y^2 = 225 \Rightarrow$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$a=5, b=3$$

The major axis is x-axis.

The center is (0,0)



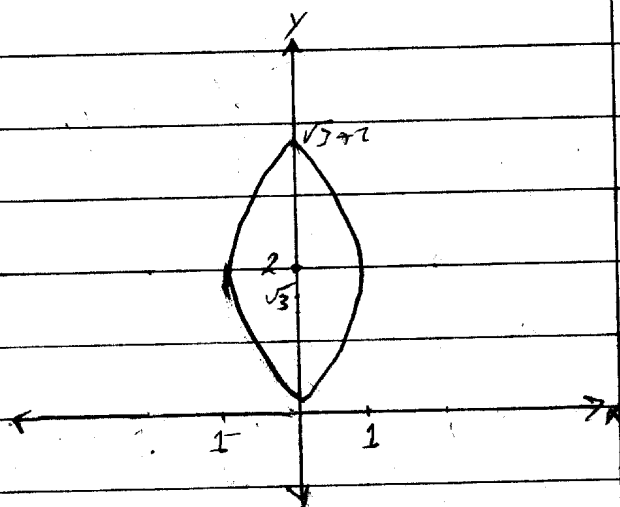
$$79) 3x^2 + (y-2)^2 = 3$$

$$\frac{x^2}{1} + \frac{(y-2)^2}{3} = 1$$

$$a=1, b=\sqrt{3}$$

The center is (0,2)

The major axis is y-axis.

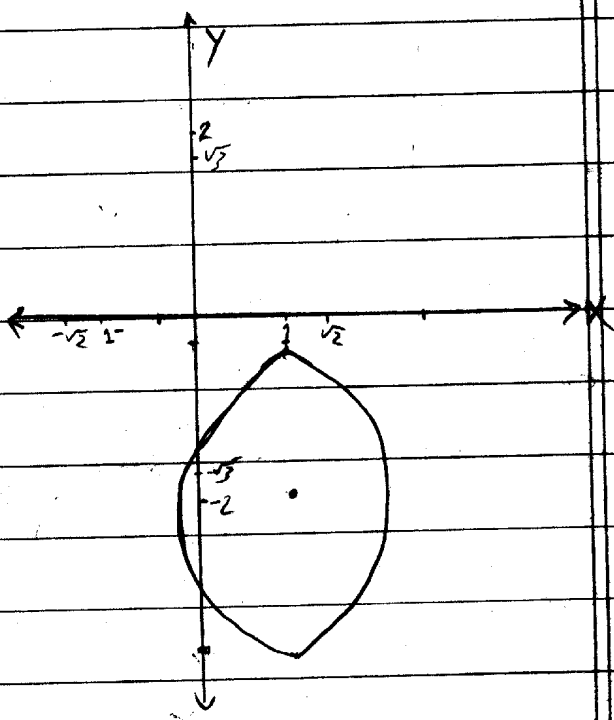


$$81) 3(x-1)^2 + 2(y+2)^2 = 6$$

$$\frac{(x-1)^2}{2} + \frac{(y+2)^2}{3} = 1$$

The center = (1, -2)

$$a=\sqrt{2}, b=\sqrt{3}$$

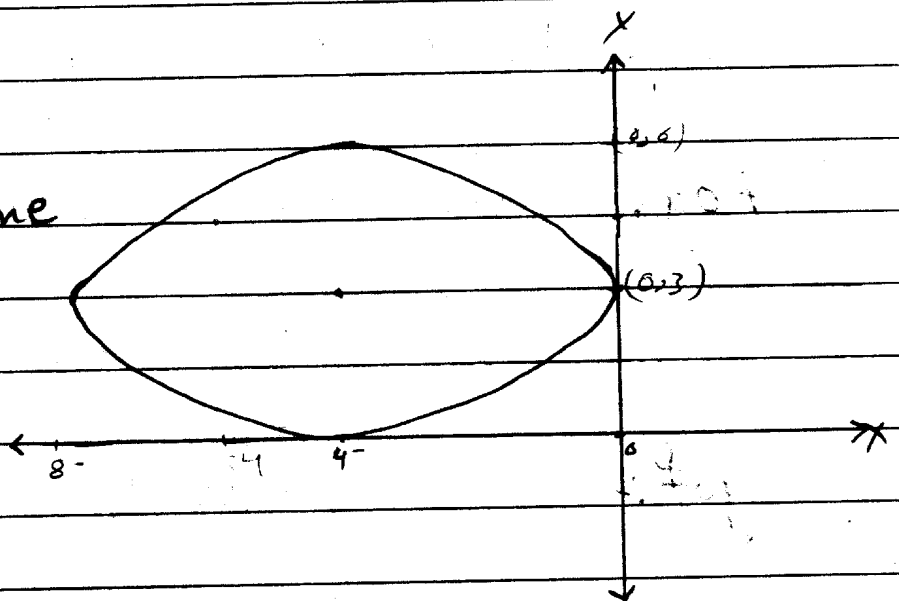


$$83) \frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow \frac{(x+4)^2}{16} + \frac{(y-3)^2}{9} = 1$$

The center is $(-4, 3)$

~~the major axis is x-axis~~

the major axis is the line segment between $(-8, 3)$ and $(0, 3)$



25) f is an even fn $\rightarrow f(x) = f(-x)$

g is an odd fn $\rightarrow g(x) = -g(-x)$

$$\textcircled{a} fg: fg(-x) = f(-x) \cdot g(-x) = f(x) \cdot -g(x) = -fg(x)$$

$\therefore fg$ is an odd fn. because $fg(-x) = -fg(x)$

$$\textcircled{b} f/g: \frac{f}{g}(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\frac{f}{g}(x)$$

f/g is an odd fn.

$\textcircled{c} g/f$ is $\neq$ $\neq$ $=$ $=$, is an odd fn

$$\textcircled{d} f^2 = f \cdot f(x) := f \cdot f(-x) = f(-x) \cdot f(-x) = f(x) \cdot f(x) = f \cdot f(x)$$

f^2 is an even fn.

⑥ g^2 : $g \cdot g(-x) = g(-x) \cdot g(-x) = -g(x) \cdot -g(x)$
 $= g(x) \cdot g(x) = g \cdot g(x)$
 g^2 is an even fn.

⑦ $f \circ g$: $f \circ g(-x) = f(g(-x)) = f(-g(x)) = f(g(x))$
 $= f \circ g(x)$
 So $f \circ g$ is an even fn.

⑧ $g \circ f$: $g \circ f(-x) = g(f(-x)) = g(-f(x)) = -g(f(x))$
 $= -g \circ f(x)$
 So $g \circ f$ is an odd fn.

⑨ $f \circ f$: $f \circ f(-x) = f(f(-x)) = f(f(x)) = f \circ f(x)$
 So $f \circ f$ is an even fn.

⑩ $g \circ g$: $g \circ g(-x) = g(g(-x)) = g(-g(x)) = -g(g(x))$
 $= -g \circ g(x)$
 So $g \circ g$ is an odd fn.

Exercises 1.3.2

$$1) \text{ a) } s = r\theta = 10 \cdot \frac{4\pi}{5} = \underline{\underline{8\pi}} \text{ m}$$

$$\text{b) } s = r\theta = 10 \cdot 110 \times \frac{\pi}{180} = \underline{\underline{\frac{55\pi}{9}}} \text{ m}$$

$$3) r = \frac{1}{2} \text{ diameter} = \frac{1}{2} \cdot 12 = 6 \text{ inch}$$

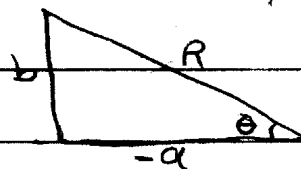
$$s = r\theta = 6 \cdot \frac{80 \cdot \pi}{180} = \underline{\underline{83.73}} \text{ inch}$$

5) θ	$-\pi$	$-\frac{2\pi}{3}$	0	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$\sin \theta$	0	1	0	1	$\frac{1}{\sqrt{2}}$
$\cos \theta$	-1	0	1	0	$-\frac{1}{\sqrt{2}}$
$\tan \theta$	0	∞	0	∞	-1
$\cot \theta$	∞	0	∞	0	-1
$\sec \theta$	-1	∞	1	∞	$-\sqrt{2}$
$\csc \theta$	∞	1	∞	1	$\sqrt{2}$

$$7) \sin x = \frac{3}{5}, x \in [\frac{\pi}{2}, \pi]$$

$$R^2 = a^2 + b^2 \Rightarrow a^2 = 25 - 9$$

$$\therefore \underline{\underline{a = -4}}$$

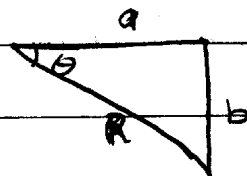


$$\cos x = \frac{-4}{5}, \tan x = \frac{5}{-4} = -\frac{5}{4}$$

[9] $\cos X = \frac{1}{3}, X \in [-\frac{\pi}{2}, 0]$.

$$R^2 = a^2 + b^2$$

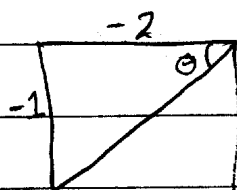
$$9 = 1 + b^2 \rightarrow b = -\sqrt{8} = -2\sqrt{2}$$



$$\sin X = \frac{-2\sqrt{2}}{3}, \quad \tan X = \frac{-2\sqrt{2}}{1} = -2\sqrt{2}.$$

[11] $\tan X = \frac{1}{2}, X \in [\pi, \frac{3\pi}{2}]$

$$R^2 = a^2 + b^2 = (-1)^2 + (-2)^2 = 5 \Rightarrow R = \sqrt{5}$$



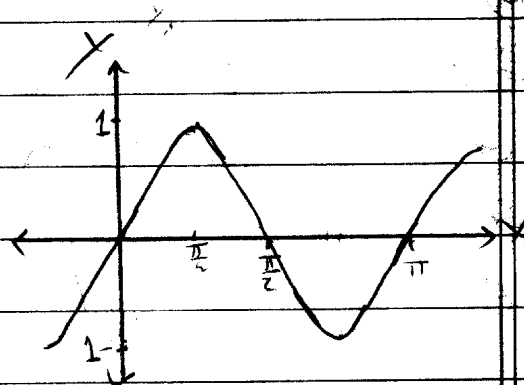
$$\sin X = \frac{-1}{\sqrt{5}}, \quad \cos X = \frac{-2}{\sqrt{5}}.$$

[13] $\sin 2X$

the period of $\sin X$ is 2π

$$\text{so } 2X = 2\pi \Rightarrow X = \pi$$

the period is π

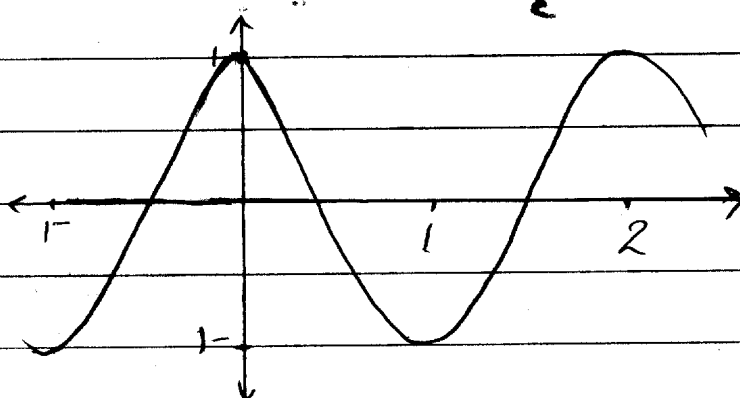


[15] $\cos \pi x$

$$\pi x = 2\pi$$

$$x = 2$$

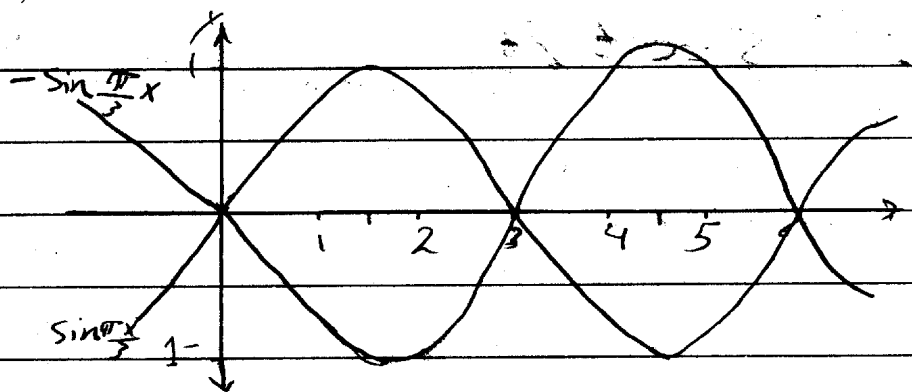
Period = 2.



$$[17] -\sin \frac{\pi}{3} x.$$

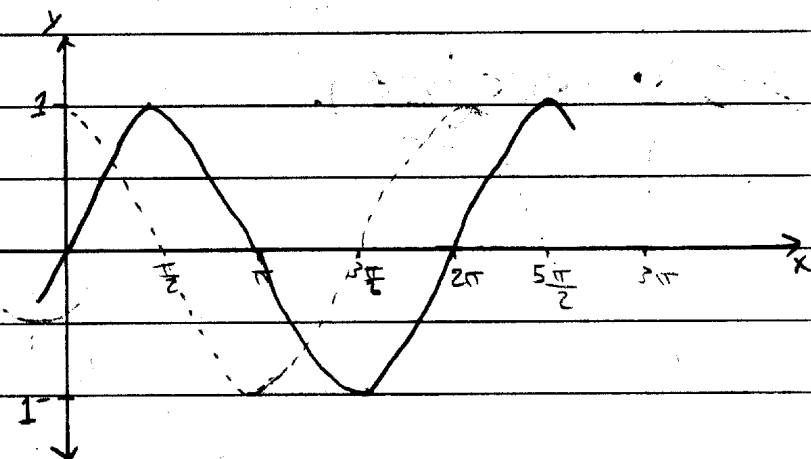
$$\frac{\pi x}{3} = 2\pi$$

$$\Rightarrow x = 6$$



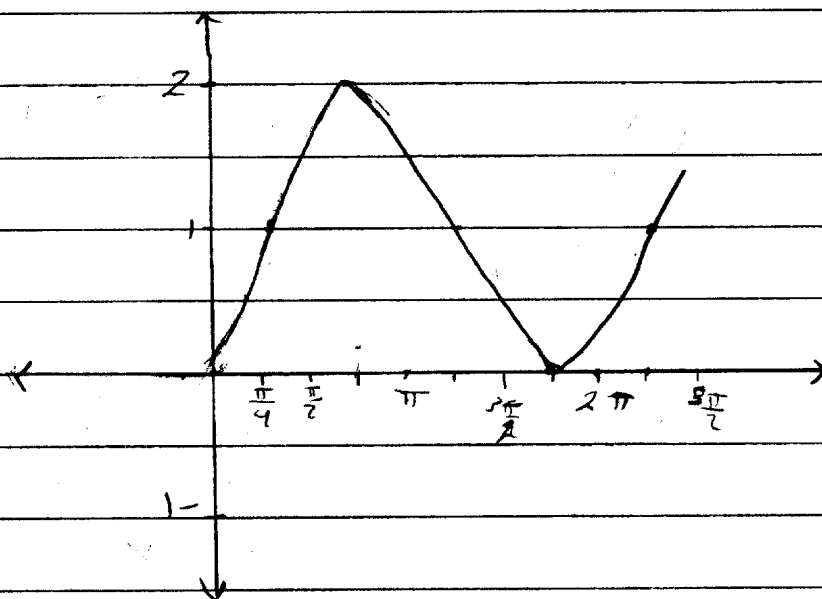
$$[19] \cos(x - \frac{\pi}{2}).$$

$$x = 2\pi$$



$$[21] \sin(x - \frac{\pi}{4}) + 1.$$

$$\text{the period} = 2\pi$$

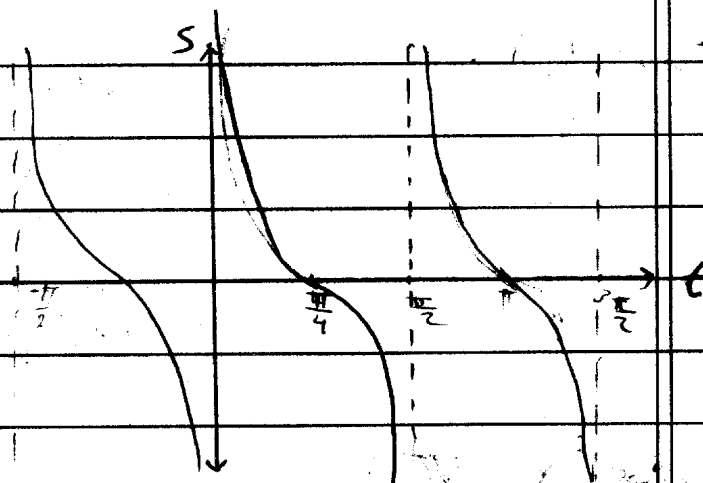


[23] $s = \cot 2t$

$$2t = \pi$$

$$\Rightarrow t = \frac{\pi}{2}$$

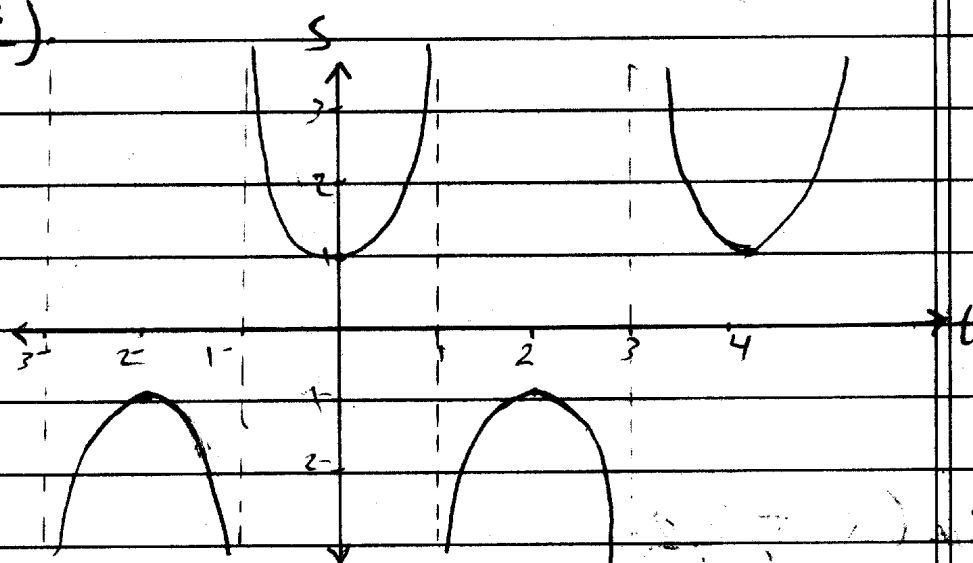
The period = $\frac{\pi}{2}$



[25] $s = \sec(\pi t)$

$$\frac{\pi t}{2} = 2\pi$$

$$\Rightarrow t = 4$$



From the graph of Trigonometric fns, we have.

① $\cos(x)$ and $\sec(x)$ are symmetric about the y-axis, that is, $\cos(-x) = \cos(x)$ and $\sec(-x) = \sec(x) \Rightarrow \cos(x)$ and $\sec(x)$ are even fns.

② $\sin(x)$, $\tan(x)$, $\csc(x)$ and $\cot(x)$ are symmetric about the origin, that is, $\tan(-x) = -\tan(x)$, $\sin(-x) = -\sin(x)$, $\cot(-x) = -\cot(x)$, $\csc(-x) = -\csc(x) \Rightarrow \tan(x)$, $\sin(x)$, $\csc(x)$, and $\cot(x)$ are odd fns.

Example/Determine whether the following fns are even, odd or neither.

① $f(x) = x^2 \cos(2x)$. ② $f(x) = \frac{|x| + \sin^2 x}{\sqrt[3]{x}}$

③ $f(x) = \frac{1 + \tan(x + 2\pi)}{\sec(x - 2\pi)}$

① $f(-x) = (-x)^2 \cos(2(-x)) = x^2 \cos(-2x)$
 $= x^2 \cos 2x = f(x)$
 So $f(x)$ is an even fn.

② $f(-x) = \frac{|-x| + (\sin(-x))^2}{\sqrt[3]{-x}} = \frac{|x| + (-\sin x)^2}{-\sqrt[3]{x}}$
 $= \frac{|x| + \sin^2 x}{-\sqrt[3]{x}} = -f(x)$

So $f(x)$ is an odd fn.

$$\textcircled{1} f(x) = \frac{1 + \tan(x)}{\sec(x)}$$

$$f(-x) = \frac{1 + \tan(-x)}{\sec(-x)} = \frac{1 - \tan(x)}{\sec(x)} \neq f(x)$$

So $f(x)$ is not an even fn.

$$f(-x) = \frac{1 - \tan(x)}{\sec(x)} \neq -\left(\frac{-1 + \tan(x)}{\sec(x)}\right) \neq -f(x)$$

So $f(x)$ is not an odd fn.

$\Rightarrow f$ is neither even nor odd.

Trigonometric Identities

$$\text{I}] \cos^2 x + \sin^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\text{2}] \sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \cdot \tan(y)}$$

If $x=y$, we get

$$3) \sin 2x = 2 \sin(x) \cos(x)$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2 \cos^2 x - 1 \\ &= 1 - 2 \sin^2 x \end{aligned}$$

$$4) \cos^2 x = \frac{1}{2} (1 + \cos 2x).$$

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x).$$

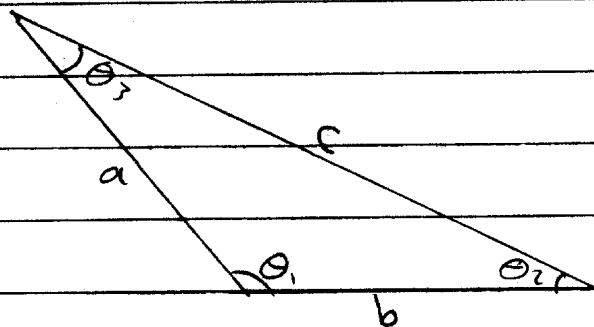
$$5) \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

6) Law of cosine:-

$$\cos \theta_1 = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos \theta_2 = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos \theta_3 = \frac{a^2 + c^2 - b^2}{2ac}.$$



7) For any angle θ measured in Radians, we have

$$-1 \leq \sin \theta \leq 1$$

$$-1 \leq 1 - \cos \theta \leq 1$$

Example / Find the Domain and Range of the following fns.

① $f(x) = \sqrt{\cos^2 x}$ ② $f(x) = 1 - \cos(x)$.

③ $f(x) = 1 + \tan^2 x$.

④ $f(x) = |\cos x|$. $\rightarrow$ period = π

$\text{Dom}(f) = (-\infty, \infty) \rightarrow \text{Rng}(f) = [0, 1]$.

⑤ $\text{Dom}(f) = (-\infty, \infty) \rightarrow \text{Rng}(f) = [0, 2]$.

⑥ $\text{Dom}(f) = \mathbb{R} - \left\{ n \cdot \frac{\pi}{2}, n \text{ is an odd} \right\}$
 $\text{Rng}(f) = [1, \infty)$

Example / Write the following fns in terms of $\sin x$ and $\cos x$.

① $\sin\left(\frac{3\pi}{2} - x\right)$ ② $\cos(3\pi + x)$ ③ $f(x) = \frac{1}{\tan(x) + \cot(x)}$

④ $\cos^2\left(\frac{x}{2}\right)$ ⑤ $\sin(3x)$.

① $\sin\left(\frac{3\pi}{2}\right)\cos(x) - \cos\left(\frac{3\pi}{2}\right)\sin x = -1 \cos x - 0$
 $= -\cos(x)$.

② $\cos(2\pi + (\pi + x)) = \cos(\pi + x) = \cos(\pi)\cos(x) - \sin(\pi)\sin(x)$
 $= -1 \cos(x) - 0 = -\cos(x)$

③ $\frac{1}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{1}{\frac{\cos^2 x + \sin^2 x}{\cos x \cdot \sin x}} = \frac{\cos x \cdot \sin x}{1} = \cos x \cdot \sin x$.

$$\textcircled{a} \cos X = 2 \cos^2 \frac{X}{2} - 1$$

$$2 \cos^2 \left(\frac{X}{2} \right) = 1 + \cos X \Rightarrow \cos^2 \left(\frac{X}{2} \right) = \frac{1}{2} (1 + \cos X)$$

$$\therefore \cos^2 \left(\frac{X}{2} \right) = \frac{1}{2} (1 + \cos X)$$

$$\textcircled{c} \sin(3X) = \sin(X+2X) = \sin(X) \cos(2X) + \cos(X) \sin(2X)$$

$$\sin(X)(2 \cos^2 X - 1) + \cos(X)(2 \cos$$

Example/Solve for θ , where $0 \leq \theta < 2\pi$.

$$\textcircled{a} \sqrt{3} \sec \theta = 2. \quad \textcircled{b} \sin 2\theta - \cos \theta = 0.$$

$$\textcircled{a} \sec \theta = \frac{2}{\sqrt{3}} \Rightarrow \cos \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$\textcircled{b} 2 \sin \theta \cos \theta - \cos \theta = 0$$

$$\cos \theta (2 \sin \theta - 1) = 0$$

$$\Rightarrow \cos \theta = 0 \quad , \quad 2 \sin \theta - 1 = 0 \Rightarrow \sin \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Example/Show that $\frac{\sin \theta + \cos 2\theta - 1}{\cos \theta - \sin 2\theta} = \tan \theta$.

$$\frac{\sin \theta - 2 \sin^2 \theta}{\cos \theta - 2 \sin \theta \cos \theta} = \frac{\sin \theta (1 - 2 \sin \theta)}{\cos \theta (1 - 2 \sin \theta)}$$

$$= \frac{\sin \theta}{\cos \theta} = \tan \theta$$

$$b) \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} = \tan\left(\frac{\theta}{2}\right)$$

$$\frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} = \frac{1 - \cos^2 \theta}{\sin \theta (1 + \cos \theta)} = \frac{\sin^2 \theta}{\sin \theta (1 + \cos \theta)}$$

$$= \frac{\sin \theta}{1 + \cos \theta} \quad \neq$$

$$\text{And } \frac{\sin \theta}{1 + \cos \theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{1 + 2 \cos^2 \frac{\theta}{2} - 1} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan\left(\frac{\theta}{2}\right)$$

#

Chapter 2 :- Limits and Continuity :-

2.1 Rate of change and tangents to curves.

II] Rates of change :-

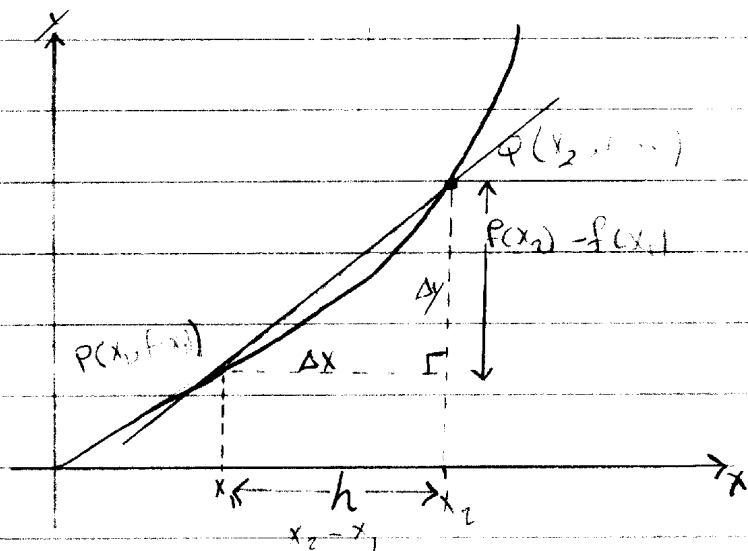
Def. The average speed = $\frac{\text{distance traveled}}{\text{time interval}}$

Def. the average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is :

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h}, h \neq 0$$

= Slope of the secant PQ.



If the point Q approaches the point P along the curve $y = f(x)$, then the secant PQ becomes a tangent ~~to~~ to the curve at the point P.

$\Rightarrow$ Slope of the curve at P = Slope of the tangent at P.

Example/ Find the average rate of change of the fn.

$f(x) = \sqrt{4x+1}$ over the interval $[0, 2]$.

$$f(0) = \sqrt{4(0)+1} = 1.$$

$$f(2) = \sqrt{4(2)+1} = 3.$$

$$\text{average rate of change} = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2} = \boxed{1}$$

Example/ Consider the curve $y = x^2 - 4x$.

① Find the slope of the curve at the point $P(1, -3)$.

② = an equation of the tangent line at the point $P(1, -3)$.

$$\textcircled{1} \frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h} = \frac{(x_1+h)^2 - 4(x_1+h) - (x_1^2 - 4x_1)}{h}$$

$$= \frac{(1+h)^2 - 4(1+h) - 1 + 4}{h} = \frac{1 + 2h + h^2 - 4 - 4h + 3}{h}$$

$$= \frac{h^2 - 2h}{h} = \underline{\underline{h-2}}$$

If h approaches 0, then the slope of curve = $\boxed{-2}$

= the slope of tangent at $P(1, -3)$.

$$\textcircled{2} y = m(x - x_0) + y_0 \Rightarrow y = -2(x - 1) - 3$$

Exercises 1.3.1

$$\boxed{31} \quad \cos\left(X - \frac{\pi}{2}\right) = \sin X.$$

$$\cos X \cdot \cos \frac{\pi}{2} + \sin X \cdot \sin \frac{\pi}{2} =$$

$$0 + \sin X \cdot 1 = \sin X \quad \#$$

$$\boxed{33} \quad \sin\left(X + \frac{\pi}{2}\right) = \cos X.$$

$$\sin X \cdot \cos \frac{\pi}{2} + \cos X \cdot \sin \frac{\pi}{2} = 0 + \cos X \cdot 1$$

$$= \cos X \quad \#$$

$$\boxed{35} \quad \cos(A - B) = \cos A$$

$$\boxed{37}$$

2.2 Limit of a fn and Limit laws:-

The concept of a limit.

Example/How does the following fn behave near $x=3$?

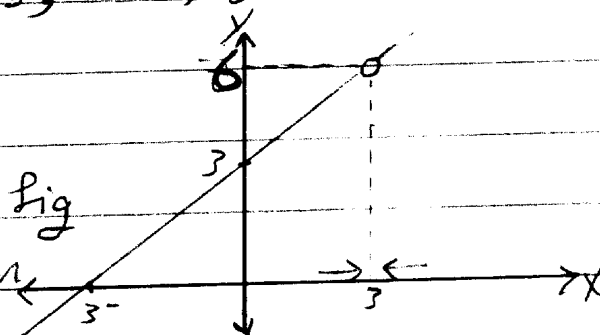
① $f(x) = \frac{x^2-9}{x-3}$

② $g(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$

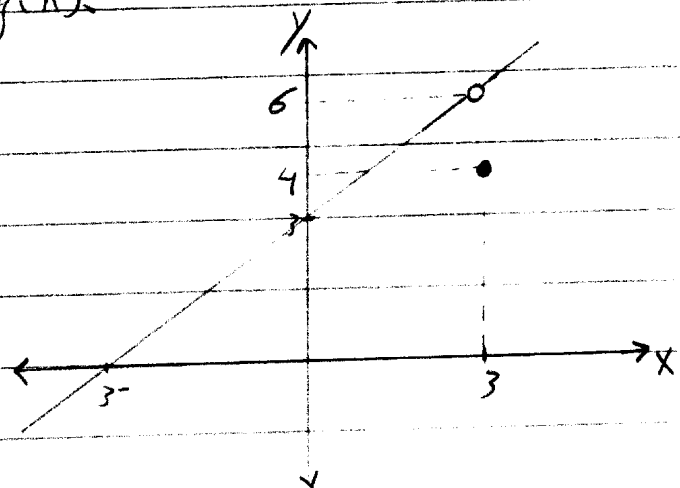
③ $h(x) = x+3$

① f is undefined at $x=3$, that is, $f(3)$ does not exist, but $f(x) = \frac{(x+3)(x-3)}{(x-3)} = x+3, x \neq 3$

The graph of the fn f is shown in the fig with removed point at $x=3$ but we can make the value of $f(x)$ vary close to 6 by choose x very close to 3. This means that f approaches 6 as x approaches 3, and we write $\lim_{x \rightarrow 3} \frac{x^2-9}{x-3} = 6$.



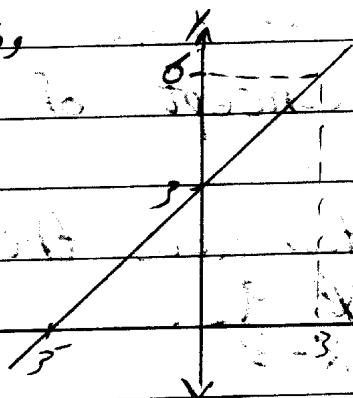
② g is defined at $x=3$, that is, $g(3)=4$ but $\lim_{x \rightarrow 3} g(x) = 6$. So $g(3) \neq \lim_{x \rightarrow 3} g(x)$



© h is defined at $x=3$, that is,

$$f(3) = 3 + 3 = 6, \text{ and } \lim_{x \rightarrow 3} h(x) = 6$$

$$\Rightarrow h(3) = \lim_{x \rightarrow 3} h(x)$$

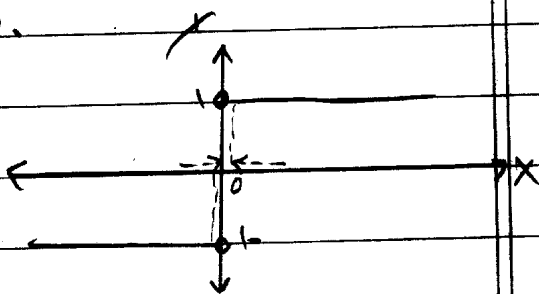


Example / Discuss the behavior of the following fnc

① $f(x) = \frac{x}{|x|}$ ② $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

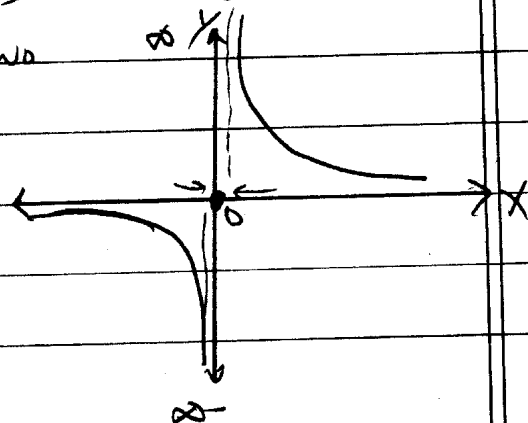
$$\text{① } f(x) = \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{x}{-x}, & x < 0 \end{cases} \Rightarrow \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

f is undefined at $x=0$, $\lim_{x \rightarrow 0} \frac{x}{|x|}$ does not exist because f jumps from -1 to 1 at $x=0$.



