

Chapter 7

Transcendental Functions

Functions can be classified into two broad complementary groups called algebraic functions and transcendental functions. In this chapter we study the calculus of important transcendental functions, including the logarithmic, exponential, inverse trigonometric, and hyperbolic functions. These functions occur frequently in many mathematical setting and scientific applications.

7.1. Inverse Functions and Their Derivatives

A function that inverts the effect of a function f is called the inverse of f . Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in formulas for antiderivatives and solutions of differential equations.

Recall: additive inverse of a real number $x + (-x) = 0$
multiplicative inverse of a non-zero real number $xx^{-1} = 1$.

One-to-one Functions

As we know, a function is a rule that assigns a unique value from its range to each point in its domain. Some functions assign the same value to more than one point.

Example. $f(x) = \sqrt{x}$ assigns 2 only to 4.

$f(x) = x^2$ assigns 1 to both 1 and -1 ,

$f(x) = \sin x$ assigns 0 to $0, \pi, 2\pi, \dots$

Definition. (One-to-one function)

A function $f(x)$ is one-to-one on a domain D if for all $x_1, x_2 \in D$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Example 1. Show that $f(x) = x + 4$ is one-to-one on $\mathbb{R}$.

Solution. Let $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. We have to show that $x_1 = x_2$.

$$f(x_1) = f(x_2) \Rightarrow x_1 + 4 = x_2 + 4 \Rightarrow x_1 = x_2$$

Thus $f(x)$ is one-to-one on $\mathbb{R}$.

Example 2. Show that $f(x) = \sqrt{x+1}$ is one-to-one on $[-1, \infty)$.

Solution. Let $x_1, x_2 \in [-1, \infty)$ such that $f(x_1) = f(x_2)$. Then we have $\sqrt{x_1+1} = \sqrt{x_2+1}$.

Squaring both sides we get $x_1 + 1 = x_2 + 1$ and hence $x_1 = x_2$.

Thus $f(x) = \sqrt{x+1}$ is one-to-one on $[-1, \infty)$.

Example 3. Show that $f(x) = 3 - 2x^2$ is not one-to-one on its domain.

Solution. $f(1) = 1 = f(-1)$.

Example 4. If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $g(x) = \frac{4}{f(x)} - 1$? Give reasons for your answer.

Solution. Let $g(x_1) = g(x_2)$. Then we have $f(x_1) = f(x_2)$ and hence $x_1 = x_2$. Therefore $g(x)$ is one-to-one.

Increasing and decreasing functions:

Theorem 1. (Increasing and decreasing functions are one-to-one)

(1) If a function $f(x)$ is increasing on an interval I , then $f(x)$ is one-to-one on I .

(2) If a function $f(x)$ is decreasing on an interval I , then $f(x)$ is one-to-one on I .

Proof. (1) $f(x)$ is increasing on $I \Rightarrow f(x)$ is one-to-one on I .

Let x_1 and x_2 be in I such that $f(x_1) = f(x_2)$. Then $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$; contradiction.

$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$; contradiction.

Thus $x_1 = x_2$. □

Example 1. Show that $f(x) = x^{5/4}$ is one-to-one on its domain.

Solution. The domain of $f(x)$ is $[0, \infty)$.

$f'(x) = \frac{5}{4}x^{1/4} > 0$ for $x \in (0, \infty)$. Thus $f(x)$ is increasing on $[0, \infty)$ and hence it is one-to-one on its domain $[0, \infty)$.

Example 2. Determine whether $f(x) = 1 - x^2$ is one-to-one on $(0, \infty)$.

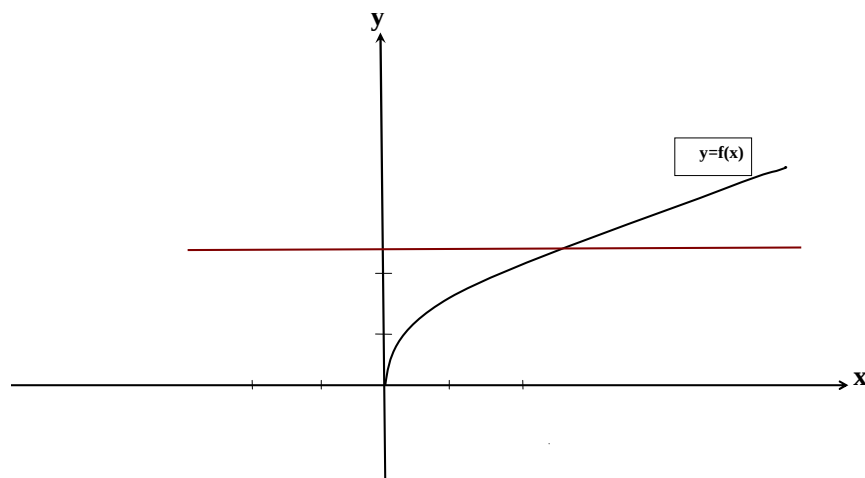
Solution. $f'(x) = -2x < 0$ for all $x \in (0, \infty)$. Thus $f(x)$ is one-to-one on $(0, \infty)$.

The horizontal line test

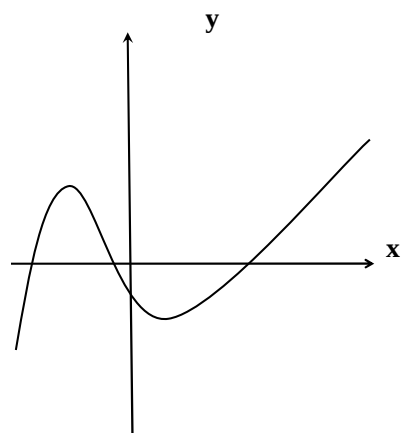
A function $y = f(x)$ is one-to-one if and only if its graph intersect each horizontal line at most once.

Example. Determine whether the functions graphed below are one-to-one?

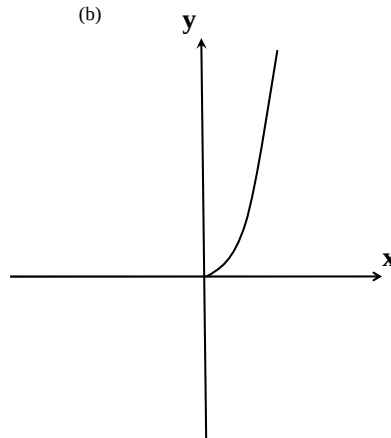
Solution.



(a)



(b)



Inverses

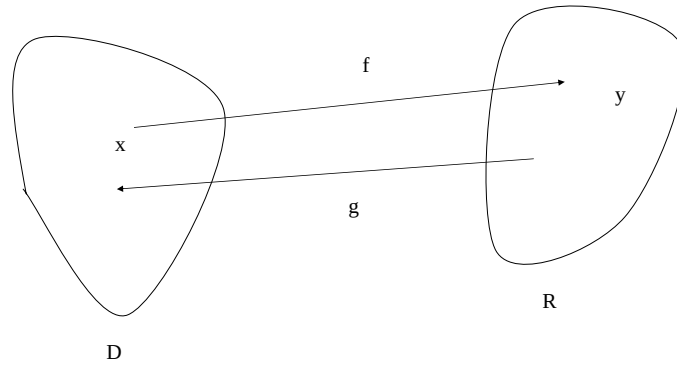
Since a one-to-one function sends distinct numbers to distinct values, it is possible to define a function which reverse f so that it sends number from the range of f back to the number from which it came.

Definition. (Inverse Function)

A function $f(x)$ has an inverse if there is a function $g(x)$ such that $f(g(x)) = g(f(x)) = x$.

Remarks.

- (1) If $f(x)$ has an inverse $g(x)$, then we write $g = f^{-1}$ (read f inverse)
- (2) If f^{-1} is the inverse of f , then f^{-1} has an inverse and $(f^{-1})^{-1} = f$.
- (3) f has an inverse if and only if f is one-to-one.



(4) The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f .

Example 1. Show that $f(x) = 4 - \frac{3}{5}x$ has an inverse and find it.

