

Chapter 10

Infinite Sequences and Series

Everyone knows how to add two numbers together, or even several. But how do you add infinitely many numbers together? In this chapter we answer this question, which is a part of the theory of infinite sequences and series.

An important application of this theory is a method of representing a known differentiable function $f(x)$ as an infinite sum of powers of x , so it looks like a “polynomial with infinitely many terms.” Moreover, the method extends our knowledge of how to evaluate, differentiate, and integrate polynomials, so we can work with even more general functions than those encountered so far. These new functions are often solutions of important problems in science and engineering.

10.1. Sequences

Sequences are fundamental to the study of infinite series and many applications of mathematics. We will explore general sequences of numbers and their convergence.

Definition. (Infinite sequence)

An infinite sequence of numbers is a function whose domain is the set of positive integers.

Notation. If we have a sequence $a(n)$, then we will write $a(n) = a_n$ or $\{a_n\}$.

Example 1. $a_n = -4n$ is a sequence.

Example 2. $a_n = 3$ is a sequence.

Example 3. $a_n = (-1)^n$ is a sequence.

Example 4. $a_n = n!$, $n \geq 0$ is a sequence.

Example 5. Find a formula for the n th term of the sequence $1, -4, 9, -16, 25, \dots$.

Solution. $a_n = (-1)^{n+1}n^2$

Convergence and divergence

Definition. (Limit, convergence, and divergence)

Consider a sequence $\{a_n\}$.

- (1) We say that $\{a_n\}$ converges to the number L if to every positive number ϵ there corresponds an integer N such that for all n

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

(2) If $\{a_n\}$ converges to L , then we write

$$\lim_{n \rightarrow \infty} a_n = L$$

and we call L the limit of the sequence.

(3) If no such number L exists, we say that $\{a_n\}$ diverges.

Theorem 1. (Calculating limits)

Let $\{a_n\}$ be a sequence of real numbers and let f defined for all $x \geq n_0$ such that $f(n) = a_n$ for $n \geq n_0$. Then

$$\lim_{n \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} f(x),$$

provided that $\lim_{x \rightarrow \infty} f(x) = L$ or $\pm \infty$

Example 1. Find

$$\lim_{n \rightarrow \infty} \frac{1 - 2n}{3n - 2}.$$

Solution. $-2/3$

Example 2. Determine whether $a_n = \frac{n-1}{1+\sqrt{n}}$ converges or diverges.

Solution. diverges

Example 3. Discuss the convergence of the sequence $a_n = r^n$, $r \in \mathbb{R}$.

Solution. • If $r = 0$, then $a_n = 0$. Thus a_n converges when $r = 0$.

$$\bullet \text{ If } r > 0, \text{ then } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} e^{n \ln r} = \begin{cases} \infty, & \text{if } r > 1, \\ 1, & \text{if } r = 1 \\ 0, & \text{if } 0 < r < 1. \end{cases}$$

$$\bullet \text{ If } r < 0, \text{ then } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n e^{n \ln(-r)} = \begin{cases} \text{does not exist,} & \text{if } r \leq -1, \\ 0, & \text{if } -1 < r < 0. \end{cases}.$$

Therefore $a_n = r^n$ converges if $-1 < r \leq 1$ and diverges otherwise.

Commonly occurring limits

Theorem 2.

$$(1) \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$$

$$(2) \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

$$(3) \lim_{n \rightarrow \infty} x^{1/n} = 1, \quad (x > 0).$$

$$(4) \lim_{n \rightarrow \infty} x^n = 0, \quad (|x| < 1).$$

$$(5) \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x, \quad (x \in \mathbb{R}).$$

$$(6) \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \quad (x \in \mathbb{R}).$$

Proof. (1) $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$

(2)

(3)

(4)

(5)

(6)

□

Example 1. Determine whether $a_n = \left(\frac{n+3}{n-1}\right)^n$ converges or diverges.