② f is defined at $x=0$, that is, $f(0)=0$, but $\lim_{x \rightarrow 0}$ does not exist because f has two

different infinite limits, that is, ∞ and ∞^- .



Example/ From the corresponding graph, state the following limits if it exists.

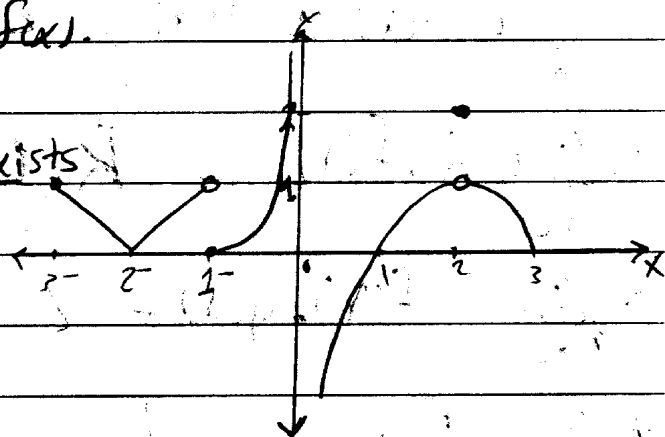
[1] $\lim_{x \rightarrow 1} f(x)$ [2] $\lim_{x \rightarrow 2} f(x)$ [3] $\lim_{x \rightarrow 0} f(x)$ [4] $\lim_{x \rightarrow -1} f(x)$

[5] $\lim_{x \rightarrow 2} f(x)$ [6] $\text{Dom } f(x)$ [7] $\text{Rng } f(x)$

[1] 0 [2] 1 [3] doesn't exist

[4] doesn't exist [5] 0

[6] $\mathbb{R} - \{0\}$ [7] $(-\infty, \infty)$



Some Limits:-

① For the identity fn $f(x) = x$, we have $\lim_{x \rightarrow x_0} x = x_0$

② = constant fn $f(x) = k$, we have $\lim_{x \rightarrow x_0} k = k$

③ = polynomial $\overset{\text{fn}}{p(x)} = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, we have $\lim_{x \rightarrow c} p(x) = p(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$

④ For the Rational fn, we have $\lim_{x \rightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, $q(c) \neq 0$

⑤ = trigonometric fn, we have $\lim_{x \rightarrow \theta} \sin x = \sin \theta$, $\lim_{x \rightarrow \theta} \csc x = \csc \theta$

$$\lim_{x \rightarrow \theta} \frac{8 - \cos x}{\tan x} = \frac{8 - \cos \theta}{\tan \theta}, \quad \tan \theta \neq 0$$

The Limit Laws:

If L, M, C and K are real numbers and
 $\lim_{x \rightarrow C} f(x) = L$, $\lim_{x \rightarrow C} g(x) = M$, then:

$$\textcircled{1} \lim_{x \rightarrow C} f(x) \pm g(x) = L \pm M$$

$$\textcircled{2} \lim_{x \rightarrow C} k \cdot f(x) = k \cdot L$$

$$\textcircled{3} \lim_{x \rightarrow C} f(x) \cdot g(x) = L \cdot M$$

$$\textcircled{4} \lim_{x \rightarrow C} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0.$$

$$\textcircled{5} \lim_{x \rightarrow C} [f(x)]^n = (L)^n, n \text{ is a positive integer.}$$

$$\textcircled{6} \lim_{x \rightarrow C} \sqrt[n]{f(x)} = \sqrt[n]{L}, n = 2, 3, 4, 5, \dots$$

Example/Find the following limits:

$$\textcircled{a} \lim_{x \rightarrow 0} (5x - 8)^{\frac{1}{3}} = \left(\lim_{x \rightarrow 0} 5x - 8 \right)^{\frac{1}{3}} = (5(0) - 8)^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = \boxed{-2}$$

$$\textcircled{b} \lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x - 5)}{(x - 2)}$$

$$= \lim_{x \rightarrow 2} x - 5 = 2 - 5 = \boxed{-3}$$

$$\textcircled{a} \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^4 - 16} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2+2x+4)}{(x^2-4)(x^2+4)}$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)(x^2+4)} = \frac{(2)^2+2(2)+4}{(2+2)((2)^2+4)}$$

$$= \frac{12}{32} = \boxed{\frac{3}{8}}$$

$$\textcircled{a} \lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(4-x)}{2 - \sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(2 - \sqrt{x})(2 + \sqrt{x})}{2 - \sqrt{x}}$$

$$\lim_{x \rightarrow 4} x(2 + \sqrt{x}) = 4(2 + \sqrt{4}) = \boxed{16}$$

$$\textcircled{a} \lim_{x \rightarrow 4} \frac{4-x}{5 - \sqrt{x^2+9}} = \lim_{x \rightarrow 4} \frac{4-x}{5 - \sqrt{x^2+9}} \cdot \frac{5 + \sqrt{x^2+9}}{5 + \sqrt{x^2+9}} = \lim_{x \rightarrow 4} \frac{4-x(5 + \sqrt{x^2+9})}{25 - (x^2+9)}$$

$$\lim_{x \rightarrow 4} \frac{4-x(5 + \sqrt{x^2+9})}{16 - x^2} = \lim_{x \rightarrow 4} \frac{(4-x)(5 + \sqrt{x^2+9})}{(4-x)(4+x)}$$

$$\lim_{x \rightarrow 4} \frac{5 + \sqrt{x^2+9}}{4+x} = \frac{5 + \sqrt{16+9}}{4+4} = \frac{10}{8} = \boxed{\frac{5}{4}}$$

$$\textcircled{a} \lim_{x \rightarrow 0} \frac{\tan(x) - \sin(x)}{\sin^3(x)} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{\cos x \cdot \sin^3 x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x(1 - \cos x)}{\cos x \cdot \sin^3 x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\cos x(1 - \cos x)(1 + \cos x)}$$

$$\lim_{x \rightarrow 0} \frac{1}{\cos x(1 + \cos x)} = \frac{1}{\cos 0(1 + \cos 0)}$$

$$= \boxed{\frac{1}{2}}$$

Example/If $\lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} = 3$, find $\lim_{x \rightarrow 2} f(x)$.

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} f(x) - 5 + 5$$

$$= \left(\lim_{x \rightarrow 2} f(x) - 5 \right) + \lim_{x \rightarrow 2} 5 = \lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \cdot (x-2) + 5$$

$$= \left(\lim_{x \rightarrow 2} \frac{f(x)-5}{x-2} \right) \times \lim_{x \rightarrow 2} x-2 + 5$$

$$= 3 \times 0 + 5 = \boxed{5}$$

The Sandwich Theorem.

Suppose that $g(x) \leq f(x) \leq h(x) \quad \forall x$ in some ~~open~~ open interval containing c , except possibly at $x=c$ itself. If $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, the $\lim_{x \rightarrow c} f(x) = L$.

Example/Find $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$.

$$\text{If } x \neq 0 \rightarrow$$

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

$$\lim_{x \rightarrow 0} -x^2 \leq \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} \leq \lim_{x \rightarrow 0} x^2$$

$$0 \leq \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} \leq 0$$

By Sandwich Theorem $\Rightarrow \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$

Example / Given $|g(x) - 4| \leq 5(x-2)^2 \forall x$, find $\lim_{x \rightarrow 2} g(x)$.

$$-5(x-2)^2 \leq g(x) - 4 \leq 5(x-2)^2$$

$$4 - 5(x-2)^2 \leq g(x) \leq 4 + 5(x-2)^2$$

$$\lim_{x \rightarrow 2} 4 - 5(x-2)^2 \leq \lim_{x \rightarrow 2} g(x) \leq \lim_{x \rightarrow 2} 4 + 5(x-2)^2$$

$$4 - 5(0) \leq \lim_{x \rightarrow 2} g(x) \leq 4 + 5(0)$$

$$4 \leq \lim_{x \rightarrow 2} g(x) \leq 4$$

by sandwich theorem $\Rightarrow \lim_{x \rightarrow 2} g(x) = \underline{\underline{4}}$

Exercises 2.1 :-

[3] $h(t) = \cot(t)$.

on $[\pi/4, 3\pi/4]$

average rate of change = $\frac{h(\frac{3\pi}{4}) - h(\frac{\pi}{4})}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-1 - 1}{\frac{\pi}{2}} = \boxed{\frac{-4}{\pi}}$

[6] $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$, $[1, 2]$.

average rate of change = $\frac{P(2) - P(1)}{2 - 1} = \frac{(2^3 - 4(2)^2 + 5(2)) - (1^3 - 4(1)^2 + 5(1))}{1} = 0$

[7] $y = x^2 - 3$, $P(2, 1)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h} = \frac{[(x_1 + h)^2 - 3] - [x_1^2 - 3]}{h}$$

$$= \frac{x_1^2 + 2hx_1 + h^2 - 3 - x_1^2 + 3}{h} = \frac{2*2h + h^2}{h} = \frac{4h + h^2}{h} = \underline{\underline{4 + h}}$$

If h approaches to 0, then the slope of curve = $\boxed{4}$
 = the slope of tangent at P.

$$* Y = m(X - x_0) + y_0$$

$$Y = 4(X - 2) + 1 \Rightarrow Y = 4X - 7$$

$$ii) Y = x^3, P(2, 8).$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h} = \frac{(x_1+h)^3 - x_1^3}{h} = \frac{(2+h)^3 - (2)^3}{h}$$

$$\frac{8 + 6h + h^3 - 8}{h} = 6 + h^2.$$

If h approaches to 0, then the slope of curve =
 6 = the slope of tangent at P.

$$* Y = m(X - x_0) + y_0 \Rightarrow Y = 6(X - 2) + 8$$

$$Y = 6X - 4.$$

$$(13) Y = x^3 - 12x, P(1, -11).$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h} = \frac{[(x_1+h)^3 - 12(x_1+h)] - [x_1^3 - 12x_1]}{h}$$

$$\frac{(1+h)(1-h+h^2) - 12(1+h) - 1 + 12}{h}$$

14) $\cancel{f(x)} = x^3 - 3x^2 + 4, P(2,0)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h} = \frac{(2+h)^3 - 3(2+h)^2 + 4 - (2)^3 + 3(2)^2 - 4}{h}$$

Exercises 2.2:

1) a) does not exist, because f jumps from 0 to 1 at $x=1$.

b) 1

c) 0

d) $1/2$

3) a - true b - true c - false d - false

e - false f - true g - true

6) $\lim_{x \rightarrow 1} \frac{1}{x-1}$

$$11) \lim_{x \rightarrow 7} (2x + 5), \quad \lim_{x \rightarrow 7} 2x + \lim_{x \rightarrow 7} 5 = 2 * 7 + 5 = \boxed{19}$$

$$13) \lim_{t \rightarrow 6} 8(t-5)(t-7), \quad 8 \left(\lim_{t \rightarrow 6} t - 5 * \lim_{t \rightarrow 6} t - 7 \right) \\ = 8(6-5)(6-7) = \boxed{-8}$$

$$17) \lim_{x \rightarrow 2} 3(2x-1)^2 = 3(\lim_{x \rightarrow 2} 2x-1)^2 = 3(2(2)-1)^2 = \boxed{3}$$

$$21) \lim_{h \rightarrow 0} \frac{3}{\sqrt{3h+1}+1} = 3 \lim_{h \rightarrow 0} \frac{1}{\sqrt{3h+1}+1} = 3 \frac{1}{\sqrt{\lim_{h \rightarrow 0} 3h+1}+1}$$

$$\frac{3}{\sqrt{3(0)+1}+1} = \frac{3}{2} = \boxed{1.5}$$

$$23) \lim_{x \rightarrow 5} \frac{x-5}{x^2-25} = \lim_{x \rightarrow 5} \frac{(x-5)}{(x-5)(x+5)} \quad x \neq -5$$

$$\lim_{x \rightarrow 5} \frac{1}{x+5} = \frac{1}{5+5} = \boxed{\frac{1}{10}}$$

$$27) \lim_{t \rightarrow 1} \frac{t^2+t-2}{t^2-1} = \lim_{t \rightarrow 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \rightarrow 1} \frac{t+2}{t+1}$$

$$\lim_{t \rightarrow 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2} = \boxed{1.5}$$

$$29) \lim_{x \rightarrow -2} \frac{-2x-4}{x^3+2x^2} = \lim_{x \rightarrow -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \rightarrow -2} \frac{-2}{x^2} = \frac{-2}{(-2)^2} = \boxed{-\frac{1}{2}}$$

$$30) \lim_{y \rightarrow 0} \frac{5y^3+8y^2}{3y^4-16y^2} = \lim_{y \rightarrow 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y \rightarrow 0} \frac{5y+8}{3y^2-16}$$

$$\lim_{y \rightarrow 0} \frac{5y+8}{3y^2-16} = \frac{5(0)+8}{3(0)^2-16} = \boxed{-\frac{1}{2}}$$

$$32) \lim_{x \rightarrow 0} \frac{1}{x-1} + \frac{1}{x+1} \quad \lim_{x \rightarrow 0} \frac{(x-1) + (x+1)}{x(x-1)(x+1)}$$

$$\lim_{x \rightarrow 0} \frac{2x}{x(x^2-1)} = \lim_{x \rightarrow 0} \frac{2}{x^2-1} = \frac{2}{-1} = \boxed{-2}$$

$$33) \lim_{u \rightarrow 1} \frac{u^4-1}{u^3-1} \quad \lim_{u \rightarrow 1} \frac{(u-1)(u+1)(u^2+1)}{(u-1)(u^2+u+1)}$$

$$\lim_{u \rightarrow 1} \frac{(u+1)(u^2+1)}{(u^2+u+1)} = \frac{(1+1)(1^2+1)}{(1^2+1+1)} = \boxed{\frac{4}{3}}$$

$$37) \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x+3}-2} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x+3}+2)}{x+3-4}$$

$$\lim_{x \rightarrow 1} \sqrt{x+3} + 2 = \sqrt{\lim_{x \rightarrow 1} x+3} + \lim_{x \rightarrow 1} 2$$

$$= 2 + 2 = \boxed{4}$$

$$48) \lim_{x \rightarrow 0} (x^2-1)(2-\cos x) = \left(\lim_{x \rightarrow 0} x^2-1 \right) \left(\lim_{x \rightarrow 0} 2-\cos x \right)$$

$$(0^2-1)(2-\cos 0) = -1 \cdot 1 = \boxed{-1}$$

$$49) \lim_{x \rightarrow -\pi} \sqrt{x+4} \cos(x+\pi) = \sqrt{\lim_{x \rightarrow -\pi} x+4} \cdot \lim_{x \rightarrow -\pi} \cos(x+\pi)$$

$$\sqrt{-\pi+4} \cdot \cos(0) = \sqrt{-\pi+4}$$

$$53) \lim_{x \rightarrow c} f(x) = 5, \lim_{x \rightarrow c} g(x) = -2.$$

$$\textcircled{a} \lim_{x \rightarrow c} f(x) \cdot g(x) = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = 5 \cdot -2 = \boxed{-10}$$

$$\textcircled{b} \lim_{x \rightarrow c} 2f(x) \cdot g(x) = 2 \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) = 2 \cdot 5 \cdot -2 = \boxed{-20}$$

$$56) \lim_{x \rightarrow -2} p(x) = 4, \lim_{x \rightarrow -2} r(x) = 0, \lim_{x \rightarrow -2} s(x) = -3.$$

$$\begin{aligned} \textcircled{c} \lim_{x \rightarrow -2} \frac{-4p(x) + 5r(x)}{s(x)} &= \frac{-4 \lim_{x \rightarrow -2} p(x) + 5 \lim_{x \rightarrow -2} r(x)}{\lim_{x \rightarrow -2} s(x)} \\ &= \frac{-4 \cdot 4 + 5 \cdot 0}{-3} = \boxed{\frac{16}{3}} \end{aligned}$$

$$57) f(x) = x^2, x = 1.$$

$$63) \sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2} \quad x$$

$$\lim_{x \rightarrow 0} \sqrt{5-2x^2} \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} \sqrt{5-x^2}$$

$$\sqrt{5} \leq \lim_{x \rightarrow 0} f(x) \leq \sqrt{5}$$

By Sandwich Theorem: $\lim_{x \rightarrow 0} f(x) = \sqrt{5}$.

65)

Remark / For any f in $F(x)$, if $\lim_{x \rightarrow c} |f(x)| = 0$, then $\lim_{x \rightarrow c} f(x) = 0$.

Proof / $-|f(x)| \leq f(x) \leq |f(x)|, \forall x$
 $\lim_{x \rightarrow c} -|f(x)| \leq \lim_{x \rightarrow c} f(x) \leq \lim_{x \rightarrow c} |f(x)|$

$$0 \leq \lim_{x \rightarrow c} f(x) \leq 0$$

By Sandwich theorem, we have $\lim_{x \rightarrow c} f(x) = 0$

Example / Find $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$0 \leq \left| \sin \frac{1}{x} \right| \leq 1$$

$$0 \leq |x| \left| \sin \frac{1}{x} \right| \leq |x|$$

$$0 \leq \left| x \sin \frac{1}{x} \right| \leq |x|$$

$$\lim_{x \rightarrow 0} 0 \leq \lim_{x \rightarrow 0} \left| x \sin \frac{1}{x} \right| \leq \lim_{x \rightarrow 0} |x|$$

$$0 \leq \lim_{x \rightarrow 0} \left| x \sin \frac{1}{x} \right| \leq 0$$

By Sandwich theorem, we have $\lim_{x \rightarrow 0} \left| x \sin \frac{1}{x} \right| = 0$

$$\therefore \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

2.3. / The precise def. of a limit

In this section we do not tell how to find a limit of a fn but we verify that the suspected limit is correct.

Def. Let f be a fn defined on an open interval containing x_0 except possibly at x_0 itself. $\lim_{x \rightarrow x_0} f(x) = L$ means that for every $\epsilon > 0$, $\exists \delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$.

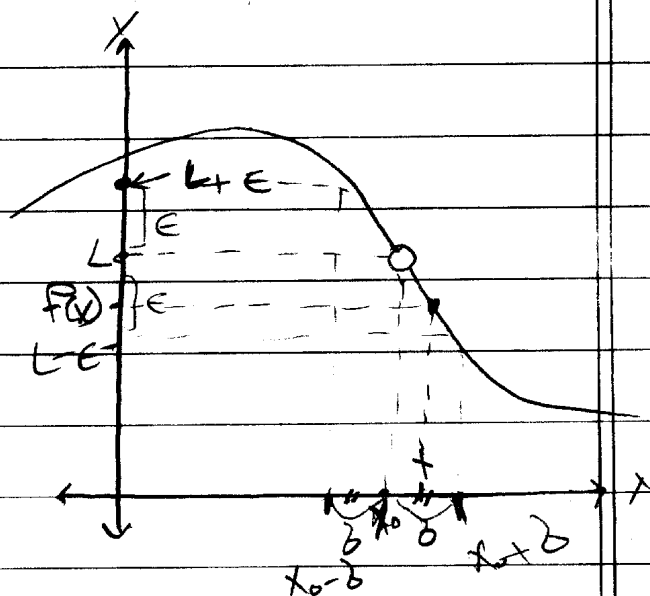
$\epsilon > 0$, $\exists \delta > 0$ such that if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \epsilon$

$$-\delta < x - x_0 < \delta$$

$$x_0 - \delta < x < \delta + x_0 \Rightarrow$$

$$-\delta < f(x) - L < \delta$$

$$L - \delta < f(x) < L + \delta$$



2.4 One-Sided Limits:-

Theorem A fn $f(x)$ has a limit as x approaches c iff it has left-hand and right-hand limits there and these limits are equal, that is,

$$\lim_{x \rightarrow c} f(x) = L \text{ iff } \lim_{x \rightarrow c^-} f(x) = L \text{ and } \lim_{x \rightarrow c^+} f(x) = L$$

Example/ For the graph of a fn as shown find the following:-

a) $\lim_{x \rightarrow -1^+} f(x) = 1$

b) $\lim_{x \rightarrow 0} f(x) = 0$

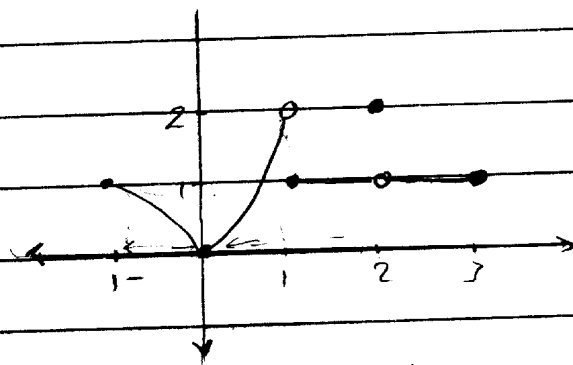
c) $\lim_{x \rightarrow 1^-} f(x) = 2$

d) $\lim_{x \rightarrow 1^+} f(x) = 1$

e) $\lim_{x \rightarrow 1} f(x)$ doesn't exist

f) $\lim_{x \rightarrow 3^+} f(x)$ doesn't exist

g) $\lim_{x \rightarrow c} f(x)$ where $c \in (1, 3)$



Example/ Find the following limits if it exists:-

a) $\lim_{x \rightarrow 3} \frac{|x-3|}{x^2-x-6} = 0$

$$|x-3| = \begin{cases} x-3 & x \geq 3 \\ -x+3 & x < 3 \end{cases}$$

$$\lim_{x \rightarrow 3^-} \frac{-(x-3)}{x^2-x-6} = -\lim_{x \rightarrow 3^-} \frac{x-3}{(x-3)(x+2)} = \boxed{-\frac{1}{5}}$$

$$\lim_{x \rightarrow 3^+} \frac{x-3}{(x-3)(x+2)} = \boxed{\frac{1}{5}}$$

So $\lim_{x \rightarrow 3} \frac{|x-3|}{x^2-x-6}$ does not exist, because

$$\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$$

(b) $\lim_{x \rightarrow 3} \frac{\sqrt{3x}(x-3)}{|x-3|}$

$$\lim_{x \rightarrow 3^-} \frac{\sqrt{3x}(x-3)}{|x-3|} = \lim_{x \rightarrow 3^-} \frac{\sqrt{3x}(x-3)}{-(x-3)} = \boxed{-3}$$

$$\lim_{x \rightarrow 3^+} \frac{\sqrt{3x}(x-3)}{(x-3)} = \lim_{x \rightarrow 3^+} \sqrt{3x} = \boxed{3}$$

So $\lim_{x \rightarrow 3} \frac{\sqrt{3x}(x-3)}{|x-3|}$ does not exist because

$$\lim_{x \rightarrow 3^+} \neq \lim_{x \rightarrow 3^-}$$

(c) $\lim_{x \rightarrow 3} \frac{x - \lfloor x \rfloor}{\lceil x - 2 \rceil}$

$$\lim_{x \rightarrow 3^-} \frac{x - \lfloor x \rfloor}{\lceil x - 2 \rceil} = \frac{3 - 2}{1} = \underline{\underline{1}}$$

$$\lim_{x \rightarrow 3^+} \frac{x - \lfloor x \rfloor}{\lceil x - 2 \rceil} = \frac{3 - 3}{2} = 0$$

So $\lim$ does not exist

(d) $\lim_{x \rightarrow 3.6} \frac{\lceil x \rceil}{x}$

$$\lim_{x \rightarrow 3.6^-} \frac{\lceil x \rceil}{x} = \frac{4}{3.6} = \frac{10}{9}$$

$$\lim_{x \rightarrow 3.6^+} \frac{\lceil x \rceil}{x} = \frac{4}{3.6} = \frac{10}{9}$$

$\Rightarrow$ So $\lim_{x \rightarrow 3.6} = \underline{\underline{\frac{10}{9}}}$

② $\lim_{x \rightarrow 3} f(x)$ where $f(x) = \begin{cases} x^2 - 5 & , x \leq 3 \\ \sqrt{x+13} & , x > 3 \end{cases}$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} x^2 - 5 = 9 - 5 = 4$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \sqrt{x+13} = \sqrt{3+13} = 4$$

$$\therefore \lim_{x \rightarrow 3} f(x) = 4$$

~~Example~~ Example / Let $f(x) = \begin{cases} ax+1 & , x \leq 3 \\ ax^2-1 & , x > 3 \end{cases}$
Find the value of A (constant) such that $\lim_{x \rightarrow 3} f(x)$ exists

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} ax+1 = 3a+1$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} ax^2-1 = 9a-1$$

Since $\lim_{x \rightarrow 3} f(x)$ exists, we have

$$\begin{aligned} 3a+1 &= 9a-1 \\ \Rightarrow \boxed{a = \frac{1}{3}} \end{aligned}$$

Limits Involving $\sin \theta / \theta$

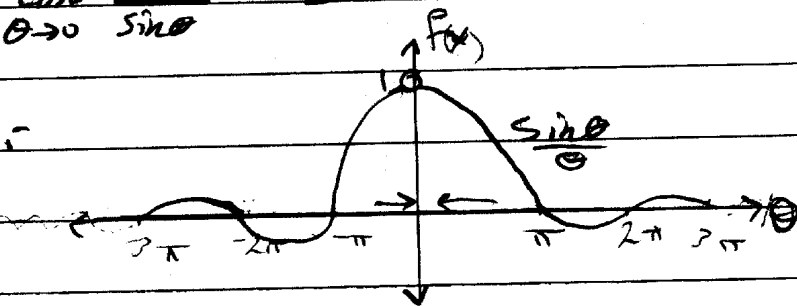
Theorem $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ and $\lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1$.

From the graph we have:-

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 \quad \text{and}$$

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$$

$$\text{So } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$



Example/Find the following limits.

$$\textcircled{1} \lim_{x \rightarrow 0} \frac{\sin(\frac{x}{2})}{\frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} \sin(\frac{x}{2})}{\frac{x}{2}}$$

$$\frac{1}{2} \lim_{\frac{x}{2} \rightarrow 0} \frac{\sin(\frac{x}{2})}{(\frac{x}{2})} = \boxed{\frac{1}{2}}$$

$$\textcircled{2} \lim_{x \rightarrow 0} \frac{\tan 5x}{3x} = \lim_{x \rightarrow 0} \frac{\frac{5}{3} \tan 5x}{\frac{5}{3} 3x} = \frac{5}{3} \lim_{5x \rightarrow 0} \frac{\tan 5x}{5x}$$

$$\frac{5}{3} \lim_{5x \rightarrow 0} \frac{\sin 5x}{5x} \times \frac{1}{\cos 5x} = \frac{5}{3} (1) \times \frac{1}{\cos 0} = \boxed{\frac{5}{3}}$$

$$\textcircled{3} \lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cdot \cos x} = \lim_{x \rightarrow 0} \frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x}$$

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} \times \lim_{x \rightarrow 0} \frac{1}{\cos x} + \lim_{x \rightarrow 0} \frac{x}{\sin x}$$

$$1 \left(\frac{1}{1} \right) + 1 = \boxed{2}$$

$$4) \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 8x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos 3x}$$

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 8x} \cdot \frac{1}{1} = \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 8x}$$

$$\lim_{3x \rightarrow 0} \frac{3 \sin 3x}{3x} = \frac{3(1)}{8(1)} = \boxed{\frac{3}{8}}$$

$$5) \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2+x-2} = \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x-1)(x+2)}$$

$$\lim_{x-1 \rightarrow 0} \frac{\sin(x-1)}{(x-1)} \cdot \lim_{x \rightarrow 1} \frac{1}{(x+2)} = (1) \cdot \frac{1}{3} = \boxed{\frac{1}{3}}$$

$$6) \lim_{x \rightarrow 0} \frac{\sin(1-\cos x)}{x} = \lim_{x \rightarrow 0} \frac{\sin(1-\cos x)}{x} \cdot \frac{1-\cos x}{1-\cos x}$$

$$\lim_{x \rightarrow 0} \frac{\sin(1-\cos x)}{(1-\cos x)} \cdot \frac{(1-\cos x)}{x}$$

$$\lim_{1-\cos x \rightarrow 0} \frac{\sin(1-\cos x)}{(1-\cos x)} \cdot \lim_{x \rightarrow 0} \frac{1-\cos x}{x}$$

$x \rightarrow 0$
 $\cos x \rightarrow \cos 0$
 $\cos x \rightarrow 1$
 $\cos x - 1 \rightarrow 0$
 $1 - \cos x \rightarrow 0$

$$(1) \cdot \lim_{x \rightarrow 0} \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \lim_{x \rightarrow 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

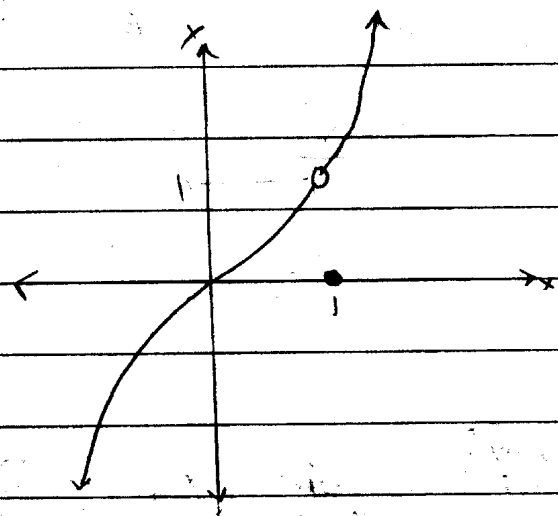
$$\lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{(1+\cos x)}$$

$$= 1 \left(\frac{0}{2} \right) = \underline{\underline{0}}$$

Exercises / 2.4

$$7) \lim_{x \rightarrow 1^+} f(x) = 1, \quad \lim_{x \rightarrow 1^-} f(x) = 1$$

Yes, $\lim_{x \rightarrow 1} f(x) = 1$

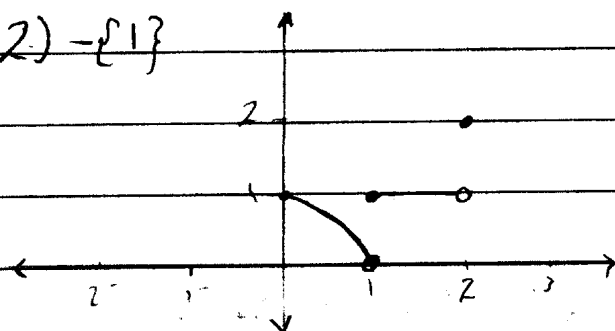


$$9) \text{a) } \text{Dom}(f) = [0, 2], \quad \text{Rng}(f) = (0, 2]$$

$$\text{b) } [0, 1) \cup (1, 2) \text{ or } [0, 2) - \{1\}$$

$$\text{c) } x = 2$$

$$\text{d) } x = 0$$



$$11) \lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{3}$$

$$13) \lim_{x \rightarrow -2^+} \left(\frac{x}{x+1} \right) \left(\frac{2x+5}{x^2+x} \right) = \left(\frac{-2}{-2+1} \right) \left(\frac{2(-2)+5}{(-2)^2-2} \right) = \boxed{1}$$

$$15) \lim_{h \rightarrow 0^+} \frac{\sqrt{h^2+4h+5} - \sqrt{5}}{h} \cdot \frac{\sqrt{h^2+4h+5} + \sqrt{5}}{\sqrt{h^2+4h+5} + \sqrt{5}}$$

$$\lim_{h \rightarrow 0^+} \frac{h^2+4h+5-5}{h[\sqrt{h^2+4h+5} + \sqrt{5}]} = \lim_{h \rightarrow 0^+} \frac{h(h+4)}{h(\sqrt{h^2+4h+5} + \sqrt{5})}$$

$$\lim_{h \rightarrow 0^+} \frac{(h+4)}{\sqrt{h^2+4h+5} + \sqrt{5}} = \frac{4}{\sqrt{0+0+5} + \sqrt{5}} = \boxed{\frac{2}{\sqrt{5}}}$$

$$\boxed{16} \lim_{h \rightarrow 0} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h} \cdot \frac{\sqrt{6} + \sqrt{5h^2 + 11h + 6}}{\sqrt{6} + \sqrt{5h^2 + 11h + 6}}$$

$$\lim_{h \rightarrow 0} \frac{6 - (5h^2 + 11h + 6)}{h(\sqrt{6} + \sqrt{5h^2 + 11h + 6})} = \lim_{h \rightarrow 0} \frac{h(5h + 11)}{h(\sqrt{6} + \sqrt{5h^2 + 11h + 6})}$$

$$\lim_{h \rightarrow 0} \frac{5h + 11}{\sqrt{6} + \sqrt{5h^2 + 11h + 6}} = \frac{11}{\sqrt{6} + \sqrt{6}} = \boxed{\frac{11}{2\sqrt{6}}}$$

$$\boxed{17} \text{ a) } \lim_{x \rightarrow 2^+} (x+3) \frac{|x+2|}{(x+2)}$$

$$\lim_{x \rightarrow 2^+} (x+3) \cdot \lim_{x \rightarrow 2^+} \frac{x+2}{x+2} = \lim_{x \rightarrow 2^+} (x+3) \cdot \lim_{x \rightarrow 2^+} 1$$

$$= 1 \cdot 1 = \boxed{1}$$

$$\text{b) } \lim_{x \rightarrow 2^-} (x+3) \frac{|x+2|}{(x+2)} = \lim_{x \rightarrow 2^-} (x+3) \cdot \lim_{x \rightarrow 2^-} \frac{-(x+2)}{(x+2)}$$

$$= 1 \cdot -1 = \boxed{-1}$$

$$\boxed{19} \text{ a) } \lim_{\theta \rightarrow \pi^+} \frac{L\theta}{\theta} = \frac{3}{3} = \boxed{1}$$

$$\text{b) } \lim_{\theta \rightarrow \pi^-} \frac{L\theta}{\theta} = \boxed{\frac{2}{3}}$$

$$\boxed{21} \lim_{\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$$

$$\frac{\theta \rightarrow 0}{\sqrt{2}\theta \rightarrow 0}$$

$$\lim_{\sqrt{2}\theta \rightarrow 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta} = \boxed{1}$$

$$[25] \lim_{x \rightarrow 0} \frac{\tan 2x}{x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{\cos 2x}$$

$$\lim_{x \rightarrow 0} \frac{2 \sin 2x}{2x} \cdot \frac{1}{\cos 0} = 2 \lim_{2x \rightarrow 0} \frac{\sin 2x}{2x} = 2(1) = \underline{\underline{2}}$$

$$[29] \lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x} = 2 \lim_{x \rightarrow 0} \frac{x(1 + \cos x)}{2 \sin x \cos x} = 2 \lim_{x \rightarrow 0} \frac{x(1 + \cos x)}{\sin 2x}$$

$$2 \lim_{x \rightarrow 0} \frac{1 + \cos x}{\frac{\sin 2x}{x}} = 2 \lim_{x \rightarrow 0} \frac{1 + \cos x}{\frac{\sin 2x}{2}} = \frac{1 + \cos 0}{1} = \underline{\underline{2}}$$

$$[40] \lim_{y \rightarrow 0} \frac{(\sin 3y) \cot 5y}{y(\cot 4y)} = \lim_{y \rightarrow 0} \frac{\sin 3y}{y} \cdot \frac{\cos 5y}{\sin 5y} \cdot \frac{\sin 4y}{\cos 4y}$$

$$[41] \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta^2 \cot 3\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta} \cdot \frac{\sin 3\theta}{\theta} \cdot \frac{1}{\cos 3\theta}$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} \cdot 3 \lim_{3\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos 3\theta} = 1(1)(3(1))(1) = \underline{\underline{3}}$$

$$[42] \lim_{\theta \rightarrow 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cdot \cot^2 2\theta}$$

[2.5] Continuity

① Continuity at a point :-

Def. A fn $f(x)$ is continuous at an interior point c of its Dom if $\lim_{x \rightarrow c} f(x) = f(c)$.