Solution. $f'(x) = -\frac{3}{5} < 0$ for all $x \in D(f) = \mathbb{R}$. Thus f is decreasing and hence is one-to-one. Therefore f has an inverse.

How to find f^{-1} ?

First Method:

(1) Solve $y = f(x)$ for x in terms of y

$$y = 4 - \frac{3}{5}x \Rightarrow x = \frac{3}{5}(4 - y)$$

(2) Interchange x and y

$$y = \frac{3}{5}(4 - x)$$

(3) Write $f^{-1}(x) = y$

Second Method:

Let $y = f^{-1}(x)$. Using $f(f^{-1}(x)) = x \Rightarrow f(y) = x \Rightarrow 4 - \frac{3}{5}y = x \Rightarrow y = \frac{3}{5}(4 - x)$.

As a check: $f((f^{-1}(x))) = x = f^{-1}(f(x))$?

Example 2. Show that $y = \frac{1+x}{2-x}$, $x \neq 2$, is one-to-one and find its inverse.

Solution. $f'(x) = \frac{1}{(2-x)^2} > 0 \Rightarrow f \nearrow \Rightarrow f$ is one-to-one.

$$y = f^{-1}(x). \text{ Then } f(y) = x \Rightarrow y = \frac{1+y}{2-y} = x \Rightarrow y = \frac{2x-1}{x+1}$$

Example 3. Find $f^{-1}(x)$ if $f(x) = x^2 + 4x + 1$, $x \geq -2$.

Solution. Let $y = f^{-1}(x)$. Then $f(y) = x \implies y^2 + 4y + 1 = x$
 $\implies y = \frac{-4 \pm \sqrt{16 - 4(1 - x)}}{2} = -2 \pm \sqrt{3 + x}$
 $R(f^{-1}) = D(f) = [-2, \infty) \implies f^{-1}(x) = -2 + \sqrt{3 + x}$

The relation between the graphs of f and f^{-1}

The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$.

$$\begin{aligned} f(1) = 2 &\implies f^{-1}(2) = 1 \\ f(2) = 4 &\implies f^{-1}(4) = 2 \end{aligned}$$

Example. $f(x) = x^2$, $x \geq 0$, and $f^{-1}(x) = \sqrt{x}$.

Derivatives of Inverses of Differentiable Functions

Consider the identity

$$f(f^{-1}(x)) = x.$$

Differentiating both sides, we obtain

$$\begin{aligned} f'(f^{-1}(x)) \frac{d}{dx} f^{-1}(x) &= 1 \\ \implies \frac{d}{dx} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

Evaluating $\frac{d}{dx} f^{-1}(x)$ at $x = f(a) \implies$

$$\left. \frac{df^{-1}}{dx} \right|_{x=f(a)} = \frac{1}{f'(a)}$$

Theorem 2. (The derivative rule for inverses)

If f is differentiable at every point of an interval I and $f'(x) \neq 0$ for all $x \in I$, then f^{-1} is differentiable at every point of the interval $f(I)$ and

$$\implies \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

or

$$\left. \frac{df^{-1}}{dx} \right|_{x=f(a)} = \frac{1}{f'(a)}.$$

Example 1. (Exam)

Let $f(x) = x^3 + 3x + 2$.

(a) Show that $f(x)$ has an inverse.

(b) Find $\left. \frac{df^{-1}}{dx} \right|_{x=6} = f(1)$

Solution.

$$\begin{aligned}\left. \frac{df^{-1}}{dx} \right|_{x=f(1)} &= \frac{1}{f'(1)} \\ f'(x) &= 3x^2 + 3 \Rightarrow f'(1) = 6 \\ \Rightarrow \left. \frac{df^{-1}}{dx} \right|_{x=f(1)} &= \frac{1}{6}.\end{aligned}$$

Example 2. Let $f(x) = \int_2^x \sqrt{u^4 + 1} du$. Evaluate $(f^{-1})'$ at $x = f(\sqrt{3})$.

Solution.

$$\begin{aligned}f'(x) &= \sqrt{x^4 + 1} \Rightarrow f'(\sqrt{3}) = \sqrt{10} \\ \Rightarrow (f^{-1})'(f(\sqrt{3})) &= \frac{1}{f'(\sqrt{3})} = \frac{1}{\sqrt{10}}\end{aligned}$$

Example 3. Find $\left. \frac{df^{-1}}{dx} \right|_{x=9}$, if $f(x) = x^2 - 2x + 1$, $x \leq 1$.

Solution. $f'(x) = 2x - 2$

$$\left. \frac{df^{-1}}{dx} \right|_{x=9} = \frac{1}{f'(f^{-1}(9))}$$

$$\text{Let } f^{-1}(9) = a \Rightarrow f(a) = 9 \Rightarrow a^2 - 2a + 1 = 9 \Rightarrow a = 4 \text{ or } a = -2$$

$$D(f) = (-\infty, 1] \Rightarrow f^{-1}(9) = a = -2$$

$$\text{Thus } \left. \frac{df^{-1}}{dx} \right|_{x=9} = \frac{1}{f'(-2)} = -\frac{1}{6}.$$

7.2. Natural Logarithms

The most important function-inverse pair in mathematics and science is the pair consisting of the natural logarithm $\ln x$ and exponential function e^x . We start by defining $\ln x$ as an integral using the Fundamental Theorem of Calculus. Then we will define e^x as the inverse function of $\ln x$.

The natural logarithm function

Definition. (Natural logarithm)

For any $x > 0$, the natural logarithm of x is

$$\ln x = \int_1^x \frac{1}{t} dt.$$

The value of $\ln x$ for any $x > 1$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$. For $0 < x < 1$, $\ln x$ is the negative of the area under the curve $y = 1/t$ from $t = x$ to $t = 1$.

Remarks.

(1) $\ln x$ is not defined for $x \leq 0$, that is $Dom(\ln x) = (0, \infty)$,

(2) $\ln 1 = 0$.

Example. Find $Dom(f)$ if $f(x) = \ln(3x - 9)$.

Solution. $\ln(3x - 9)$ is defined only if $3x - 9 > 0$. Thus $Dom(f) = (3, \infty)$.

The derivative of $y = \ln x$

Using the first part of the fundamental theorem of calculus, we have

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Example 1. Find $\frac{dy}{dx}$ if $y = \ln(x^2 + 3x + 1)$.

Solution.

Example 2. Find y' if $y = \sec \ln x$.

Solution.

Example 3. Find y' if $y = \frac{1 + \ln(2x)}{x^2}$.

Solution.

Properties of logarithms

Theorem 3. (Properties of logarithms)

For any numbers $a > 0$ and $x > 0$, we have

$$1. \ln(ax) = \ln a + \ln x.$$

$$2. \ln\left(\frac{a}{x}\right) = \ln a - \ln x.$$

$$3. \ln\left(\frac{1}{x}\right) = -\ln x.$$

$$4. \ln x^n = n \ln x \text{ for any real number } n.$$

Proof. 1. $\frac{d}{dx} \ln(ax) = \frac{1}{x} = \frac{d}{dx} \ln(x)$. Thus $\ln(ax) = \ln x + C$

$$x = 1 \Rightarrow C = \ln a.$$

$$2. \frac{d}{dx} \ln\left(\frac{a}{x}\right) = -\frac{1}{x} = -\frac{d}{dx} \ln(x). \text{ Thus } \ln\left(\frac{a}{x}\right) = -\ln x + C$$

$$x = 1 \Rightarrow C = \ln a.$$

□

Example 1. Express $\ln(4.5)$ in terms of $\ln 2$ and $\ln 3$.

Solution.

Example 2. Express $\ln \sqrt{15}$ in terms of $\ln 3$ and $\ln 5$.

Solution.

Example 3. Use the properties of logarithms to simplify the expression $\ln \sec \theta + \ln(5 \sin \theta)$.