Solution. $a_n = \left(\frac{1 + \frac{3}{n}}{1 - \frac{1}{n}}\right)^n$. It follows, by Theorem 2 (part (5)), that $\lim_{n \rightarrow \infty} a_n = \frac{e^3}{e^{-1}} = e^4$.

Example 2. Find the limit of the sequence $a_n = \sqrt[n]{3n^4}$.

Solution. $a_n = \sqrt[n]{3n^4} = \sqrt[n]{3} (\sqrt[n]{n})^4$. It follows, by Theorem 2 (part (2) and part (3)), that $\lim_{n \rightarrow \infty} a_n = 1$.

Recursive definition

Sequences may be defined by given a_n directly. Alternatively sequences may be defined by a rule for calculating any later term from terms that precede it and by giving the necessary initial terms.

Example 1. Let $a_1 = 2$, $a_{n+1} = n - a_n$. Find the first 5 terms of this sequence.

Solution. $a_1 = 2$, $a_2 = -1$, $a_3 = 3$, $a_4 = 0$, $a_5 = 4$

Example 2. Find the first 5 terms of the sequence $a_1 = 1$, $a_2 = 2$, $a_{n+2} = a_n + a_{n+1}$.

Solution. $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 7$

10.2. Infinite Series

Definition. (Infinite Series)

An infinite series is a sum of an infinite sequence of numbers

$$a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k.$$

Example 1. $\sum_{k=1}^{\infty} k = 1 + 2 + 3 \cdots$ is a series.

Example 2. $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \cdots$ is a series.

Definition. (Sequence of partial sums)

Let $\sum_{k=1}^{\infty} a_k$ be a series and define

$$S_1 = a_1, S_2 = a_1 + a_2, \dots, S_n = a_1 + \dots + a_n = \sum_{k=1}^n a_k.$$

Then the sequence $\{S_n\}$ is called the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$.

Example 1. Consider $\sum_{k=1}^{\infty} k$. Find S_1, S_2, S_3 and the n-th partial sum S_n .

Solution. $a_k = k$

$$S_1 = a_1 = 1, S_2 = a_1 + a_2 = 3, \dots, S_n = \frac{n(n+1)}{2}$$

Example 2. Given the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, find S_1, S_2, S_3 .

Solution.

$$a_k = 1/k^2$$

$$S_1 = 1, S_2 = 5/4, S_3 = 49/36$$

Definition. (Convergence and divergence)

Let $\{S_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$.

(1) If $\lim_{n \rightarrow \infty} S_n = L$ exist, then we say that $\sum_{k=1}^{\infty} a_k$ converges and we write $\sum_{k=1}^{\infty} a_k = L$.

(2) If $\lim_{n \rightarrow \infty} S_n$ does not exist, then we say that $\sum_{k=1}^{\infty} a_k$ diverges.

Example 1. Show that $\sum_{k=1}^{\infty} k$ diverges.

Solution. $S_n = \frac{n(n+1)}{2} \rightarrow \infty$

Example 2. (Exam)

Show that the telescoping series $\sum_{k=1}^{\infty} \frac{2}{(2k+1)(2k+3)}$ converges and find its sum.

Solution. $a_k = \frac{1}{2k+1} - \frac{1}{2k+3}$

Geometric Series

A geometric series is a series of the form

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \cdots,$$

where r and c are numbers and $c \neq 0$.

Example 1. $\sum_{n=0}^{\infty} \frac{3^n}{5}$ is a geometric series with $r = 3$ and $c = \frac{1}{5}$.

Example 2. $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$ is geometric series with $c = \frac{-3}{2}$ and $r = \frac{-3}{2}$.

Convergence of Geometric Series

The sequence of partial sums for a geometric series $\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \cdots$ is given by

$$S_n = \begin{cases} \frac{c(1-r^n)}{1-r}, & r \neq 1 \\ nc, & r = 1. \end{cases}$$

Therefore, the geometric series

$$\sum_{n=0}^{\infty} cr^n$$

converges if and only if $|r| < 1$. Moreover, if $|r| < 1$, then $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$.