* f is continuous at left-end point $x = a$ of its Dom

If $\lim_{x \rightarrow a^+} f(x) = f(a)$. (Right continuous)

* f is continuous at right-end point $x = b$ of its Dom If

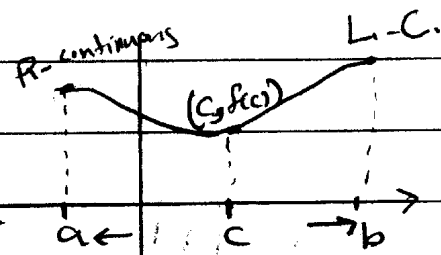
$\lim_{x \rightarrow b^-} f(x) = f(b)$. (Left continuous)

⇒ Continuity Test :-

A fn f is continuous of $x = c$ if f the

following three conditions are satisfied:

- ① $f(c)$ exists.
- ② $\lim_{x \rightarrow c} f(x)$ exists.
- ③ $\lim_{x \rightarrow c} f(x) = f(c)$

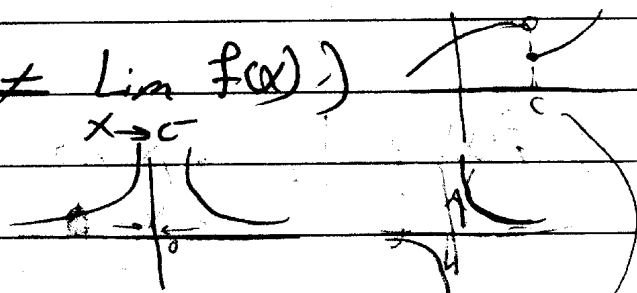


⇒ Type of Discontinuity

① Removable discontinuity ($f(c)$ exists and $\lim_{x \rightarrow c} f(x) \neq f(c)$)

② Jump discontinuity ($\lim_{x \rightarrow c^+} f(x) \neq \lim_{x \rightarrow c^-} f(x)$)

③ Infinite discontinuity.



Continuous function

Example/ Discuss the continuity of the following fns.

$$[1] f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

f is defined at $x=3$, $f(3) = 4$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \rightarrow 3} (x+3) = 6$$

$$f(3) \neq \lim_{x \rightarrow 3} f(x)$$

So f is discontinuous at $x=3$.

and So this is a removable discontinuity, but

$$F(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases} \text{ is continuous at } x=3.$$

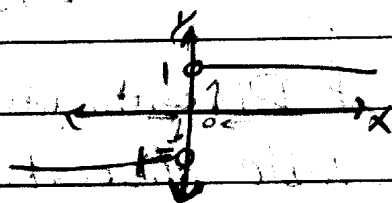
$$[2] f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$f(0)$ doesn't exist, So

f is discontinuous at $x=0$

This is a jump discontinuity

because $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$



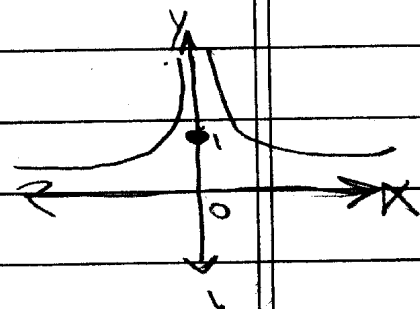
$$[3] f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

f is defined at $x=0$, $f(0)=1$, but $\lim_{x \rightarrow 0^+} f(x) = \infty$ and

$$\lim_{x \rightarrow 0^-} f(x) = \infty, \quad \lim_{x \rightarrow 0} f(x) \neq f(0)$$

So f is discontinuous at $x=0$.

This is infinite discontinuity



Example/ Is $f(x) = \begin{cases} x-1 & , x < 3 \\ 5-x & , x \geq 3 \end{cases}$ continuous at $x=3$? why?

$$f(3) = 5 - (3) = 2.$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} 5 - x = 5 - 3 = 2.$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} 5 - x = 5 - 3 = 2$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} x - 1 = 3 - 1 = 2$$

So $\lim_{x \rightarrow 3} f(x) = 2$ and $\lim_{x \rightarrow 3} f(x) = f(3)$

So f is a continuous fn.

Example/ Find the value of the constant A such that $f(x) = \begin{cases} Ax^2 & , x \leq 2 \\ 2x + A & , x > 2 \end{cases}$ is continuous at $x=2$.

Since $f(x)$ at $x=2$ is continuous, we have:

$$f(2) = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x)$$

$$A(2)^2 = \lim_{x \rightarrow 2^+} 2x + A$$

$$4A = 4 + A \Rightarrow A = \underline{\underline{\frac{4}{3}}}$$

Properties of continuous fns.:

① If f and g are continuous fns of $x=c$, then $f+g$, fg , kf where k is constant, $\frac{f}{g}$ ($g(c) \neq 0$) $> f^n$ and $\sqrt[n]{f}$, where n is a positive integer are continuous fns of $x=c$.

② Every Polynomial is continuous.

③ Rational fns are continuous for every points where its denominator is different from zero.

المقام

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- [4] $\sin x$ and $\cos x$ are continuous $\forall x$.
 $\tan x$ and $\sec x$ are continuous $\forall x \neq \frac{n\pi}{2}$,
 n is odd integer.
 $\cot x$ and $\csc x$ are continuous $\forall x \neq n\pi$, n is an integer.

[5] $f(x) = |x|$ and their shifts are continuous $\forall x$.

[6] If f is continuous at $x=c$ and g is continuous at $f(c)$ then $g \circ f$ is continuous at $x=c$.

Example/ Discuss the continuity of the following fns:-

[1] $f(x) = x+1 + \sin 2x$

$f(x)$ is continuous, because $(x+1)$ and $(\sin 2x)$ are continuous $\forall x$ and their sum is continuous. ✓

[2] $f(x) = \frac{x}{|x|-3}$ $|x|-3=0 \Rightarrow |x|=3 \therefore x = \pm 3$
 So $f(x)$ is continuous $\forall x \neq \{-3, 3\}$

[3] $f(x) = x^3 + 4x^2 + 3x - 4$ Continuous because it's Polynomial.

[4] $f(x) = \frac{x^2}{x^3+1}$ $x^3+1=0 \Rightarrow x^3=-1 \therefore x = -1$
 So f is continuous $\forall x \neq \{-1\}$

[5] $f(x) = \sqrt[3]{2x+3}$ $2x+3$ is continuous $\forall x$, So
 $f(x)$ is continuous $\forall x$.

continuous $\forall x$ continuous $\forall x$

take $h(x) = \sqrt[3]{x}$ and $g(x) = 2x+3$

$$(h \circ g)(x) = h(2x+3) = \sqrt[3]{2x+3} = f(x)$$

So f is continuous because g is continuous $\forall x$ and $h(x)$ also.

Continuous extension to a point.

Def / If $f(x)$ is undefined at $x=c$ and if $\lim_{x \rightarrow c} f(x) = L$ "exists", we ~~to~~ define a new fn $F(x)$ as follow.

$$F(x) = \begin{cases} f(x) & , x \neq c \\ L & , x = c \end{cases}$$

the $F(x)$ is continuous at $x=c$ and its called the continuous extension of f to $x=c$.

Example / Define $f(x)$ in away that extends $f(x) = \frac{x^2-16}{x^2-3x-4}$ to be continuous at $x=4$.

f is undefined at $x=4$, $f(4) = 0$

$$\lim_{x \rightarrow 4} \frac{x^2-16}{x^2-3x-4} = \lim_{x \rightarrow 4} \frac{(x-4)(x+4)}{(x-4)(x+1)} = \frac{8}{5} \text{ exists}$$

$$\text{Put } f(4) = \frac{8}{5}, \quad F(x) = \begin{cases} \frac{x^2-16}{x^2-3x-4} & , x \neq 4 \\ \frac{8}{5} & , x = 4 \end{cases}$$

$F(x)$ is continuous extension of f at $x=4$

Continuity on an interval:-

Def/ A fn f is continuous on (a, b) if it's continuous $\forall$ every point c in (a, b) .

* f is continuous on $[a, b]$ if it's continuous on (a, b) and $\lim_{x \rightarrow a^+} f(x) = f(a)$ and $\lim_{x \rightarrow b^-} f(x) = f(b)$.

$\underbrace{\lim_{x \rightarrow a^+} f(x) = f(a)}_{R\text{-continuous}} \quad \quad \quad \underbrace{\lim_{x \rightarrow b^-} f(x) = f(b)}_{L\text{-continuous}}$

Example/ Show that the fn $f(x) = 2x + \sqrt{x-3}$ is continuous on $[3, \infty)$.

$2x$ and $\sqrt{x-3}$ are continuous, $\therefore$ their sum $f(x)$ is continuous $\forall x \in (3, \infty)$ $\left\{ \begin{array}{l} x-3 \geq 0 \\ x \geq 3 \end{array} \right.$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} 2x + \sqrt{x-3} = 2(3) + \sqrt{3-3} = 6$$

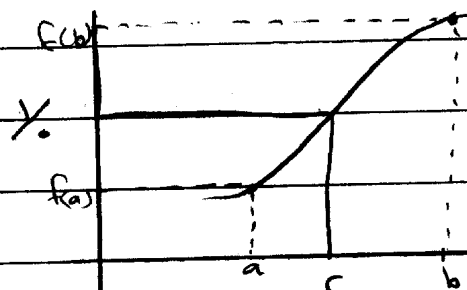
$$f(3) = 2(3) + \sqrt{3-3} = 6$$

$$\therefore \lim_{x \rightarrow 3^+} f(x) = f(3) \Rightarrow f \text{ is continuous on } [3, \infty)$$

Let $c \in (3, \infty)$ then $f(c) = 2c + \sqrt{c-3}$.
 $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} 2x + \sqrt{x-3} = 2c + \sqrt{c-3}$.
 $\therefore f$ is continuous $\forall x \in (3, \infty)$.

Intermediate Value Theorem for Continuous Functions

If f is a continuous fn on $[a, b]$ and If y_0 is any value between $f(a)$ and $f(b)$ then $y_0 = f(c)$ for some $c \in [a, b]$



Example/ Show that there is a root for the eqn $x^3 - x - 1 = 0$ between 1 and 2.

Let $f(x) = x^3 - x - 1$ continuous $\forall x$, because it's a polynomial

So $f(x)$ is continuous on $[1, 2]$

$$f(1) = (1)^3 - 1 - 1 = -1, \quad f(2) = (2)^3 - 2 - 1 = 5$$

$y_0 = 0$ lies between $f(1)$ and $f(2)$.

By the I.M.V, $\exists c \in [1, 2]$ such that $f(c) = 0$.

That means that c is a root "solution" for $f(x) = x^3 - x - 1$.

Example/ Use the I.M.V to Prove that the eqn $\sqrt{2x+5} = 4 - x^2$ has a solution.

The given eqn is written as $\sqrt{2x+5} + x^2 = 4$

$$\text{Suppose } f(x) = \underbrace{\sqrt{2x+5}}_{g_1} + \underbrace{x^2}_{g_2} = g_1(x) + g_2(x)$$

$$\text{Dom } g_1 = \{2x+5 \geq 0\} = \left\{x \geq \frac{-5}{2}\right\} = \left[-\frac{5}{2}, \infty\right)$$

$$\text{Dom } g_2 = (-\infty, \infty)$$

$$\text{Dom } f(x) = \text{Dom } g_1 \cap \text{Dom } g_2 = \left[-\frac{5}{2}, \infty\right)$$

So $f(x)$ is continuous $\forall x \in \left[-\frac{5}{2}, \infty\right)$

$$\text{If } x=0 \rightarrow f(0) = \sqrt{2(0)+5} + (0)^2 = \sqrt{5}$$

$$\text{If } x=2 \rightarrow f(2) = 7$$

Now, 4 lies between $f(0)$ and $f(2)$.

and f is continuous on $[0, 2] \subset [-\frac{5}{2}, \infty)$.

By I.M.V there's c in $[0, 2]$ such that $f(c) = 4$.

The eqn $\sqrt{5x+5} = 4 - x^2$ has a solution on $[0, 2]$

Exercise **2.5** :-

$$\text{[13]} y = \frac{1}{x-2} - 3x. \quad \rightarrow \forall x \neq \{2\}$$

y is continuous $\forall x \neq \{x : x-2 = 0\} = \{x : x=2\}$

$$\text{[15]} y = \frac{x+1}{x^2-4x+3}$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

y is continuous $\forall x \neq \{3, 1\}$ $x = 3$ or 1

$$\text{[17]} y = |x-1| + \sin x$$

y is continuous $\forall x$.

$$\text{[18]} y = \frac{1}{|x|+1} - \frac{x^2}{2}$$

y is continuous $\forall x$.

[21] $y = \csc 2x \Rightarrow y = \frac{1}{\sin 2x}$
 y is continuous $\forall x \neq \{n\frac{\pi}{2}, n \text{ is integer}\}$ $\left\{ \begin{array}{l} \sin 2x = 0 \\ 2x = 0, \pi, 2\pi \Rightarrow x = 0, \frac{\pi}{2}, \pi \end{array} \right.$

[23] $y = \frac{x \tan x}{x^2 + 1} = \frac{x}{x^2 + 1} \cdot \frac{\sin x}{\cos x}$ $\left\{ \begin{array}{l} \cos x = 0 \\ x = \frac{\pi}{2}, 3\frac{\pi}{2} \end{array} \right.$
 y is continuous $\forall x \neq \{n\frac{\pi}{2}, n \text{ is odd}\}$

[24] $y = \frac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$ continuous $\forall x$

[25] $y = \sqrt{2x + 3}$ $2x + 3 \geq 0$
 Continuous $\forall x \geq -\frac{3}{2}$ $x \geq -\frac{3}{2}$

[27] $y = (2x - 1)^{\frac{1}{3}}$ continuous $\forall x$

[29] $g(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$
 $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 2)}{(x - 3)} = \lim_{x \rightarrow 3} x + 2 = \boxed{5} = f(3)$

So y is continuous $\forall x$.

(37) $g(x) = \frac{x^2 - 9}{x - 3}$ $g(3)$ is undefined

put $G(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$

$G(x)$ is continuous $\forall x$ So $G(x)$ continuous at $x=3$
 extension at $x=3$

[39] $f(1)$ is undefined.

$$\lim_{s \rightarrow 1} \frac{s^3 - 1}{s^2 - 1} = \lim_{s \rightarrow 1} \frac{(s-1)(s^2 + s + 1)}{(s-1)(s+1)}$$

$$\lim_{s \rightarrow 1} \frac{s^2 + s + 1}{s + 1} = \frac{1 + 1 + 1}{1 + 1} = \frac{3}{2}$$

Put $f(1) = \frac{3}{2}$, $F(x) = \begin{cases} \frac{s^3 - 1}{s^2 - 1}, & s \neq 1 \\ \frac{3}{2}, & s = 1 \end{cases}$

$F(x)$ is continuous extension at $s = 1$.

[41] Since f is continuous $\forall x$, f is continuous at $x = 3$.

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = f(3).$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} 2ax = 2a(3) = \underline{\underline{6a}}$$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} x^2 - 1 = 9 - 1 = 8$$

$$\text{So } 6a = 8 \rightarrow \boxed{a = \frac{4}{3}}$$

(52) Let $f(x) = \cos x - x$.

(53) Roots of a cubic

Let $f(x) = x^3 - 15x + 1$.

f is continuous $\forall x \Rightarrow f$ is continuous on $[-4, 4]$.

$$f(-4) = (-4)^3 - 15(-4) + 1 = -3$$

$$f(-3) = (-3)^3 - 15(-3) + 1 = 19$$

$-3 < 0$ lies between $f(-4)$ and $f(-3)$, so there's root ₁ for $f(x)$.

$$f(-2) = (-2)^3 - 15(-2) + 1 = 23$$

$$f(-1) = (-1)^3 - 15(-1) + 1 = 15$$

$$f(0) = (0)^3 - 15(0) + 1 = 1$$

$$f(1) = (1)^3 - 15(1) + 1 = -13$$

$0 < 1$ lies between $f(0)$ and $f(1)$, so there's root ₂ for $f(x)$.

$$f(2) = (2)^3 - 15(2) + 1 = -21$$

$$f(3) = (3)^3 - 15(3) + 1 = -17$$

$$f(4) = (4)^3 - 15(4) + 1 = 5$$

$-17 < 0$ lies between $f(3)$ and $f(4)$, so there's root ₃ for $f(x)$.

(54) A function value:

2.6 Limits involving infinity, asymptotes of Graphs.

I Limits involving infinity:-

Example / Find the following limits:-

$$\textcircled{a} \lim_{x \rightarrow \infty} \pi - \frac{2}{x^2} = \lim_{x \rightarrow \infty} \pi - \lim_{x \rightarrow \infty} \frac{2}{x^2} = \pi - 0 = \boxed{\pi}$$

$$\textcircled{b} \lim_{x \rightarrow \infty} \frac{x + \sin x}{3x + 5 - 2\sin x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{\sin x}{x}}{3 + \frac{5}{x} - 2\frac{\sin x}{x}}$$

$$= \frac{1 + 0}{3 + 0 - 2(0)}$$

$$= \boxed{\frac{1}{3}}$$

$$\begin{aligned} -1 &\leq \sin x \leq 1 \\ \frac{-1}{x} &\leq \frac{\sin x}{x} \leq \frac{1}{x} \\ \lim_{x \rightarrow \infty} \frac{-1}{x} &\leq \lim_{x \rightarrow \infty} \frac{\sin x}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x} \\ 0 &\leq 0 \leq 0 \end{aligned}$$

By Sandwich theorem,

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

$$\textcircled{c} \lim_{x \rightarrow \infty} \frac{2x+5}{x^2-3} = \lim_{x \rightarrow \infty} \frac{\frac{2x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{3}{x^2}} = \lim_{x \rightarrow \infty} \frac{2+0}{1-0} = 0$$

$$\textcircled{d} \lim_{x \rightarrow -\infty} \frac{9x^4 + x}{3x^4 + 5x^2 - x + 6} = \lim_{x \rightarrow -\infty} \frac{9 + \frac{1}{x^3}}{3 + \frac{5}{x^2} - \frac{1}{x} + \frac{6}{x^4}} = \boxed{3}$$

$$\textcircled{e} \lim_{x \rightarrow \infty} \sqrt[3]{\frac{x^2 + x - 1}{8x^2 - 3}} = \sqrt[3]{\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}}} = \sqrt[3]{\frac{1}{8}} = \boxed{\frac{1}{2}}$$

$$\begin{aligned} \textcircled{f} \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{x+1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2(4 + \frac{1}{x^2})}}{x(1 + \frac{1}{x})} = \lim_{x \rightarrow \infty} \frac{x\sqrt{4 + \frac{1}{x^2}}}{x(\frac{1}{x} + 1)} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{4 + \frac{1}{x^2}}}{1 + \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{4}}{1} = \boxed{2} \end{aligned}$$

2.6 Limits involving infinity, asymptotes of Graphs.

I Limits involving infinity:-

Example / Find the following limits:-

$$\textcircled{a} \lim_{x \rightarrow \infty} \pi - \frac{2}{x^2} = \lim_{x \rightarrow \infty} \pi - \lim_{x \rightarrow \infty} \frac{2}{x^2} = \pi - 0 = \boxed{\pi}$$

$$\textcircled{b} \lim_{x \rightarrow \infty} \frac{x + \sin x}{3x + 5 - 2 \sin x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{\sin x}{x}}{3 + \frac{5}{x} - 2 \frac{\sin x}{x}}$$

$$= \frac{1 + 0}{3 + 0 - 2(0)}$$

$$= \boxed{\frac{1}{3}}$$

$$-1 \leq \sin x \leq 1$$

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \frac{-1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin x}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin x}{x} \leq 0$$

By Sandwich theorem,

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

$$\textcircled{c} \lim_{x \rightarrow \infty} \frac{2x + 5}{x^2 - 3} = \lim_{x \rightarrow \infty} \frac{\frac{2x}{x} + \frac{5}{x}}{\frac{x^2}{x^2} - \frac{3}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 + 0}{1 - 0} = 0$$

$$\textcircled{d} \lim_{x \rightarrow -\infty} \frac{9x^4 + x}{3x^4 + 5x^2 - x + 6} = \lim_{x \rightarrow -\infty} \frac{9 + \frac{1}{x^3}}{3 + \frac{5}{x^2} - \frac{1}{x} + \frac{6}{x^4}} = \boxed{3}$$

$$\textcircled{e} \lim_{x \rightarrow \infty} \sqrt[3]{\frac{x^2 + x - 1}{8x^2 - 3}} = \sqrt[3]{\lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} - \frac{1}{x^2}}{8 - \frac{3}{x^2}}} = \sqrt[3]{\frac{1}{8}} = \boxed{\frac{1}{2}}$$

$$\begin{aligned} \textcircled{f} \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(\frac{4}{x^2} + \frac{1}{x^2})}}{x(1 + \frac{1}{x})} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{4 + \frac{1}{x^2}}}{x(\frac{1}{x} + 1)} \\ &= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{4 + \frac{1}{x^2}}}{x(1 + \frac{1}{x})} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{4 + \frac{1}{x^2}}}{1 + \frac{1}{x}} = \boxed{-2} \end{aligned}$$

$$\text{eg } \lim_{x \rightarrow -\infty} \frac{2x + \sqrt{4x^2 + 3x - 2}}{2x - \sqrt{4x^2 + 3x - 2}} \cdot \frac{2x - \sqrt{4x^2 + 3x - 2}}{2x - \sqrt{4x^2 + 3x - 2}}$$

$$\lim_{x \rightarrow -\infty} \frac{4x^2 - (4x^2 + 3x - 2)}{2x - \sqrt{4x^2 + 3x - 2}} = \lim_{x \rightarrow -\infty} \frac{-x(3 + \frac{2}{x})}{2x - \sqrt{x^2(4 + \frac{3}{x} - \frac{2}{x^2})}}$$

$$\lim_{x \rightarrow -\infty} \frac{-x(3 + \frac{2}{x})}{2x - |x|\sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} = \lim_{x \rightarrow -\infty} \frac{-x(3 + \frac{2}{x})}{2x + x\sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}}$$

$$\lim_{x \rightarrow -\infty} \frac{-x(3 + \frac{2}{x})}{x(2 + \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}})} = \frac{-3}{2 + \sqrt{4}} = \boxed{\frac{-3}{4}}$$

2] Asymptotes

Def 1 / A line $y=b$ is a horizontal asymptote of the graph of $y=f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$.

Def 2 / The graph of rational fn $f(x) = \frac{p(x)}{q(x)}$ has an oblique asymptote if the degree of $p(x)$ is one greater than the degree of $q(x)$.

Example / Find the horizontal asymptotes of the following fns.

(a) $f(x) = \frac{2x}{x-3}$

$$\lim_{x \rightarrow \infty} \frac{2x}{x-3} = \lim_{x \rightarrow \infty} \frac{2}{1 - \frac{3}{x}} = 2 \quad \left. \begin{array}{l} \text{So } y=2 \\ \text{is a horizontal} \\ \text{asymptote.} \end{array} \right\}$$

$$\lim_{x \rightarrow -\infty} \frac{2x}{x-3} = \frac{\infty}{\infty} = 2$$

$$[D] f(x) = \frac{\sqrt{9x^2 + 2}}{2x + 3}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 + 2}}{2x + 3} = \lim_{x \rightarrow \infty} \frac{|x| \sqrt{9 + \frac{2}{x^2}}}{x(2 + \frac{3}{x})} = \lim_{x \rightarrow \infty} \frac{x \sqrt{9 + \frac{2}{x^2}}}{x(2 + \frac{3}{x})}$$

$$= \frac{\sqrt{9}}{2} = \frac{3}{2}$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 2}}{2x + 3} = \lim_{x \rightarrow -\infty} \frac{-x \sqrt{9 + \frac{2}{x^2}}}{x(2 + \frac{3}{x})} = -\frac{3}{2}$$

So $y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes

$$[E] f(x) = \frac{x^3 + 2x - 5}{x^2 + 1}$$

$$\lim_{x \rightarrow \infty} \frac{x^3 + 2x - 5}{x^2 + 1} = \lim_{x \rightarrow \infty} \frac{x + \frac{2}{x} - \frac{5}{x^2}}{1 + \frac{1}{x^2}} = \lim_{x \rightarrow \infty} x = \infty$$

$$\lim_{x \rightarrow -\infty} \frac{x^3 + 2x - 5}{x^2 + 1} = \lim_{x \rightarrow -\infty} \frac{x + \frac{2}{x} - \frac{5}{x^2}}{1 + \frac{1}{x^2}} = \lim_{x \rightarrow -\infty} x = -\infty$$

So there's no horizontal asymptotes.

Example/ Find the oblique asymptotes of the following fns.

$$a) f(x) = \frac{x^2 - 4}{x - 1}$$

$$f(x) = x + 1 + \frac{-3}{x - 1}$$

the oblique asymptote is $y = x + 1$

$$\begin{array}{r} (x-1) \overline{) x^2 - 4} \\ \underline{x^2 - x} \\ x - 4 \\ \underline{x - 1} \\ -3 \end{array}$$

[3] Infinite Limits:-

Example/ Find the following limits if it exists:-

(1) $\lim_{x \rightarrow 2} \frac{1}{x-2}$

$$\lim_{x \rightarrow 2^+} \frac{1}{x-2} = \frac{1}{\text{V.S.P.Q}} = +\infty$$

$$\lim_{x \rightarrow 2^-} \frac{1}{x-2} = \frac{1}{\text{V.S.N.Q}} = -\infty$$

So $\lim_{x \rightarrow 2} \frac{1}{x-2}$ does not exist.

(2) $\lim_{x \rightarrow 0} \frac{1}{x^2}$

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} = \frac{1}{(\text{V.S.P.Q})^2} = \frac{1}{\text{V.S.P.Q}} = \infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \frac{1}{(\text{V.S.N.Q})^2} = \frac{1}{\text{V.S.P.Q}} = \infty$$

So $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

(3) $\lim_{x \rightarrow 3} \frac{-2}{(x-3)^2}$

$$\lim_{x \rightarrow 3^+} \frac{-2}{(x-3)^2} = \frac{-2}{(\text{V.S.P.Q})^2} = \frac{-2}{\text{V.S.P.Q}} = -\infty$$

$$\lim_{x \rightarrow 3^-} \frac{-2}{(x-3)^2} = \frac{-2}{(\text{V.S.N.Q})^2} = \frac{-2}{\text{V.S.P.Q}} = -\infty$$

So $\lim_{x \rightarrow 3} \frac{-2}{(x-3)^2} = -\infty$

[4] Vertical asymptotes:

Def A line $x=a$ is a vertical asymptote of the graph of the fn. $y=f(x)$ if either $\lim_{x \rightarrow a^-} f(x) = +\infty$ or $\lim_{x \rightarrow a^+} f(x) = +\infty$.

Example/ Find the asymptotes of the fn $f(x) = \frac{x}{x^2-4}$

① There's no oblique asymptotes because degree of $x < \text{degree of } x^2-4$

$$\textcircled{2} \lim_{x \rightarrow \infty} \frac{x}{x^2-4} = \lim_{x \rightarrow \infty} \frac{1}{x - \frac{4}{x}} = \frac{0}{1-0} = 0$$

$$\lim_{x \rightarrow -\infty} = \dots = 0$$

So $y=0$ is a horizontal asymptote.

$$\textcircled{3} \lim_{x \rightarrow 2^+} \frac{x}{(x-2)(x+2)} = \frac{2}{(V.S.P.Q)(4)} = \infty$$

$$\lim_{x \rightarrow 2^-} \frac{x}{(x-2)(x+2)} = \frac{2}{(V.S.N.Q)(4)} = -\infty$$

So $x=2$ is a vertical asymptote.

$$\lim_{x \rightarrow -2^+} \frac{x}{(x-2)(x+2)} = \frac{-2}{-4(V.S.P.Q)} = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x}{(x-2)(x+2)} = \frac{-2}{-4(V.S.N.Q)} = -\infty$$

So $x=-2$ is a vertical asymptote

Example/Sketch the graph of the fn. $y = \frac{x^2 - 1}{2x + 4}$

find the asymptotes and dominant terms.

$$\lim_{x \rightarrow \pm\infty} \frac{x^2 - 1}{2x + 4} = \lim_{x \rightarrow \pm\infty} \frac{x - \frac{1}{x}}{2 + \frac{4}{x}} = \pm\infty$$

So there is no horizontal asymptotes.

$$P(x) = \frac{x}{2} - 1 + \frac{3}{2x + 4}$$

The oblique asymptote is $y = \frac{x}{2} - 1$

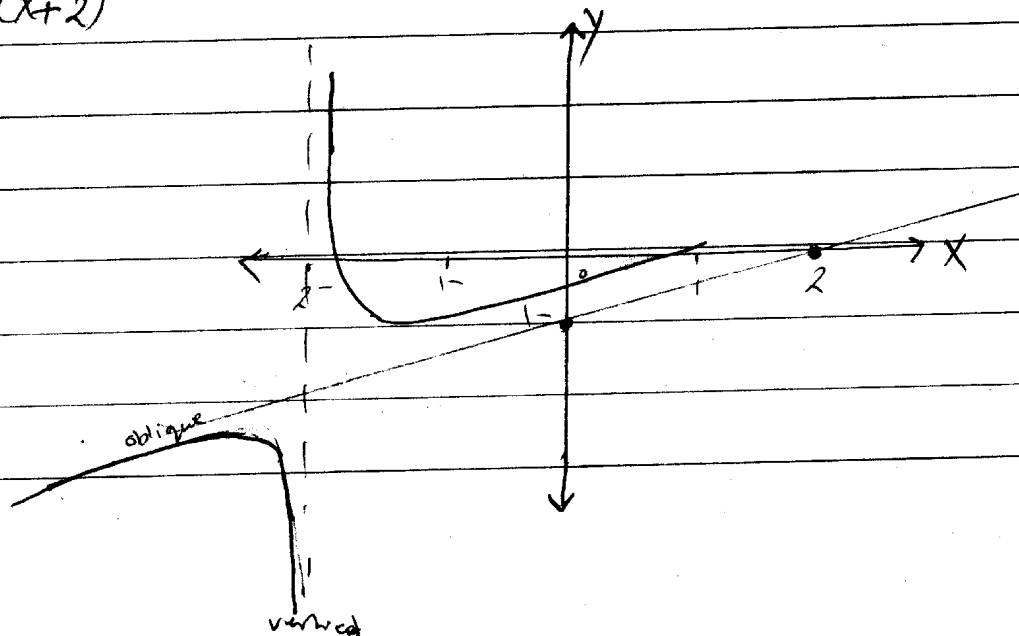
$$\lim_{x \rightarrow -2^+} \frac{x}{2} - 1 + \frac{3}{2(x+2)} = -1 - 1 + \frac{3}{2(\text{V.S.P.Q})} = -2\infty = \infty$$

$$\lim_{x \rightarrow -2^-} \frac{x}{2} - 1 + \frac{3}{2(x+2)} = -1 - 1 + \frac{3}{2(\text{V.S.N.Q})} = -\infty$$

So $x = -2$ is a vertical asymptote.

* the term $\frac{x}{2} - 1$ dominates when x is numerically large.

* the term $\frac{3}{2(x+2)}$ is near -2 .



