

Chapter 8

Techniques of Integration

The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x) dx$$

is equivalent to finding a function F such that $F'(x) = f(x)$, and then adding an arbitrary constant C :

$$\int f(x) dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to improper integrals for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

8.1 Integration By Parts

Integration by parts is a rule that transforms the integral of products of functions,

$$\int f(x)g'(x) dx,$$

into other (ideally simpler) integrals. It is useful when $f(x)$ can be differentiated repeatedly and $g'(x)$ can be integrated repeatedly.

The rule arises from the product rule of differentiation

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$$

Integrating both sides gives

$$f(x)g(x) = \int f(x)g'(x) dx + \int f'(x)g(x) dx.$$

Rearranging terms, we get

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

If $u = f(x)$, $v = g(x)$, then using the differentials $du = f'(x)dx$ and $dv = g'(x)dx$, the above formula can be written in the differential form

$$\int u dv = uv - \int v du.$$

Given a single function to integrate, the typical strategy is to carefully separate it into a product of two functions $u = f(x)$ and $\frac{dv}{dx} = g'(x)$ such that the integral produced by the integration by parts formula is easier to evaluate than the original one. It is useful to choose $f(x)$ as a function that simplifies when differentiated, and/or to choose $g'(x)$ as a function that simplifies when integrated.

Integrals involving $\ln x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\csc^{-1}x$

Integrals involving one of the seven functions $\ln x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\csc^{-1}x$ can be evaluated by using substitution or by parts. In both methods we choose the function that appear in the integral as u .

Example 1. Evaluate $\int \ln x dx$.

Solution. We will use integration by parts. We choose u and dv as follows:

$$\begin{array}{cc} u = \ln x & dv = dx \\ & \searrow \swarrow \\ du = \frac{dx}{x} & v = x \end{array}$$

$$\begin{array}{cc} u = \ln x & du = \frac{dx}{x} \\ & \searrow \swarrow \\ dv = dx & v = x \end{array}$$

Example 2. Evaluate $\int_1^e \frac{\ln x}{x^2} dx$.

Solution.

Example 3. Evaluate $\int x \tan^{-1} x \, dx$.

Solution.

The integrals $\int x^n e^{ax} dx, \int x^n \cos(ax) dx, \int x^n \sin(ax) dx$

The integrals $\int x^n e^{ax} dx, \int x^n \cos(ax) dx, \int x^n \sin(ax) dx$ in which n is a positive integer and a is any real number, can be evaluated by parts using $u = x^n$. Differentiating u will eventually lead to zero and integrating is easy.

Example 1. Evaluate $\int x \sin(3x) \, dx$.

Solution.

Example 2. Evaluate $\int x^3 \cos(x) \, dx$.

Solution.

Tabular integration by parts

Derivatives of u	Integrals of (dv/dx)
x^3	$\cos x$
$3x^2$	$\sin x$
$6x$	$-\cos x$
6	$-\sin x$
0	$\cos x$

This leads to the result

$$\int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C.$$

Example 3. Evaluate $\int x^2 e^{-5x} \, dx$.

Solution.

Example 4. Evaluate $\int (2x^3 + 6x - 3) \cosh(2x) \, dx$.

Solution.

The integrals $\int e^{ax} \sin(bx) dx$, $\int e^{ax} \cos(bx) dx$

The integrals $\int e^{ax} \sin(bx) dx$ and $\int e^{ax} \cos(bx) dx$, where a and b are real numbers, occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

Example 1. Evaluate $\int e^x \cos x dx$.

Solution. With proper understanding of the tabular method, it can be extended.

Derivatives of u	Integrals of (dv/dx)
e^x	$\cos x$
e^x	$\sin x$
e^x	$-\cos x$

In this case in the last step it is necessary to integrate the product of the two bottom cells obtaining:

$$2 \int e^x \cos x dx = e^x \sin x + e^x \cos x,$$

and yields the result:

$$\int e^x \cos x dx = \frac{e^x(\sin x + \cos x)}{2} + C$$

Example 2. Evaluate $\int 2^x \sin(4x) dx$.

Solution.

Other examples

Integration by parts can be used to evaluate several integrals in addition to the integrals considered above.

Example 1. Evaluate $\int x \operatorname{sech}^2 x dx$.