Proof.

□

Example 1. Find the sum of $\sum_{n=0}^{\infty} \frac{3^{n-1}}{4^n}$.

Solution. $\sum_{n=0}^{\infty} \frac{3^{n-1}}{4^n} = \frac{1}{3} + \frac{1}{4} + \frac{3}{4^2} + \frac{3^2}{4^3} + \cdots = \frac{1}{3} \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots \right]$ converges.

Example 2. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n 6^n}{4^{n+1}}$ converges or diverges. If it converges, then find its sum.

Solution. $\sum_{n=1}^{\infty} \frac{(-1)^n 6^n}{4^{n+1}} = \frac{-3}{8} \left[1 + \left(\frac{-3}{2}\right) + \left(\frac{-3}{2}\right)^2 + \cdots \right]$ diverges.

Example 3. (Repeating decimals)

Express the number $2.\overline{51}$ as the ratio of two integers.

Solution. $2.\overline{51} = 2 + \frac{51}{100} \left[1 + \frac{1}{100} + \cdots \right] = \frac{249}{99}$

Example 4. Find the values of x for which $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converges and find the sum of the series.

Solution. $\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{3}{3-x}$ for $|x| < 3$.

Divergent Series

One reason that a series may fail to converge is that its terms does not become small.

Theorem 3. $(\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow a_n \rightarrow 0)$

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Corollary 1. (The n th-term test for divergence)

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ **diverges**.

Remark. (Caution)

If $\lim_{n \rightarrow \infty} a_n = 0$, then the test fails. That is, the series $\sum_{n=1}^{\infty} a_n$ may converge or may diverge.

Example 1. Show that $\sum_{n=1}^{\infty} \left(1 + \frac{2}{n}\right)^n$ diverges.

Solution. The given series is neither a geometric nor a telescoping series. Thus we can not find S_n and hence we have to use a convergence test. $\lim_{n \rightarrow \infty} a_n = e^2 \neq 0$. Thus the series diverges.

Example 2. Test $\sum_{n=1}^{\infty} \frac{n^2}{n+1}$ for convergence.

Solution. $\lim_{n \rightarrow \infty} a_n = \infty \neq 0$. Thus the series diverges.

Combining series

If we have two convergent series, then we can add or subtract them term by term, or multiply them by constants to obtain new convergent series.

Theorem 4. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

$$(1) \sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$$

$$(2) \text{ For any number } c, \sum_{n=1}^{\infty} ca_n = cA.$$

Example. Show that $\sum_{n=1}^{\infty} \left(\frac{2}{4^n} - \frac{1}{8^{n-1}} \right)$ converges and find its sum.

Solution. $\sum_{n=1}^{\infty} \frac{2}{4^n} = \frac{2}{3}$ and $\sum_{n=1}^{\infty} \frac{1}{8^{n-1}} = \frac{8}{7} \cdot \sum_{n=1}^{\infty} \left(\frac{2}{4^n} - \frac{1}{8^{n-1}} \right) = \frac{-10}{21}$

Adding and deleting terms

Adding or deleting a finite number of terms to a series does not effect the convergence or divergence of the series.

Theorem 5. Let m_1 and m_2 be any integers such that $m_1 < m_2$.

(1) $\sum_{n=m_1}^{\infty} a_n$ converges if and only if $\sum_{n=m_2}^{\infty} a_n$ converges.

(2) If $\sum_{n=m_1}^{\infty} a_n$ converges, then $\sum_{n=m_1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{m_2-1} + \sum_{n=m_2}^{\infty} a_n$.

Example. Use $\sum_{n=1}^{\infty} \frac{2}{3^n} = 1$ to find the sum of $\sum_{n=4}^{\infty} \frac{2}{3^n}$.