Exercises 2.6

[9] $\lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$

$$-1 \leq \sin 2x \leq +1$$

$$\frac{-1}{x} \leq \frac{\sin 2x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \frac{-1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$0 \leq \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} \leq 0$$

By Sandwich theorem $\Rightarrow$

$$\lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = 0$$

[10] ~~$\lim_{t \rightarrow \infty} \frac{2-t+\sin t}{t+\cos t}$~~ $\lim_{t \rightarrow -\infty} \frac{\frac{2}{t} - 1 + \frac{\sin t}{t}}{1 + \frac{\cos t}{t}}$

$$\frac{0 - 1 + 0}{1 + 0} = \boxed{-1}$$

$$-1 \leq \cos t \leq +1$$

$$-1 \leq \sin t \leq +1$$

$$\frac{-1}{t} \geq \sin t \geq \frac{1}{t}$$

$$\lim_{t \rightarrow \infty} \frac{\sin t}{t} = 0$$

By sandwich theorem

$$\lim_{t \rightarrow \infty} \frac{\sin t}{t} = 0$$

[35] $\lim_{x \rightarrow \infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x \rightarrow \infty} \frac{x-3}{\sqrt{x^2(4+\frac{25}{x^2})}} = \lim_{x \rightarrow \infty} \frac{x(1-\frac{3}{x})}{|x|(\sqrt{4+\frac{25}{x^2}})}$

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{4}} = \boxed{\frac{1}{2}}$$

[43] $\lim_{x \rightarrow 7} \frac{4}{(x-7)^2}$

$$\lim_{x \rightarrow 7^+} \frac{4}{(x-7)^2} = \frac{4}{(\text{V.S.P.Q})^2} = \frac{4}{\text{V.S.P.Q}} = \infty$$

$$\lim_{x \rightarrow 7^-} \frac{4}{(x-7)^2} = \frac{4}{(\text{V.S.N.QR})^2} = \frac{4}{\text{V.S.N.QR}} = \infty$$

$$S = \lim_{x \rightarrow 7} \frac{4}{(x-7)^2}$$

$$= \infty$$

$$\boxed{63} \quad y = \frac{1}{x-1}$$

$\lim_{x \rightarrow \infty} \frac{1}{x-1} = 0 \Rightarrow y=0$ is a horizontal asymptote.
 there's no oblique asymptotes because $\text{degree}(1) < \text{degree}(x)$

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty \quad \text{V.S.P.Q.} \quad \lim_{x \rightarrow 1^+} \frac{1}{x-1} = \frac{1}{0^+} = +\infty \quad \text{V.S.N.Q.}$$

So $x=1$ is a vertical asymptote

dominant term is $\frac{1}{x-1}$

$$\boxed{67} \quad y = \frac{x+3}{x+2}$$

$$\lim_{x \rightarrow \infty} \frac{x+3}{x+2} = 1 \Rightarrow y=1 \text{ is a horizontal asymptote}$$

$$\lim_{x \rightarrow -2^+} \frac{x+3}{x+2} = \frac{-2+3}{0^+} = \frac{1}{0^+} = +\infty \Rightarrow x=-2 \text{ is a vertical asymptote, V.S.P.Q.}$$

$$\boxed{81)} \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+25} - \sqrt{x^2-1}}{\sqrt{x^2+25} + \sqrt{x^2-1}} \cdot x$$

$$\lim_{x \rightarrow \infty} \frac{x^2+25 - x^2+1}{\sqrt{x^2+25} + \sqrt{x^2-1}} = \lim_{x \rightarrow \infty} \frac{26}{|x| \sqrt{1+\frac{25}{x^2}} + |x| \sqrt{1-\frac{1}{x^2}}}$$

$$\lim_{x \rightarrow \infty} \frac{26}{x(\sqrt{1+0} + \sqrt{1-0})} = \lim_{x \rightarrow \infty} \frac{26}{1+1} = \underline{\underline{0}}$$

$$\boxed{85)} \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2+3x} - \sqrt{x^2-2x})}{\sqrt{x^2+3x} + \sqrt{x^2-2x}} \cdot x$$

$$\lim_{x \rightarrow \infty} \frac{x^2+3x - x^2+2x}{\sqrt{x^2+3x} + \sqrt{x^2-2x}} = \lim_{x \rightarrow \infty} \frac{5x}{x(\sqrt{1+\frac{3}{x}} + \sqrt{1-\frac{2}{x}})}$$

$$= \frac{5}{1+1} = \boxed{\frac{5}{2}}$$

99 $y = \frac{x^2}{x-1}$

* $\lim_{x \rightarrow \infty} \frac{x^2}{x-1} = +\infty \rightarrow$ There's no horizontal asymptotes.

* $y = x+1 + \frac{1}{x-1}$

$y = x+1$ is an oblique asymptote.

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2} \\ \underline{x^2 - x} \end{array}$$

* $\lim_{x \rightarrow 1^+} (x+1) \frac{1}{x-1} = 1+1 + \frac{1}{x-1} = \infty$.

$$\begin{array}{r} x \\ x-1 \overline{) x} \\ \underline{x-1} \\ 1 \end{array}$$

$x=1$ is a vertical asymptote. V.S.P.Q

101 $y = \frac{x^2 - 4}{x-1}$

* There's no horizontal asymptotes.

* $y = x+1 + \frac{-3}{x-1}$

$y = x+1$ is an oblique asymptote.

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2 - 4} \\ \underline{x^2 - x} \end{array}$$

* $\lim_{x \rightarrow 1^+} x+1 + \frac{-3}{x-1} = 2 + \frac{-3}{x-1} = -\infty$.

$$\begin{array}{r} x-4 \\ x-1 \overline{) x-4} \\ \underline{x-1} \\ -3 \end{array}$$

$x=1$ is a vertical asymptote. V.S.P.Q

$$\boxed{103} \quad y = \frac{x^2 - 1}{x}$$

there is no horizontal asymptotes.

$$y = x - \frac{1}{x}$$

$y = x$ is ~~not~~ oblique asymptote.

$$\lim_{x \rightarrow 0^+} x - \frac{1}{x} = \infty \quad \Rightarrow x = 0 \text{ is a vertical asymptote.}$$

Ex 3, Differentiation

[P.1] Tangents and the derivative at a point

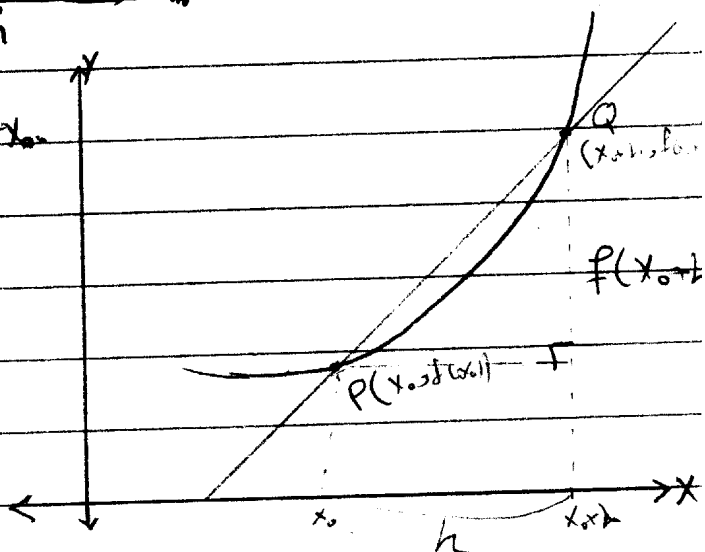
Def. The slope of the curve $y=f(x)$ at the point $(x_0, f(x_0))$ is the number $m = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$, provided the limit exists.

* The tangent line to the curve $y=f(x)$ at the point $P(x_0, f(x_0))$ is the line through P with this slope m . It's eqn is $y=m(x-x_0)+f(x_0)$.

The derivative of f_n at the point x_0 denoted by $f'(x_0)$,
 $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$, provided the limit exists.

Remark / The Limit $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$ means:

- 1) The Slope of the curve $y=f(x)$ at $x=x_0$.
- 2) $=$ $=$ $=$ $=$ tangent to the curve $y=f(x)$ at $x=x_0$.
- 3) The rate of change of $f(x)$ with respect to x at $x=x_0$.
- 4) The derivative of $f(x)$ at $x=x_0$, that is, $f'(x_0)$.



Example/Find the slope of the fn $f(x) = \sqrt{x+1}$ at the point $P(8,3)$, and find the eqn of the tangent line at $P(8,3)$.

The slope of the curve (m) = $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(8+h) - f(8)}{h}$

$$\lim_{h \rightarrow 0} \frac{\sqrt{(h+8)+1} - \sqrt{8+1}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{9+h} - 3}{h} \times \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$$

$$\lim_{h \rightarrow 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}$$

= the tangent of the curve at $P(8,3)$.

* the eqn: $y = m(x - x_0) + y_0$

$$y = \frac{1}{6}(x - 8) + 3$$

Example/Find eqns of all lines having the slope -1 that are tangents to the curve $y = \frac{1}{x-1}$.

solution $m = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = -1$

$$\lim_{h \rightarrow 0} \frac{\frac{1}{x_0+h-1} - \frac{1}{x_0-1}}{h} = -1$$

$$\lim_{h \rightarrow 0} \frac{x_0-1 - x_0-h+1}{(x_0-1)(x_0+h-1)h} = \lim_{h \rightarrow 0} \frac{-1}{(x_0-1)(x_0+h-1)} = -1$$

$$\frac{1}{(x_0-1)(x_0-1)} = 1 \Rightarrow (x_0-1)^2 = 1 \Rightarrow$$

$$x_0-1 = 1 \quad \text{or} \quad x_0-1 = -1$$

$$\Rightarrow x_{01} = 2 \Rightarrow y_1 = 1 \quad \text{the point } P_1(2,1)$$

$$\Rightarrow x_{02} = 0 \Rightarrow y_2 = -1 = P_2(0,-1)$$

$$\text{the first eqn: } y_1 = -1(x-2)+1 \Rightarrow y_1 = -x+3$$

$$\text{= second: } y_2 = -1(x-0)-1 \Rightarrow y_2 = -x-1$$

Def. The curve $y=f(x)$ has a vertical tangent $x=x_0$ at the point x_0 if $\lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = \infty$ or $-\infty$.

Example/Sketch the graph of the fn $f(x) = (x-1)^{\frac{1}{3}}$. Is the fn has a vertical tangent at $x=1$?

$$\lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h)-f(1)}{h}$$

$$\lim_{h \rightarrow 0} \frac{(1+h-1)^{\frac{1}{3}} - (1-1)^{\frac{1}{3}}}{h} = \lim_{h \rightarrow 0} \frac{h^{\frac{1}{3}}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h^{\frac{2}{3}}} \quad (\text{undefined})$$

$$\lim_{h \rightarrow 0^+} \frac{1}{(h^2)^{\frac{1}{3}}} = \frac{1}{(U.S.P.Q)^{\frac{1}{3}}} = \infty$$

$$\lim_{h \rightarrow 0^-} \frac{1}{(h^2)^{\frac{1}{3}}} = \infty \Rightarrow \text{So } \lim_{h \rightarrow 0} \frac{1}{(h^2)^{\frac{1}{3}}} = \infty$$

So $x=1$ is a vertical tangent of the curve.

Example/let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Show that $f(x)$ is continuous at $x=0$.

Does the graph has a vertical tangent at $x=0$?

$$f(0) = 0$$

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$0 \leq \left| \sin \frac{1}{x} \right| \leq 1$$

$$0 \leq |x| \left| \sin \frac{1}{x} \right| \leq |x|$$

lim

So $f(x)$ is continuous at $x=0$.

By sandwich, $\lim_{x \rightarrow 0} \left| \frac{1}{x} \right|$
so $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{K \sin \frac{1}{h} - 0}{K} = \lim_{h \rightarrow 0} \sin \frac{1}{h}$$

$\sin \frac{1}{h}$ does not exist

$\therefore$ there is no vertical tangent at $x=0$.

Exercises [3.1] ~

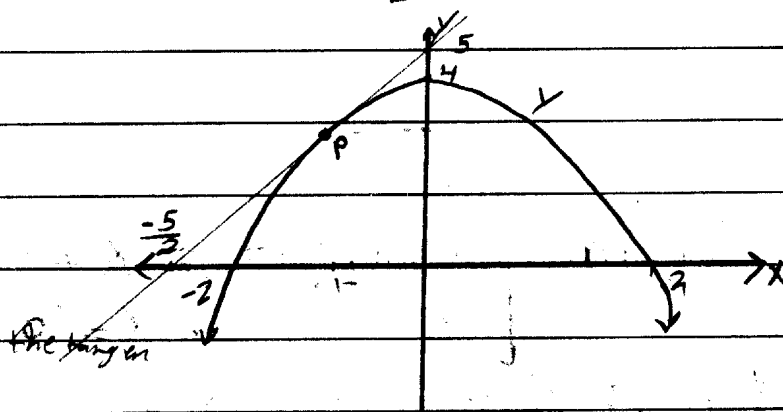
$$5) m = \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{4 - (-1+h)^2 - 4 + (-1)}{h}$$

$$\lim_{h \rightarrow 0} \frac{4 - 1 + 2h - h^2 - 3}{h} = \lim_{h \rightarrow 0} \frac{2 + 2h - h^2}{h} = \underline{\underline{2}}$$

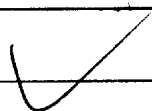
= the slope of the tangent at $(-1, 3)$.

$\therefore$ the eqn: $y = m(x - x_0) + y_0$

$$y = 2(x + 1) + 3 \Rightarrow \underline{\underline{y = 2x + 5}}$$



11)



19) $y = 5(-1)^2 = 5 \Rightarrow$ the point is $(-1, 5)$.

$$m = \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{5(-1+h)^2 - 5}{h}$$

$$\lim_{h \rightarrow 0} \frac{5 - 10h + 5h^2 - 5}{h} = \lim_{h \rightarrow 0} -10 + 5h = \underline{\underline{-10}}$$

$$22) \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h-1}{h+1} + \frac{1}{h}}{h} = \lim_{h \rightarrow 0} \frac{h-1+h+1}{h(h+1)}$$

$$\lim_{h \rightarrow 0} \frac{2h}{h(h+1)} = \frac{2}{1} = \underline{\underline{2}}$$

23) horizontal tangent $\Rightarrow f'(x_0) = 0$.

$$m = \lim_{h \rightarrow 0} \frac{f(h+x_0) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{[(h+x_0)^2 + 4(h+x_0) - 1] - [x_0^2 + 4x_0 - 1]}{h}$$

$$\lim_{h \rightarrow 0} \frac{h^2 + 2hx_0 + x_0^2 + 4h + 4x_0 - 1 - x_0^2 - 4x_0 + 1}{h}$$

$$\lim_{h \rightarrow 0} \frac{h^2 + 2x_0h + 4h}{h} = \lim_{h \rightarrow 0} h + 2x_0 + 4 = 0$$

$$2x_0 + 4 = 0 \rightarrow x_0 = -2. \Rightarrow y = (-2)^2 + 4(-2) - 1 = -5$$

the point is $(-2, -5)$

$$26) m = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h}$$

$$\frac{1}{4} = \lim_{h \rightarrow 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \times \frac{\sqrt{x_0+h} + \sqrt{x_0}}{\sqrt{x_0+h} + \sqrt{x_0}}$$

$$\frac{1}{4} = \lim_{h \rightarrow 0} \frac{x_0+h - x_0}{h(\sqrt{x_0+h} + \sqrt{x_0})} \Rightarrow \lim_{h \rightarrow 0} \frac{1}{\sqrt{x_0+h} + \sqrt{x_0}} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{x_0}} = \frac{1}{4} \rightarrow \sqrt{x_0} = 2 \rightarrow \underline{x_0 = 4} \rightarrow \underline{y = \pm 2}$$

the slope of the tangent $= \frac{1}{4}$.

the eqn. $y = \frac{1}{4}(x-4) + 2$

$$y = \frac{1}{4}x - 1 + 2 \Rightarrow \underline{\underline{y = \frac{1}{4}x + 1}}$$

$$[33] \quad m = \lim_{h \rightarrow 0} \frac{f(h+0) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0 \quad \text{since } \sin \frac{1}{h} \text{ is bounded}$$

So f has a ~~tangency~~ tangency at the origin.

$$[34] \quad m = \lim_{h \rightarrow 0} \frac{f(h+0) - f(0)}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h} \quad \text{does not exist}$$

So f ~~doesn't~~ doesn't have a tangency at the origin.

$$\boxed{35} \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{1-0}{h} = \frac{1}{0} = +\infty.$$

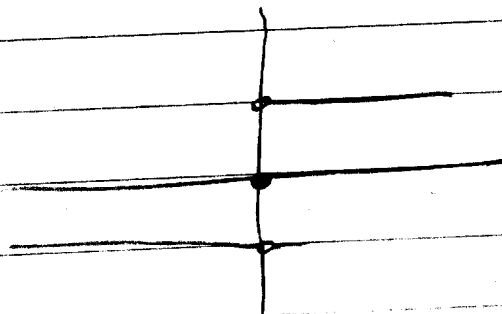
V.I.S.P.Q

when $h < 0$

$$\lim_{h \rightarrow 0^-} \frac{-1-0}{h} = \frac{-1}{0} = +\infty$$

V.I.S.N.Q

Yes, because $\lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \pm \infty.$



[3.2] The derivative as a fn.

Def. The derivative of the fn $f(x)$ with respect to x is the fn $f'(x)$, where $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, provided the limit exists.

$\text{Dom } f'$ is the set of points in $\text{Dom } f$ for which the limit exists.

If f' exists at $x=a$, we say that f is diff at $x=a$.

at every point $x \in \text{Dom } f$, we say that f is diff.

we noted $f'(x)$ by $\frac{dy}{dx}$, $\frac{dF}{dx}$, $D_x F$, y' and $\frac{dF}{dx}$ at $x=a$ is denoted by $f'(a)$, $\frac{df}{dx} \Big|_{x=a}$, $\frac{dy}{dx} \Big|_{x=a}$.

Remark / If we put $z = h + x$, then $h = z - x$, the above Def. take the form.

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}.$$

Example / If $f(x) = \frac{1-x}{2x}$. find (a) $f'(x)$ (b) $f'(-1)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1-x-h}{2(x+h)} - \frac{1-x}{2x}}{h}$$

$$\frac{1}{2} \lim_{h \rightarrow 0} \frac{x - x^2 - xh - x^2 + x^2 - h + xh}{h(x+h)(x)} = \frac{1}{2} \lim_{h \rightarrow 0} \frac{-1}{(x+h)(x)}$$

$$\frac{1}{2} \frac{-1}{(x+0)(x)} = \boxed{\frac{-1}{2x^2}}$$

$$f'(-1) = \frac{-1}{2(-1)^2} = \boxed{\frac{-1}{2}}$$

Example 2/ If $f(x) = \sqrt{x}$ find $f'(x)$ and $f'(81)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\lim_{h \rightarrow 0} \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

$$f'(81) = \frac{1}{2\sqrt{81}} = \boxed{\frac{1}{18}}$$

Example 3/ Use the formula $f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$ to find $f'(x)$ for $f(x) = x^2 - 3x + 4$.

and find the tangent at $x=2$.

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{z^2 - 3z + 4 - x^2 + 3x - 4}{z - x}$$

$$\lim_{z \rightarrow x} \frac{(z^2 - x^2) - 3(z - x)}{z - x} = \lim_{z \rightarrow x} \frac{(z - x)(z + x - 3)}{z - x}$$

$$\lim_{z \rightarrow x} z + x - 3 = \boxed{2x - 3} = \text{the slope of the tangent}$$

$$* f(2) = (2)^2 - 3(2) + 4 = 2 \text{ "the point of tangency"}$$

$$f'(2) = 2(2) - 3 = 1$$

$$\text{the eqn } y = 1(\underline{x - 2}) + 2 \Rightarrow \boxed{y = x}$$

Def. A fn $y = f(x)$ is diff on an open interval if it has a derivative at each point of the interval

Def. A fn $f(x)$ is diff on $[a, b]$ if it's diff on (a, b) and if the limits $\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$ and $\lim_{h \rightarrow 0^-} \frac{f(b+h) - f(b)}{h}$ exists
 R.H.D at $x=a$ L.H.D at $x=b$.

Example/ Is the fn $f(x) = |x-3|$ diff $\forall x$?

$$f(x) = \begin{cases} x-3, & x \geq 3 \\ 3-x, & x < 3 \end{cases}$$

$$f'(x) = \lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{x_0+h-3 - x_0}{h}$$

$$\lim_{h \rightarrow 0^+} \frac{3+h-3 - 3+3}{h} = [1]$$

$$f'(x) = \lim_{h \rightarrow 0^-} \frac{3-x_0-h - 3+x_0}{h} = \lim_{h \rightarrow 0^-} \frac{3-3-h - 3+3}{h} = [-1]$$

$f'(3)$ does not exist, $\therefore f(x)$ is not diff $\forall x$.

Remark/ If the fn is diff then it's continuous but the converse is not true.

Example/ Let $f(x) = \sqrt{x-7}$, find

(a) Dom (f) .

$$\text{Dom } f = \{x : x-7 \geq 0\} \Rightarrow \{x : x \geq 7\} \\ = [7, \infty).$$

$$(6) f'(x) = \frac{1}{2\sqrt{x-7}}$$

$$(7) \text{Dom } f' : \text{Dom } f - \{f' = 0\} = [7, \infty) - \{7\} \\ = (7, \infty)$$

Remark / The fn $f(x)$ is not diff at a point $x=c$ in the following cases:

(I) If the graph has a corner, because the one-sided derivative differ.

(2) $\curvearrowright$ a cusp (Sharp turn)

(3) $\text{---} \text{---} \text{---}$ a vertical tangent.

(4) $\text{---} \text{---} \text{---}$ a discontinuity.

Example / Show that the fn $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ is diff. at $x=0$ and find $f'(0)$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h}$$

$$\lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

$\therefore f$ is diff at $x=0$, $f'(0) = 0$.

Example / For what value of A and B will the fn be diff at $x=2$. $f(x) = \begin{cases} ax & , x < 2 \\ ax^2 - bx + 3 & , x \geq 2. \end{cases}$

$$\text{If } x < 2, f'(2) = \lim_{h \rightarrow 0^-} \frac{f(2+h) - f(2)}{h}$$

$$\lim_{h \rightarrow 0^-} \frac{a(2+h) - 2a}{h} = \lim_{h \rightarrow 0^-} \frac{2a + ah - 2a}{h} = \underline{a}$$

$$\lim_{h \rightarrow 0^+} \frac{[a(2+h)^2 - b(2+h) + 3] - [4a - 2b + 3]}{h}$$

$$\lim_{h \rightarrow 0^+} \frac{4a + 4ah + ah^2 - 2b - bh - 4a + 2b}{h}$$

$$\lim_{h \rightarrow 0^+} \frac{4ah + ah^2 - bh}{h} = \lim_{h \rightarrow 0^+} 4a + ah - b = 4a - b$$

Since $f(x)$ is diff at $x=2$, we have $f'_+(2) = f'_-(2)$

$$4a - b = a \rightarrow \boxed{3a = b}$$

Since f is diff at $x=2$, so f is continuous at $x=2$.

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

$$\lim_{x \rightarrow 2^-} f(x) - \lim_{x \rightarrow 2^-} ax = 2a$$

$$\lim_{x \rightarrow 2^+} ax^2 - bx + 3 = 4a - 2b + 3 = 4a - 6a + 3 = -2a + 3$$

$$-2a + 3 = 2a \rightarrow 4a = 3 \rightarrow \boxed{a = \frac{3}{4}}, \boxed{b = \frac{9}{4}}$$

Exercises 3.2

$$[3] g(t) = \frac{1}{t^2}$$

$$g'(t) = \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(t+h)^2} - \frac{1}{t^2}}{h}$$

$$\lim_{h \rightarrow 0} \frac{t^2 - (t+h)^2}{h(t^2)(t+h)^2} = \lim_{h \rightarrow 0} \frac{t^2 - t^2 - 2th - h^2}{h(t^2)(t+h)^2}$$

$$\lim_{h \rightarrow 0} \frac{-h(h+2t)}{h(t^2)(t+h)^2} = \lim_{h \rightarrow 0} \frac{-1(h+2t)}{t^2(t+h)^2} = \frac{-2t}{t^4} = \boxed{\frac{-2}{t^3}}$$

$$[6] r(s) = \sqrt{2s+1}$$

$$r'(s) = \lim_{h \rightarrow 0} \frac{\sqrt{2(h+s)+1} - \sqrt{2s+1}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2s+2h+1} - \sqrt{2s+1}}{h}$$

$$\lim_{h \rightarrow 0} \frac{2s+2h+1 - 2s-1}{h(\sqrt{2s+2h+1} + \sqrt{2s+1})} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2s+2h+1} + \sqrt{2s+1}}$$

$$\boxed{\frac{2}{2\sqrt{2s+1}}}$$

$$[9] \frac{ds}{dt} \text{ if } s = \frac{t}{2t+1}$$

$$\frac{ds}{dt} = \lim_{h \rightarrow 0} \frac{s\left(\frac{t}{2t+1} + h\right) - s\left(\frac{t}{2t+1}\right)}{h} = \lim_{h \rightarrow 0} \frac{\frac{t+h}{2(t+h)+1} - \frac{t}{2t+1}}{h}$$

$$\lim_{h \rightarrow 0} \frac{2t^2 + 2th + t + h - 2t^2 - 2th - t}{h(2(t+h)+1)(2t+1)}$$

$$\frac{1}{(2t+1)(2t+1)} = \frac{1}{(2t+1)^2}$$

[17] $y = f(x) = \frac{8}{\sqrt{x-2}}$, $(x, y) = (6, 4)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\frac{8}{\sqrt{x+h-2}} - \frac{8}{\sqrt{x-2}}}{h} = \lim_{h \rightarrow 0} \frac{8\sqrt{x-2} - 8\sqrt{x+h-2}}{h(\sqrt{x+h-2})(\sqrt{x-2})}$$

$$8 \lim_{h \rightarrow 0} \frac{x-2 - x-h+2}{h(\sqrt{x+h-2})(\sqrt{x-2})} \times \frac{1}{\sqrt{x-2} + \sqrt{x+h-2}}$$

$$8 \lim_{h \rightarrow 0} \frac{1}{(\sqrt{x+h-2})(\sqrt{x-2})} \times \lim_{h \rightarrow 0} \frac{1}{\sqrt{x-2} + \sqrt{x+h-2}} = \frac{1}{(\sqrt{x-2})^2} \times \frac{1}{2\sqrt{x-2}}$$

$$= \frac{1}{2\sqrt{x-2}(x-2)} = \text{the slope of the tangent}$$

$$f'(6) = \frac{1}{2\sqrt{6-2}(6-2)} = \frac{1}{4(4)} = \sqrt{\frac{1}{16}} = \text{the slope of the tangent at (6, 4)}$$

the eqn: $y = \frac{1}{16}(x-6) + 4$

$$\Rightarrow \boxed{y = \frac{1}{16}x + \frac{29}{8}}$$

[25] $f(x) = \frac{x}{x-1}$, $\lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$

$$\lim_{z \rightarrow x} \frac{\frac{z}{z-1} - \frac{x}{x-1}}{z-x} = \lim_{z \rightarrow x} \frac{z(x-1) - x(z-1)}{(x-1)(z-1)(z-x)} = \lim_{z \rightarrow x} \frac{-(z-x)}{(x-1)(z-1)(z-x)}$$

$$= \frac{-1}{(x-1)(x-1)} = \frac{-1}{(x-1)^2}$$

31) ~~$x = -4, 0, 1, 4, 6$~~

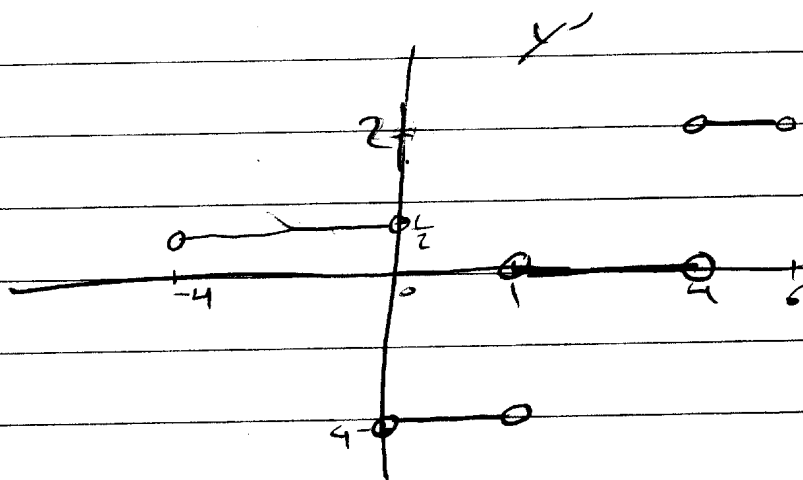
$x = -4, 6$ because it's an endpoint.

$x = 0, 1, 4$ because they are corners.

$$m_1 = \frac{2-0}{0-4} = -\frac{1}{2}$$

$$m_2 = \frac{2+2}{0-1} = -4$$

$$m_3 = 0, m_4 = 2$$



42) $f(0) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 0$

$$g'_+(x) = \lim_{h \rightarrow 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^{\frac{2}{3}} - 0}{h^{\frac{1}{3}}} = \lim_{h \rightarrow 0^+} h^{\frac{1}{3}} = (\text{V.S.P.})^{\frac{1}{3}} = \infty$$

$$= \lim_{h \rightarrow 0^+} h^{\frac{1}{3}} = (\text{V.S.P.})^{\frac{1}{3}} = \infty$$

$$g'_-(x) = \lim_{h \rightarrow 0^-} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0^-} \frac{h^{\frac{2}{3}}}{h^{\frac{1}{3}}} = (\text{V.S.N.})^{\frac{1}{3}} = -\infty$$

$$= -\infty$$

f' is undifferentiable

3.3 Differentiation Rules.

① If $f(x) = c$, where c is constant, then $f'(x) = 0$.

② If n is nonzero real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

③ If c is constant then $\frac{d}{dx}(cf(x)) = c \frac{df}{dx}$, for example $\frac{d}{dx}(cx^n) = cnx^{n-1}$.

④ If u and v are diff. fns of x , then $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$.
In general, $\frac{d}{dx}(u_1 \pm u_2 \pm \dots \pm u_n) = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \dots \pm \frac{du_n}{dx}$.

⑤ If u and v are diff fns of x , $\frac{d}{dx}(u \cdot v) = u \frac{dv}{dx} + v \frac{du}{dx}$.

⑥ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{(v)^2}$.

In particular, $\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2} \cdot \frac{dv}{dx}$.

⑦ Second and higher derivative. If $y' = \frac{dy}{dx}$ then $y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, $y^n = \frac{d^ny}{dx^n}$.

Example 1/ Find y' for each of the following.