Solution.

Example 2. Evaluate $\int \cos(3 \ln x) dx$.

Solution.

Example 3. Evaluate $\int \frac{1}{\sqrt{x}} \sin^{-1} \sqrt{x} dx$.

Solution.

Example 4. Evaluate $\int (\ln x)^2 dx$.

Solution.

Example 5. Evaluate $\int \left(\frac{1-x}{x^2} \right) e^x dx$.

Solution. Use $u = (1-x)e^x$, $dv = \frac{1}{x^2} dx$.

8.2. Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principal, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions.

Products of Powers of Sines and Cosines

Consider integral of the form

$$\int \cos^m x \sin^n x \, dx,$$

where m and n are nonnegative integers.

Case(1) If m is odd, then use the substitution

$$u = \sin x.$$

Case(2) If n is odd, then use the substitution

$$u = \cos x.$$

Case(3) If both n and m are even, then use

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \quad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

Example 1. Evaluate $\int \cos^4 x \sin^3 x \, dx$.

Solution.

Example 2. Evaluate $\int \cos^7 x \sin^5 x \, dx$.

Solution.

Example 3. Evaluate $\int \cos^4 x \sin^2 x \, dx$.

Solution.

Powers of $\tan x$ and $\sec x$

To integrate powers of $\tan x$ and $\sec x$ we use the identities

$$\tan^2 x = \sec^2 x - 1,$$

and integrate by parts when necessary to reduce the higher powers to lower powers.

Example 1. Evaluate $\int \tan^4 x \, dx$.

Solution. $\int \tan^4 x \, dx = \int \tan^2 x (\sec^2 x - 1) dx = \int (\tan^2 x \sec^2 x - \sec^2 x + 1) dx$

Example 2. Evaluate $\int \sec^3 x \, dx$.

Solution. Use integration by parts: $u = \sec x$, $dv = \sec^2 x dx$

Example 3. Evaluate $\int \csc^5 x \, dx$.

Solution.

Products of $\sin(ax)$ and $\cos(bx)$

Integrals of the forms $\int \cos(ax) \sin(bx) dx$, $\int \sin(ax) \sin(bx) dx$, $\int \cos(ax) \cos(bx) dx$, where a and b are real numbers with $a \neq b$, can be evaluated through integration by parts. However it is simpler to use the identities

$$\sin(ax) \sin(bx) = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x],$$

$$\cos(ax) \cos(bx) = \frac{1}{2}[\cos(a-b)x + \cos(a+b)x],$$

$$\cos(ax) \sin(bx) = \frac{1}{2}[\sin(a+b)x - \sin(a-b)x].$$

Example 1. Evaluate $\int \cos(5x) \sin(2x) \, dx$.

Solution.

Example 2. Evaluate $\int \cos(3x) \cos(2x) \, dx$.

Solution.

Eliminating Square Roots

Example. Evaluate $\int_0^\pi \sqrt{1 - \cos^2(x)} \, dx$.

Solution.

8.3. Trigonometric Substitutions

The method of trigonometric substitution is used to evaluate integrals that involve one of the expressions

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}, \quad \text{or} \quad \sqrt{a^2 + x^2}.$$

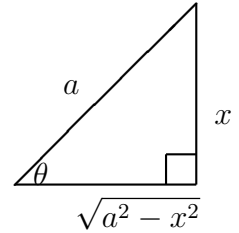
Integral containing $\sqrt{a^2 - x^2}$

If an integral contains the expression $\sqrt{a^2 - x^2}$, then we can use the substitution

$$x = a \sin \theta, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$dx = a \cos \theta \, d\theta,$$

$$\sqrt{a^2 - x^2} = a \cos \theta.$$



Example 1. Evaluate $\int \frac{x^2}{\sqrt{16 - x^2}} \, dx$.

Solution.