Solution.

Reindexing (Changing indexes)

As long as we preserve the order of the terms of a series, we can change the index of the series without altering its convergence.

Example. Write $\sum_{n=5}^{\infty} \frac{3}{9^n}$ in the form $\sum_{k=1}^{\infty} a_k$.

Solution. $\sum_{n=5}^{\infty} \frac{3}{9^n} \xrightarrow{k=n-1} \sum_{k=1}^{\infty} \frac{3}{9^{k+4}}$

10.3. The Integral Test

Given a series $\sum_{n=1}^{\infty} a_n$, we have two questions:

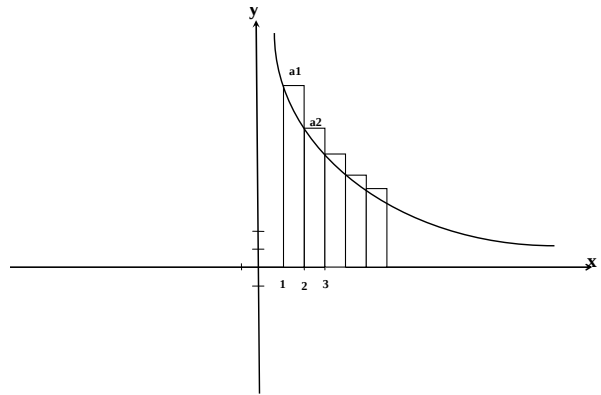
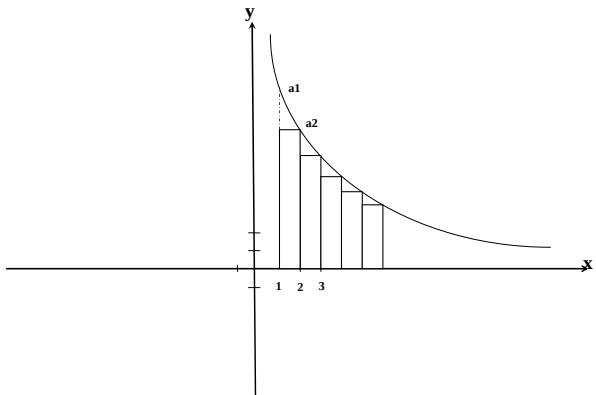
- (1) Does the series converge?
- (2) If it converges, what is its sum?

Theorem 6. (The integral test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$ for all $n \geq 1$ and let $f(n) = a_n$. Assume that there is an integer N such that

- (1) $f(x) > 0$ for all $x > N$
- (2) $f(x)$ is continuous on $[N, \infty)$
- (3) $f(x)$ is decreasing on $[N, \infty)$.

Then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)dx$ both converge or both diverge.



Proof.

From the above figure we see that the area of the rectangles $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)dx$.

On the other hand, the area of the rectangles $a_1 + a_2 + \cdots + a_n \geq \int_1^{n+1} f(x)dx$.

Thus we have

$$\int_1^{n+1} f(x)dx \leq S_n \leq a_1 + \int_1^n f(x)dx.$$

Taking limits as $n \rightarrow \infty$, we get

$$\int_1^{\infty} f(x)dx \leq \lim_{n \rightarrow \infty} S_n \leq a_1 + \int_1^{\infty} f(x)dx.$$

Thus the result of the theorem follows. □

Example 1. Test $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ for convergence.

Solution. Let $f(x) = \frac{1}{x(\ln x)^3}$. Then

- (1) $f(x) > 0$ for all $x > 2$.
- (2) $f(x)$ is continuous on $[2, \infty)$.
- (3) $f'(x) = -\frac{(3 + \ln x)}{x^2(\ln x)^4} < 0$ for all $x > 2$ implies $f(x)$ is decreasing on $[2, \infty)$.

Thus we can use the integral test.

Example 2. Test $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ for convergence.

Solution.