Solution.

$$\ln \sec \theta + \ln(5 \sin \theta) = -\ln \cos \theta + \ln 5 + \ln \sin \theta = \ln 5 + \ln \tan \theta.$$

The Graph and Range of $\ln x$

Domain of $\ln x$ is $(0, \infty)$

$$\begin{aligned}\ln 3 &= 1.0986123 \Rightarrow \ln 3^n = n(1.0986123) \\ &\Rightarrow \ln 3^{-n} = -n(1.0986123) \\ &\Rightarrow \lim_{x \rightarrow \infty} \ln x = \infty, \text{ and } \lim_{x \rightarrow 0^+} \ln x = -\infty.\end{aligned}$$

Range of $\ln x$ is $(-\infty, \infty) = \mathbb{R}$.

x -intercept: $\ln x = 0 \Rightarrow x = 1$.

y -intercept: $\ln 0$ is not defined. Thus there is no y -intercept.

$$\frac{d}{dx} \ln x = \frac{1}{x} > 0, \text{ for all } x > 0, \Rightarrow \ln x \nearrow \text{ on its domain } (0, \infty).$$

No critical points.

$$\frac{d^2}{dx^2} \ln x = -\frac{1}{x^2} < 0, \text{ for all } x > 0, \Rightarrow \ln x \text{ is concave down on its domain } (0, \infty).$$

No inflection points.

$$\lim_{x \rightarrow 0^+} \ln x = -\infty \implies x = 0 \text{ is a vertical asymptote.}$$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and power can be found more easily if we take the natural logarithm of both sides before differentiating. This enables us to use the properties of logarithms to simplify the formulas and hence the differentiation.

The method consists of the following steps:

- Step(1): Take the logarithm of both sides
- Step(2): Use the properties of logarithm to simplify the formulas
- Step(3): Differentiate both sides
- Step(4): Solve for y'

Example 1. Find $\frac{dy}{dx}$ if $y = (x^3 + x + 1)(x^4 - 3x^2 + x)^{2/3}$.

Solution.

Example 2. Find y' if $y = \frac{(x+1)^2}{x(x-1)^3}$.

Solution.

Example 3. Find $\frac{dy}{d\theta}$ if $y = \frac{(\sin \theta + 1)^2}{\sqrt{\theta} + \cot \theta}$.

Solution.

The Integral $\int (1/u) du$

If $x > 0$, then $\int \frac{1}{x} dx = \ln x + C$.

If $x < 0$, then $\ln(-x)$ is defined and $\frac{d}{dx} \ln(-x) = \frac{1}{x}$

Thus $\int \frac{1}{x} dx = \ln(-x) + C$ for $x < 0$.

Therefore, for all $x \neq 0$,

$$\int \frac{1}{x} dx = \ln |x| + C.$$

Example 1. Evaluate the integral $\int_4^{12} \frac{1}{x} dx$.

Solution. $\ln 3$

Example 2. Find $\int \frac{dx}{3-2x}$.

Solution. $-\frac{1}{2} \ln |3-2x| + C$

Example 3. Evaluate $\int_4^8 \frac{dx}{x \ln^3 x}$.

Solution. $\int_4^8 \frac{dx}{x \ln^3 x} = \int_{u=\ln 4}^{u=\ln 8} \frac{du}{u^3} = -\frac{1}{2u^2} \Big|_{u=\ln 4}^{u=\ln 8} = \frac{1}{18 \ln^2 2} - \frac{1}{8 \ln^2 2} = \frac{-5}{72 \ln^2 2}.$

Example 4. Evaluate $\int \frac{\tan^2(\ln(x+1))}{x+1} dx$.

Solution.

Example 5. Evaluate $\int \frac{x^5}{x^3+1} dx$.

Solution.

Example 6. Evaluate $\int \frac{\sin(2x)}{3+2\cos^2 x} dx$.

Solution.

The Integrals of $\tan x$ and $\cot x$

$$\int \tan x = -\ln |\cos x| + C = \ln |\sec x| + C$$

$$\int \cot x = \ln |\sin x| + C = -\ln |\csc x| + C$$

Proof.

□

Example. Evaluate $\int x \cot(x^2 + 1) dx$.

Solution.

7.3. Exponential Functions

Having defined the function $\ln x$ and studied its properties, we now introduce its inverse, the exponential function e^x .

The inverse of $\ln x$ and the number e

We know that

$$\frac{d}{dx} \ln x = \frac{1}{x} > 0 \text{ for all } x \in \text{Dom}(\ln x) = (0, \infty).$$

Thus $\ln x$ is an increasing function on its domain and hence it is one-to-one on its domain. Therefore, $\ln x$ has an inverse $\ln^{-1}(x)$.

Definition. (The number e (Euler's number))

We define the number e to be the value of $\ln^{-1}(1)$. This real number has the value $e = 2.718281828459045$ up to 15 places.

The function e^x

Since $e > 0$, $e^x > 0$ for any rational number x . Thus $\ln e^x$ is defined.

But $\ln e^x = x \ln e = x = \ln(\ln^{-1}(x))$ and $\ln$ is one-to-one. Thus

$$\ln^{-1}(x) = e^x$$

for any rational number x . This suggest the extension

$$\ln^{-1}(x) = e^x, \text{ for all } x \in \mathbb{R}.$$

Definition. (The natural exponential function)

For every real number x , we define $e^x = \ln^{-1}(x)$.

Remarks.

- (1) $e^x = \exp x$
- (2) The domain of e^x is the range of $\ln x = \mathbb{R}$.
- (3) The range of e^x is the domain of $\ln x = (0, \infty)$.
- (4) $\ln e^x = x$ for all $x \in \mathbb{R}$.
- (5) $e^{\ln x} = x$ for all $x \in (0, \infty)$.
- (6) $e^0 = 1$.
- (7) The graph of e^x

$$(8) \lim_{x \rightarrow \infty} e^x = \infty \text{ and } \lim_{x \rightarrow -\infty} e^x = 0.$$

Example 1. Simplify the expression $\ln(3e^2)$.

Solution.

Example 2. Solve for x : $e^{3\sqrt{x+1}} = 4$.

Solution. $3\sqrt{x+1} = \ln 4 \Rightarrow x = \left(\frac{\ln 4}{3}\right)^2 - 1$

Example 3. Solve the equation $\ln x^2 = 2 \ln 4 - 6 \ln 2$.

Solution. $\ln x^2 = -2 \ln 2 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$.

Example 4. Solve for y in terms of t : $\ln(4 + 3y) = 2t + 1$.

Solution. $y = \frac{1}{3} (e^{2t+1} - 4)$

Example 5. Solve for y : $\ln(y + 2) - \ln(y - 1) = \cos x$.

Solution. $y = \frac{e^{\cos x} + 2}{e^{\cos x} - 1}$

Laws of exponents

Even though e^x is defined as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra.

Theorem 4. (Laws of exponents for e^x)

For all real numbers x_1 and x_2 , we have

$$(1) \quad e^{x_1} e^{x_2} = e^{x_1+x_2}.$$

$$(2) \quad \frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}.$$

$$(3) \quad \frac{1}{e^{x_1}} = e^{-x_1}.$$

$$(4) \quad (e^{x_1})^{x_2} = e^{x_1 x_2}.$$

Proof. (1) $\ln(e^{x_1} e^{x_2}) = \ln e^{x_1} + \ln e^{x_2} = x_1 + x_2 = \ln e^{x_1+x_2}$

$$\Rightarrow e^{x_1} e^{x_2} = e^{x_1+x_2}.$$

□

Example 1. Simplify the expression $e^{2 \ln x - \ln t}$.

Solution. $e^{2 \ln x - \ln t} = \frac{x^2}{t}$

Example 2. Find $f^{-1}(x)$ if $f(x) = e^{3x+2} + 1$.

Solution. $f^{-1}(x) = \frac{1}{3}[-2 + \ln(x-1)]$

The derivative and integral of e^x

Since $\ln x$ is differentiable for all $x \in (0, \infty)$ and $\frac{d}{dx} \ln x = \frac{1}{x} \neq 0$ for all $x \in (0, \infty)$, $e^x = \ln^{-1} x$ is differentiable at each $x \in \ln((0, \infty)) = \mathbb{R}$ and

$$\frac{d}{dx} e^x = \frac{1}{\frac{d}{dx} \ln x|_{x=e^x}} = e^x.$$

Therefore

$$\frac{d}{dx} e^x = e^x$$

The only functions that are the derivatives of themselves are e^x or a constant multiple of e^x .