① $y = 4\sqrt{x} - \frac{5}{x}$

$$y' = 4 \cdot \frac{1}{2\sqrt{x}} - \frac{-5}{x^2} (1) = \frac{2}{\sqrt{x}} + \frac{5}{x^2}$$

② $y = 3\sqrt[3]{x^2} - \frac{2}{\sqrt{x^3}} = 3(x)^{\frac{2}{3}} - 2x^{-\frac{3}{2}}$

$$y' = 3 \cdot \frac{2}{3} x^{-\frac{1}{3}} - 2 \cdot \frac{-3}{2} x^{-\frac{5}{2}}$$

$$y' = \frac{2}{\sqrt[3]{x}} + \frac{3}{\sqrt{x^5}}$$

$$\textcircled{a} y = \frac{5 + 2x + x^2}{\sqrt{x}} = 5x^{-\frac{1}{2}} + 2x^{\frac{1}{2}} + x^{\frac{3}{2}}$$

$$y' = -\frac{5}{2}x^{-\frac{3}{2}} + x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}}$$

$$\textcircled{d} y = \frac{x^2 + 2x}{x^2 - 1} \quad y' = \frac{(x^2 - 1)(2x + 2) - (x^2 + 2x)(2x)}{(x^2 - 1)^2}$$

$$y' = \frac{2x^3 - 2x + 2x^2 - 2 - 2x^3 - 4x^2}{(x^2 - 1)^2} = \frac{-2x - 2x^2 - 2}{(x^2 - 1)^2}$$

$$\textcircled{e} y = (x^2 + 3)(\sqrt{x} - 3)(3x^{\frac{2}{3}} - 2)$$

$$y' = 2x(\sqrt{x} - 3)(3x^{\frac{2}{3}} - 2) + (x^2 + 3)x \cdot \frac{1}{2\sqrt{x}}(3x^{\frac{2}{3}} - 2) + (x^2 + 3)(\sqrt{x} - 3)(2x^{-\frac{1}{3}})$$

Example 2/ $\textcircled{a}$ Find the second derivative of $y = x^2 - x^{\frac{4}{3}}$

$$y' = 2x - \frac{4}{3}x^{-\frac{1}{3}} \Rightarrow y'' = 2 + \frac{4}{9}x^{-\frac{4}{3}}$$

$\textcircled{a}$ Find y^n for the fn $y = \frac{1}{x+3}$

$$y' = \frac{-1}{(x+3)^2} (1) = -1(x+3)^{-2} \Rightarrow y' = -1!(x+3)^{-2}$$

$$y'' = (-1)(-2)(x+3)^{-3} = 2(x+3)^{-3} \Rightarrow y'' = +2!(x+3)^{-3}$$

$$y''' = (2)(-3)(x+3)^{-4} \Rightarrow y''' = -3!(x+3)^{-4}$$

$$y^{(4)} = (2)(-3)(-4)(x+3)^{-5} \Rightarrow y^{(4)} = +4!(x+3)^{-5}$$

$$\therefore y^n = (-1)^n n! (x+3)^{-n-1}$$

Example 3/ If $f(2)=2$ and $f'(2)=3$. Find $\frac{dy}{dx}$ for the fn

$$y = \frac{f(x)}{x^2 + f(x)} \text{ at } x=2$$

$$\frac{dy}{dx} = \frac{(x^2 + f(x))(f'(x)) - f(x)(2x + f'(x))}{(x^2 + f(x))^2}$$

$$\left. \frac{dy}{dx} \right|_{x=2} = \frac{(4 + f(2))(f'(2)) - f(2)(4 + f'(2))}{(4 + f(2))^2} = \boxed{\frac{1}{9}}$$

Example 4/ (A) Find an eqn for the line perpendicular to the tangent of the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$

$$y' = 3x^2 - 4 = \text{the slope of the tangent}$$

$$\text{the slope of the tangent at } (2, 1) = 3(2)^2 - 4 = \underline{8} \text{ (m}_1\text{)}$$

$$\text{Since } m_1 \cdot m_2 = -1 \Rightarrow \boxed{m_2 = -\frac{1}{8}} \text{ (the slope of the perpendicular line to the tangent)}$$

$$\text{eqn: } y = -\frac{1}{8}(x-2) + 1 \Rightarrow \boxed{8y = -x + 10}$$

(B) Find eqns for the tangents to the curve at the points where the slope is 8.

$$m = y' = 3x^2 - 4 = 8$$

$$3x^2 = 12 \rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$$\text{when } x = 2 \rightarrow y = 1$$

$$\text{when } x = -2 \rightarrow y = 1$$

the points of the tangency are $(2, 1)$ and $(-2, 1)$.

$$\text{eqn}_1: y = 8(x-2) + 1 \Rightarrow \underline{y_1 = 8x - 15}$$

$$\text{eqn}_2: y = 8(x+2) + 1 \Rightarrow \underline{y_2 = 8x + 17}$$

Example 5/ Find the values of a and b if the tangent to the curve $y = ax^2 + bx$ has slope 8 at $(1, 5)$ at the point $(1, 5)$, we have

$$5 = a(1)^2 + b(1) \Rightarrow \underline{a + b = 5}$$

$$y' = 2ax + b \quad \text{-- the slope of the tangent}$$

$$y'|_{x=1} = \underline{2a + b = 8}$$

Subtracting the two eqns, we have $\boxed{a = 3}$ and $\boxed{b = 2}$

Exercises 3.3.1

$$\boxed{28} \quad y = \frac{(x+1)(x+2)}{(x-1)(x-2)}$$

$$y' = \frac{(x-1)(x-2)(x+2) + (x+1)(x+2)(x-1)(x-2)}{((x-1)(x-2))^2}$$

$$\boxed{39} \quad p = \left(\frac{q^2 + 3}{12q} \right) \left(\frac{q^4 - 1}{q^3} \right)$$

$$p = \left(\frac{q}{12} + \frac{1}{4q} \right) \left(q - \frac{1}{q^3} \right)$$

$$p' = \left(\frac{1}{12} + \frac{-4}{16q^2} \right) \left(q - \frac{1}{q^3} \right) + \left(\frac{q}{12} + \frac{1}{4q} \right) \left(1 + \frac{3}{q^4} \right)$$

$$\frac{-1}{q^4} \times 3q^2$$

$$\boxed{41)} \quad u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

$$\textcircled{a} \quad \frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \frac{du}{dx} = uv' + vu'$$

$$\left. \frac{d}{dx}(uv) \right|_{x=0} = u(0) \cdot v'(0) + v(0) u'(0) = 5(2) + (-1)(-3) = \boxed{13}$$

$$\textcircled{b} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\left. \frac{d}{dx}\left(\frac{u}{v}\right) \right|_{x=0} = \frac{v(0) \cdot u'(0) - u(0) \cdot v'(0)}{(v(0))^2} = \frac{(-1)(-3) - (5)(2)}{(-1)^2} = \boxed{-7}$$

$$\textcircled{c} \quad \frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u \cdot v' - v \cdot u'}{(u)^2}$$

$$\left. \frac{d}{dx}\left(\frac{v}{u}\right) \right|_{x=0} = \frac{u(0) \cdot v'(0) - v(0) u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(-3)}{(5)^2} = \boxed{\frac{7}{25}}$$

$$\textcircled{d} \quad \frac{d}{dx}(7v - 2u) = 7v' - 2u'$$

$$= 7(v'(0)) - 2(u'(0)) = 7(2) - 2(-3) = \boxed{20}$$

$\boxed{44} \textcircled{A}$ horizontal tangent means that the slope of the tangent = 0.

$$y' = 3x^2 - 3 = 0$$

$$3x^2 - 3 \quad x = \pm 1$$

$$\text{when } x = 1 \rightarrow y = -4$$

$$\text{when } x = -1 \rightarrow y = 0$$

$$\text{first eqn: } y = 0(x - 1) - 4 \Rightarrow \boxed{y = -4}$$

$$\text{second eqn: } y = 0(x + 1) + 0 \Rightarrow \boxed{y = 0}$$

Since $m_1 m_2 = -1 \Rightarrow m_2 = \frac{-1}{0}$ the slope of the perpendicular line to the tangent

* first eqn: $y = \frac{-1}{0}(x-1) - 4$
 $0 = -x + 1 - 4 \Rightarrow \boxed{x = -3}$

* Second eqn: $y = \frac{-1}{0}(x+1) + 0$
 $0 = -x - 1 \Rightarrow \boxed{x = -1}$

[45] $m = \frac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2}$

$m = \frac{4-4x^2}{(x^2+1)^2}$ the slope of the tangent

* the slope of tangent at the origin $m = \frac{4-4(0)}{(0+1)^2} = 4$

the eqn $y = 4(x-0) + 0 \Rightarrow \boxed{y = 4x}$

* the slope of tangent at (1,2) $m_1 = \frac{4-4(1)^2}{(1+1)^2} = 0$

the eqn $y = 0(x-1) + 2 \Rightarrow \boxed{y = 2}$

[47] At the origin, we have.

$0 = a(0)^2 + b(0) + c \Rightarrow \boxed{c = 0}$

$y' = 2ax + b$ the slope of the tangent

$y'|_{x=0} = 2a(0) + b = b$ the slope of tangent at the origin.

the eqn - $y = b(x-a)+0 \Rightarrow y = bx$ [17]

Since $y = x$ we have $bx = x \rightarrow \boxed{b=1}$

at the point $(1,2)$, we have

$$2 = a(1)^2 + 1(1) + 0 \Rightarrow \boxed{a=1}$$

[48] Since the two curves have a common tangent, we have

• the point $(1,a)$ lies on the two curves.

• the two curves have the same tangent "the same slope"

now: $0 = (1)^2 + a(1) + b \Rightarrow \underline{a+b = -1}$

$$0 = c(1) - (1)^2 \Rightarrow \boxed{c=1}$$

$$y' = 2x + a \Rightarrow y' = 2(1) + a = 2 + a$$

$$y' = 1 - 2x \Rightarrow y' = 1 - 2(1) = -1$$

$$\text{so } -1 = 2 + a \rightarrow \boxed{a = -3}, \boxed{b = 2}$$

[49] the tangent lines parallel to the line, this means that the slope of the tangents of the curve and line are the same.

So y' for the curve = y' for the line

$$6x - 4 = 8 \rightarrow 6x = 12 \rightarrow \underline{\underline{x=2}}$$

$$\text{the point } (x,y) = (2, (2(2)^2 - 4(2))) = (2, 4)$$

[51] The tangent line perpendicular to the line $\Rightarrow$

$$m_{\text{tangent}} \cdot m_{\text{line}} = -1.$$

$$y' \text{ For the curve} = \frac{(x-2)(1) - x(1)}{(x-2)^2} = \frac{x-2-x}{(x-2)^2} = \frac{-2}{(x-2)^2}$$

$$y' \text{ For the line} = 2.$$

$$\text{So } \frac{-2}{(x-2)^2} = -1 \Rightarrow (x-2)^2 = 2$$

$$x^2 - 4x + 4 = 2$$

$$x^2 - 4x + 2 = 0$$

$$(x - 2) \pm \sqrt{2} = 0$$

$$\frac{2x-2}{(x-2)^2} = -1$$

$$4 - (x-2)^2 = 0$$

$$x-2 = \pm 2$$

$$x-2 = 2 \Rightarrow x = 4$$

$$x = 4$$

$$x = 0$$

[52] If $x > -1$, we have

$$\lim_{h \rightarrow 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0^+} \frac{a(-1+h) + b - a(-1) - b}{h}$$

$$\lim_{h \rightarrow 0^+} \frac{-a + ah + b + a - b}{h} = a.$$

If $x < -1$

$$\lim_{h \rightarrow 0^-} \frac{b(-1+h)^2 - b(-1)^2}{h} = \lim_{h \rightarrow 0^-} \frac{b - 2hb + bh^2 - b}{h}$$

$$\lim_{h \rightarrow 0^-} \frac{h(-2b + bh)}{h} = \lim_{h \rightarrow 0^-} -2b + bh = \underline{\underline{-2b}}$$

Since f is diff., we have $f'_+(x) = f'_-(x)$.

So $\boxed{a = -2b}$

Since f is diff. at $x = -1$, f is continuous at $x = -1$.

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^-} f(x) = f(-1).$$

$$\left. \begin{aligned} \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} ax + b = -a + b = 2b + b = \underline{\underline{3b}} \\ \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} bx^2 - 3 = \underline{\underline{b-3}} \end{aligned} \right\}$$

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} bx^2 - 3 = \underline{\underline{b-3}}$$

$$3b = b - 3 \rightarrow 2b = -3 \Rightarrow \boxed{b = -\frac{3}{2}}, \boxed{a = 6}$$

[5a] $P'(x) = a_n(n)x^{n-1} + a_{n-1}(n-1)x^{n-2} + \dots + a_2(2)x + a_1$

Exercises 3.5

$$[9] y = (\sec x + \tan x)(\sec x - \tan x) = (\sec x - \tan x)^2$$

~~$$y' = \sec^2 x + \tan^2 x - 2 \tan x \sec x$$~~

~~$$y' = 2(\sec x - \tan x)(\sec x \tan x - \sec^2 x)$$~~

$$[14] y = \frac{\cos x}{x} + \frac{x}{\cos x}$$

$$y' = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$$

$$[33] \textcircled{a} y = \csc x$$

$$y' = -\csc x \cot x$$

$$y'' = -\csc x (-\csc^2 x) + \cot x (+\csc x \cot x)$$

$$= \csc^3 x + \cot^2 x \csc x$$

$$\textcircled{b} y = \sec x$$

$$y' = \tan x \sec x$$

$$y'' = \sec x (\sec^2 x) + \tan x (\sec x \tan x)$$

$$= \sec^3 x + \tan^2 x \sec x$$

$$[35] y = \sin x, -\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$$

$$x = 0, -\pi, \frac{3\pi}{2}$$

$$y' = \cos x$$

$$y' \Big|_{x=-\pi} = \cos -\pi = -1$$

$$\text{eqn: } y = -1(x - \pi)$$

[38]

[39] $y = x + \sin x$

$$y' = 1 + \cos x$$

$$0 = 1 + \cos x \rightarrow \cos x = -1 \Rightarrow \boxed{x = \pi}$$

Yes, at the point (π, π)

[40] $y = x - \cot x$

$$y' = 1 + \csc^2 x$$

$$0 = 1 + \csc^2 x \Rightarrow \csc^2 x = -1 \Rightarrow \csc x = \sqrt{-1}$$

No.

(undefined.)

[43] $y' = \sec^2 x$. "the slope of the tangent of the curve"

$y' = 2$ "the slope = 2 = line =

the tangent // to the line $\Rightarrow 2 = \sec^2 x$

$\sec x = \pm \sqrt{2} \rightarrow x = \frac{\pi}{4}, -\frac{\pi}{4}$ $\frac{1}{\cos x} = \sqrt{2}$

the points are $(\frac{\pi}{4}, f(\frac{\pi}{4}))$ and $(-\frac{\pi}{4}, f(-\frac{\pi}{4}))$.

at the point $(\frac{\pi}{4}, 1)$, the eqn

$$y = 2(x - \frac{\pi}{4}) + 1$$

$$y = 2x - \frac{\pi}{2} + 1$$

at the point $(-\frac{\pi}{4}, -1)$, the eqn

$$y = 2(x + \frac{\pi}{4}) - 1$$

$$y = 2x + \frac{\pi}{2} - 1$$

[45] @ $y' = -\csc^2 x + 2 \csc x \cot x$

$$y' \Big|_{x=\frac{\pi}{2}} = -\csc^2(\frac{\pi}{2}) + 2 \csc \frac{\pi}{2} \cot \frac{\pi}{2} = -1 + 2(1)(0) = -1$$

the slope of the tangent at $(\frac{\pi}{2}, 2)$

the eqn $y = -1(x - \frac{\pi}{2}) + 2$
 $= -x + \frac{\pi}{2} + 2$

(b) horizontal tangent $\Rightarrow$ the slope of the tangent = 0.

$$y' = -\csc^2 x + 2 \csc x \cot x = 0$$

$$\csc x (-\csc x + 2 \cot x) = 0$$

$$\csc x = 0 \rightarrow x = \text{undefined}$$

$$-\csc x + 2 \cot x = 0$$

$$\frac{-1}{\sin x} + \frac{2 \cos x}{\sin x} = \frac{-1 + 2 \cos x}{\sin x} = 0 \Rightarrow -1 + 2 \cos x = 0$$

$$\cos x = \frac{1}{2} \Rightarrow \boxed{x = \frac{\pi}{3}}$$

$$R\left(\frac{\pi}{3}\right) = 4 + \cot \frac{\pi}{3} - 2 \csc \frac{\pi}{3} = 4 + \frac{1}{\sqrt{3}} - \frac{2 \times 2}{\sqrt{3}} \\ = 4 + \frac{-3}{\sqrt{3}}$$

the eqn: $y = 0\left(x - \frac{\pi}{3}\right) + 4 - \frac{3}{\sqrt{3}}$
 $\Rightarrow y = \underline{\underline{4 - \frac{3}{\sqrt{3}}}}$

$$\boxed{47} \lim_{x \rightarrow 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right) = \sin\left(\lim_{x \rightarrow 2} \frac{1}{x} - \frac{1}{2}\right) \\ = \sin(0) = \underline{\underline{0}}$$

$$\boxed{48} \lim_{x \rightarrow -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)}$$

$$\sqrt{\lim_{x \rightarrow -\frac{\pi}{6}} 1 + \cos(\lim_{x \rightarrow -\frac{\pi}{6}} \pi \csc x)} = \sqrt{1 + \cos(\pi \times \csc(-\frac{\pi}{6}))} \\ = \sqrt{1 + \cos(-2\pi)} = \sqrt{1+1} = \boxed{\sqrt{2}}$$

$$\left. \begin{array}{l} -1 \\ \sin \frac{\pi}{6} \\ = -\frac{1}{1} \end{array} \right\}$$

$$\boxed{49} \lim_{\theta \rightarrow \frac{\pi}{6}} \frac{\sin \theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$$

put $X = \theta - \frac{\pi}{6}$
 $\theta = X + \frac{\pi}{6}$

$$\lim_{x \rightarrow 0} \frac{\sin(x + \frac{\pi}{6}) - \frac{1}{2}}{x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x - \frac{1}{2}}{x}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{3}}{2} \frac{\sin x}{x} + \lim_{x \rightarrow 0} \frac{1}{2} \left(\frac{\cos x - 1}{x} \right)$$

$$\frac{\sqrt{3}}{2} (1) + \frac{-1}{2} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \times \frac{1 + \cos x}{1 + \cos x} = \frac{\sqrt{3}}{2} - \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$\frac{\sqrt{3}}{2} - \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} = \frac{\sqrt{3}}{2} - \frac{1}{2} (1) \left(\frac{0}{2} \right)$$

$$= \boxed{\frac{\sqrt{3}}{2}}$$

$$\boxed{51} \lim_{x \rightarrow 0} \sec \left[\cos x + \pi \tan \left(\frac{\pi}{4 \sec x} \right) - 1 \right]$$

$$\sec \left[\lim_{x \rightarrow 0} \cos x + \pi \lim_{x \rightarrow 0} \tan \left(\frac{\pi}{4 \sec x} \right) - 1 \right]$$

$$\sec \left[\cos 0 + \pi \tan \left(\lim_{x \rightarrow 0} \frac{\pi}{4 \sec x} \right) - 1 \right]$$

$$\sec \left(1 + \pi \tan \left(\frac{\pi}{4 \sec 0} \right) - 1 \right) = \sec \left(\pi \tan \frac{\pi}{4} \right) = \sec(\pi) = \boxed{-1}$$

$$\boxed{52} \lim_{t \rightarrow 0} \tan \left(1 - \frac{\sin t}{t} \right) = \tan \left(\lim_{t \rightarrow 0} 1 - \lim_{t \rightarrow 0} \frac{\sin t}{t} \right)$$

$$= \tan(1 - 1) = \tan 0 = \boxed{0}$$

$$\boxed{53}$$

[57]

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin^2 3x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin 3x}{x}$$

$$= 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} \cdot 3 \lim_{3x \rightarrow 0} \frac{\sin 3x}{3x} = 3 \cdot 3 = \underline{\underline{9}}$$

Yes, we can put $\boxed{c=9}$ to make the curve continuous at $x=0$. ($\lim_{x \rightarrow 0} f(x) = f(0)$)

$$[58] * f(0) = \cos(0) = 1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \cos x = 1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x + b = b.$$

to be continuous, we can put $\boxed{b=1}$
 ($\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$).

$$* f'(x) = \int x$$

[B.5] Derivative of Trigonometric

It should be noted the $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Example/ Find $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} = - \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} \lim_{x \rightarrow 0} \frac{1}{\cos x + 1}$$

$$= - \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \sin x \cdot \lim_{x \rightarrow 0} \frac{1}{\cos x + 1} = -1(0)\left(-\frac{1}{2}\right) = 0$$

[I] If $f(x) = \sin x$, then $f'(x) = \cos x$.

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$\lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1)}{h} \cdot \frac{\cos h + 1}{\cos h + 1} + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h}$$

$$\sin x \lim_{h \rightarrow 0} \frac{\sin^2 h}{h} \lim_{h \rightarrow 0} \frac{1}{\cos h + 1} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h}$$

$$(\sin x) \sin 0 \lim_{h \rightarrow 0} \frac{\sin h}{h} \frac{1}{(\cos 0 + 1)} + \cos x \cdot x(1) = \underline{\underline{\cos x}}$$

[2] If $f(x) = \cos x$ then $f'(x) = -\sin x$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$\lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1)}{h} - \lim_{h \rightarrow 0} \sin x \frac{\sin h}{h}$$

$$\cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \underline{\underline{-\sin x}}$$

Example 1 / Find y' for each of the following:-

① $y = x \cos x - x^2 \sin x$.

$$y' = (1) \cos x + x(-\sin x) - 2x \sin x - x^2 \cos x$$

$$\cos x - 3x \sin x - x^2 \cos x.$$

② $y = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x$

$$y' = -\sin x$$

③ $y = x \sin x \tan x$.

$$y' = 1(\sin x \tan x) + x(\cos x) \tan x + (x \sin x) \sec^2 x.$$

④ $y = \frac{1 + \tan x}{1 - \tan x}$.

$$y' = \frac{(1 - \tan x)(\sec^2 x) + (1 + \tan x)(\sec^2 x)}{(1 - \tan x)^2}$$

$$= \frac{\sec^2 x - \cancel{\tan x \sec^2 x} + \sec^2 x + \cancel{\tan x \sec^2 x}}{(1 - \tan x)^2} = \frac{2\sec^2 x}{(1 - \tan x)^2}$$

⑤ $y = \tan x \cdot x$

$$y' = \sec^2 x - 1 = \underline{\tan^2 x}$$

⑥ $y = \frac{\sin x}{1 + \cos x}$

$$y' = \frac{(1 + \cos x)(\cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2} = \frac{1}{(\cos x + 1)}$$

Example 1/ Find y' for each of the following:-

① $y = x \cos x - x^2 \sin x$.

$$y' = (1) \cos x + x(-\sin x) - 2x \sin x - x^2 \cos x$$

$$\cos x - 3x \sin x - x^2 \cos x.$$

② $y = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x$

$$y' = -\sin x$$

③ $y = x \sin x \tan x$.

$$y' = 1(\sin x \tan x) + x(\cos x) \tan x + (x \sin x) \sec^2 x.$$

④ $y = \frac{1 + \tan x}{1 - \tan x}$.

$$y' = \frac{(1 - \tan x)(\sec^2 x) + (1 + \tan x)(\sec^2 x)}{(1 - \tan x)^2}$$

$$= \frac{\sec^2 x - \cancel{\tan x \sec^2 x} + \sec^2 x + \cancel{\tan x \sec^2 x}}{(1 - \tan x)^2} = \frac{2\sec^2 x}{(1 - \tan x)^2}$$

⑤ $y = \tan x - x$

$$y' = \sec^2 x - 1 = \underline{\tan^2 x}$$

⑥ $y = \frac{\sin x}{1 + \cos x}$

$$y' = \frac{(1 + \cos x)(\cos x) - \sin x(-\sin x)}{(1 + \cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2} = \frac{1}{(\cos x + 1)}$$

Example 2/ Find y'' For each of the following --

① $y = x^2 \sin x$.

$$y' = 2x \sin x + x^2 \cos x \Rightarrow y'' = 2 \sin x + 2x \cos x + 2x \cos x - x^2 \sin x$$

$$y'' = 2 \sin x + 4x \cos x - x^2 \sin x.$$

② $y = \csc x$.

$$y' = -\cot x \csc x.$$

$$y'' = -(-\csc^2 x \csc x + \cot x (-\csc x \cot x))$$

$$= \csc^3 x + \cot^2 x \csc x.$$

Example 3/ Find the following Limits.

① $\lim_{x \rightarrow 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right) = \sin\left(\lim_{x \rightarrow 0} \frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$

$$\sin\left(\frac{\pi + 0}{0 - 2(1)}\right) = \sin\left(-\frac{\pi}{2}\right) = \boxed{-1}$$

② $\lim_{x \rightarrow 0} \cos\left(\frac{\pi x}{\sin x}\right) = \cos\left(\lim_{x \rightarrow 0} \frac{\pi x}{\sin x}\right) = \cos\left(\pi\right) = \boxed{-1}$

③ $\lim_{\theta \rightarrow \frac{\pi}{4}} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}} = \lim_{\theta \rightarrow \frac{\pi}{4}} \frac{\tan \theta - 1}{\theta - \frac{\pi}{4}}$

$$\lim_{x \rightarrow 0} \frac{\tan\left(x + \frac{\pi}{4}\right) - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{\tan x + \tan \frac{\pi}{4}}{1 - \tan x \tan \frac{\pi}{4}} - 1}{x}$$

$$\lim_{x \rightarrow 0} \frac{\tan x + 1 - 1 + \tan x}{x(1 - \tan x)} = \lim_{x \rightarrow 0} \frac{2 \tan x}{x(1 - \tan x)} = 2 \lim_{x \rightarrow 0} \frac{\tan x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{1 - \tan x}$$

$$2 \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{1 - 0} = 2(1)\left(\frac{1}{1}\right)(1) = \boxed{2}$$

3.6 The Chain Rule

Theorem: If $y = f(u)$ is a diff. fn of u and $u = g(x)$ is a diff. fn of x , then the composite fn $(f \circ g)(x) = f(g(x))$ is a diff. fn of x and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. also, If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Proof: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d f(u)}{du} \cdot g'(x) = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x) = (f \circ g)'(x)$.

Example 1 / Find y' for each of the following.

① $y = |x^2 - 1| = \begin{cases} x^2 - 1, & |x| \geq 1 \\ 1 - x^2, & |x| < 1 \end{cases}$

$y' = \begin{cases} 2x, & |x| > 1 \end{cases}$

$\begin{cases} -2x, & |x| < 1 \end{cases}$

undefined, $x = 1$

$y'(1) = -2(1) = -2$

$y'(-1) = 2(-1) = -2$

② $y = \tan u$ and $u = \sqrt{x^2 + 5}$

$y' = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot \frac{1}{2\sqrt{x^2 + 5}} \cdot 2x$

$= \sec^2 \sqrt{x^2 + 5} \cdot \frac{x}{\sqrt{x^2 + 5}}$

Or: $y = \tan \sqrt{x^2 + 5}$

$y' = \sec^2 \sqrt{x^2 + 5} \cdot \frac{1}{2\sqrt{x^2 + 5}} \cdot 2x$

$$(c) y = f(\cot(x^2))$$

$$y' = f'(\cot(x^2)) \cdot -\csc^2 x^2 \cdot 2x \\ = -2x \csc^2 x^2 f'(\cot(x^2))$$

Example 2/ Find y' for each of the following:

$$(a) y = (x^3 + 8)^8$$

$$y' = 8(x^3 + 8)^7 (3x^2) = 24x^2(x^3 + 8)^7$$

$$(b) y = \sin^8\left(\frac{\pi x}{2}\right)$$

$$y' = 8 \sin^7\left(\frac{\pi x}{2}\right) \cdot \cos\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2} \\ = 4\pi \cos\left(\frac{\pi x}{2}\right) \sin^7\left(\frac{\pi x}{2}\right)$$

$$(c) y = \sin(\tan 5x)$$

$$y' = \cos(\tan 5x) \cdot (\sec^2 5x) \cdot 5 = 5 \sec^2 5x \cos(\tan 5x)$$

$$(d) y = \sec \sqrt{x^2 + 1} \cdot \tan \frac{1}{x}$$

$$y' = \sec \sqrt{x^2 + 1} \cdot \sec^2 \frac{1}{x} \cdot \frac{-1}{x^2} + \sec \sqrt{x^2 + 1} \cdot \tan \sqrt{x^2 + 1} \cdot \frac{2x}{2\sqrt{x^2 + 1}}$$

$$(e) y = x^3 \cos^3\left(\frac{x}{3}\right) + \cos^{-3}\left(\frac{x}{3}\right)$$

$$y' = 3x^2 \cos^3\left(\frac{x}{3}\right) + x^3 3 \cos^2\left(\frac{x}{3}\right) \left(-\sin \frac{x}{3}\right) \frac{1}{3} + -3 \cos^{-2}\left(\frac{x}{3}\right) \left(-\sin \frac{x}{3}\right) \frac{1}{3}$$

$$\textcircled{P} \quad y = \csc(f(\sin \pi x)).$$

$$y' = -\csc(f(\sin \pi x)) \cot(f(\sin \pi x)) \cdot f'(\sin \pi x) \cdot \cos(\pi x) \cdot \pi.$$

Example 3 / Given that $f'(x) = \frac{x}{x^2+1}$ and $g(x) = \sqrt{3x-1}$.

Find $(f \circ g)'(x)$.

$$\begin{aligned} (f \circ g)'(x) &= f'(g(x)) \cdot g'(x) = f'(\sqrt{3x-1}) \cdot \frac{3}{2\sqrt{3x-1}} \\ &= \frac{\sqrt{3x-1}}{3x-1+1} \cdot \frac{3}{2\sqrt{3x-1}} = \frac{1}{2X} \end{aligned}$$

Example 4 / If $f(2) = 8$, $g(2) = 2$, $f'(2) = \frac{1}{3}$ and $g'(2) = 3$.

Find.

$$\textcircled{1} \quad \frac{d}{dx} (f(g(x))) \Big|_{x=2}.$$

$$\begin{aligned} f'(g(x)) \cdot g'(x) \Big|_{x=2} &= f'(g(2)) \cdot g'(2) \\ &= f'(2) \cdot 3 = 3 \left(\frac{1}{3} \right) = \boxed{1} \end{aligned}$$

$$\textcircled{2} \quad \frac{d}{dx} (\sqrt{f^2(x) + g^2(x)}) \Big|_{x=2}.$$

$$\frac{1}{2\sqrt{f^2(x) + g^2(x)}} \cdot (2f(x)f'(x) + 2g(x)g'(x)) \Big|_{x=2}$$

$$\frac{1}{2\sqrt{f^2(2) + g^2(2)}} (2f(2)f'(2) + 2g(2)g'(2)) =$$

$$= \frac{-5}{3\sqrt{17}}$$

$$\textcircled{3} \frac{d}{dx} \left(\frac{1}{2} x^2 f(2x - g(x)) \right) \Big|_{x=2}$$

$$\frac{1}{2} 2x f(2x - g(x)) + \frac{1}{2} x^2 f'(2x - g(x)) (2 - g'(x)) \Big|_{x=2}$$

$$2(f(4) - g(2)) + \frac{1}{2} (2)^2 f'(4 - g(2)) (2 - g'(2)) = \boxed{\frac{58}{3}}$$

Exercices 3.6

[3.7] Implicit Differentiation.

Example 1 / Find y' for each of the following.

(a) $x^3 - xy + y^3 = 1$

$$3x^2 - (1)y - xy' + 3y^2y' = 0$$

$$y'(3y^2 - x) = y - 3x^2 \rightarrow y' = \frac{y - 3x^2}{3y^2 - x}$$

(b) $x \cos(2x + 3y) = y \sin x$

$$x(-\sin(2x + 3y))(2 + 3y') + 1(\cos(2x + 3y)) = y' \sin x + y \cos x$$

(c) $x^3 \tan^3\left(\frac{1}{y}\right) = y^{-\frac{2}{3}} + \sin^3 \sqrt{y}$

$$3x^2 \tan^3\left(\frac{1}{y}\right) + x^3 \left(3 \tan^2\left(\frac{1}{y}\right) \sec^2\left(\frac{1}{y}\right) \left(-\frac{1}{y^2}\right) y'\right) = -\frac{2}{3} y^{-\frac{5}{3}} y' + 3 \sin^2 \sqrt{y} \cos \sqrt{y} \cdot \frac{1}{2\sqrt{y}} y'$$

(d) $\sqrt{xy} + \sec\left(\frac{x}{y}\right) = (x^3 + y^2)^{\frac{2}{3}}$

$$\frac{1}{2\sqrt{xy}} (1y + xy') + \sec\left(\frac{x}{y}\right) \tan\left(\frac{x}{y}\right) \cdot \frac{y(1 - xy')}{y^2} = \frac{2}{3} (x^3 + y^2)^{\frac{2}{3}} (3x^2 + 2yy')$$

Example 2 / Find y'' for the following.

(a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2\sqrt{y}$ at the point $(1, 1)$.

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} y' = 2 \cdot \frac{1}{2\sqrt{y}} y'$$

$$\text{at point } (1, 1): \frac{2}{3}(1) + \frac{2}{3}(1)y' = \frac{1}{1} y'$$

$$\frac{2}{3} + \frac{2}{3}y' = y' \rightarrow y' = 2$$

$$\frac{2}{3} x^{-\frac{1}{3}} - \frac{1}{3} x^{-\frac{4}{3}} + \frac{2}{3} x^{-\frac{1}{3}} y^{-\frac{1}{3}} y' y' + \frac{2}{3} y^{-\frac{1}{3}} y'' = -\frac{1}{2} y^{-\frac{3}{2}} y' y' + (y)^{\frac{1}{2}} y''$$

$$\text{at point } (1, 1): \frac{-2}{9}(1) + \frac{-2}{9}(1)(2)(2) + \frac{2}{3}(1)y'' = -\frac{1}{2}(1)(2)(2) + (1)y''$$

$$y'' = \frac{8}{3}$$

Example 3/ (a) Verify that the point $(\frac{\pi}{4}, \frac{\pi}{2})$ is on the curve $x \sin 2y = y \cos 2x$.

$$L.H.S := \frac{\pi}{4} \sin 2\left(\frac{\pi}{2}\right) = \frac{\pi}{4} \sin \pi = 0$$

$$R.H.S := \frac{\pi}{2} \cos 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \cos \frac{\pi}{2} = 0.$$

This means that the point $(\frac{\pi}{4}, \frac{\pi}{2})$ lie on the curve.

(b) Find the tangent and normal lines to the given curve at the point.

$$x(\cos 2y) 2y' + \sin 2y = y(-\sin 2x) 2 + y' \cos 2x$$

$$\frac{\pi}{4}(\cos \pi) 2y' + \sin \pi = \frac{\pi}{2}(-\sin \frac{\pi}{2}) 2 + y' \cos \frac{\pi}{2} \quad \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$-\frac{\pi}{2} y' = -\pi \Rightarrow y' = +2$$

So the slope of the tangent = +2

$$\text{Eqn of tangent: } y = 2\left(x - \frac{\pi}{4}\right) + \frac{\pi}{2} \Rightarrow \boxed{y = 2x}$$

$$m_1 \cdot m_2 = -1 \Rightarrow m_2 = -\frac{1}{2} = \text{the slope of the normal line.}$$

$$\text{Eqn of normal line: } y = -\frac{1}{2}\left(x - \frac{\pi}{4}\right) + \frac{\pi}{2} \Rightarrow \boxed{y = -\frac{1}{2}x + \frac{8\pi}{5}}$$

3.9 Linearization and differentials:-

Def. If f is a diff. fn at $x=a$ then the approximating fn $L(x) = f(a) + f'(a)(x-a)$ is the linearization of f at $x=a$. $L(x)$ is called the standard linear approximation of f at $x=a$.

* The point $x=a$ is called the center of approximation.