Example 3. Test $\sum_{n=1}^{\infty} n e^{-n^2}$ for convergence.

Solution.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$a_n = \frac{1}{n^p} \Rightarrow f(x) = \frac{1}{x^p}.$$

$f(x)$ satisfies the conditions of the integral test only if $p > 0$.

It follows that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p < 1$.

If $p \leq 0$, then $\lim_{n \rightarrow \infty} a_n \neq 0$. Thus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 0$.

Theorem 7. (The p-series)

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Remark. If $p = 1$, then $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series.

10.4. Comparison Tests

We can test the convergence of many series by comparing their terms to those of a series whose convergence is known.

Theorem 8. (Direct Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with

$$0 \leq a_n \leq b_n \quad \text{for all } n > N$$

for some integer N . Then

$$(1) \sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges},$$

$$(2) \sum_{n=1}^{\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}.$$

Example 1. Test $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{5^n}$ for convergence.

Solution. $a_n \leq \frac{1}{5^n}$

Example 2. Test $\sum_{n=2}^{\infty} \frac{1}{\sqrt{\ln n}}$ for convergence.

Solution. $\ln n < n \Rightarrow \frac{1}{\sqrt{\ln n}} > \frac{1}{\sqrt{n}}$

Example 3. Test $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2 + n + 1}$ for convergence.

Solution. For $n \geq 1$, $0 < \tan^{-1} n < \frac{\pi}{2}$

Moreover $n^2 + n + 1 > n^2$. Therefore $\frac{\tan^{-1} n}{n^2 + n + 1} \leq \frac{\pi}{2n^2}$.

Theorem 9. (Limit Comparison Test)

Let a_n and b_n be positive for all $n > N$ for some integer N . Then

(1) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$, implies $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

(2) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges implies $\sum_{n=1}^{\infty} a_n$ converges.

(3) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges implies $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1. Test $\sum_{n=1}^{\infty} \frac{\sqrt{2n-1}}{n^2 - n + 7}$ for convergence.

Solution. $a_n \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

Example 2. Test $\sum_{n=3}^{\infty} \frac{1}{\ln^2 n - 1}$ for convergence.

Solution. We have $\ln n < n^r$ for any positive real number r . It follows that

$$a_n \approx \frac{1}{\ln^2 n} < \frac{1}{n} = b_n$$

Example 3. Test $\sum_{n=1}^{\infty} \frac{1 + \ln n}{\sqrt{n^4 + 1}}$ for convergence.

Solution.

Example 4. Test $\sum_{n=0}^{\infty} \frac{n^2 + 3^n}{\pi^n}$ for convergence.

Solution.

Example 5. Test $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n}$ for convergence.

Solution.

Example 6. Test $\sum_{n=1}^{\infty} \frac{\tanh(n)}{n^3 + 1}$ for convergence.

Solution.

Example 7. Test $\sum_{n=1}^{\infty} n \tan\left(\frac{1}{n}\right)$ for convergence.

Solution.

10.5. The Ratio and Root Tests

Comparison tests are useful, but we may not be able to find the series to make comparison work. The ratio and root tests depend only on the series itself.

Theorem 10. (*The ratio test*)

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$ for all $n \geq 1$ and let

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

Then

$$(1) \quad \rho < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$(2) \quad \rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

$$(3) \quad \rho = 1 \Rightarrow \text{the test fails.}$$

Example 1. Test $\sum_{n=1}^{\infty} \frac{n+2}{n!}$ for convergence.

Solution.

Example 2. Test $\sum_{n=0}^{\infty} \frac{(2n)!}{3^n n!}$ for convergence.

Solution.

Example 3. Test $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$ for convergence.

Solution.

Example 4. (Exam)

Test $\sum_{n=1}^{\infty} \frac{(n!)^2}{e^{n^2}}$ for convergence.

Solution.

Example 5. Test $\sum_{n=2}^{\infty} \frac{(n+1)!}{n(\ln n) n!}$ for convergence.