Example 2. Evaluate $\int_0^1 \frac{1}{(9 - x^2)^{3/2}} \, dx$.
 Note that $x = a \sin \theta$ requires $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ with $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution. $1/18\sqrt{2}$

Integrals containing $\sqrt{x^2 - a^2}$

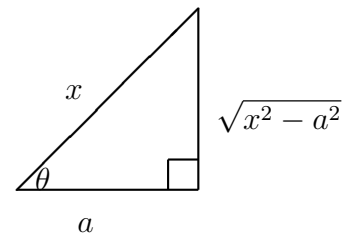
Note that $x = a \sec \theta$ implies $\theta = \sec^{-1}\left(\frac{x}{a}\right) \in \begin{cases} \left[0, \frac{\pi}{2}\right), & \frac{x}{a} \geq 1; \\ \left(\frac{\pi}{2}, \pi\right], & \frac{x}{a} \leq -1. \end{cases}$

To simplify the calculations we assume $x \geq a$ and use the substitution

$$x = a \sec \theta, \quad \theta \in \left[0, \frac{\pi}{2}\right)$$

$$dx = a \sec \theta \tan \theta \, d\theta,$$

$$\sqrt{x^2 - a^2} = a \tan \theta.$$



Example 1. Evaluate $\int \frac{1}{x^2 \sqrt{x^2 - 4}} \, dx$.

Solution.

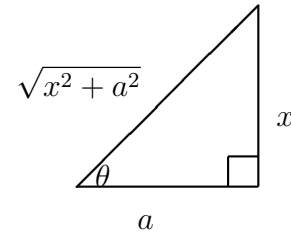
Example 2. Evaluate $\int_3^4 \frac{1}{(9x^2 - 25)^{1/2}} \, dx$.

Solution.

Integrals containing $\sqrt{a^2 + x^2}$

If an integral involves the expression $\sqrt{a^2 + x^2}$, then it can be simplified by using the substitution

$$\begin{aligned} x &= a \tan \theta, & \theta &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ dx &= a \sec^2 \theta d\theta, \\ \sqrt{a^2 + x^2} &= a \sec \theta. \end{aligned}$$



Example 1. Evaluate $\int \frac{1}{(x^2 + 2)^2} dx$.

Solution.

Example 2. Evaluate $\int_1^e \frac{1}{y\sqrt{1 + \ln^2 y}} dy$.
 Note that $x = a \tan \theta$ requires $\theta = \tan^{-1} \left(\frac{x}{a} \right)$ with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. $\ln(1 + \sqrt{2})$

8.4. Integration of Rational Functions by Partial Fractions

Our aim in this section is to integrate rational functions,

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}.$$

Any rational function of a real variable can be written as the sum of a polynomial function and a finite number of partial fractions. Each partial fraction has as its denominator a polynomial function of degree 1 or 2, or some positive integer power of such a function. If the denominator is a 1st-degree polynomial or a power of such a polynomial, then the numerator is a constant. If the denominator is a 2nd-degree polynomial or a power of such a polynomial, then the numerator is a 1st-degree polynomial.

Success in writing a rational function $f(x) = \frac{P(x)}{Q(x)}$ as a sum of partial fractions depends on two things:

- (1) The degree of $P(x)$ must be less than the degree of $Q(x)$. That is, the fraction must be proper. If it is not, divide $P(x)$ by $Q(x)$ and work with the remainder term.
- (2) We must know the factors of $Q(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

The Method of Partial Fractions

- (1) Write $f(x)$ in proper form. That is, use long division to write $f(x)$ as a polynomial plus a fraction $\frac{P(x)}{Q(x)}$ with degree of $P(x)$ less than the degree $Q(x)$.
- (2) Factor the denominator $Q(x)$ into linear factors and irreducible quadratic factors.
- (3) Write out the form of the partial fractions.
- (4) Find the unknown constants in the partial fractions.

Example 1. Find $\int \frac{x^3 - 8x + 22}{x^2 + 3x - 4} dx$.

Solution. $\frac{x^3 - 8x + 22}{x^2 + 3x - 4} = x - 3 + \frac{2}{x + 4} + \frac{3}{x - 1}$.

Example 2. Find $\int \frac{x^2 - 3x + 1}{(x^2 - 7x + 12)(x - 4)} dx$.

Solution. $\frac{x^2 - 3x + 1}{(x - 3)(x - 4)^2} = \frac{1}{x - 3} + \frac{5}{(x - 4)^2}$

Example 3. Find $\int \frac{10x^2 + 12x + 20}{x^3 - 8} dx$.

Solution. $\frac{10x^2 + 12x + 20}{x^3 - 8} = \frac{7}{x - 2} + \frac{3x + 4}{x^2 + 2x + 4}$

Example 4. Find $\int \frac{x + 8}{x^3(x^2 + 4)} dx$.