Example 1/II Find the linearization $L(x)$ of the following fns at the given points:-

(a) $f(x) = \sqrt{x^2+9}$ at $x=-4$.

$$f(-4) = \sqrt{(-4)^2+9} = \sqrt{25} = 5.$$

$$f'(x) = \frac{1}{2\sqrt{x^2+9}} \cdot 2x = \frac{x}{\sqrt{x^2+9}}$$

$$f'(-4) = \frac{-4}{\sqrt{25}} = \frac{-4}{5}.$$

$$L(x) = f(-4) + f'(-4)\left(\frac{x}{-4} - (-4)\right)$$

$$= 5 + \frac{-4}{5}(x+4) = \frac{9}{5} - \frac{4}{5}x.$$

(b) $f(x) = \tan x$ at $\alpha = \pi$.

$$f(\pi) = \tan \pi = 0$$

$$f'(x) = \sec^2 x$$

$$f'(\pi) = 1$$

$$L(x) = f(\pi) + f'(\pi)(x-\pi)$$

$$= 0 + 1(x-\pi) = \underline{x - \pi}$$

Example 2) a) Show that the linearization of $f(x) = (1+x)^K$ at $x=0$ is $L(x) = 1 + Kx$.

b) Use part a) to find an approximation for the following
 Ans: $f(x) = \frac{1}{\sqrt{1+x}}$ and $f(x) = (4+3x)^{\frac{1}{3}}$

c) Use approximation $(1+x)^K \approx 1 + Kx$ to estimate $\sqrt[3]{1.009}$.

a) $f(0) = (1+0)^K = 1$

$$f'(x) = K(1+x)^{K-1}$$

$$f'(0) = K(1+0)^{K-1} = K(1) = K.$$

$$L(x) = f(0) + f'(0)(x-0)$$

$$= 1 + K(x)$$

$$= 1 + Kx.$$

b) $f(x) = (1+x)^{\frac{1}{2}}$

from part a) $L(x) = 1 + \frac{1}{2}x$

$$f(x) = (4(1+\frac{3}{4}x))^{\frac{1}{3}} = 4^{\frac{1}{3}}(1+\frac{3}{4}x)^{\frac{1}{3}}$$

from part a)

$$L(x) = 4^{\frac{1}{3}}(1 + \frac{1}{3} \cdot \frac{3}{4}x) = 4^{\frac{1}{3}}(1 + \frac{x}{4})$$

c) $\sqrt[3]{1.009} = (1+0.009)^{\frac{1}{3}}$

$$L(x) = 1 + \frac{1}{3}(0.009) = \boxed{1.003}$$

Def. Let $y = f(x)$ be a diff. fn. the differentiation ~~dx~~ is an independent variable and the differentiation dy is the dependent variable. $dy = f'(x) dx$.

Example 3 / Find the differentiation dy for the following fns.

① $y = x\sqrt{1-x^2}$.

$$dy = \sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot -2x = \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{1-x^2-x^2}{\sqrt{1-x^2}} dx = \frac{1-2x^2}{\sqrt{1-x^2}} dx.$$

② $y = \sec(x^2-1)$

$$dy = (\sec(x^2-1) \cdot \tan(x^2-1) \cdot 2x) dx$$

③ $xy^2 = 4x^{\frac{3}{2}} - y = 0$

$$x \cdot 2y \frac{dy}{dx} + y^2 = 4 \cdot \frac{3}{2} x^{\frac{1}{2}} - \frac{dy}{dx} = 0$$

$$2xy dy + y^2 dx = 6x^{\frac{1}{2}} dx - dy = 0$$

$X dx$

$$dy(2xy-1) = dx(6x^{\frac{1}{2}}-y^2)$$

$$dy = \frac{6x^{\frac{1}{2}}-y^2}{2xy-1} dx.$$

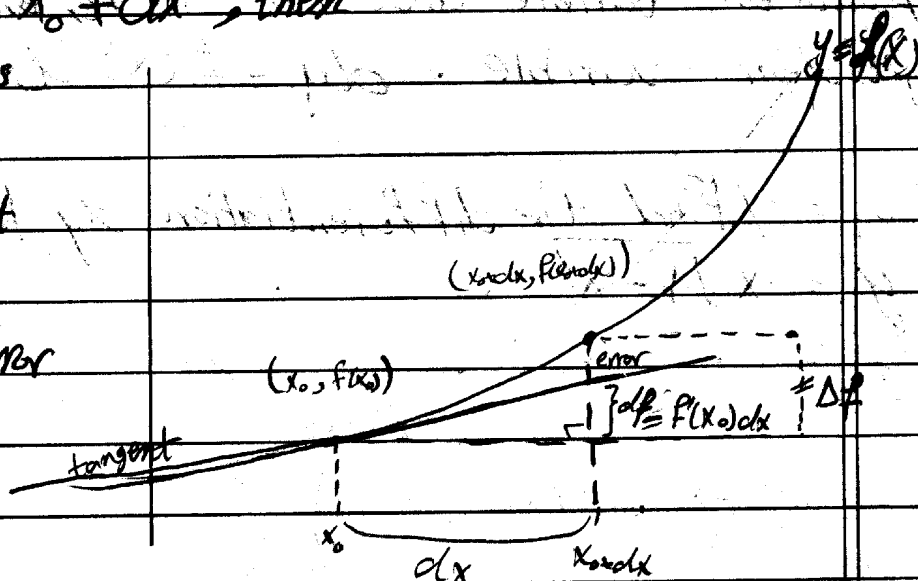
Remark/ If $y = f(x)$ and if the variable x is changed at x_0 from x_0 to $x_0 + dx$, then

[1] The change in f is

$$\Delta f = f(x_0 + dx) - f(x_0)$$

[2] The value of df at x_0 is $df = f'(x_0) dx$.

[3] The approximation error is $|\Delta f - df|$.



Example 4/ Let $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$. Find :-

(a) df at $x_0 = 1$. (b) Δf . (c) error $|\Delta f - df|$

$$(a) \Delta f = f(x_0 + dx) - f(x_0)$$

$$= f(1 + 0.1) - f(1) = f(1.1) - f(1)$$

$$= (1.1)^2 + 2(1.1) - (1)^2 - 2(1) = \boxed{0.41}$$

$$(b) df = f'(x_0) dx.$$

$$f(x) = x^2 + 2x$$

$$f'(x) = 2x + 2$$

$$f'(1) = 2(1) + 2 = 4$$

$$df|_{x_0=1} = 4 \times 0.1 = \boxed{0.4}$$

$$(c) \text{ error} = |\Delta f - df| = |0.41 - 0.4| = \boxed{0.01}$$

Chapter 4. Applications of derivatives:-

[4.1] Extreme values of fns.

Def Let f be a fn with Domain D , then f has an abs. max value on D at $x=c \in D$ if $f(x) \leq f(c) \forall x \in D$ and has abs. min value on D at $x=c \in D$ if $f(x) \geq f(c) \forall x \in D$.

The abs. max and the abs. min values ^{are} called abs. extrema, and max and min values are called extreme values of the fn.

Example 1/ Find the abs extrema for the following fns on the given interval and graph the fns.

① $f(x) = \begin{cases} \frac{1}{x} & -1 \leq x < 0 \\ \sqrt{x} & 0 \leq x \leq 4 \end{cases}$

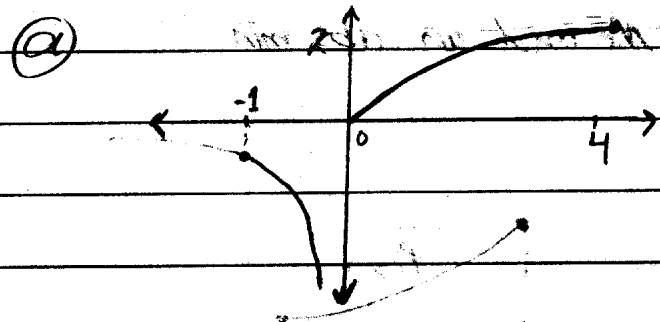
② $f(x) = 4 - x^2, -3 \leq x \leq 1$

③ $f(x) = \sqrt{4-x^2}, -2 < x < 2$

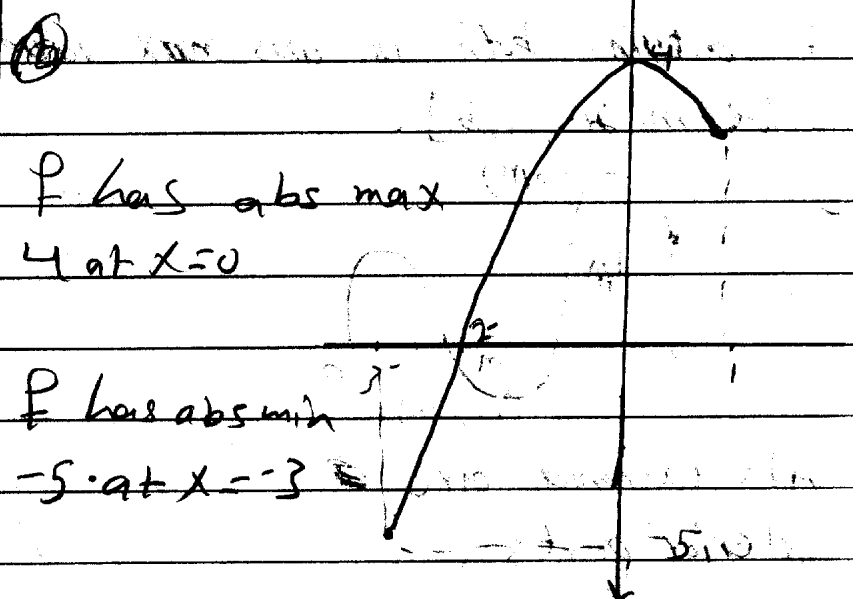
④ $f(x) = 3 \sin x, 0 \leq x \leq 2\pi$

⑤ $f(x) = |x|/2, -2 \leq x < \infty$

⑥ $f(x) = x^3 + 1$



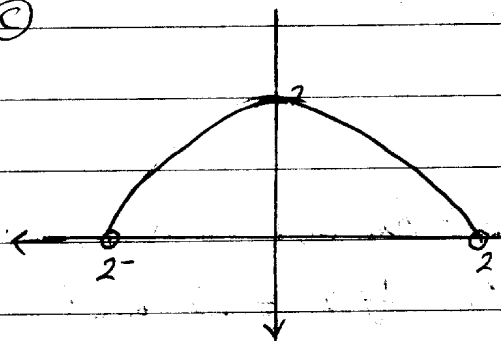
f has abs. max 2 at $x=4$
 f has no abs min



f has abs max
 4 at $x=0$

f has abs min
 -5 at $x=-3$

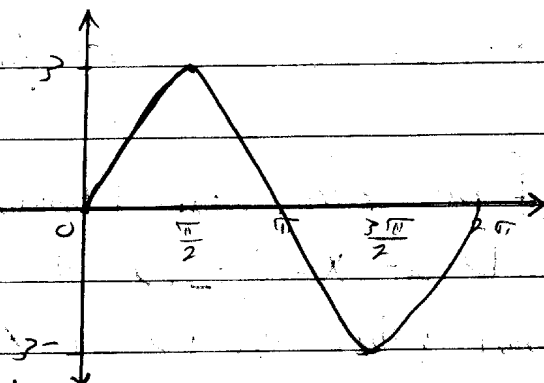
③



f has abs max 2 at $x=0$

f has no abs min

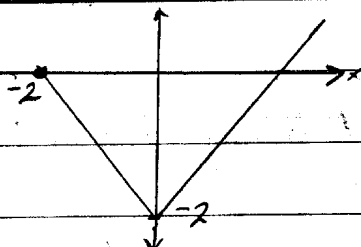
④



f has abs max 3 at $x = \frac{\pi}{2}$

abs min -3 at $x = \frac{3\pi}{2}$

⑤

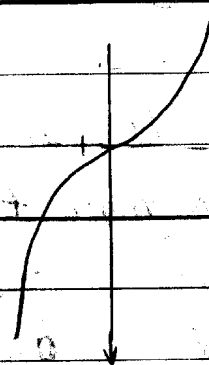


f has abs min -2 at $x=0$

f has no abs max

⑥

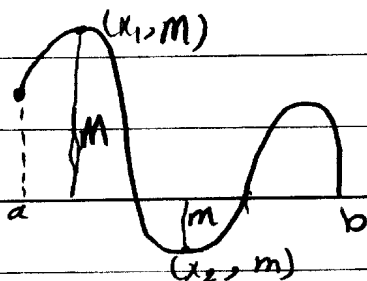
f has no extrema



Theorem: The extreme value theorem:

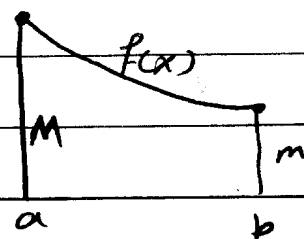
If f is continuous on a closed interval $[a, b]$, then f attains both an abs max value M and an abs min value m in $[a, b]$.

⑦

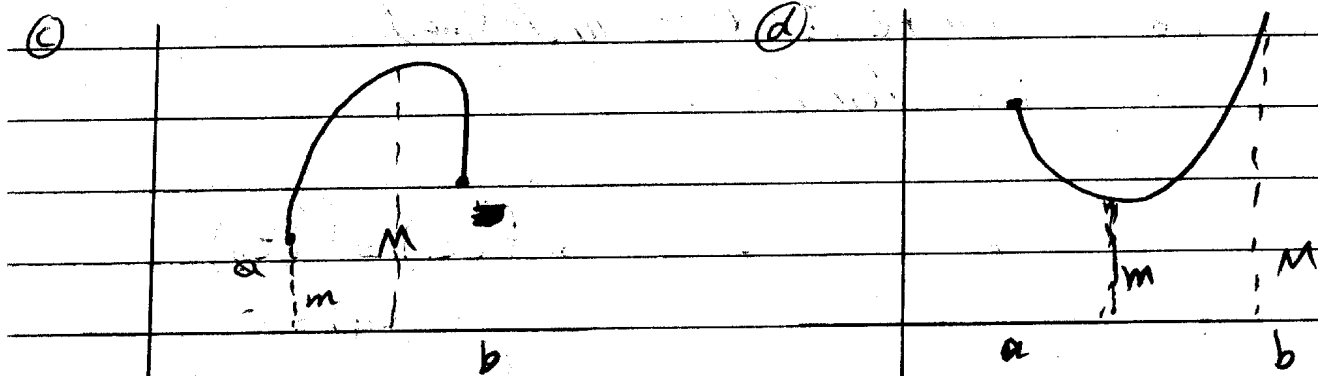


Abs extrema are interior points.

⑧



Abs extrema are end points.



Abs max is interior point

Abs min is endpoint

Abs max is endpoint

Abs min is interior point

Def. A fn has a local max (local min) value at an interior point $c \in \text{Dom}(f)$ if $f(x) < f(c)$ ($f(x) > f(c)$) $\forall x$ in some open interval containing c .

Remark / If f is continuous on $[a, b]$ then :

- (1) f has a local min (local max) at $x=a$ if $f(x) \geq f(a)$ ($f(x) \leq f(a)$) $\forall x$ in the interval $[a, a+\delta]$
- (2) f has a local max (local min) at $x=b$ if $f(x) \leq f(b)$ ($f(x) \geq f(b)$) $\forall x$ in the interval $[b-\delta, b]$

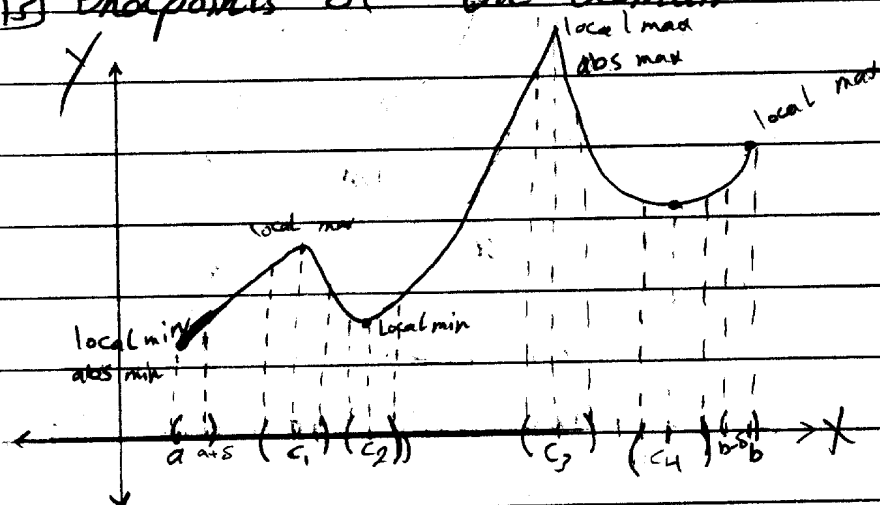
Theorem / If f has local max (or a local min) value at an interior point c of its Domain and if f' is defined at c , then $f'(c) = 0$.

Remark / The fn f has extreme values only in the following places:

- (1) Interior points where $f'(x) = 0$.

[2] Interior points where $f'(x)$ is undefined

[3] endpoints of the domain of f .



$$\begin{aligned}
 & f'(c_1) = f'(c_2) \\
 & = f'(c_4) = 0 \\
 & f'(c_3) \text{ is} \\
 & \text{undefined} \\
 & (\text{cusp})
 \end{aligned}$$

Def. An interior point of the domain of a fn f where f' is zero or f' is undefined is called a critical point of f .

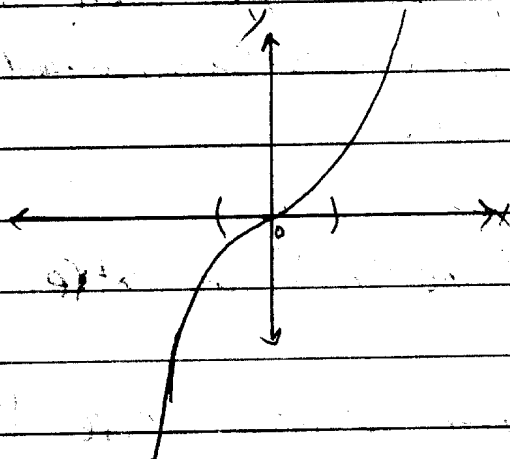
$\Rightarrow$ From the above remark and def, we say that the extreme values occur only at critical points and endpoints.

Remark/ the fn f may have a critical point at $x=c$ and this point need not be a local extreme value.

Example 2/ (A) $f(x) = x^3$

$$f'(x) = 3x^2 = 0 \rightarrow x = 0$$

So $x=0$ is a critical point, but $x=0$ is not an extreme value.

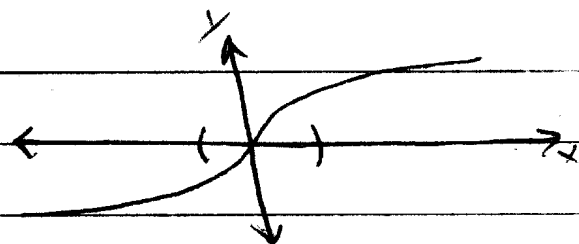


⑧ $f(x) = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = 0 \Rightarrow \frac{1}{x^{\frac{2}{3}}} = 0$$

f' is undefined at $x=0$, but f is defined

at $x=0$, So $x=0$ is critical point but not extreme value.



How to find the abs extreme values of continuous f on $[a, b]$?

① Evaluate f at all critical points and endpoints.

② Take the largest and smallest of these values.

Example 3/ ① Find the abs extreme values of the following f s on the given interval and graph the f s.

① $f(x) = 4 - x^2$, $-3 \leq x \leq 1$

$$f'(x) = -2x = 0 \Rightarrow x = 0, \text{ the only critical point of } f.$$

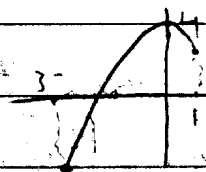
$$f(0) = 4 - 0 = \boxed{4}, \quad f(-3) = 4 - (-3)^2 = \boxed{-5}, \quad f(1) = 4 - 1 = \boxed{3}$$

critical point

end points

f has abs max 4 at $x=0$

f has abs min -5 at $x=-3$



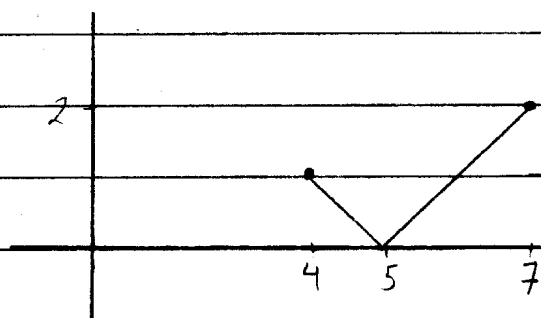
② $f(x) = |x-5|$, $4 \leq x \leq 7$

$$f(x) = \begin{cases} x-5, & 5 \leq x \leq 7 \\ 5-x, & 4 \leq x \leq 5 \end{cases}$$

$$f'(x) = \begin{cases} 1, & 5 < x < 7 \\ -1, & 4 < x < 5 \end{cases}$$

at $x=5$, f' is undefined

So $x=5$ is a critical point



⑤

$$f(5) = |5-5| = \boxed{0}$$

, f has abs max 2

$$f(4) = |4-5| = \boxed{1}$$

at $x=7$

$$f(7) = |7-5| = \boxed{2}$$

f has abs min 0 at $x=5$

Example/Determine the critical point of the fn

$$f(x) = \frac{x^2}{x-2}$$

f is undefined at $x=2 \rightarrow \text{Dom}(f) = \mathbb{R} - \{2\}$

$$f'(x) = \frac{(x-2)(2x) - x^2(1)}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = 0$$

$$x(x-4) = 0 \rightarrow x=0 \text{ and } x=4$$

f is undefined at $x=2$, but f is undefined at $x=2$

So $x=2$ is not a critical point.

So the only critical points are $\boxed{x=0 \text{ and } x=4}$

Example/Find the extreme values (abs and local) of the fn $f(x) = \sqrt{3+2x-x^2}$

$$\text{Dom } f = \{x: 3+2x-x^2 \geq 0\} = \{x: (3-x)(1+x) \geq 0\}$$

$$\text{Dom } f = \underline{[-1, 3]} \leftarrow \begin{array}{c} \text{Sign } (3-x) \quad + + + + - - - \\ \text{Sign } (1+x) \quad - - - + + + + + \\ \text{Sign } f \quad - - - + + + + - - - \end{array}$$

$$f'(x) = \frac{1}{2\sqrt{3+2x-x^2}} \cdot (2-2x) = \frac{1-x}{\sqrt{3+2x-x^2}} = 0$$

$$\text{only when } 1-x=0 \Rightarrow \boxed{x=1}$$

f' is undefined at $x = -1$ and $x = 3$.

the critical points are $x = -1$, $x = 1$ and $x = 3$.

$$f(-1) = f(3) = \underline{0}, f(1) = \underline{2}$$

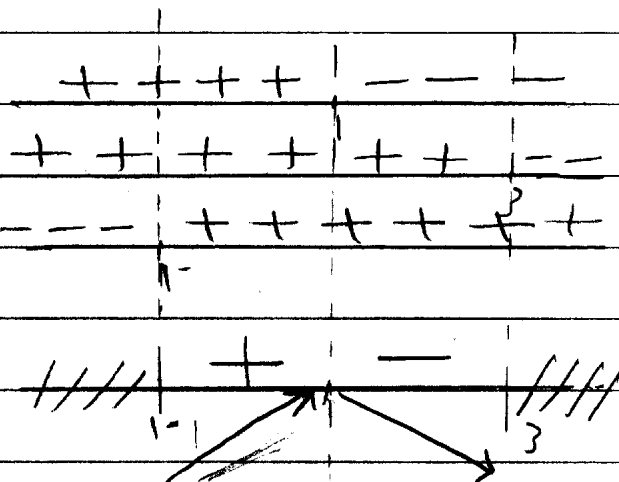
$$f'(x) = \frac{1-x}{\sqrt{(3-x)(1+x)}}$$

sign $(1-x)$

sign $(3-x)$

sign $(1+x)$

sign f'

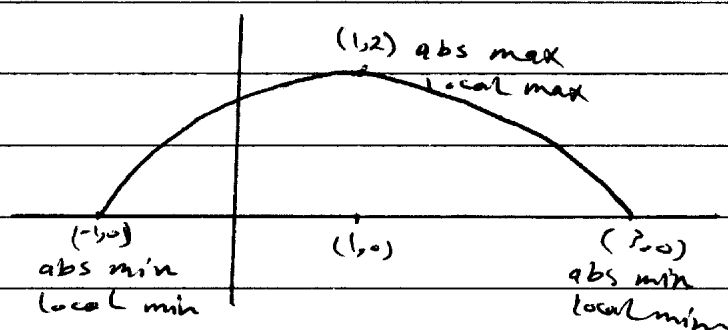


f has abs max and local max 2 at $x = 1$

f has min 0 at $x = -1$ and $x = 3$.

$$y = \sqrt{3+2x-x^2} = \sqrt{(3+1)-1+2x-x^2} = \sqrt{4-(x^2-2x+1)} = \sqrt{4-(x-1)^2} \Rightarrow y^2 = 4-(x-1)^2 \Rightarrow (x-1)^2 + y^2 = 4$$

the center ~~of the circle~~ is $(1, 0)$.



Example / Find the domain, critical points and the extreme values (abs and local) for the fn $f(x) = \underbrace{x^2}_{f_1} \underbrace{\sqrt{3-x}}_{f_2}$

* Dom $f_1 = \mathbb{R}$

Dom $f_2 = \{x: 3-x \geq 0\} = (-\infty, 3]$

Dom $f = \text{Dom } f_1 \cap \text{Dom } f_2 = (-\infty, 3]$

* $f'(x) = x^2 \cdot \frac{1}{2\sqrt{3-x}} \cdot -1 + 2x\sqrt{3-x}$

$= \frac{4x(3-x) - x^2}{2\sqrt{3-x}} = \frac{12x - 5x^2}{2\sqrt{3-x}} = 0$

$x(12-5x) = 0 \rightarrow x=0 \text{ and } x=\frac{12}{5}$

f is undefined at $x=3$

So the points $x=3$, $x=0$, and $x=\frac{12}{5}$ are the critical points.

$f(0) = f(3) = [0]$, $f(\frac{12}{5}) = \frac{144}{25} \sqrt{\frac{3}{5}}$

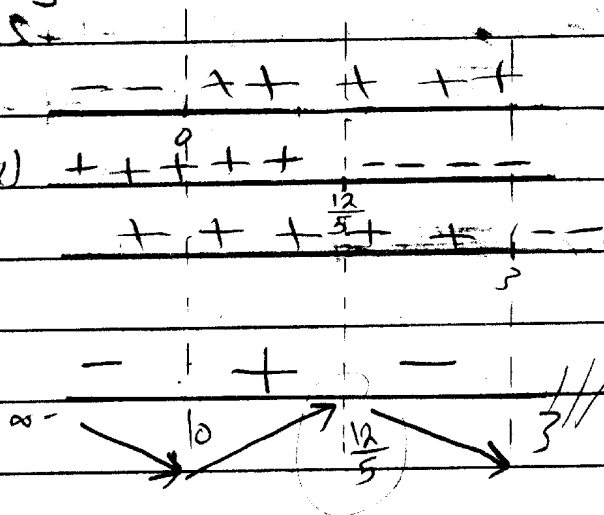
$f'(x) = \frac{x(12-5x)}{2\sqrt{3-x}}$

sign x

sign $(12-5x)$

sign $\sqrt{3-x}$

sign f



f has no abs max, but has local max $\frac{144}{25} \sqrt{\frac{3}{5}}$ at $x = \frac{12}{5}$

f has local and abs min 0 at $x=0$ and $x=3$

3.9 Linearization and differentials:-

Def. If f is a diff. fn at $x=a$ then the approximating fn $L(x) = f(a) + f'(a)(x-a)$ is the linearization of f at $x=a$. $L(x)$ is called the standard linear approximation of f at $x=a$.

* The point $x=a$ is called the center of approximation.

Example 1/II Find the linearization $L(x)$ of the following fns at the given points:-

(a) $f(x) = \sqrt{x^2+9}$ at $x=-4$.

$$f(-4) = \sqrt{(-4)^2+9} = \sqrt{25} = 5.$$

$$f'(x) = \frac{1}{2\sqrt{x^2+9}} \cdot 2x = \frac{x}{\sqrt{x^2+9}}$$

$$f'(-4) = \frac{-4}{\sqrt{25}} = \frac{-4}{5}.$$

$$L(x) = f(-4) + f'(-4)\left(\frac{x}{-4} - (-4)\right)$$

$$= 5 + \frac{-4}{5}(x+4) = \frac{9}{5} - \frac{4}{5}x.$$

(b) $f(x) = \tan x$ at $\alpha = \pi$.

$$f(\pi) = \tan \pi = 0$$

$$f'(x) = \sec^2 x$$

$$f'(\pi) = 1$$

$$L(x) = f(\pi) + f'(\pi)(x-\pi)$$

$$= 0 + 1(x-\pi) = \underline{x - \pi}$$

Example 2) a) Show that the linearization of $f(x) = (1+x)^K$ at $x=0$ is $L(x) = 1 + Kx$.

b) Use part a) to find an approximation for the following
 fns: $f(x) = \frac{1}{\sqrt{1+x}}$ and $f(x) = (4+3x)^{\frac{1}{3}}$

c) Use approximation $(1+x)^K \approx 1 + Kx$ to estimate $\sqrt[3]{1.009}$.

a) $f(0) = (1+0)^K = 1$

$$f'(x) = K(1+x)^{K-1}$$

$$f'(0) = K(1+0)^{K-1} = K(1) = K.$$

$$L(x) = f(0) + f'(0)(x-0)$$

$$= 1 + K(x)$$

$$= 1 + Kx.$$

b) $f(x) = (1+x)^{-\frac{1}{2}}$

from part a) $L(x) = 1 + \frac{-1}{2}x = 1 - \frac{1}{2}x$

$$f(x) = (4(1+\frac{3}{4}x))^{\frac{1}{3}} = 4^{\frac{1}{3}}(1+\frac{3}{4}x)^{\frac{1}{3}}$$

from part a)

$$L(x) = 4^{\frac{1}{3}}(1 + \frac{1}{4}x)$$

c) $\sqrt[3]{1.009} = (1+0.009)^{\frac{1}{3}}$

$$L(x) = 1 + \frac{1}{3}(0.009) = \sqrt[3]{1.003}$$

Def. Let $y = f(x)$ be a diff. fn. the differentiation ~~dx~~ is an independent variable and the differentiation dy is the dependent variable. $dy = f'(x) dx$.

Example 3 / Find the differentiation dy for the following fns.

① $y = x\sqrt{1-x^2}$,

$$dy = \sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot -2x = \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{1-x^2-x^2}{\sqrt{1-x^2}} dx = \frac{1-2x^2}{\sqrt{1-x^2}} dx.$$

② $y = \sec(x^2-1)$

$$dy = (\sec(x^2-1) \cdot \tan(x^2-1) \cdot 2x) dx$$

③ $xy^2 = 4x^{\frac{3}{2}} - y = 0$

$$x \cdot 2y \frac{dy}{dx} + y^2 = 4 \cdot \frac{3}{2} x^{\frac{1}{2}} - \frac{dy}{dx} = 0$$

$$2xy dy + y^2 dx = 6x^{\frac{1}{2}} dx - dy = 0$$

$X dx$

$$dy(2xy-1) = dx(6x^{\frac{1}{2}}-y^2)$$

$$dy = \frac{6x^{\frac{1}{2}}-y^2}{2xy-1} dx.$$

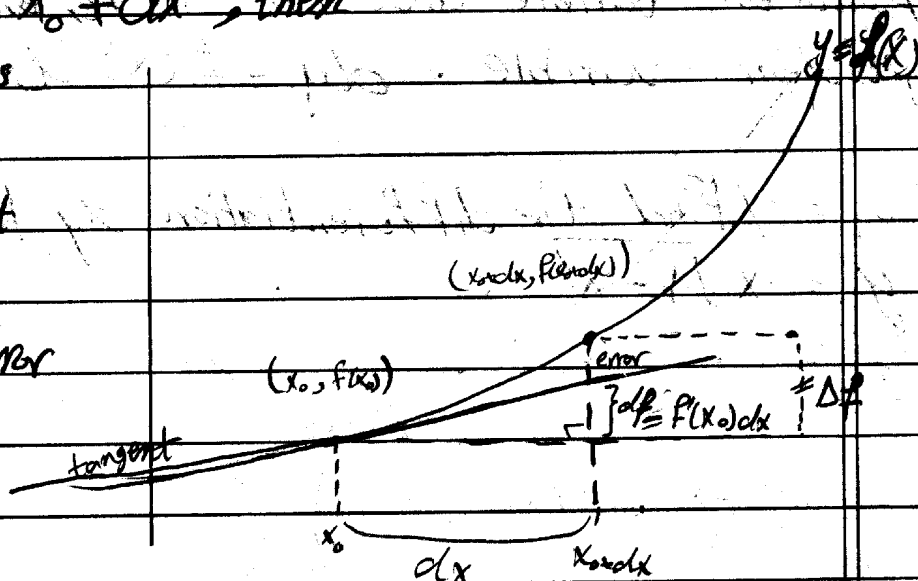
Remark/ If $y = f(x)$ and if the variable x is changed at x_0 from x_0 to $x_0 + dx$, then

[1] The change in f is

$$\Delta f = f(x_0 + dx) - f(x_0)$$

[2] The value of df at x_0 is $df = f'(x_0) dx$.

[3] The approximation error is $|\Delta f - df|$.



Example 4/ Let $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$. Find :-

(a) df at $x_0 = 1$. (b) Δf . (c) error $|\Delta f - df|$

(a) $\Delta f = f(x_0 + dx) - f(x_0)$

$$= f(1 + 0.1) - f(1) = f(1.1) - f(1)$$

$$= (1.1)^2 + 2(1.1) - (1)^2 - 2(1) = \boxed{0.41}$$

(b) $df = f'(x_0) dx$.

$$f(x) = x^2 + 2x$$

$$f'(x) = 2x + 2$$

$$f'(1) = 2(1) + 2 = 4$$

$$df|_{x_0=1} = 4 \times 0.1 = \boxed{0.4}$$

(c) error $= |\Delta f - df| = |0.41 - 0.4| = \boxed{0.01}$

Chapter 4. Applications of derivatives:-

[4.1] Extreme values of fns.

Def Let f be a fn with Domain D , then f has an abs. max value on D at $x=c \in D$ if $f(x) \leq f(c) \forall x \in D$ and has abs. min value on D at $x=c \in D$ if $f(x) \geq f(c) \forall x \in D$.

The abs. max and the abs. min values ^{are} called abs. extrema, and max and min values are called extreme values of the fn.

Example 1/ Find the abs extrema for the following fns on the given interval and graph the fns.