Solution.

Example 6. Test $\sum_{n=1}^{\infty} a_n$ for convergence, where a_n is defined by $a_1 = 5$, $a_{n+1} = \frac{1+2n}{1+n} a_n$.

Solution.

Theorem 11. (The root test)

Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$ for all $n \geq 1$ and let

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}.$$

Then

$$(1) \rho < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$(2) \rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

$$(3) \rho = 1 \Rightarrow \text{the test fails.}$$

Example 1. Test $\sum_{n=1}^{\infty} \frac{n^4}{3^n}$ for convergence.

Solution.

Example 2. Test $\sum_{n=0}^{\infty} \left(\frac{2n+1}{n+4} \right)^n$ for convergence.

Solution.

Example 3. Test $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ for convergence.

Solution.

10.6. Alternating Series, Absolute and Conditional Convergence

The integral test, the comparison tests, the ratio test, and the root test are applicable only to series with positive terms. In this section we will study series involving both positive and negative terms.

Definition. (Alternating series)

A series in which the terms are alternately positive and negative is called an alternating series. That is, an alternating series is a series $\sum_{n=1}^{\infty} a_n$ with $a_n a_{n+1} < 0$.

Example 1. $\sum_{n=1}^{\infty} (-1)^n$ is an alternating series.

Example 2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n+1}$ is an alternating series.

Theorem 12. (The alternating series test (Leibniz's theorem))

Let $\sum_{n=1}^{\infty} (-1)^n u_n$ be an alternating series. Then $\sum_{n=1}^{\infty} (-1)^n u_n$ converges if the following three conditions are satisfied

- (1) $u_n > 0$ for all $n \geq 1$.
- (2) u_n is decreasing ($u_{n+1} < u_n$ for all $n \geq 1$.)
- (3) $\lim_{n \rightarrow \infty} u_n = 0$.

Example 1. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Solution.

Example 2. Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+4}}$ converges.

Solution.

Absolute and conditional convergence

Definition. (Absolute convergence)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Example. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

Solution.

Definition. (Conditional convergence)

If $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Example. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

Solution.

Theorem 13. (The absolute convergence test)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Example 1. Test $\sum_{n=1}^{\infty} \frac{\cos n}{n^2 + 1}$ for convergence.

Solution.

Example 2. Test $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^2 + 1}$ for absolute and conditional convergence.

Solution.

Example 3. Test $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1}$ for absolute and conditional convergence.

Solution.

Example 4. Test $\sum_{n=1}^{\infty} \frac{(-1)^n n 3^n}{4^n}$ for absolute and conditional convergence.

Solution.

Example 5. Test $\sum_{n=1}^{\infty} \frac{(-5)^n}{n + 5^n}$ for absolute and conditional convergence.

Solution.

Example 6. Test $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(3n)!}$ for absolute and conditional convergence.

Solution.

Error estimation

Theorem 14. (The alternating series estimation theorem)

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ satisfies the three conditions of the alternating series test, then for $n \geq N$,

$$S_n = u_1 - u_2 + \cdots + (-1)^{n+1} u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1} . Furthermore, the remainder $L - S_n$ has the same sign as u_{n+1} .

Example. Consider the geometric series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{n-1}}{3^{n-1}}$. Its sum is $L = 0.6$. On the other hand

$$S_5 = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} = \frac{81 - 54 + 36 - 24 + 16}{81} = \frac{55}{81} = 0.679$$

Thus $|L - S_5| = 0.079 < u_6 = 0.13$.

10.7. Power Series

In this section we will study infinite polynomials or power series. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

Definition. (Power series)

A power series about $x = x_0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots$$

The constant x_0 is the center and the constants $c_0, c_1, \cdots$ are the coefficients.

Example 1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ is a power series about $x = 0$.