As a consequence of the above formula, we have

$$\int e^x dx = e^x + C$$

Example 1. (Exam)

Find y' if $y = \ln(x^2 e^x)$.

Solution.

Example 2. Find $\frac{dy}{dx}$ if $y = \frac{e^{\tan x}}{e^{2x} + \ln x}$.

Solution.

Example 3. (Exam)

Let $f(x) = e^x + x$.

(a) Show that $f(x)$ is 1-1.

(b) Find $\frac{df^{-1}}{dx}$ at $x = f(\ln 2)$.

Solution.

Example 4. Evaluate the integral $\int (e^{2x} - e^{-x}) dx$.

Solution.

Example 5. Evaluate the integral $\int_{-1}^4 x e^{x^2} dx$.

Solution.

Example 6. Evaluate the integral $\int_0^{\pi/4} e^{\sec x} \sec x \tan x dx$.

Solution.

Example 7. Evaluate the integral $\int \frac{1}{e^{-x} + 4} dx$.

Solution.

The general exponential function a^x

As we have seen above, we can define e^x for every real number x . This definition enable us to define the general exponential function a^x for every real number x provided that $a > 0$. Now we will study the derivatives and integrals of a^x and we will define the general logarithmic function $\log_a x$ as the inverse of a^x .

Definition. (General exponential function)

For any $a > 0$ and any $x \in \mathbb{R}$, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

Example. Evaluate 3^π .

Solution.

The power rule (general form)

Since we can define x^n for any $x \in \mathbb{R}$ as long as $x > 0$, the formula for differentiating x^n has the final form

$$\frac{d}{dx} x^n = nx^{n-1}, \quad \text{for any real number } n.$$

The derivative of a^x

Using $a^x = e^{x \ln a}$, we obtain

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a.$$

Therefore we have

$$\frac{d}{dx} a^x = a^x \ln a$$

The above formula shows why is the natural exponential function preferred in calculus.

Example 1. Find $\frac{dy}{dx}$ if $y = 7^x$.

Solution.

Example 2. Find $\frac{dy}{dx}$ if $y = 3^{\csc x}$.

Solution.

Other power functions

As long as $x > 0$, we can define $x^{f(x)} = e^{f(x) \ln x}$, for any function $f(x)$. To find $\frac{d}{dx} x^{f(x)}$, we use the definition

$$\frac{d}{dx} x^{f(x)} = \frac{d}{dx} e^{f(x) \ln x}.$$

Example 1. Find $\frac{dy}{dx}$ if $y = x^{x^2}$.

Solution.

Example 2. Find $\frac{dy}{dx}$ if $y = (\ln x)^{e^x}$.

Solution.

Example 3. Find $\frac{dy}{dx}$ if $y = (\cos x)^{\ln x + e^x}$.

Solution.

Example 4. (Exam)

Find $\frac{dy}{dx}$ if $y = \left(\frac{1}{x}\right)^x + \ln(\sec e^{3x})$.

Solution.

Laws of exponents for a^x

For any real number $a > 0$ and all real numbers x_1 and x_2 , we have

$$(1) \quad a^{x_1} a^{x_2} = a^{x_1+x_2}.$$

$$(2) \quad \frac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}.$$

$$(3) \quad \frac{1}{a^{x_1}} = a^{-x_1}.$$

$$(4) \quad (a^{x_1})^{x_2} = a^{x_1 x_2}.$$

The Integral of a^x

$$\int a^x dx = \int e^{x \ln a} dx = \frac{1}{\ln a} a^x + C, \quad \text{provided that } a \neq 1.$$

Example 1. Evaluate the integral $\int_2^4 x 3^{x^2} dx$.

Solution.

Example 2. Find the value of $\int \csc^2 x 7^{\cot x} dx$.

Solution.

Logarithms with Base a

If $a > 0$, then $f(x) = a^x$ is defined for all $x \in \mathbb{R}$. Does f have an inverse?

$$f'(x) = (\ln a) a^x \left\{ \begin{array}{l} > 0, \text{ if } a > 1 \\ < 0, \text{ if } a < 1. \end{array} \right.$$

Thus $f(x) = a^x$ is one-to-one for any $0 < a \neq 1$, and hence it has an inverse. We will denote this inverse by $\log_a x$ (the logarithm of x with base a).

Definition. (The general logarithmic function)

For any positive number $a \neq 1$, we define

$$\log_a x = \text{the inverse function of } a^x.$$

Inverse equations for a^x and $\log_a x$

$$(1) \log_a(a^x) = x, \text{ for all } x \in \mathbb{R}.$$

$$(2) a^{\log_a x} = x, \text{ for all } x > 0.$$

Example 1. Simplify $\log_5(5^{-6})$.

Solution.

Example 2. Simplify $3^{\log_3 4}$.

Solution.

Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$. Since $a^{\log_a x} = x$, we have $\log_a x \ln a = \ln x$. Therefore

$$\log_a x = \frac{\ln x}{\ln a}$$

Remark. The above formula shows that the arithmetic rules satisfied by $\log_a x$ are the same as that for $\ln x$. That is, for any numbers $x > 0$, $y > 0$, we have

$$(1) \log_a(xy) = \log_a x + \log_a y.$$

$$(2) \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

$$(3) \log_a(x^y) = y \log_a x.$$

Example 1. Simplify $9^{\log_3 4}$.

Solution.

Example 2. Express $\frac{\log_5 x}{\log_4(x^2 + 1)}$ in terms of natural logarithms.

Solution.

Example 3. If $x > 0$, then find the exact value of $\frac{1}{x} \log_2(8^x)$.

Solution.

Derivatives and integrals involving $\log_a x$

Using $\log_a x = \frac{\ln x}{\ln a}$, we find

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

Example 1. Find $\frac{dy}{dx}$ if $y = \frac{e^{2x^3+5x}}{8^x \log_4 x}$.

Solution.

Example 2. Find $\int \frac{dx}{x \log_6 x}$.

Solution.

Example 3. Find $\int_e^{e^2} \frac{(\log_{10} x)^3}{x} dx$.

Solution.

7.5. Indeterminate Forms and L'Hôpital's Rule

L'Hôpital's Rule was discovered by Bernoulli but written for the first time in a text by L'Hôpital.

Indeterminate Forms $0/0$ and ∞/∞

If $f(x_0) = g(x_0) = 0$, then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ can not be found by substituting $x = x_0$. The substitution produces $0/0$, a meaningless expression known as an indeterminate form. Similarly, if $\lim_{x \rightarrow x_0} f(x) = \pm\infty = \lim_{x \rightarrow x_0} g(x)$, then $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$ produces the indeterminate form $\frac{\infty}{\infty}$.

Sometimes, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations. Limits involving transcendental functions often can not be found by these classical methods. L'Hôpital's Rule is an alternative method to find limits that lead to indeterminate forms.

Theorem 5. (*L'Hôpital's Rule*)

Suppose that

- (1) $f(x_0) = g(x_0) = 0$ or $\lim_{x \rightarrow x_0} f(x) = \pm\infty = \lim_{x \rightarrow x_0} g(x)$,
- (2) f and g are differentiable on an open interval I containing x_0 ,
- (3) $g'(x) \neq 0$ for all $x \in I \setminus \{x_0\}$.

Then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)},$$

provided the limit on the right hand side exists.

Proof. (partial)

$$\frac{f'(x_0)}{g'(x_0)} = \frac{\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}}{\lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0}} = \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}.$$

□

Remark. L'Hôpital's Rule can be applied to one-sided limits and when x_0 is a real number or $\pm\infty$.

Example 1. Use L'Hôpital's rule to show that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

Solution.

Example 2. Find $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2}$.

Solution. 2/3

Example 3. Find $\lim_{x \rightarrow 1} \frac{x^3 - 1}{4x^3 - x - 3}$.

Solution. 3/11

Example 4. Find $\lim_{x \rightarrow 0} \frac{\cos x - \cos(3x)}{x^2}$.

Solution. 4

Example 5. Find $\lim_{x \rightarrow \infty} \frac{\ln x}{5 + 2 \ln x}$.