① $f(x) = \begin{cases} \frac{1}{x} & -1 \leq x < 0 \\ \sqrt{x} & 0 \leq x \leq 4 \end{cases}$

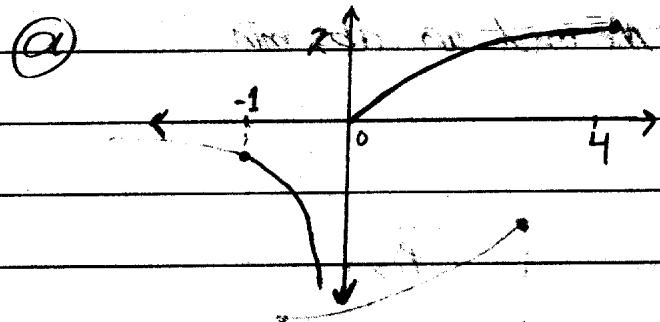
② $f(x) = 4 - x^2, -3 \leq x \leq 1$

③ $f(x) = \sqrt{4-x^2}, -2 \leq x \leq 2$

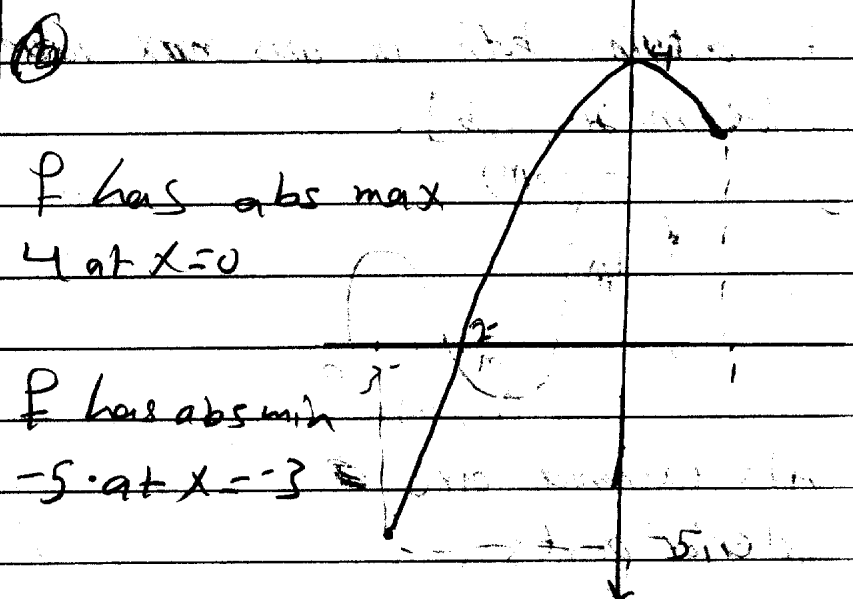
④ $f(x) = 3 \sin x, 0 \leq x \leq 2\pi$

⑤ $f(x) = |x|/2, -2 \leq x \leq \infty$

⑥ $f(x) = x^3 + 1$



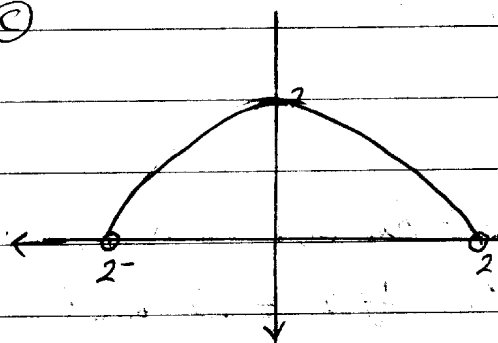
f has abs. max 2 at $x=4$
 f has no abs min



f has abs max
 4 at $x=0$

f has abs min
 -5 at $x=-3$

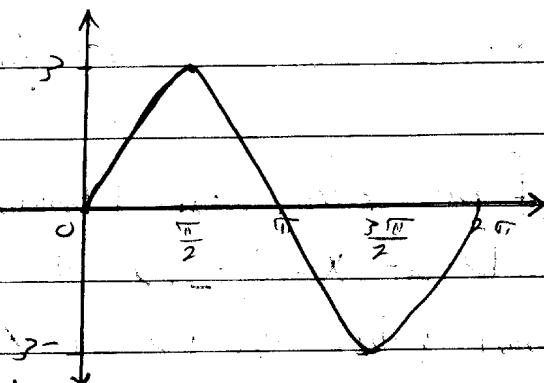
③



f has abs max 2 at $x=0$

f has no abs min

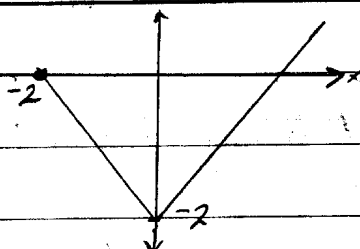
④



f has abs max 3 at $x=\frac{\pi}{2}$

f has abs min -3 at $x=\frac{3\pi}{2}$

⑤

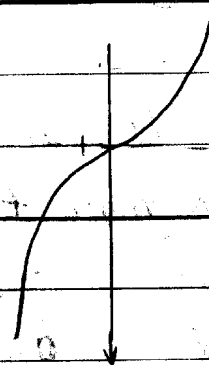


f has abs min -2 at $x=0$

f has no abs max

⑥

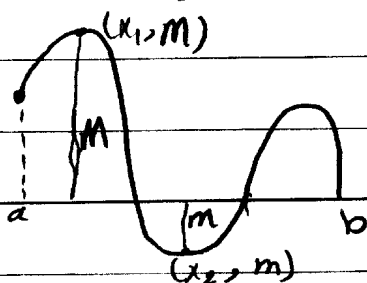
f has no extrema



Theorem: The extreme value theorem:

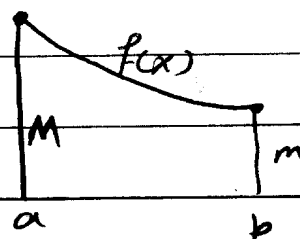
If f is continuous on a closed interval $[a, b]$, then f attains both an abs max value M and an abs min value m in $[a, b]$.

⑦

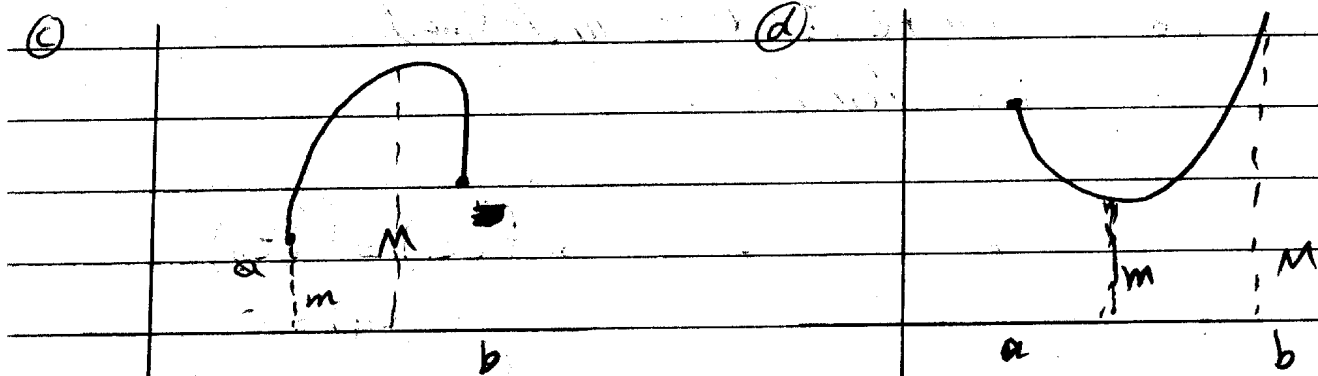


Abs extrema are interior points.

⑧



Abs extrema are end points.



Abs max is interior point

Abs min is endpoint

Abs max is endpoint

Abs min is interior point

Def. A fn has a local max (local min) value at an interior point $c \in \text{Dom}(f)$ if $f(x) < f(c)$ ($f(x) > f(c)$) $\forall x$ in some open interval containing c .

Remark / If f is continuous on $[a, b]$ then :

- (1) f has a local min (local max) at $x=a$ if $f(x) \geq f(a)$ ($f(x) \leq f(a)$) $\forall x$ in the interval $[a, a+\delta]$
- (2) f has a local max (local min) at $x=b$ if $f(x) \leq f(b)$ ($f(x) \geq f(b)$) $\forall x$ in the interval $[b-\delta, b]$

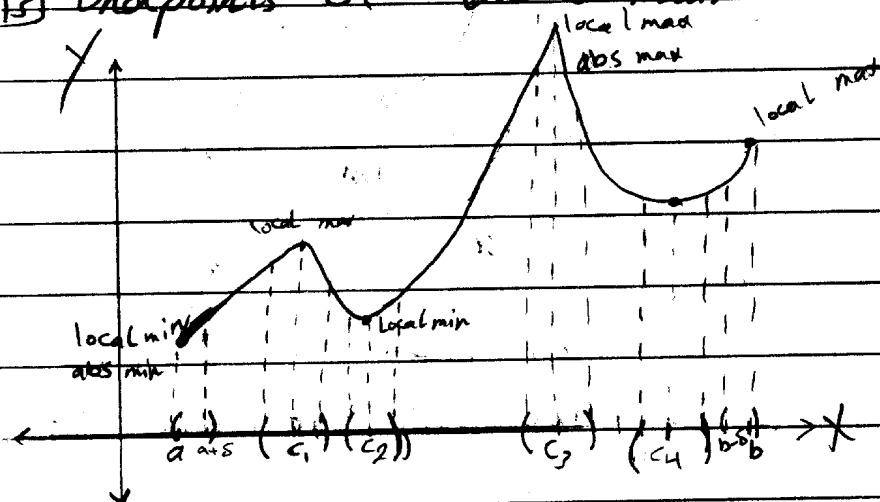
Theorem / If f has local max (or a local min) value at an interior point c of its Domain and if f' is defined at c , then $f'(c) = 0$.

Remark / The fn f has extreme values only in the following places:

- (1) Interior points where $f'(x) = 0$.

[2] Interior points where $f'(x)$ is undefined

[3] endpoints of the domain of f .



$$\begin{aligned}
 & f'(c_1) = f'(c_2) \\
 & = f'(c_4) = 0 \\
 & f'(c_3) \text{ is} \\
 & \text{undefined} \\
 & (\text{cusp})
 \end{aligned}$$

Def. An interior point of the domain of a fn f where f' is zero or f' is undefined is called a critical point of f .

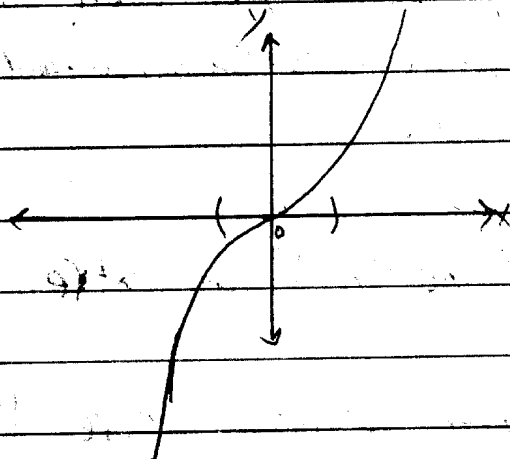
$\Rightarrow$ From the above remark and def, we say that the extreme values occur only at critical points and endpoints.

Remark/ the fn f may have a critical point at $x=c$ and this point need not be a local extreme value.

Example 2/ (A) $f(x) = x^3$

$$f'(x) = 3x^2 = 0 \rightarrow x = 0$$

So $x=0$ is a critical point, but $x=0$ is not an extreme value.

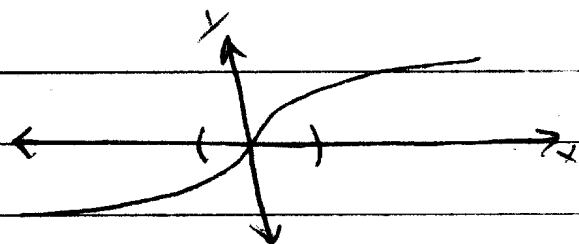


⑧ $f(x) = x^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = 0 \Rightarrow \frac{1}{x^{\frac{2}{3}}} = 0$$

f' is undefined at $x=0$, but f is defined

at $x=0$, So $x=0$ is critical point but not extreme value.



How to find the abs extreme values of continuous f on $[a, b]$?

① Evaluate f at all critical points and endpoints.

② Take the largest and smallest of these values.

Example 3/ ① Find the abs extreme values of the following f s on the given interval and graph the f s.

① $f(x) = 4 - x^2$, $-3 \leq x \leq 1$

$$f'(x) = -2x = 0 \Rightarrow x = 0, \text{ the only critical point of } f.$$

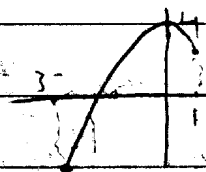
$$f(0) = 4 - 0 = \boxed{4}, \quad f(-3) = 4 - (-3)^2 = \boxed{-5}, \quad f(1) = 4 - 1 = \boxed{3}$$

critical point

end points

f has abs max 4 at $x=0$

f has abs min -5 at $x=-3$



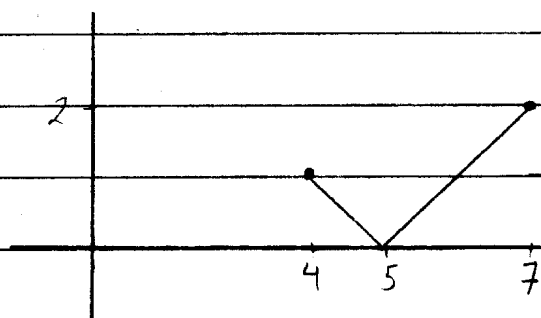
② $f(x) = |x-5|$, $4 \leq x \leq 7$

$$f(x) = \begin{cases} x-5, & 5 \leq x \leq 7 \\ 5-x, & 4 \leq x \leq 5 \end{cases}$$

$$f'(x) = \begin{cases} 1, & 5 < x < 7 \\ -1, & 4 < x < 5 \end{cases}$$

at $x=5$, f' is undefined

So $x=5$ is a critical point



⑤

$$f(5) = |5-5| = \boxed{0}$$

, f has abs max 2

$$f(4) = |4-5| = \boxed{1}$$

at $x=7$

$$f(7) = |7-5| = \boxed{2}$$

 f has abs min 0 at $x=5$

Example/Determine the critical point of the fn

$$f(x) = \frac{x^2}{x-2}$$

 f is undefined at $x=2 \rightarrow \text{Dom}(f) = \mathbb{R} - \{2\}$

$$f'(x) = \frac{(x-2)(2x) - x^2(1)}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = 0$$

$$x(x-4) = 0 \rightarrow x=0 \text{ and } x=4$$

 f is undefined at $x=2$, but f is undefined at $x=2$ So $x=2$ is not a critical point.So the only critical points are $\boxed{x=0 \text{ and } x=4}$ Example/Find the extreme values (abs and local) of the fn $f(x) = \sqrt{3+2x-x^2}$

$$\text{Dom } f = \{x: 3+2x-x^2 \geq 0\} = \{x: (3-x)(1+x) \geq 0\}$$

$$\text{Dom } f = \underline{[-1, 3]} \leftarrow \begin{array}{c} \text{Sign } (3-x) \quad + + + + - - - \\ \text{Sign } (1+x) \quad - - - + + + + + \\ \text{Sign } f \quad - - - + + + + - - - \end{array}$$

$$f'(x) = \frac{1}{2\sqrt{3+2x-x^2}} \cdot (2-2x) = \frac{1-x}{\sqrt{3+2x-x^2}} = 0$$

$$\text{only when } 1-x=0 \Rightarrow \boxed{x=1}$$

f' is undefined at $x = -1$ and $x = 3$.

the critical points are $x = -1$, $x = 1$ and $x = 3$.

$$f(-1) = f(3) = \underline{0}, f(1) = \underline{2}$$

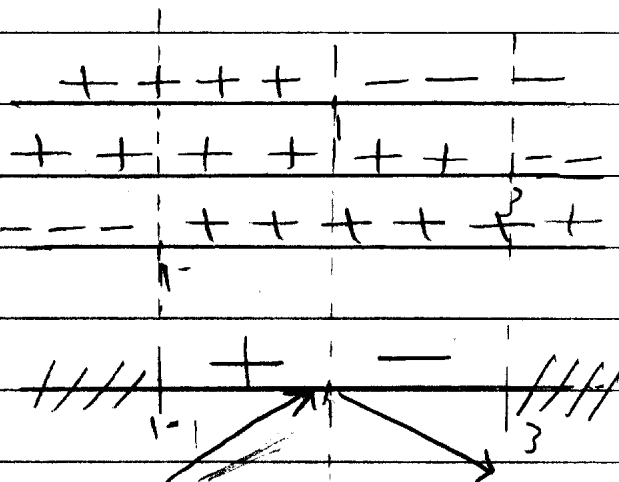
$$f'(x) = \frac{1-x}{\sqrt{(3-x)(1+x)}}$$

sign $(1-x)$

sign $(3-x)$

sign $(1+x)$

sign f'

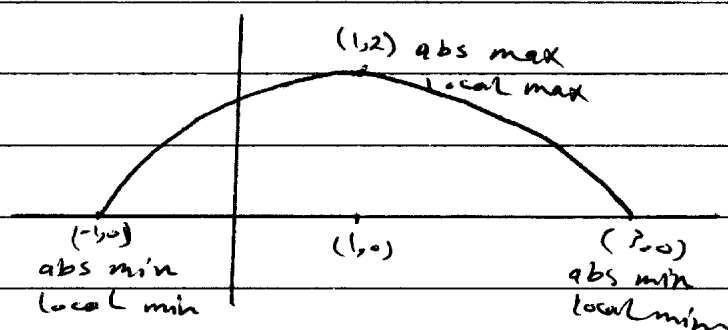


f has abs max and local max 2 at $x = 1$

f has min 0 at $x = -1$ and $x = 3$.

$$y = \sqrt{3+2x-x^2} = \sqrt{(3+1)-1+2x-x^2} = \sqrt{4-(x^2-2x+1)} = \sqrt{4-(x-1)^2} \Rightarrow y^2 = 4-(x-1)^2 \Rightarrow (x-1)^2 + y^2 = 4$$

the center ~~of the circle~~ is $(1, 0)$.



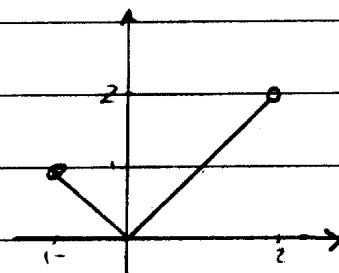
Exercises 4.1

[15] $f(x) = |x|$, $-1 < x < 2$.

f has abs. min 0 at $x=0$ because

$$f(0) \leq f(x) \quad \forall x \in (-1, 2)$$

No abs max. because there is no $f(x) \geq f(x) \quad \forall x, x \in (-1, 2)$.



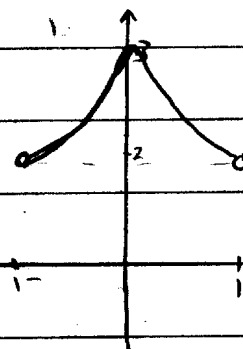
[16] $y = \frac{6}{x^2+2}$, $-1 < x < 1$

y has no oblique asymptotes and no vertical asymptotes.

$$\lim_{x \rightarrow \infty} \frac{6}{x^2+2} = 0 \rightarrow y=0 \text{ is a horizontal asymptote.}$$

f has abs max 3 at $x=0$

f has no abs min



[21] $f(x) = \frac{2}{3}x - 5$, $-2 \leq x \leq 3$.

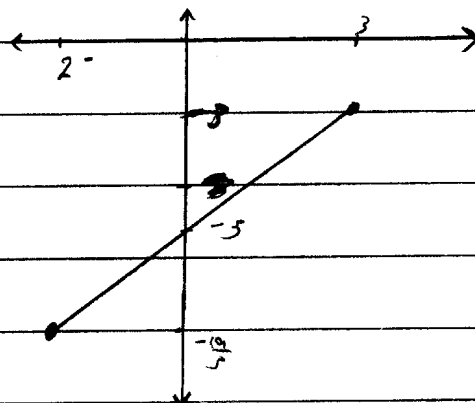
$$f'(x) = \frac{2}{3} \neq 0 \rightarrow \text{no critical point}$$

$$f(-2) = \frac{2}{3}(-2) - 5 = -\frac{19}{3}$$

$$f(3) = \frac{2}{3}(3) - 5 = -3$$

f has abs max -3 at $x=3$

$$= = = \text{min } -\frac{19}{3} \text{ at } x=-2$$



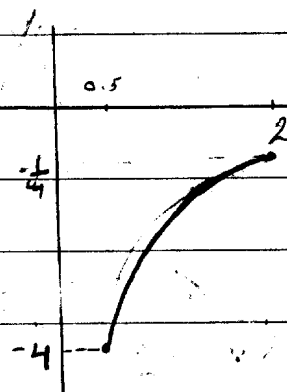
[25] $F(x) = \frac{-1}{x^2}$, $0.5 \leq x \leq 2$
 F is undefined at $x=0$

$F'(x) = \frac{1}{x^4} \cdot 2x = \frac{2}{x^3} \neq 0$ no critical point.

$F(0.5) = -4$, $F(2) = \frac{1}{4}$

F has abs max $\frac{1}{4}$ at $x=2$

$\therefore \therefore \therefore \therefore -4$ at $x = \frac{1}{2}$



[29] $g(x) = \sqrt{4-x^2}$, $-2 \leq x \leq 1$

$\text{Dom}(g) = \{x: 4-x^2 \geq 0\} = \{x: (4-x)(4+x) \geq 0\}$

$\text{Dom } g = [-4, 4]$

and $[-2, 1] \subset [-4, 4]$

$g'(x) = \frac{1}{2\sqrt{4-x^2}} \cdot -2x$

Sign $(4-x)$ $\begin{array}{cccc} + & + & + & - \end{array}$

Sign $(4+x)$ $\begin{array}{cccc} - & + & + & + \end{array}$

Sign $g(x)$ $\begin{array}{cccc} - & + & + & + \end{array}$

$g'(x) = \frac{-x}{\sqrt{4-x^2}} = 0 \Rightarrow \underline{\underline{x=0}}$

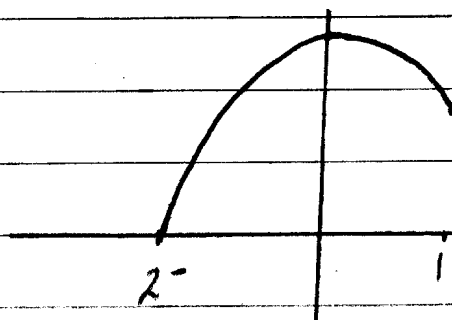
$g'(x)$ is undefined at $\underline{\underline{x=-2}}$ but g is defined.

$\therefore \therefore \therefore \therefore x=2$ and g is undefined

$g(-2) = 0$, $g(0) = 2$, $g(1) = \sqrt{3}$.

g has abs min 0 at $x=-2$

$\therefore \therefore \therefore \therefore$ max 2 at $x=0$



$$[33] \quad g(x) = \csc x, \quad \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$$

$$g'(x) = -\cot x \csc x = 0$$

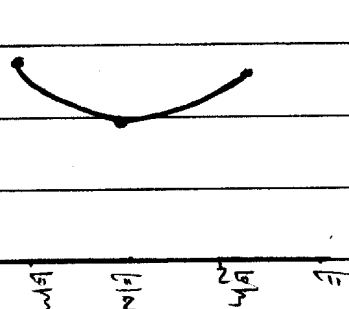
$$\cot x = 0 \rightarrow x = \frac{\pi}{2}$$

$$\csc x = 0 \rightarrow x \notin \mathbb{R}$$

$$g\left(\frac{\pi}{3}\right) = \csc \frac{\pi}{3} = \frac{1}{\sin \frac{\pi}{3}} = \frac{2}{\sqrt{3}}$$

$$g\left(\frac{\pi}{2}\right) = \frac{1}{\sin \frac{\pi}{2}} = 1$$

$$g\left(\frac{2\pi}{3}\right) = \frac{1}{\sin \frac{2\pi}{3}} = \frac{2}{\sqrt{3}}$$



f has abs min 1 at $x = \frac{\pi}{2}$

$\Rightarrow$ max $\frac{2}{\sqrt{3}}$ at $x = \frac{2\pi}{3}$ and $\frac{\pi}{3}$

$$[35] \quad f(t) = 2 - |t|, \quad -1 \leq t \leq 3$$

$$f(t) = \begin{cases} 2-t & , 0 \leq t \leq 3 \\ 2+t & , -1 \leq t < 0 \end{cases}$$

$$f(0) = 2 - 0 = 2 = \lim_{t \rightarrow 0} f(t)$$

$$\lim_{t \rightarrow 0^-} f(t) = \lim_{t \rightarrow 0^-} 2+t = 2 \quad \lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow 0^+} 2-t = 2 \Rightarrow f \text{ is continuous at } t=0$$

So f is continuous on $[-1, 3]$.

$$f'_-(t) = 1 \Rightarrow f'_-(0) = 1 \Rightarrow f'(0) \text{ is undefined}$$

$$f'_+(t) = -1 \Rightarrow f'_+(0) = -1$$

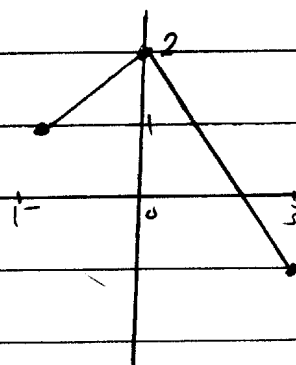
$$f(-1) = 2 - |-1| = 1$$

$$f(0) = 2 - |0| = 2$$

$$f(3) = 2 - |3| = -1$$

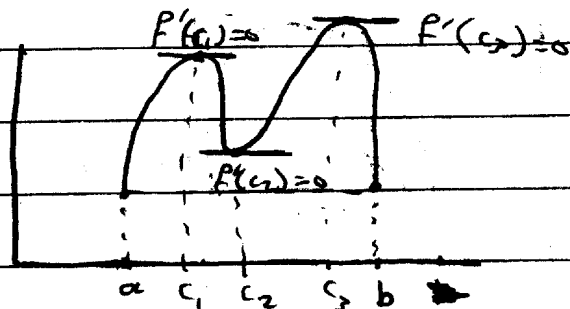
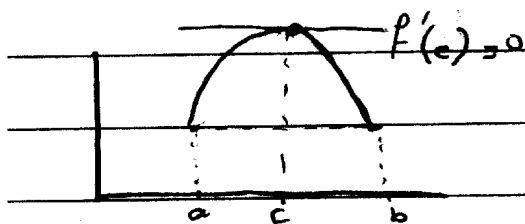
f has abs min -1 at $x=3$

f has max 2 at $x=0$.



4.2 The mean value theorem:

~~Rolle's Theorem~~ Suppose that $y=f(x)$ is continuous at every point of a closed interval $[a,b]$ and diff at every point at (a,b) . If $f(a)=f(b)$, then there is at least one num c in (a,b) at which $f'(c)=0$.



Example 1 / Show that the fn $f(x) = \frac{x}{2} - \sqrt{x}$ satisfy the hypotheses of the Rolle's theorem on $[0, 4]$. Find the values of $c \in (0, 4)$ such that $f'(c)=0$.

$\text{Dom}(f) = [0, \infty) \Rightarrow f$ is continuous on $[0, 4]$

$$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}} \Rightarrow \text{Dom } f' = (0, 4) \Rightarrow$$

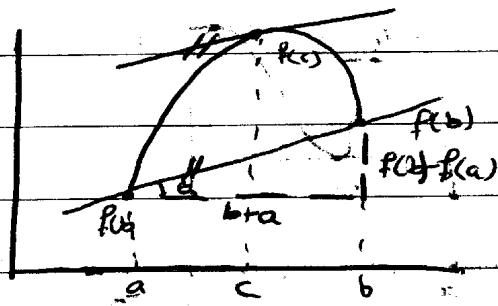
f is diff. on $(0, 4)$

$$f(0) = 0, \quad f(4) = \frac{4}{2} - \sqrt{4} = 0 \Rightarrow f(0) = f(4)$$

So the fn $f(x) = \frac{x}{2} - \sqrt{x}$ satisfy the hypotheses of the Rolle's theorem $\Rightarrow \exists c \in (0, 4)$ such that $f'(c)=0$.

$$\frac{1}{2} - \frac{1}{2\sqrt{c}} = 0 \Rightarrow \frac{1}{\sqrt{c}} = 1 \Rightarrow \boxed{c=1}$$

The mean value theorem / Suppose that the fn $f(x) = y$ is continuous on a closed interval $[a, b]$ and diff on (a, b) then there is at least one point c in (a, b) at which $f'(c) = \frac{f(b) - f(a)}{b - a}$.



$$\tan \theta = f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 2 / If the fn $f(x) = \begin{cases} 2x - 3 & , 0 \leq x \leq 2 \\ 6x - x^2 - 7 & , 2 < x \leq 3 \end{cases}$

satisfy the hypotheses of the M.V.T.?

$$\text{Dom } f = [0, 3]$$

* $2x - 3$ is a fn $f(x)$, and it's continuous on $[0, 2]$

* $6x - x^2 - 7 = f(x)$, = = = on $(2, 3]$

$$f(2) = 2(2) - 3 = 1 = \lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 6x - x^2 - 7 = 6(2) - (2)^2 - 7 = 1$$

$\Rightarrow \underline{f}$ is continuous on $\underline{[0, 3]}$

$$f'_-(x) = 2$$

$$f'_+(x) = 6 - 2x \Rightarrow f'_+(2) = 6 - 2(2) = 2$$

So f is diff at $x=2 \Rightarrow f$ is diff on $(0, 3)$

$\Rightarrow f$ satisfy the hypotheses of the M.V.T.

Example 3 / Show that the fn $f(x) = x + \frac{1}{x}$ satisfy the hypot of the M.V.Th in the interval $[\frac{1}{2}, 2]$ and find all values c that satisfy the M.V.Th.

f is undefined at $x=0 \notin [\frac{1}{2}, 2] \Rightarrow f$ is continuous on $[\frac{1}{2}, 2]$.

$f'(x) = 1 - \frac{1}{x^2}$. f' is undefined at $x=0 \notin (\frac{1}{2}, 2)$.

f is diff on $(\frac{1}{2}, 2)$

So f satisfy the hypotheses of the M.V.Th.

By the M.V.Th, $\exists c \in (\frac{1}{2}, 2)$ such that $\frac{f(2) - f(\frac{1}{2})}{2 - \frac{1}{2}} = f'(c)$

$$f(2) = 2 + \frac{1}{2} = \frac{5}{2}, \quad f(\frac{1}{2}) = \frac{1}{2} + 2 = \frac{5}{2}$$

$$f'(c) = \frac{\frac{5}{2} - \frac{5}{2}}{2 - \frac{1}{2}} = 0 \Rightarrow 1 - \frac{1}{c^2} = 0 \Rightarrow c^2 = 1$$

$$\boxed{c = +1} \in (\frac{1}{2}, 2) \quad \text{or } c = -1 \notin (\frac{1}{2}, 2) \quad (\text{refused})$$

Example 4 / Show that $|\sin b - \sin a| \leq |b - a|$ for any num a and b . Let $f(x) = \sin x \Rightarrow f$ is continuous $\forall x$ and f is continuous on $[a, b]$.

$f'(x) = \cos x \Rightarrow f'$ is ~~not~~ defined $\forall x \neq 0 = (x)$

So f is diff on (a, b)

$f(x)$ satisfy the hypotheses of the M.V.Th $\Rightarrow \exists c \in (a, b)$

such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

$$\frac{|f(b) - f(a)|}{|b - a|} = |f'(c)| = |\cos(c)| \leq 1$$

$$|\sin(b) - \sin(a)| \leq |b - a|$$

Example 5 / Suppose that f is continuous on $[0, 4]$, $f(0) = 1$ and $2 \leq f'(x) \leq 5 \quad \forall x \in (0, 4)$. Show that $9 \leq f(4) \leq 21$.

f is continuous on $[0, 4]$

$2 \leq f'(x) \leq 5 \quad \forall x \in (0, 4) \Rightarrow f$ is diff on $(0, 4)$

f satisfy the hypotheses of the M.V.Th $\Rightarrow$

$\exists c \in (0, 4)$ such that $\frac{f(4) - f(0)}{4 - 0} = f'(c)$

$\frac{f(4) - 1}{4} = f'(c)$, since $2 \leq f'(x) \leq 5 \quad \forall x \in (0, 4)$

$$\Rightarrow 2 \leq f'(c) \leq 5 \Rightarrow 2 \leq \frac{f(4) - 1}{4} \leq 5$$

$$2 \leq f(4) - 1 \leq 20$$

$$\underline{9 \leq f(4) \leq 21} \quad \neq$$

Corollary II) If $f'(x) = 0$ at each point x of an open interval (a, b) then $f(x) = c \quad \forall x \in (a, b)$ where c is constant.

Proof / let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.

$f'(x) = 0$ on $(a, b) \Rightarrow f$ is diff on $(a, b) \Rightarrow$

f is diff on $[x_1, x_2]$

So f is continuous on $[x_1, x_2]$ and diff on (x_1, x_2)

f satisfy the hypotheses of the M.V.Th $\Rightarrow$

$\exists c \in (x_1, x_2)$ such that $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$

but $f'(x) = 0 \quad \forall x \in (x_1, x_2) \Rightarrow$

$$f'(c) = 0$$

So $\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \Rightarrow f(x_2) = f(x_1) \quad \forall x_1, x_2 \in (a, b)$

$\exists c \in (0,4)$ such
that $\frac{f(4)-f(0)}{4-0} = f'(c)$

Example 5 / Suppose that f is continuous on $[0,4]$, $f(0)=1$ and $2 \leq f'(x) \leq 5 \quad \forall x \in (0,4)$. Show that $9 \leq f(4) \leq 21$.

f is continuous on $[0,4]$

$2 \leq f'(x) \leq 5 \quad \forall x \in (0,4) \Rightarrow f$ is diff on $(0,4)$

f satisfy the hypotheses of the M.V.Th $\Rightarrow$

$\exists c \in (0,4)$ such that $\frac{f(4)-f(0)}{4-0} = f'(c)$

$\frac{f(4)-1}{4} = f'(c)$, since $2 \leq f'(x) \leq 5 \quad \forall x \in (0,4)$

$$\Rightarrow 2 \leq f'(c) \leq 5 \Rightarrow 2 \leq \frac{f(4)-1}{4} \leq 5$$

$$8 \leq f(4)-1 \leq 20$$

$$\underline{9 \leq f(4) \leq 21}$$

Corollary II / If $f'(x)=0$ at each point x of an open interval (a,b) then $f(x)=c \quad \forall x \in (a,b)$ where c is constant.