Example 2. $\sum_{n=0}^{\infty} \frac{n+2}{2^n} (x-1)^n = 2 + \frac{3}{2}(x-1) + (x-1)^2 + \cdots$ is a power series about $x = 1$.

Example 3. $\sum_{n=0}^{\infty} x^{n/2} = 1 + x^{1/2} + x + \cdots$ is NOT a power series.

Convergence of power series

Example. For what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n+1}$ converge?

Solution. It converges absolutely for $|x| < 1$.

It converges conditionally at $x = -1$.

It diverges at $x = 1$.

It diverges for $|x| > 1$.

Theorem 15. (Convergence of power series)

Let $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ be a power series, and let $L = \lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|}$.

(1) If $0 < L < \infty$, then the series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ converges for $|x - x_0| < \frac{1}{L}$ and diverges for

$$|x - x_0| > \frac{1}{L}.$$

(2) If $L = 0$, then $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ converges for all $x \in \mathbb{R}$.

(3) If $L = \infty$, then $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ converges only for $x = x_0$.

Definition. (Radius and interval of convergence)

Let $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ be a power series.

(1) If $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$, then R is called the radius of convergence of the power series.

(2) The set of all points at which $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ converges is called the interval of convergence.

How to find the radius and interval of convergence of a power series

In order to find the radius and interval of convergence of a power series $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ use the following steps:

(1) Use the ratio test (or root test) and the condition $\rho < 1$ to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - x_0| < R \quad \text{or} \quad x_0 - R < x < x_0 + R.$$

(2) If the interval of absolute convergence is finite, test the end points for convergence. (Use a comparison test, the integral test, or the alternating series test.)

(3) If the interval of absolute convergence is $I = (x_0 - R, x_0 + R)$, then the series diverges for $|x - x_0| > R$ because the n th term does not approach zero for those value of x .

Example 1. Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2 + 1}}$.

Solution. In this example the n th term of the series is $a_n = \frac{x^n}{\sqrt{n^2 + 1}}$. Using the ratio test we obtain

$\rho = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = |x| < 1$ implies that the series converges absolutely in the interval $(-1, 1)$. It follows that the radius of convergence is $R = 1$.

End points: When $x = 1$ we have $a_n = \frac{1}{\sqrt{n^2 + 1}} > \frac{1}{\sqrt{4n^2}} = \frac{1}{2n}$. Thus the series diverges at $x = 1$.

When $x = -1$ we have $a_n = \frac{(-1)^n}{\sqrt{n^2 + 1}}$. $|a_n| = \frac{1}{\sqrt{n^2 + 1}}$ implies the series does not converges absolutely at $x = -1$. Now using the alternating series test we see that s the series converges at $x = -1$. Therefore the interval of convergence is $[-1, 1)$.

Example 2. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n(x+3)^n}{n^4}$.

Solution.

Example 3. Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3x-2)^n}{(2n)!}$.

Solution.

Example 4. Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n!x^n}{n+1}$.

Solution.

Example 5. Find the radius and interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (2x+1)^n}{\sqrt{n^2+1}}$.

Solution.

Term-by-Term Differentiation

Theorem 16. If $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ has radius of convergence $R > 0$, then it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n, \quad x \in (x_0 - R, x_0 + R).$$

This function f has derivatives of all orders inside the interval $(x_0 - R, x_0 + R)$, and we obtain the derivatives by differentiating the original series term by term:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n c_n (x-x_0)^{n-1}, \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) c_n (x-x_0)^{n-2}, \end{aligned}$$

and so on. Each of these derived series converges at each point of the interval $(x_0 - R, x_0 + R)$.

Example 1. Find $f'(x)$ and $f''(x)$ if $f(x) = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

Solution.

Example 2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n}$.

Solution. $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n} = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{d}{dx} \left(\frac{3}{3-x} \right) = \frac{3}{(3-x)^2}$.