Solution. 1/2

Example 6. (Exam) Find $\lim_{x \rightarrow 0} \frac{3^x - 1}{x}$.

Solution. $\ln 3$

Example 7. Find $\lim_{x \rightarrow 0} \frac{2^{\sin x} - 1}{e^x - 1}$.

Solution. $\ln 2$

Example 8. Evaluate $\lim_{x \rightarrow \infty} \frac{5^x - 1}{3^x - 1}$.

Solution. ∞

Example 9. Find $\lim_{x \rightarrow \infty} \frac{x^2}{\ln x}$.

Solution. ∞

Example 10. Find $\lim_{x \rightarrow \infty} \frac{\log_2 x}{\log_3(x + 1)}$.

Solution. $\ln 3 / \ln 2$

Example 11. (Exam) Find the value of the constant a such that $a > 0$ and $\lim_{x \rightarrow 0} \frac{x - \ln(x + 1)}{1 - \cos(ax)} = \frac{1}{4}$.

Solution. $a = 1/2$

Indeterminate Forms $\infty \cdot 0$ and $\infty - \infty$

We can handle the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$ by using algebra to convert them to a $0/0$ or ∞/∞ form and then we apply L'Hôpital's rule.

Example 1. Evaluate $\lim_{x \rightarrow 0^+} x \cot x$.

Solution. 1

Example 2. Find $\lim_{x \rightarrow 0^+} \left(\csc x - \frac{1}{x} \right)$.

Solution. 0

Example 3. $\lim_{x \rightarrow 2^+} \left(\frac{x}{x-2} - \frac{1}{\ln(x-1)} \right)$

Solution. ∞

Example 4. (Exam)

Find $\lim_{x \rightarrow \infty} e^{-x}(3x+1)$.

Solution. 0

Indeterminate Powers (1^∞ , 0^0 , and ∞^0)

1^∞ , 0^0 , ∞^0 are indeterminate forms. The limits that lead to one of these forms can be handled by L'Hôpital's rule as follows:

$$\lim_{x \rightarrow a} [f(x)]^{g(x)} \longrightarrow \lim_{x \rightarrow a} g(x) \ln f(x).$$

$$\text{If } \lim_{x \rightarrow a} \ln F(x) = L, \text{ then } \lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} e^{\ln F(x)} = e^L$$

Remark. a may be a real number or $\pm\infty$.

Example 1. Evaluate $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x} \right)^x$.

Solution. e^{-2}

Example 2. Find $\lim_{x \rightarrow 0^+} (\sin x)^x$.

Solution. 1

Example 3. Find $\lim_{x \rightarrow \infty} (\ln x)^{1/x}$.

Solution. 1

Example 4. Find $\lim_{x \rightarrow 0^+} (e^x + x^2)^{1/x}$.

Solution. e

Example 5. Evaluate $\lim_{x \rightarrow 0^+} \left(e^{\frac{1}{x}} \right)^{\tan x}$.

Solution. 1

Example 6. Evaluate $\lim_{x \rightarrow 0^+} (\tan x)^x$.

Solution. e

Example 7. (Exam)

Find $\lim_{x \rightarrow \infty} (1 + 3x)^{1/x}$.

Solution. 1

Example 8. (Exam)

Find $\lim_{x \rightarrow 0^+} \left(\frac{1}{x}\right)^x$.

Solution. 1

Example 9. (Exam)

Find $\lim_{x \rightarrow \infty} (x^3 + e)^{1/\ln x}$.

Solution. e^3

Example 10. Evaluate $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$.

Solution. $e^{-1/2}$

Example 11. Evaluate $\lim_{x \rightarrow 0^+} \left(1 + \frac{3}{x}\right)^x$.

Solution. 1

7.6. Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how these functions are defined, graphed, and evaluated, how their derivatives are computed, and why they appear as important antiderivatives.

Defining the inverses

The six basic trigonometric functions are not one-to-one in their domains, but we can restrict their domains to intervals on which they are one-to-one.

The arcsine function

The function $f(x) = \sin x$ is one-to-one on any one of the intervals $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{2}]$, $\dots$. But we choose $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to satisfy the relationship $\sin^{-1}(1/x) = \csc^{-1} x$.

Definition. (The arcsine function)

For any $x \in [-1, 1]$, $y = \sin^{-1} x$ is the number $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $\sin y = x$.

Example. Find $\sin^{-1}(0)$.

Solution.

Remarks.

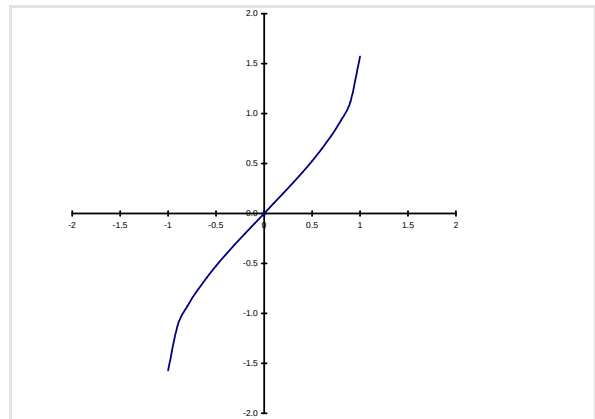
$$(1) \sin^{-1} x \neq \frac{1}{\sin x}.$$

$$(2) \frac{1}{\sin x} = (\sin x)^{-1} = \csc x.$$

The graph of $y = \sin^{-1} x$

The domain of $\sin^{-1} x$ is $[-1, 1]$.

The range of $\sin^{-1} x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.



Remark. The graph of $y = \sin^{-1} x$ is symmetric about the origin. Therefore $y = \sin^{-1} x$ is an odd function; that is

$$\sin^{-1}(-x) = -\sin^{-1} x.$$

Example 1. Find $\sin^{-1} \left(\frac{-1}{\sqrt{2}} \right)$.

Solution. $\sin^{-1}(-1/\sqrt{2}) = -\sin^{-1}(1/\sqrt{2}) = -\pi/4$.

Example 2. Find the value of $\tan \left(\sin^{-1} \left(\frac{3}{7} \right) \right)$.

Solution. $3/\sqrt{40}$

Remarks.

(a) $\sin^{-1}(\sin x) = x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(b) $\sin(\sin^{-1} x) = x$ for $x \in [-1, 1]$.

Example. Find $\sin(\sin^{-1} 2)$ and $\sin^{-1}(\sin 2\pi)$.

Solution. $\sin(\sin^{-1} 2)$ is not defined.

$$\sin^{-1}(\sin 2\pi) = \sin^{-1} 0 = 0.$$

The arccosine function

The function $f(x) = \cos x$ is one-to-one on the interval $[0, \pi]$.

Definition. (The arccosine function)

For any $x \in [-1, 1]$, $y = \cos^{-1} x$ is the number $y \in [0, \pi]$ for which $\cos y = x$.

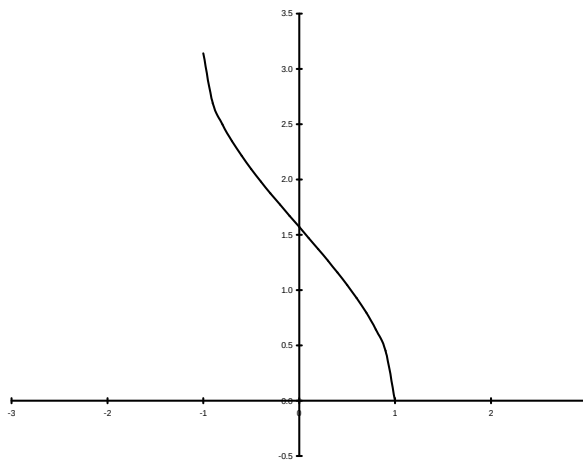
Example. Find $\cos^{-1} \left(\frac{1}{2} \right)$.

Solution. $\pi/3$.

The graph of $y = \cos^{-1} x$

The domain of $\cos^{-1} x$ is $[-1, 1]$.

The range of $\cos^{-1} x$ is $[0, \pi]$.



Remarks.