Proof / let $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$.

$f'(x)=0$ on $(a,b) \Rightarrow f$ is diff on $(a,b) \Rightarrow$

f is diff on $[x_1, x_2]$

So f is continuous on $[x_1, x_2]$ and diff on (x_1, x_2)

f satisfy the hypotheses of the M.V.Th $\Rightarrow$

$\exists c \in (x_1, x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$

but $f'(x)=0 \quad \forall x \in (x_1, x_2) \Rightarrow$

$$f'(c)=0$$

So $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0 \Rightarrow f(x_2)=f(x_1) \quad \forall x_1, x_2 \in (a,b)$

f is constant fn on $(a,b) \Rightarrow f(x) = c$ on (a,b)

Corollary (2) If $f'(x) = g'(x)$ at each point in an open interval (a,b) , then $\exists$ a constant c such that $f(x) = g(x) + c \forall x \in (a,b)$. That is $f(x) - g(x)$ is constant.

Proof/ let $f'(x) = g'(x) \forall x \in (a,b)$

$$f'(x) - g'(x) = 0$$

$$\text{Let } h(x) = f(x) - g(x)$$

$$h'(x) = f'(x) - g'(x) \text{ so } h'(x) = 0$$

By corollary (1) - we have $h(x) = c$, c is constant

$$h(x) = f(x) - g(x) = c$$

$$\Rightarrow f(x) = g(x) + c$$

So $f - g$ is a constant fn

Note / The above corollaries are true on (a, ∞) , $(-\infty, b)$

Example 1 / Suppose that $f(-1) = 3$ and $f'(x) = 0 \forall x$ must $f(x) = 3 \forall x$? give reasons for your answer.

$f'(x) = 0 \rightarrow$ By corollary one we have

$f(x) = c \forall x$ where c is constant

Since $f(-1) = 3 \Rightarrow f(x) = 3 \forall x$

Example 2/ Find the fn $f(x)$ if $f'(x) = 8 - \csc^2 x$ and the graph of f passing through the point $(\frac{\pi}{4}, 0)$

let $g(x) = 8x + \cot x$

$$g'(x) = 8 - \csc^2 x = f'(x)$$

$$\Rightarrow f(x) - g(x) = c \Rightarrow f(x) = g(x) + c$$

$$f(x) = 8x + \cot x + c$$

Since f ~~through~~ passes through the point $\rightarrow$, we have

$$0 = 8\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right) + c \Rightarrow c = -2\pi - 1$$

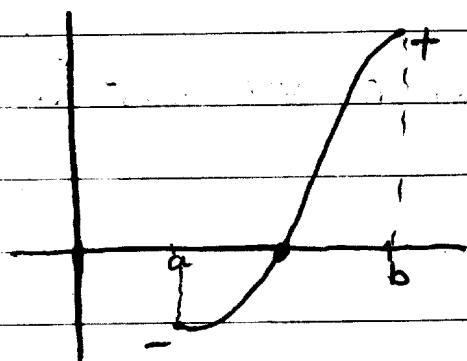
So $f(x) = 8x + \cot x - 2\pi - 1$.

Remark / Counting zeros:-

Let f be a continuous fn on $[a, b]$ and diff on (a, b) and

① If $f(a)$ and $f(b)$ have opposite signs.

② $f'(x) > 0$ on (a, b) or $f'(x) < 0$ on (a, b) , then f has exactly one zero between a and b .



Example 3/ Show that the fn $f(x) = \frac{1}{1-x} + \sqrt{1+x} - 3.1$ has one zero on $(-1, 1)$

$f(x)$ is continuous on $[-1, 1)$

So f is continuous $[-0.9, +0.9]$

$$f'(x) = \frac{+1}{(1-x)^2} + \frac{1}{2\sqrt{1+x}}$$

f is diff on $(-1, 1) \rightarrow f$ is diff on $(-0.9, +0.9)$

$$f(-0.9) = \frac{1}{1+0.9} + \sqrt{1-0.9} - 3.1 < 0$$

$$f(+0.9) = \frac{1}{1-0.9} + \sqrt{1+0.9} - 3.1 > 0$$

$$f'(x) = \frac{1}{(1-x)^2} + \frac{1}{2\sqrt{1+x}} > 0 \quad \forall x \in (-0.9, +0.9)$$

By the above remark, $\exists c \in (-1, 1)$ such that $f(c) = 0$

So f has one zero on $(-1, 1)$.

4.3 ^{المتزايد} Monotonic fns and the First derivative test

Def. A fn that is increasing or decreasing on an interval I is called monotonic on I .

For example, $f(x) = x^2$ is monotonic on $(-\infty, 0]$ and on $[0, \infty)$ But it is not monotonic on $(-\infty, \infty)$

Corollary [3] - First derivative test for monotonic fns.

Suppose that $f(x)$ is continuous on $[a, b]$ and diff on (a, b) .

① If $f'(x) > 0$ at each point $x \in (a, b)$ then f is increasing on $[a, b]$.

② If $f'(x) < 0$ $\dots \dots \dots$ decreasing on $[a, b]$.

Example. If $f(x) = x^4 - 4x^3 + 4x^2$. Find the critical points of f and the interval, on which the fn is increasing or decreasing.

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2)$$

$$4x(x-2)(x-1) = 0$$

$\Rightarrow x=0, x=2$, and $x=1$ are the critical points

f is increasing on

$[0, 1]$ and on $[2, \infty)$

f is decreasing on

and on $[1, 2]$

sign $4x$	--	+	+	+
sign $(x-1)$	--	--	+	+
sign $(x-2)$	--	--	--	+
sign f'	--	+	--	+

$\swarrow$ 0 $\nearrow$ 1 $\searrow$ 2 $\nearrow$

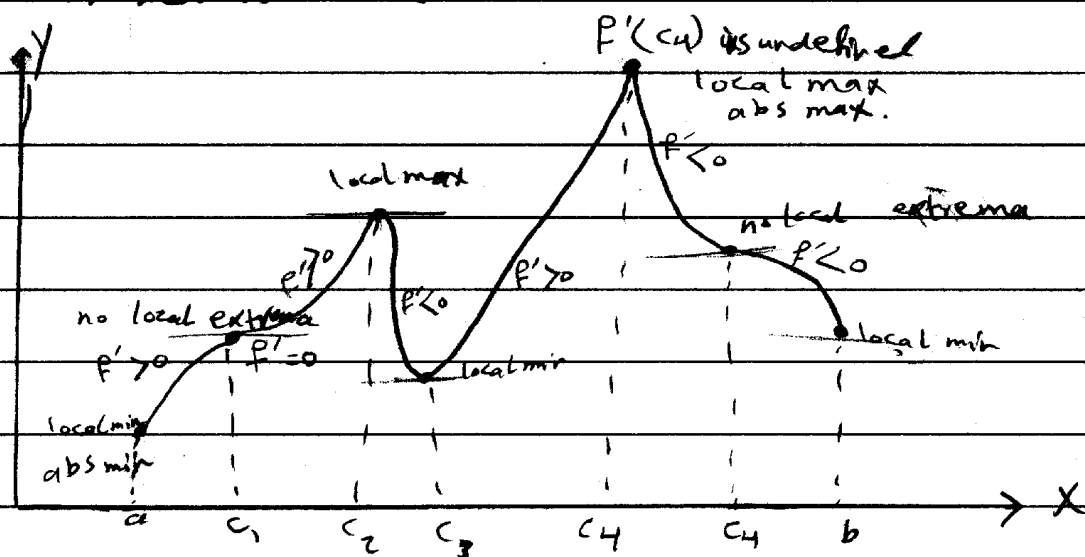
First derivative test for local extrema

Suppose that c is a critical point of a continuous fn f .
 f is diff at every point in some interval containing c or
 possibility at c itself

① If f' changes from -ve to +ve at c , then f has a local
 min at c

② $= = = = +ve$ to $-ve = = = =$
 max at c .

③ If f' does not change its sign at c , then f has no
 extrema at c .



Example / For the following Pns. 2

① Find the intervals in which the fn is increasing and decreasing

② Identify the local extrema, if any, saying where they occur

③ which, if any, of the extreme values are abs.

[1] $f(x) = 2x^3 - 18x$

Dom $f = (-\infty, \infty)$

$f'(x) = 6x^2 - 18 = 6(x^2 - 3) = 6(x - \sqrt{3})(x + \sqrt{3})$

So $x = -\sqrt{3}$ and $x = \sqrt{3}$ are the only critical points

$$\text{sign } 6(x-\sqrt{3}) \quad \begin{array}{c} - & - & - & - & + \\ | & | & | & | & | \\ -\sqrt{3} & & & & \sqrt{3} \end{array}$$

$$\text{sign } (x+\sqrt{3}) \quad \begin{array}{c} - & - & - & + & + & + & + \\ | & | & | & | & | & | & | \\ -\sqrt{3} & & & & & & \sqrt{3} \end{array}$$

$$\text{sign } f' \quad \begin{array}{c} + & - & - & + \\ | & | & | & | \\ -\sqrt{3} & & & \sqrt{3} \end{array}$$

(a) f is increasing on $(-\infty, -\sqrt{3}]$ and $[\sqrt{3}, \infty)$
 f is decreasing on $[-\sqrt{3}, \sqrt{3}]$

$$(b) f(-\sqrt{3}) = 2(-\sqrt{3})^3 - 18(-\sqrt{3}) = -6\sqrt{3} + 18\sqrt{3} = \underline{\underline{12\sqrt{3}}}$$

$$f(\sqrt{3}) = 6\sqrt{3} - 18\sqrt{3} = \underline{\underline{-12\sqrt{3}}}$$

f has local max $12\sqrt{3}$ at $x = -\sqrt{3}$

f has local min $-12\sqrt{3}$ at $x = \sqrt{3}$.

(c) f has no abs extrema because f has no endpoints

$$[2] f(x) = \frac{x^2 - 3}{x - 2}, x \neq 2$$

$$\text{Dom } f = (-\infty, 2) \cup (2, \infty)$$

$$f(x) = x + 2 + \frac{1}{x - 2}$$

$$\begin{array}{r} x + 2 \\ x^2 - 3 \\ \hline x^2 - 2x \\ \hline 2x - 3 \\ \hline 2x - 4 \\ \hline 1 \end{array}$$

$$f'(x) = 1 + \frac{-1}{(x-2)^2} = \frac{(x-2)^2 - 1}{(x-2)^2}$$

$$= \frac{x^2 - 4x + 3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}$$

$$\begin{array}{c} \text{sign } (x-1) \quad \begin{array}{c} - & - & + & + & + & + \\ | & | & | & | & | & | \\ 1 & & & & & 3 \end{array} \\ (x-3) \quad \begin{array}{c} - & - & - & - & + & + & + \\ | & | & | & | & | & | & | \\ 1 & & & & & & 3 \end{array} \\ (x-2)^2 \quad \begin{array}{c} + & + & + & + & + & + & + \\ | & | & | & | & | & | & | \\ 1 & & & & & & 3 \end{array} \\ f' \quad \begin{array}{c} + & - & - & - & + \\ | & | & | & | & | \\ 1 & & & & 3 \end{array} \end{array}$$

a) f is increasing on $(-\infty, 1]$ and on $[3, \infty)$.

$=$ decreasing on $[1, 2) \cup (2, 3]$

b) $f(1) = \frac{(1)^2 - 3}{1 - 2} = \frac{2}{-1} = -2$, $f(3) = \frac{9 - 3}{3 - 2} = \frac{6}{1} = 6$

f has local max 2 at $x=1$

$=$ min 6 at $x=3$.

c) f has no abs extrema because f has no endpoints.

[3] $f(x) = x^{\frac{2}{3}}(x+5)$

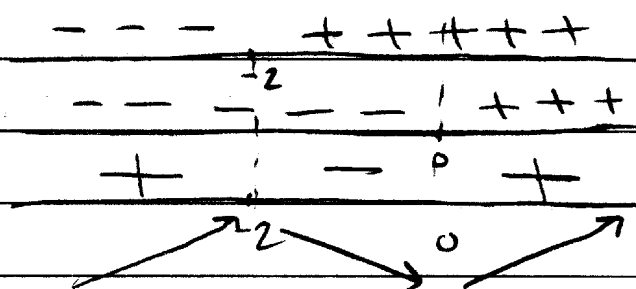
$f(x) = x^{\frac{2}{3}} + 5x^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{10}{3}x^{-\frac{1}{3}} = \frac{2}{3}x^{-\frac{1}{3}} + \frac{10}{3}x^{-\frac{1}{3}} = \frac{12}{3}x^{-\frac{1}{3}} = \frac{4}{x^{\frac{1}{3}}}$

$5(x+2)$

$3x^{\frac{1}{3}}$

f'



a) f is increasing on $(-\infty, -2]$ and on $[0, \infty)$

$=$ decreasing on $[-2, 0]$

b) $f(-2) = 2\sqrt[3]{4}$, $f(0) = 0$

f has local min 0 at $x=0$

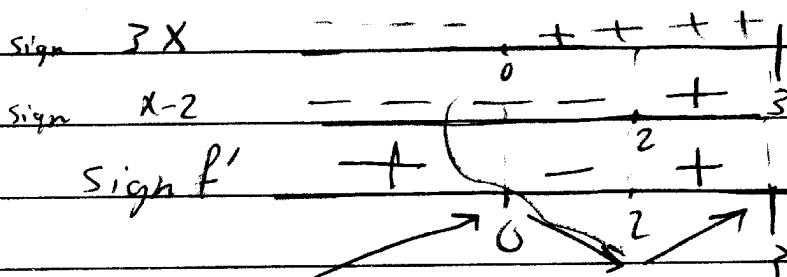
f has local max $2\sqrt[3]{4}$ at $x=-2$.

c) f has no abs extrema.

$$\begin{array}{l|l} (1-3a) \text{ odd} & (61-70) \\ (41-51) \text{ odd} & \\ (53, 59) \text{ odd} & \end{array}$$

[4] $f(x) = x^3 - 3x^2, -\infty < x \leq 3.$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$



a) f is increasing on $(-\infty, 0]$ and on $[2, 3]$
 $\quad \quad \quad =$ decreasing on $[0, 2]$

b) $f(0) = 0, f(2) = 8 - 3(4) = -4$
 $f(3) = 0$

So f has a local ~~max~~ max 0 at $x=0$ and $x=3$
 $\quad \quad \quad =$ min -4 at $x=2$.

c) f has abs max 0 at $x=0$ and $x=3$
 $\quad \quad \quad =$ no abs min because has no d. E.P

4.4: Concavity and curve sketching:

Def. The graph of a diff fn $y=f(x)$ is

① concave up on an open interval I if f' is increasing in I .

② concave down = = = = = decreasing = = =

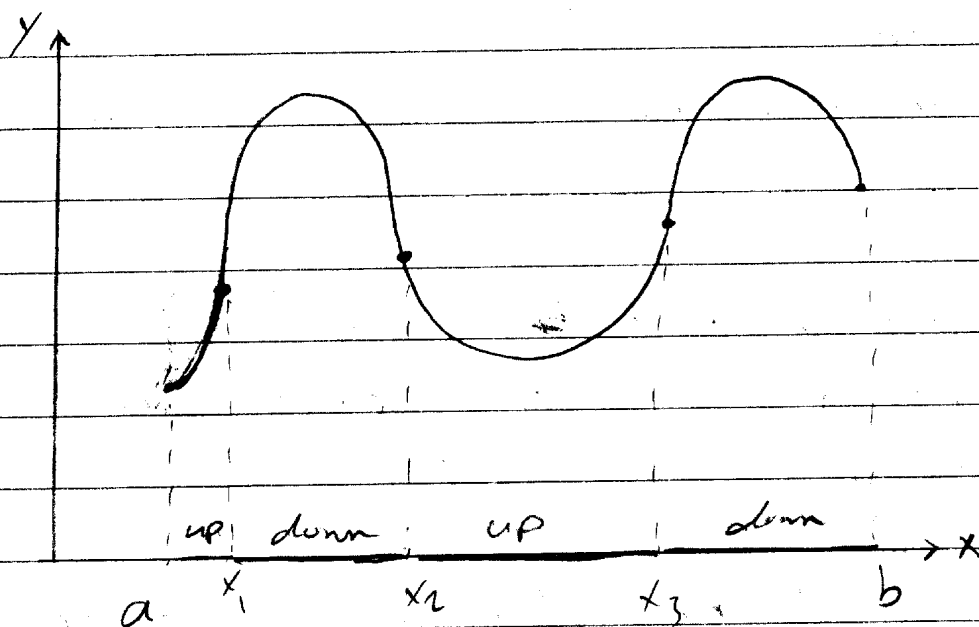
That is,

① If $f'' > 0$ on I then the graph of f over I is concave up.

② If $f'' < 0$ = = = = = down.

Def. If f is continuous on an open interval containing x_0 and if f changes the direction of its concavity at x_0 , then the point $(x_0, f(x_0))$ is called an inflection point of f .

Note / An inflection point may occur when $y'' = 0$ or y'' does not exist.



x_1, x_2, x_3 are inflection point.

Remark / If $y'' = 0$, then it's not true that must there is an inflection point

For example $f(x) = x^4 \rightarrow f' = 4x^3 \rightarrow f'' = 12x^2 = 0$
 $\rightarrow x = 0$

but it is not an inflection point because y'' does not change its sign

Remark / (second derivative test for local extrema)
 Suppose that f'' is continuous on an open interval containing c .

① If $f'(c) = 0$ and $f''(c) < 0$ then f has a local max value at $x = c$.

② If $f'(c) = 0$ and $f''(c) > 0$ then f has a local min value at $x = c$.

③ If $f'(c) = 0$ and $f''(c) = 0$ then the test fails. In this case the f may have local max, local min, or neither.

How to graphing with y' and y'' ?

① Find the intercepts with the coordinate axes.

② Find the asymptotes if it exists

③ Find y' and y'' .

④ Find the local extrema and intervals of increasing and decreasing.

⑤ Find the concavity of the curve and the inflection points

⑥ Sketch the curve by the above informations.

Example 1 / Sketch the graph of the following fns.:

II) $y = x^3 - 3x + 3$.

If $x=0 \rightarrow y=3 \Rightarrow (0,3)$ lies on the graph of

$$y' = 3x^2 - 3 = 3(x-1)(x+1).$$

f is increasing on $(-\infty, -1]$ and
on $[1, \infty)$

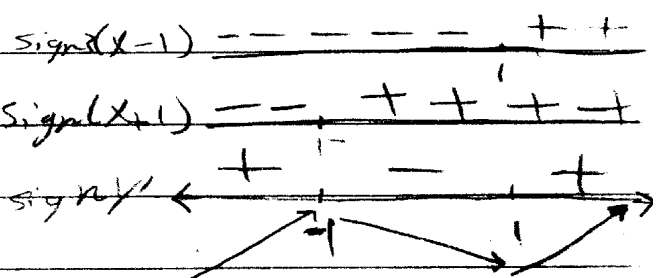
f is decreasing on $[-1, 1]$.

$$f(-1) = (-1)^3 - 3(-1) + 3 = \underline{\underline{5}}$$

$$f(1) = (1)^3 - 3(1) + 3 = \underline{\underline{1}}$$

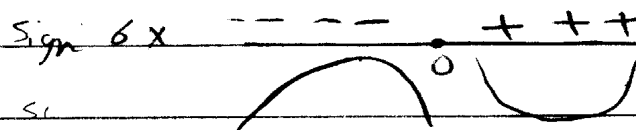
f has local max 5 at $x=-1$

$= = =$ min 1 at $x=1$



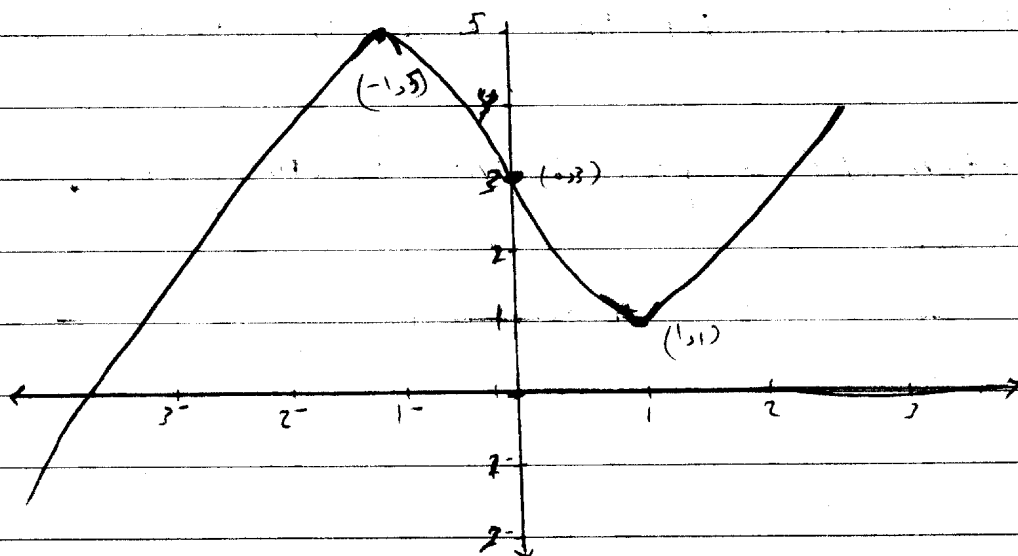
$$y'' = 6x = 0 \Rightarrow x=0$$

The graph is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$



f is continuous $\forall x \Rightarrow f$ is continuous at $x=0$ and f change its concavity at $x=0 \Rightarrow$ there's an inflection point at $x=0$

$f(0) = 3 \Rightarrow (0, 3)$ is an inflection point.



~~Example 2~~ $y = \frac{1+x^2}{1-x^2}$, $x \neq 1, x \neq -1$

If $x=0 \Rightarrow y=1 \Rightarrow (0,1)$ lie on the graph f .

$$y = -1 + \frac{2}{1-x^2}$$

$$\frac{-x^2 + 1}{x^2 - 1} = \frac{-x^2 + 1}{x^2 - 1} = -1$$

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{1+x^2}{1-x^2} = \lim_{x \rightarrow \pm\infty} \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2} - 1} = -1$$

So $y = -1$ is a horizontal asymptote.

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} -1 + \frac{2}{(1-x)(1+x)} = -1 + \frac{2}{(1+1)(\text{V.S.P.Q})} = \infty$$

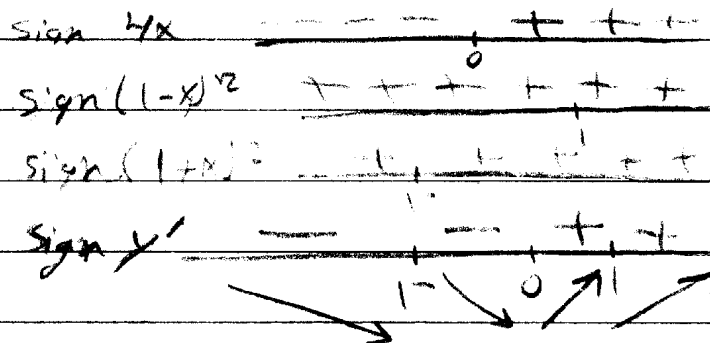
$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} -1 + \frac{2}{(1-x)(1+x)} = -1 + \frac{2}{(\text{V.S.P.Q})} = \infty$$

So $x = -1$ and $x = 1$ are vertical asymptotes.

$$y' = \frac{-2}{(1-x^2)^2} \cdot -2x = \frac{4x}{(1-x^2)^2} = \frac{4x}{(1-x)^2(1+x)^2}$$

f is decreasing on $(-\infty, -1) \cup (-1, 0]$

f is increasing on $[0, 1) \cup (1, \infty)$



f has local min 1 at $x=0$

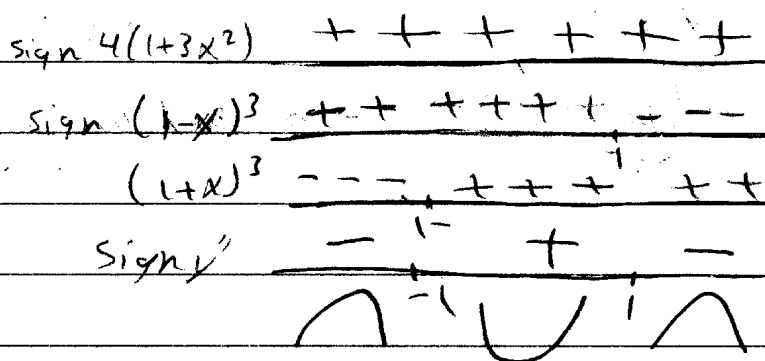
$$y'' = \frac{(1-x^2)^2(4) - 4x(2(1-x^2)(-2x))}{(1-x^2)^4}$$

$$= \frac{4(1-x^2)[(1-x^2) + 4x^2]}{(1-x^2)^4} = \frac{4[(1-x^2) + 4x^2]}{(1-x^2)^3}$$

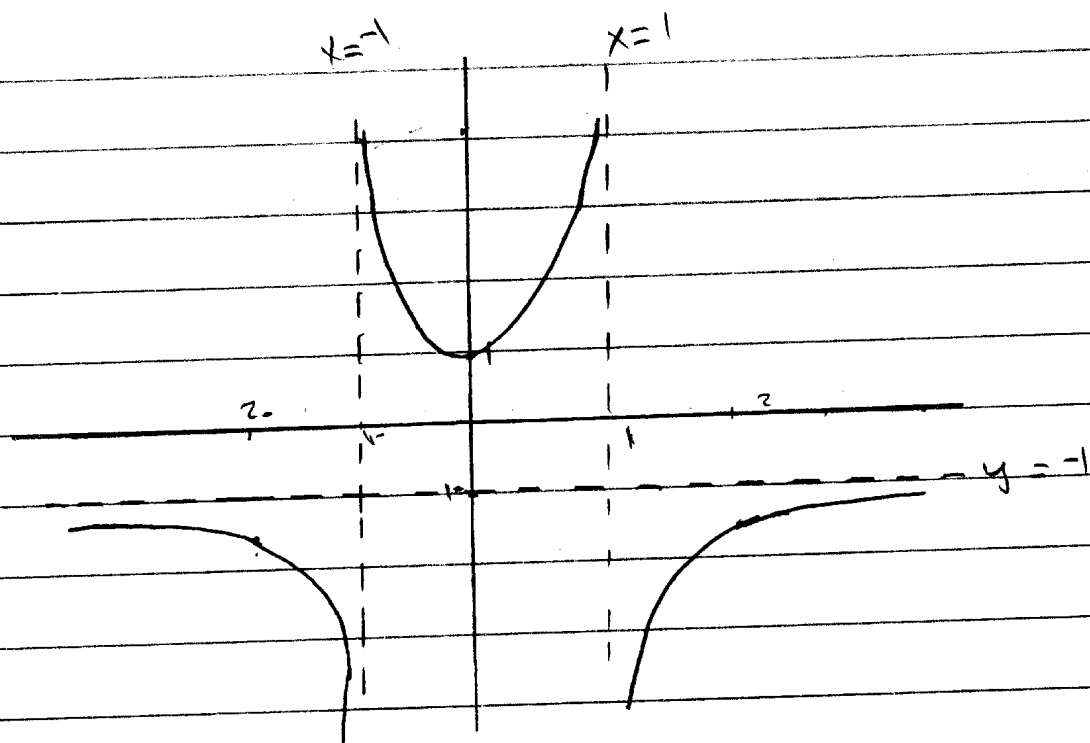
$$= \frac{4(1+3x^2)}{(1-x)^3(1+x)^3} = 0 \text{ (has no solution)}$$

the graph of f is
concave down on $(-\infty, -1)$ and
on $(1, \infty)$

the graph is concave up
on $(-1, 1)$.



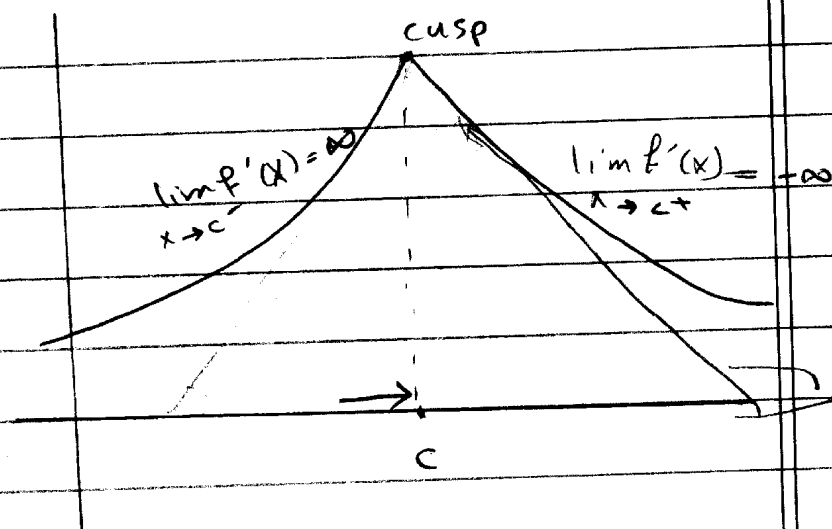
There's no inflection points at $x = -1, x = 1$ because
 f is not continuous at $x = \pm 1$.

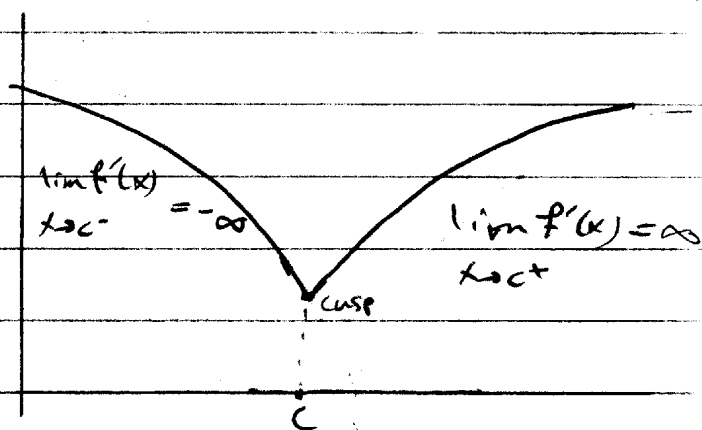


Def. The graph of the continuous fn $y = f(x)$ have a cusp at $x = c$ if the concavity is the same on both sides of c and either

$$\textcircled{1} \lim_{x \rightarrow c^-} f(x) = \infty \text{ and } \lim_{x \rightarrow c^+} f'(x) = -\infty \quad \text{or}$$

$$\textcircled{2} \lim_{x \rightarrow c^-} f'(x) = -\infty \text{ and } \lim_{x \rightarrow c^+} f'(x) = \infty$$





Example 3/ Sketch the graph of the fn $f(x) = x^{\frac{2}{3}}(x-5)$
 $f(x) = x^{\frac{2}{3}} - 5x^{\frac{2}{3}}$

If $x=0 \rightarrow y=0$

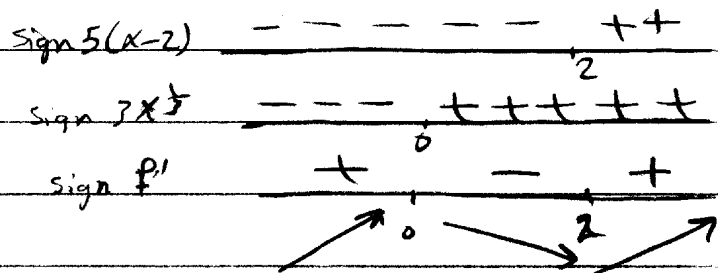
If $y=0 \rightarrow x^{\frac{2}{3}}(x-5) = 0 \rightarrow x=0, x=5$

The points $(0,0)$ and $(5,0)$ lie on the graph.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = \frac{2x^{\frac{2}{3}}}{3} - \frac{10x^{\frac{2}{3}}}{3} = \frac{2x^{\frac{2}{3}} - 10x^{\frac{2}{3}}}{3} = \frac{-8x^{\frac{2}{3}}}{3}$$

f is increasing on $(-\infty, 0]$ and
on $[2, \infty)$

f is decreasing on $[0, 2]$



$f(0) = 0, f(2) = -4.6$

f has local max 0 at $x=0$ and local min -4.6 at $x=2$.

$$f''(x) = \frac{10}{9}x^{-\frac{1}{3}} + \frac{10}{9}x^{-\frac{4}{3}} = \frac{10}{9x^{\frac{1}{3}}} + \frac{10}{9x^{\frac{4}{3}}}$$

$$= \frac{10}{9} \left[\frac{1}{x^{\frac{1}{3}}} + \frac{1}{x^{\frac{4}{3}}} \right] = \frac{10}{9} \left(\frac{x+1}{x^{\frac{4}{3}}} \right)$$

f is concave ~~down~~ ^{down} on $(-\infty, -1)$ Sign $- - -$ $+$ $+$ $+$
 and concave on $(-1, 0) \cup (0, \infty)$ Sign $+$ $+$ $+$ $+$ $+$ $+$ $+$
 Sign f'' $- -$ $+$ $+$

$$f(-1) = 8$$

The graph has an inflection point $(-1, -6)$

$$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} \frac{5(x-2)}{3x^{\frac{1}{3}}} = \frac{5}{3} \cdot \frac{-2}{\text{V.S.N.Q.}} = -\infty$$

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} \frac{5(x-2)}{3x^{\frac{1}{3}}} = \frac{5}{3} \cdot \frac{-2}{\text{V.S.P.Q.}} = -\infty$$

There's a cusp at $x=0$..