Solution. $\frac{x + 8}{x^3(x^2 + 4)} = \frac{-1/2}{x} + \frac{1/4}{x^2} + \frac{2}{x^3} + \frac{(1/2)x - 1/4}{x^2 + 4}$

Example 5. Find $\int \frac{11x^2 + x + 81}{x(x^2 + 9)^2} dx$.

Solution. $\frac{11x^2 + x + 81}{x(x^2 + 9)^2} = \frac{1}{x} - \frac{x}{x^2 + 9} + \frac{2x + 1}{(x^2 + 9)^2}$

Example 6. Find $\int x^3 \tan^{-1} x dx$.

Solution.

Example 7. Find $\int \frac{e^{4x} + 2e^{2x} - e^x}{e^{2x} + 4e^x + 3} dx$.

Solution.

Example 8. Find $\int \frac{1}{x\sqrt{x+1}} dx$.

Solution.

Example 9. Find $\int \frac{1}{x(x^{2018} + 1)} dx$.

Solution.

8.5 Integral Tables and Computer Algebra Systems

There are extensive tables appear in compilations such as CRC Mathematical Tables, which contain thousands of integrals.

Example 1. Evaluate $\int x\sqrt{x+1} \, dx$.

Solution.

Example 2. Evaluate $\int \sqrt{e^x + 1} \, dx$

Solution. Use $u = \sqrt{e^x + 1}$.

Example 3. Evaluate $\int \sin^{-1}(\sqrt{x}) \, dx$.

Solution. Use integration by parts:

$$u = \sin^{-1}(\sqrt{x}), \quad dv = dx$$

Then use the substitution $\sqrt{x} = \sin \theta$

Example 4. Evaluate $\int_2^8 \frac{\ln x}{(x+1)^2} \, dx$

Solution.

Example 5. Evaluate $\int_1^e \sqrt{x} \ln x \, dx$

Solution.

Example 6. Evaluate $\int \sin^4 x \tan x \, dx$.

Solution.

Example 7. Evaluate $\int_0^{\ln \sqrt{3}} e^{-x} \tan^{-1}(e^x) \, dx$.

Solution.

8.7. Improper Integrals

When we defined definite integrals

$$\int_a^b f(x) \, dx$$

we required two things. First, we required that the interval of integration $[a, b]$ is closed and finite. Second, we required that the range of $f(x)$ to be finite on $[a, b]$. In this section we will see how to define integrals of unbounded functions and integrals over infinite intervals.

Definition. (Improper integral)

Integrals with infinite limits of integrations and integrals of functions that becomes infinite at a point within the interval of integration are called improper integrals.

Example 1. $\int_1^{\infty} x \, dx$ is an improper integral because the upper limit of integration is $b = \infty$.

Example 2. $\int_0^1 \frac{1}{x} \, dx$ is an improper integral because $f(x) = 1/x$ becomes infinite at $x = 0 \in [0, 1]$.

Example 3. $\int_1^3 \frac{1}{x-2} \, dx$ is an improper integral because $f(x) = 1/(x-2)$ becomes infinite at $x = 2 \in [1, 3]$.

Example 4. $\int_1^2 \frac{1}{x^2} \, dx$ is NOT an improper integral because $f(x) = 1/x^2$ is finite at any $x \in [1, 2]$ and the interval of integration is finite.

Evaluation of improper integrals

In order to understand the main idea let us consider the following example.

Example. Find the area under the curve $y = 1/x^2$ from $x = 1$ to $x = \infty$.

Solution.

Definition. (Basic improper integrals)

(1) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x) \, dx = \lim_{c \rightarrow \infty} \int_a^c f(x) \, dx.$$

(2) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) \, dx = \lim_{c \rightarrow -\infty} \int_c^b f(x) \, dx.$$

(3) If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x) \, dx = \lim_{c \rightarrow a^+} \int_c^b f(x) \, dx.$$

(4) If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x) \, dx = \lim_{c \rightarrow b^-} \int_a^c f(x) \, dx.$$

In each case, if the limit is finite, we say that the improper integral converges and the limit is the value of the improper integral. If the limit is infinite or does not exist, then the improper integral diverges.