- (a) The function $\cos^{-1} x$ is neither even nor odd.
- (b) $\cos^{-1}(\cos x) = x$ for $x \in [0, \pi]$.
- (c) $\cos(\cos^{-1} x) = x$ for $x \in [-1, 1]$.

Example. Find $\cos^{-1}(\cos(\pi/2))$ and $\cos^{-1}(\cos 5\pi)$.

Solution. $\cos^{-1}(\cos(\pi/2)) = \cos^{-1} 0 = \pi/2$.
 $\cos^{-1}(\cos 5\pi) = \cos^{-1}(-1) = \pi$.

Identities involving arcsine and arccosine

For all $x \in [-1, 1]$, we have

$$(1) \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$(2) \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Proof. (1) We know that $\cos(\pi - \theta) = -\cos \theta$. Thus if $x = \cos \theta$, then $\cos(\pi - \theta) = -x$.

$$\Rightarrow \pi - \theta = \cos^{-1}(-x).$$

Now substitute $\theta = \cos^{-1} x$, we get the result.

(2) Assume $x > 0$. Then $\theta = \sin^{-1} x \in (0, \pi/2)$ and $\phi = \cos^{-1} x \in (0, \pi/2)$.

Thus from the triangle we have $\theta + \phi = \pi/2$.

□

Example 1. Find $\cos^{-1}(-\sqrt{3}/2)$.

Solution. $\cos^{-1}(-\sqrt{3}/2) = \pi - \cos^{-1}(\sqrt{3}/2) = \pi - \pi/6 = 5\pi/6$.

Example 2. Find the value of $2^{\log_4 \pi^2} - \cos^{-1}(-1/\sqrt{2})$.

Solution. $2^{\log_4 \pi^2} - \cos^{-1}(-1/\sqrt{2}) = (\sqrt{4})^{2\log_4 \pi} - [\pi - \cos^{-1}(1/\sqrt{2})]$
 $= 4^{\log_4 \pi} - \pi + \pi/4 = \pi/4$.

Example 3. Solve for x : $\ln(e^{2x} + 2e^x + 1) = 2 \sin[\cos^{-1}(-x) - \sin^{-1} x]$.

Solution.

$$\begin{aligned} \ln(e^{2x} + 2e^x + 1) &= 2 \sin[\cos^{-1}(-x) - \sin^{-1} x] \\ \Rightarrow \ln(e^x + 1)^2 &= 2 \sin[\cos^{-1}(-x) + \sin^{-1}(-x)] \\ &\Rightarrow 2 \ln(e^x + 1) = 2 \sin \frac{\pi}{2} \\ &\Rightarrow e^x + 1 = e \Rightarrow x = \ln(e - 1) \end{aligned}$$

Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The function $\tan x$ is one-to-one on the interval $(-\pi/2, \pi/2)$ and the function $\cot x$ is one-to-one on the interval $(0, \pi)$. Thus they have inverses $\tan^{-1} x$ and $\cot^{-1} x$.

Definition. (The arctangent and arccotangent functions)

- (1) For any $x \in \mathbb{R}$, $y = \tan^{-1} x$ is the number $y \in (-\pi/2, \pi/2)$ for which $\tan y = x$.
- (2) For any $x \in \mathbb{R}$, $y = \cot^{-1} x$ is the number $y \in (0, \pi)$ for which $\cot y = x$.

Remarks.

- (1) The domain of $\tan^{-1} x$ is $\mathbb{R}$ and domain of $\cot^{-1} x$ is $\mathbb{R}$.
- (2) The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$ and the range of $\cot^{-1} x$ is $(0, \pi)$.
- (3) The graph of $y = \tan^{-1} x$ is symmetric about the origin. Therefore $y = \tan^{-1} x$ is an odd function; that is

$$\tan^{-1}(-x) = -\tan^{-1} x.$$

Example 1. (Exam)

Simplify the expression $\cos(\tan^{-1}(3/x))$.

Solution. Let $\theta = \tan^{-1}(3/x)$. This implies $\tan \theta = 3/x$ and hence $\cos \theta = \frac{x}{\sqrt{x^2 + 9}}$.

Example 2. Evaluate $\lim_{x \rightarrow \infty} \tan^{-1} x$.

Solution. From the graph, we have $\lim_{x \rightarrow \infty} \tan^{-1} x = \pi/2$.

Definition. (The arcsecant and arccosecant functions)

- (1) For any $x \in (-\infty, -1] \cup [1, \infty)$, $y = \sec^{-1} x$ is the number $y \in [0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.
- (2) For any $x \in (-\infty, -1] \cup [1, \infty)$, $y = \csc^{-1} x$ is the number $y \in [-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

Useful Identities

- (1) $\cot^{-1}(x) = \pi/2 - \tan^{-1} x$.
- (2) $\sec^{-1} x = \cos^{-1}(1/x)$.
- (3) $\csc^{-1} x = \sin^{-1}(1/x)$.

Remark. In general $\cot^{-1}(x) \neq \tan^{-1}(1/x)$. We have the following identity

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}(1/x), & x > 0, \\ \tan^{-1}(1/x) + \pi, & x < 0. \end{cases}$$

Example 1. Find $\sec^{-1} 2$.

Solution. $\sec^{-1} 2 = \cos^{-1}(1/2) = \pi/3$.

Example 2. Find $\sin \left(\cos^{-1}(1/3) + \csc^{-1} 3 \right)$.

Solution. $\sin \left(\cos^{-1}(1/3) + \csc^{-1} 3 \right) = \sin \left(\cos^{-1}(1/3) + \sin^{-1}(1/3) \right) = \sin(\pi/2) = 1$.

Example 3. Find $\csc \left(\tan^{-1} \frac{\sqrt{4-x^2}}{x} \right)$.

Solution. Let $\theta = \tan^{-1} \frac{\sqrt{4-x^2}}{x}$. Then $\tan \theta = \frac{\sqrt{4-x^2}}{x}$ and hence $\csc \left(\tan^{-1} \frac{\sqrt{4-x^2}}{x} \right) = \csc \theta = \frac{2}{\sqrt{4-x^2}}$.

Example 4. (Exam)

Solve for x : $\log_5(25)^x + \tan \left(\sin^{-1} \left(\frac{2x}{\sqrt{4x^2+1}} \right) \right) = 4 \cos(\sec^{-1} x)$

Solution. First of all, $\log_5(25)^x = \ln(25)^x / \ln 5 = 2x$.

Next, if $\theta = \sin^{-1} \left(\frac{2x}{\sqrt{4x^2+1}} \right)$, then $\sin \theta = \frac{2x}{\sqrt{4x^2+1}}$ and as a result

$$\tan \left(\sin^{-1} \left(\frac{2x}{\sqrt{4x^2+1}} \right) \right) = \tan \theta = 2x.$$

Moreover, $\cos(\sec^{-1} x) = \cos(\cos^{-1}(1/x)) = 1/x$.

Therefore, the equation becomes

$$2x + 2x = 4/x \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

Example 5. Evaluate $\lim_{x \rightarrow -\infty} \sec^{-1} x$.

Solution. $\lim_{x \rightarrow -\infty} \sec^{-1} x = \lim_{x \rightarrow -\infty} \cos^{-1}(1/x) = \cos^{-1}(0) = \pi/2$.

The derivatives of inverse trigonometric functions

Inverse trigonometric functions provide antiderivatives for a verity of functions.

The derivatives of $y = \sin^{-1} x$ and $y = \cos^{-1} x$

$f(x) = \sin x$ is differentiable on the interval $(-\pi/2, \pi/2)$ and $f'(x) = \cos x > 0$ in this interval. Therefore, $f^{-1}(x) = \sin^{-1} x$ is differentiable on the interval $(-1, 1)$ and

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{f'(\sin^{-1} x)} = \frac{1}{\cos(\sin^{-1} x)}.$$

Using $\cos y = \sqrt{1 - \sin^2 y}$ and $\sin(\sin^{-1} x) = x$, we get

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

and hence

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

Since $\cos^{-1} x = \pi/2 - \sin^{-1} x$, we have

$$\frac{d}{dx} \cos^{-1} x = -\frac{d}{dx} \sin^{-1} x = \frac{-1}{\sqrt{1-x^2}}, \quad |x| < 1$$

Example 1. Find y' if $y = \sin^{-1}(e^{x^2} + 3x)$.