Term-by-Term Integration

Theorem 17. If $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ has radius of convergence $R > 0$, then

$$\sum_{n=0}^{\infty} \frac{c_n(x - x_0)^{n+1}}{n+1}$$

converges for $x \in (x_0 - R, x_0 + R)$ and

$$\int f(x)dx = \sum_{n=0}^{\infty} \frac{c_n(x - x_0)^{n+1}}{n+1} + C$$

for $x \in (x_0 - R, x_0 + R)$.

Example. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$.

Solution. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \int \sum_{n=0}^{\infty} (-1)^n x^n = \int \frac{dx}{1+x} = \ln|x+1| + C.$

$$x = 0 \Rightarrow C = \ln 1 = 0$$

10.8. Taylor and Maclaurin Series

In this section we will show how functions that are infinitely differentiable generate power series called Taylor series. In many cases, these series can provide useful polynomial approximations of the generating functions.

Series representation

We know that the sum of a power series is a continuous function with derivatives of all order within its interval of convergence. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval I , can it be expressed as a power series on I ? And if it can, what will its coefficients be?

We can answer the last question readily if we assume that

$$f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots$$

with radius of convergence $R > 0$. By term-by-term differentiation within the interval of convergence $I = (x_0 - R, x_0 + R)$ we have

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \cdots \\ f''(x) &= 2c_2 + 6c_3(x - x_0) + \cdots \end{aligned}$$

Since all these equations holds at $x = x_0$, we have

$$\begin{aligned} f'(x_0) &= c_1 \\ f''(x_0) &= 2c_2, \end{aligned}$$

and in general

$$f^{(n)}(x_0) = n!c_n.$$

Thus, if $f(x)$ has a series representation, then the series must be

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \cdots$$

Definition. (Taylor and Maclaurin series)

Let f be a function with derivatives of all orders throughout some interval I and let x_0 be an interior point of I .

(1) The Taylor series generated by f at $x = x_0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \cdots$$

(2) The Maclaurin series generated by f is the Taylor series generated by f at $x = 0$.

Example 1. Find the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 1$.

Solution.
$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} (x-1)^n = 1 - 2(x-1) + 3(x-1)^2 + \cdots$$

Example 2. Find the Maclaurin series generated by $f(x) = \sin x$.

Solution.
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Example 3. Find the first four nonzero terms of the Taylor series generated by $f(x) = \ln(2+x)$ at $x = -1$.

Solution.
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} (x+1)^n = (x+1) + \cdots$$

Taylor polynomials

Definition. (Taylor polynomial of order n)

Let f be a function with derivatives of order $k \leq N$ in some interval I and let x_0 be an interior point of I . Then for any n , $0 \leq n \leq N$, the Taylor polynomial of order n generated by f at $x = x_0$ is

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Example 1. Find the Taylor polynomial of order 4 generated by $f(x) = e^x$ at $x = 0$.

Solution.

Example 2. Find the Taylor polynomials of orders 0, 1, 2 generated by $f(x) = \cos x$ at $x = \pi$.

Solution.

10.10. The Binomial Series and Applications of Power Series

The binomial series of powers and roots

The Maclaurin series generated by $f(x) = (1+x)^m$, where m is a real number, is

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$$

This series is called the binomial series. It converges absolutely for $|x| < 1$.

Theorem 18. (*Binomial theorem*)

For $|x| < 1$,

$$(x+1)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where the binomial coefficient $\binom{m}{k}$ is defined by

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!}, \quad \binom{m}{k} = \frac{m(m-1)(m-2) \cdots (m-k+1)}{k!}$$

Example 1. Find the first four terms of the binomial series $(1+x)^{-1/3}$

Solution.

$$(1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \dots$$

Example 2. Find the first four terms of the binomial series $(1-x^3)^{3/2}$

Solution.

$$(1-x^3)^{3/2} = 1 - \frac{3}{2}x^3 + \frac{3}{8}x^6 + \frac{3}{48}x^9 + \dots$$