Example 1. Evaluate the improper integral $\int_1^{\infty} \frac{1}{x^3} \, dx$.

Solution.

Example 2. Discuss the convergence of $\int_2^{\infty} \frac{1}{x \ln x} \, dx$.

Solution.

Example 3. Test the convergence of $\int_0^1 \frac{1}{\sqrt{x}} dx$.

Solution.

Example 4. Test the convergence of $\int_{-\infty}^{-1} \frac{1}{x(x^2+1)} dx$.

Solution.

Definition.

(1) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx,$$

where a is any real number.

(2) If $f(x)$ is continuous on $[a, x_0) \cup (x_0, b]$ and discontinuous at x_0 , then

$$\int_a^b f(x) dx = \int_a^{x_0} f(x) dx + \int_{x_0}^b f(x) dx.$$

In each case, if both integrals on the right hand side converge, we say that the integral on the left hand side converges. Otherwise it diverges.

Example 1. Evaluate the improper integral $\int_{-\infty}^{\infty} \frac{1}{x^2+4} dx$.

Solution.

Example 2. Test $\int_{-\infty}^{\infty} \cosh(2x) dx$ for convergence.

Solution.

Example 3. Test the convergence of $\int_0^4 \frac{1}{\sqrt{|x-3|}} dx$.

Solution.

Example 4. Test $\int_{-1}^1 \frac{1}{x^2} dx$ for convergence.

Solution.

The integral $\int_1^{\infty} \frac{dx}{x^p}$

The integral $\int_1^{\infty} \frac{dx}{x^p}$ converges to $\frac{1}{p-1}$ if $p > 1$ and diverges if $p \leq 1$.

Remark. If $a > 0$, then

$$\int_a^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x^p} \pm \int_1^a \frac{dx}{x^p}.$$

This implies that $\int_a^{\infty} \frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Tests for convergence and divergence

When an improper integral can not be evaluated directly, we use tests for convergence and divergence to determine whether it converges or diverges.

Theorem 1. (Direct Comparison Test)

Let f and g be continuous on $[a, \infty)$ such that

$$0 \leq f(x) \leq g(x), \quad \text{for all } x \geq a.$$

Then

$$(1) \int_a^\infty g(x)dx \text{ converges} \Rightarrow \int_a^\infty f(x)dx \text{ converges},$$

$$(2) \int_a^\infty f(x)dx \text{ diverges} \Rightarrow \int_a^\infty g(x)dx \text{ diverges}.$$

Example 1. Test $\int_1^\infty \frac{\cos^2(x)}{x^3} dx$ for convergence.

Solution.

Example 2. Test $\int_\pi^\infty \frac{2 + \sin(x)}{x} dx$ for convergence.

Solution.

Example 3. Test $\int_2^\infty \frac{1}{\ln x} dx$ for convergence.

Solution. Use $0 \leq \ln x \leq x$.

Example 4. Test $\int_{-\pi/2}^{\pi/2} \frac{\cos(x)}{(\pi - 2x)^{1/3}} dx$ for convergence.

Solution. Use $0 \leq \cos x \leq 1$.

Example 5. Test $\int_0^1 \frac{dx}{x - \sin x}$ for convergence.

Solution. For $x \in [0, 1]$, $0 \leq x - \sin x \leq x$.

$$\text{Thus } 0 \leq \frac{1}{x} \leq \frac{1}{x - \sin x}.$$

Limit Comparison Test

Theorem 2. (Limit Comparison Test)

Let f and g be positive and continuous on $[a, \infty)$. Then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

implies $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

Example 1. Test $\int_1^{\infty} \frac{1}{\sqrt{x^3 + 2x^2 + 5}} dx$ for convergence.

Solution.

Example 2. Test $\int_2^{\infty} \frac{3x^2 - 1}{x^3 - x^2} dx$ for convergence.

Solution.

Example 3. Test $\int_1^{\infty} \frac{1}{\sqrt{e^{4x} + x}} dx$ for convergence.

Solution.

Example 4. Test $\int_0^{\infty} \frac{1}{e^x + x} dx$ for convergence.

Solution.

Example 5. Test $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx$ for convergence.

Solution. $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx = 2 \int_0^{\infty} \frac{1}{e^x + e^{-x}} dx$ converges.