Solution.

Example 2. Find y' if $y = 9^{\sin^{-1}(3x)} + \cos^{-1}(x^2)$.

Solution.

Example 3. Find $\int \frac{dx}{\sqrt{25-x^2}}$.

Solution. Note that $\sqrt{25-x^2} = 5\sqrt{1-\left(\frac{x}{5}\right)^2}$.

Using $u = \frac{x}{5}$, we find

$$\int \frac{dx}{\sqrt{25-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} \left(\frac{x}{5} \right) + C.$$

Example 4. Find $\int \frac{dx}{\sqrt{6x-x^2}}$.

Solution. Note that $\sqrt{6x-x^2} = \sqrt{9-(x-3)^2}$.

Using $u = \frac{x-3}{3}$, we find

$$\int \frac{dx}{\sqrt{6x-x^2}} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} \left(\frac{x-3}{3} \right) + C.$$

Example 5. Find $\int_{1/2}^1 \frac{dx}{\sqrt{3+4x-4x^2}}$.

Solution. Since $\sqrt{3+4x-4x^2} = 2\sqrt{1-(x-\frac{1}{2})^2}$, the substitution $u = x - \frac{1}{2}$ transforms the integral into

$$\int_{1/2}^1 \frac{dx}{\sqrt{3+4x-4x^2}} = \int_{u=0}^{u=1/2} \frac{du}{2\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u \Big|_{u=0}^{u=1/2} = \pi/12.$$

Example 6. Find $\int \frac{dy}{\sin^{-1} y \sqrt{1-y^2}}$.

Solution. Using the substitution $u = \sin^{-1} y$ we find

$$\int \frac{dy}{\sqrt{1-y^2} \sin^{-1} y} = \int \frac{du}{u} = \ln |\sin^{-1} y| + C.$$

The derivatives of $y = \sec^{-1} x$ and $y = \csc^{-1} x$

Since $\sec^{-1} x = \cos^{-1}(1/x)$, we have

$$\frac{d}{dx} \sec^{-1} x = \frac{d}{dx} \cos^{-1}(1/x) = \frac{-1}{\sqrt{1-x^{-2}}} \left(\frac{-1}{x^2} \right).$$

Simplifying we get

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

and hence

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} |x| + C$$

Since $\csc^{-1} x = \pi/2 - \sec^{-1} x$, we have

$$\frac{d}{dx} \csc^{-1} x = -\frac{d}{dx} \sec^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

Example 1. Find y' if $y = \sec^{-1}(2x^2 - x)$.

Solution.

Example 2. (Exam)

Find y' if $y = 3^x + \sec^{-1}(3x)$.

Solution.

Example 3. Find $\int_1^2 \frac{dx}{x\sqrt{4x^2-1}}$.

Solution. Let $u = 2x$. Then

$$\int_1^2 \frac{dx}{x\sqrt{4x^2-1}} = \int_2^4 \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| \Big|_2^4 = \sec^{-1} 4 - \sec^{-1} 2.$$

Example 4. Find $\int \frac{dx}{(x-4)\sqrt{x^2-8x+7}}$.

Solution. Since $x^2 - 8x + 7 = (x-4)^2 - 9$, using $u = \frac{x-4}{3}$, we obtain

$$\int \frac{dx}{(x-4)\sqrt{x^2-8x+7}} = \int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{3} \sec^{-1} \left| \frac{x-4}{3} \right| + C.$$

Example 5. Find $\lim_{x \rightarrow 1^+} \frac{\sec^{-1} x}{\sqrt{x^2-1}}$.

Solution. Using L'Hopital's rule, we find

$$\lim_{x \rightarrow 1^+} \frac{\sec^{-1} x}{\sqrt{x^2-1}} = \lim_{x \rightarrow 1^+} \frac{\frac{1}{|x|\sqrt{x^2-1}}}{\frac{x}{\sqrt{x^2-1}}} = \lim_{x \rightarrow 1^+} \frac{1}{x|x|} = 1.$$

The derivatives of $y=\tan^{-1} x$ and $y=\cot^{-1} x$

To find $\frac{d}{dx} \tan^{-1} x$ we can use the theorem in section 7.1. Alternatively we may process as follows:

$$y = \tan^{-1} x \Rightarrow \tan y = x.$$

Differentiating both sides we get

$$\sec^2 y \frac{dy}{dx} = 1.$$

Thus

$$\frac{dy}{dx} = \frac{d}{dx} \tan^{-1} x = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2(\tan^{-1} x)}.$$

Therefore we have the following formula

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2},$$

and hence

$$\int \frac{1}{1 + x^2} dx = \tan^{-1} x + C$$

Since $\cot^{-1} x = \pi/2 - \tan^{-1} x$, we have

$$\frac{d}{dx} \cot^{-1} x = -\frac{d}{dx} \tan^{-1} x = \frac{-1}{x^2 + 1},$$

Example 1. Find y' if $y = \log_5(\tan^{-1}(5x))$.

Solution.

Example 2. Find y' if $y = x^{\cot^{-1} x} + \sin(\tan^{-1}(x^2))$.

Solution.

Example 3. Find $\int_1^2 \frac{\cot^{-1} x}{x^2 + 1} dx$.

Solution. Using $u = \cot^{-1} x$, we find

$$\int_1^2 \frac{\cot^{-1} x}{x^2 + 1} dx = -\int_1^2 u du = -\frac{1}{2} (\cot^{-1} x)^2 \Big|_1^2.$$

Example 4. Find $\int \frac{dx}{4x^2 + 10x + 7}$.

Solution. We have $4x^2 + 10x + 7 = \left(2x + \frac{5}{2}\right)^2 + \frac{3}{4} = \frac{3}{4} \left[\left(\frac{4x+5}{\sqrt{3}}\right)^2 + 1\right]$.

Thus using $u = \frac{4x+5}{\sqrt{3}}$ we get

$$\int \frac{dx}{4x^2 + 10x + 7} = \int \frac{du}{\sqrt{3}(u^2 + 1)} = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{4x+5}{\sqrt{3}} \right) + C.$$

Example 5. Find $\int \frac{x^2 dx}{1+x^6}$.

Solution. Let $u = x^3$. Then $du = 3x^2 dx$ and hence

$$\int \frac{x^2 dx}{1+x^6} = \int \frac{du}{3(1+u^2)} = \frac{1}{3} \tan^{-1} x^3 + C.$$

Example 6. Find $\lim_{x \rightarrow 0} \frac{\tan^{-1}(4x)}{x}$.

Solution. Using L'Hopital's rule, we find

$$\lim_{x \rightarrow 0} \frac{\tan^{-1}(4x)}{x} = \lim_{x \rightarrow 0} \frac{4}{16x^2 + 1} = 4.$$

7.7. Hyperbolic Functions

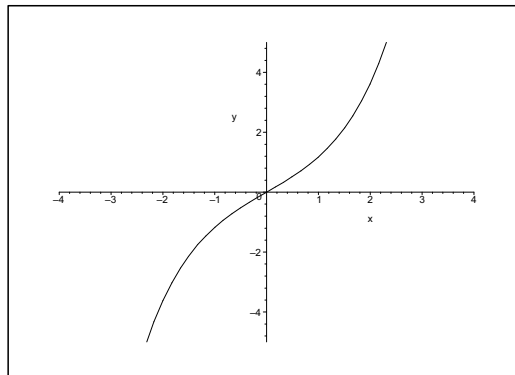
The hyperbolic functions are defined as combinations of the exponential functions e^x and e^{-x} . They simplify many mathematical expressions and occur frequently in mathematical applications.

Definition. (The six basic hyperbolic functions)

(1) Hyperbolic sine of x : $\sinh x = \frac{e^x - e^{-x}}{2}$.

Domain: $\mathbb{R}$

Range: $\mathbb{R}$



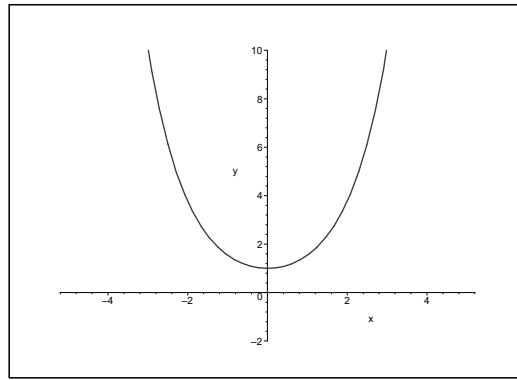
(3) Hyperbolic tangent of x : $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

(2) Hyperbolic cosine of x : $\cosh x = \frac{e^x + e^{-x}}{2}$.

(4) Hyperbolic cotangent of x : $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.

Domain: $\mathbb{R}$

Range: $[1, \infty)$



(5) Hyperbolic secant of x : $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}.$

(6) Hyperbolic cosecant of x : $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$

Remarks.

(1) $\sinh x$ is an odd function.

(2) $\cosh x$ is an even function.

Identities for hyperbolic functions

(1) $\cosh^2 x - \sinh^2 x = 1.$

(2) $\sinh(2x) = 2 \sinh x \cosh x.$

(3) $\cosh(2x) = \cosh^2 x + \sinh^2 x.$

(4) $\cosh^2 x = \frac{\cosh(2x) + 1}{2}.$

Example 1. Simplify $\cosh(2x) + \sinh(2x).$

Solution.

Example 2. Simplify $\tanh(2 \ln x).$

Solution.

Example 3. If $\sinh x = 4/3$, then find the value of the other five hyperbolic functions.

Solution. $\sinh x = 4/3 \Rightarrow e^x - e^{-x} = 8/3 \Rightarrow e^x = 3 \Rightarrow \cosh x = 5/3.$

OR $\cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{25/9}$

Derivatives and Integrals

(1)	$\frac{d}{dx} \sinh x = \cosh x$	$\int \cosh(kx) dx = \frac{1}{k} \sinh(kx) + C$
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(2)	$\frac{d}{dx} \cosh x = \sinh x$	$\int \sinh(kx) dx = \frac{1}{k} \cosh(kx) + C$
-----	----------------------------------	---

(3)	$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$	$\int \operatorname{sech}^2 x dx = \tanh x + C$
-----	--	---

(4)	$\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$	$\int \operatorname{csch}^2 x dx = -\coth x + C$
-----	---	--

(5)	$\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$	$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$
-----	---	--

(6)	$\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$	$\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + C$
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Example 1. (Exam)

Find y' if $y = x^x + \coth x$.

Solution.

Example 2. Find y' if $y = \ln(\cosh x^2)$.

Solution.

Example 3. Find $\int_4^9 \frac{\sinh \sqrt{x}}{\sqrt{x}} dx$.

Solution.

Example 4. Find $\int \cosh^2 x dx$.

Solution.

Inverse Hyperbolic Functions

- (1) Since $\frac{d}{dx} \sinh x = \cosh x > 0$ for all $x \in \mathbb{R}$, $\sinh x$ is an increasing function on $\mathbb{R}$. Thus it has an inverse. We denote this inverse by $\sinh^{-1} x$.

Domain of $\sinh^{-1} x$ is $\mathbb{R}$ and its range is $\mathbb{R}$.

- (2) $\cosh x$ is not one-to-one on its domain, but it becomes one-to-one if we take the restriction $\cosh x, \quad x \in [0, \infty)$. Thus it has an inverse $\cosh^{-1} x$.

Domain of $\cosh^{-1} x$ is $[1, \infty)$ and its range is $[0, \infty)$.

- (3) $\tanh x$ is one-to-one on $\mathbb{R}$, and hence it has an inverse $\tanh^{-1} x$.

Domain of $\tanh^{-1} x$ is $(-1, 1)$ and its range is $\mathbb{R}$.

$$(4) \quad \operatorname{sech}^{-1} x = \cosh^{-1} \left(\frac{1}{x} \right).$$

$$(5) \quad \operatorname{csch}^{-1} x = \sinh^{-1} \left(\frac{1}{x} \right).$$

$$(6) \quad \operatorname{coth}^{-1} x = \tanh^{-1} \left(\frac{1}{x} \right).$$

Evaluating inverse hyperbolic functions

Let us find a formula to evaluate $\sinh^{-1} x$.

$$y = \sinh^{-1} x \Rightarrow \sinh y = x$$

$$\Rightarrow \frac{e^y - e^{-y}}{2} = x$$

$$\Rightarrow (e^y - x)^2 = x^2 + 1$$

$$\Rightarrow e^y - x = \pm \sqrt{x^2 + 1}$$

But $e^y > 0$ for all $y \Rightarrow$

$$e^y = x + \sqrt{x^2 + 1}$$

and hence

$$y = \sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right).$$

Similarly,

$$\cosh^{-1} x = \ln \left(x + \sqrt{x^2 - 1} \right).$$

Example. Find $\cosh^{-1}(7/4)$.

Solution.

Derivatives and Integrals

$$(1) \quad \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$$

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C$$

$$(2) \quad \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1.$$

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + C$$

$$(3) \quad \frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}, \quad |x| < 1.$$

$$\int \frac{dx}{1 - x^2} = \tanh^{-1} x + C$$

Example 1. Find $\frac{d}{dx} \sinh^{-1}(5^x)$.

Solution.

Example 2. Find $\int \frac{dx}{\sqrt{1 + 16x^2}}$.

Solution. Let $u = 4x$. Then $du = 4dx$ and hence

$$\int \frac{dx}{\sqrt{1 + 16x^2}} = \int \frac{du}{4\sqrt{1 + u^2}} = \sinh^{-1}(4x) + C.$$

Example 3. (Exam)

Solve for x : $\cosh^{-1}(2x) = \sec^2(\cot^{-1}(2/\pi)) - 4^{\log_2 \sqrt{\frac{\pi^2 + 4}{4}}}$

Solution. Let $\theta = \cot^{-1}(2/\pi)$. Then $\cot \theta = 2/\pi$ and hence $\sec \theta = \sqrt{\pi^2 + 4}/2$. Therefore

$$\begin{aligned} \cosh^{-1}(2x) &= \sec^2(\cot^{-1}(2/\pi)) - 4^{\log_2 \sqrt{\frac{\pi^2 + 4}{4}}} \\ \Rightarrow \cosh^{-1}(2x) &= \sec^2 \theta - 2^{\log_2 \frac{\pi^2 + 4}{4}} = \frac{\pi^2 + 4}{4} - \frac{\pi^2 + 4}{4} = 0. \end{aligned}$$

Thus we have

$$2x = \cosh(0) = 1.$$

7.8. Relative Rates of Growth

It is often important in mathematics, computer science, and engineering to compare the rates at which functions of x grow as x becomes large. Exponential functions are important in these comparisons because of their very fast growth, and logarithmic functions because of their very slow growth. In this section we show how to compare the rates at which functions of x grow as $x \rightarrow \infty$. We restrict our attention to functions whose values eventually become and remain positive as $x \rightarrow \infty$.

Definition. (Rates of growth as $x \rightarrow \infty$)

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

- (1) If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$, then we say that $f(x)$ grows faster than $g(x)$ as $x \rightarrow \infty$.
- (2) If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$, then we say that $f(x)$ grows slower than $g(x)$ as $x \rightarrow \infty$.
- (3) If $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L > 0$, then we say that $f(x)$ and $g(x)$ grow at the same rate as $x \rightarrow \infty$.

Example 1. Show that e^{5x} grows faster than $x^2 + 5x + 4$ as $x \rightarrow \infty$.

Solution.

Example 2. Determine whether $f(x) = \ln x$ grows faster, slower, or at the same rate as $g(x) = \log_{10} x$ as $x \rightarrow \infty$.

Solution.

Example 3. Let $f(x) = \sqrt{x}$ and $g(x) = \ln(x^2 + 1)$. Does $f(x)$ grow faster, slower, or at the same rate as $g(x)$ as $x \rightarrow \infty$?

Solution.