

Chapter 11

Parametric Equations and Polar Coordinates

In this chapter we study new ways to define curves in the plane, give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. These curves are called conic sections, or conics, and model the paths traveled by planets, satellites, and other bodies whose motions are driven by inverse square forces. Planetary motion is best described with the help of polar coordinates, so we also investigate curves, derivatives, and integrals in this new coordinate system.

11.1 Parametrizations of Plane Curves

In previous chapters, we have studied curves as the graphs of functions or equations involving the two variables x and y . We are now going to introduce another way to describe a curve by expressing both coordinates as functions of a third variable t .

Parametric Equations

Let a curve be the path of a particle moving in the plane. If the curve fails the vertical line test, then it cannot be described with a formula $y = f(x)$. However, we can sometimes describe the curve by a pair of equations, $x = f(t)$ and $y = g(t)$.

Definition. If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval I of t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve. The equations $x = f(t)$, $y = g(t)$ are the parametric equations for the curve.

Remarks.

- (1) The variable t is a parameter for the curve and its domain I is the parameter interval.
- (2) If $I = [a, b]$ is a closed interval, then the point $(f(a), g(a))$ is the initial point of the curve and $(f(b), g(b))$ is the terminal point.
- (3) When we give parametric equations and a parameter interval for a curve, we say that we have parametrized the curve.

Example 1. Sketch the curve defined by the parametric equations

$$x = 1 + t, \quad y = 2 - t, \quad t \in \mathbb{R}.$$

Solution. We construct a brief table of values of (x, y) and then we plot these points and draw a smooth curve through them.

t	x	y
0	1	2
1	2	1
-1	0	3

Example 2. Identify the curve $x = \sqrt{t}$, $y = t$, $t \geq 0$ by finding a Cartesian equation for it. Sketch the curve and indicate the direction of motion.

Solution. $y = x^2$

It is the right side of the parabola $y = x^2$.

Example 3. Identify the curve $x = 2 \cos t$, $y = 3 \sin t$, $0 \leq t \leq 2\pi$ by finding a Cartesian equation for it. Sketch the curve and indicate the direction of motion.

Solution. $x^2/4 + y^2/9 = 1$ ellipse.

Example 4. Find a parametrization of the line segment with endpoints $(-1, 3)$ and $(-2, -4)$.

Solution. $y = 7x + 10$

$$x = t, \quad y = 7t + 10, \quad t \in [-1, -2].$$

$$\text{OR } x = t - 1, \quad y = 7t + 3, \quad t \in [-1, 0].$$

Example 5. Find a parametrization of the upper half of the parabola $x = y^2 + 2$.

Solution. $y = t$, $x = t^2 + 2$, $t \in [0, \infty)$.

11.2 Calculus with Parametric Curves

In this section we find slopes, length, and areas associated with parametrized curves.

Slopes of Parametrized Curves

A parametrized curve $x = f(t)$, $y = g(t)$ is differentiable at a point t if f and g are differentiable at t . If y is a differentiable function of x , then the chain rule gives:

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

Thus, if $dx/dt \neq 0$, we have

$$\text{Slope of the curve} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Example 1. Find the tangent to the curve $x = 1 + \tan^{-1} t$, $y = t + \ln(t^2 + 1)$, $0 \leq t \leq 1$ at the point $t = 0$.

Solution. The slope is given by $m = \frac{dy}{dx} \Big|_{t=0} = \frac{dy/dt}{dx/dt} \Big|_{t=0} = 1$.

Example 2. Find the slope of curve $4t = e^x + t^2$, $y + 2t^3 = x + t$ at $t = 1$.

Solution. The slope is given by $m = \frac{dy}{dx} \Big|_{t=0} = \frac{dy/dt}{dx/dt} \Big|_{t=1} = -13/2$.

Areas

The area of the region bounded from above by the curve $x = f(t)$, $y = g(t)$ and from below by the x -axis is given by

$$\text{Area} = \int_{x_1}^{x_2} y(x) dx = \int_a^b g(t) f'(t) dt,$$

where a and b are such that $x_1 = x(a)$, $x_2 = x(b)$ and $a < b$.

Example 1. Find the area enclosed by the circle $x = \cos t$, $y = -\sin t$.

Solution. Area = $2 \int_{-1}^1 y dx = \int_{t=-\pi}^{t=0} \sin^2 t dt = t - \frac{1}{2} \sin(2t) \Big|_{-\pi}^0 = \pi$.

Example 2. Find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

Solution. Let us graph the curve.

t	$x = t - \sin t$	$y = 1 - \cos t$
0.00	0.00	0.00
$\pi/4$	0.08	0.29
$\pi/2$	0.57	1.00
$3\pi/4$	1.65	1.71
π	3.14	2.00
$5\pi/4$	4.63	1.71
$3\pi/2$	5.71	1.00
$7\pi/4$	6.20	0.29
2π	6.28	0.00
$9\pi/4$	6.36	0.29
$5\pi/2$	6.85	1.00
$11\pi/4$	7.93	1.71
3π	9.42	2.00
$13\pi/4$	10.92	1.71
$7\pi/2$	12.00	1.00
$15\pi/4$	12.49	0.29
4π	12.57	0.00

$$\text{Area} = \int_0^1 y dx = \int_{t=0}^{t=2\pi} (1 - \cos t)^2 dt = \frac{3}{2}t - 2 \sin t + \frac{1}{4} \sin(2t) \Big|_0^{2\pi} = 3\pi.$$

Length of a Parametrically Defined Curve

Definition. Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b,$$

where f' and g' are continuous and not simultaneously zero on $[a, b]$. If C is traversed exactly once as t increases from $t = a$ to $t = b$, then the length of C is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

Example 1. Find the length of the curve given by $x = 1 + t^2$, $y = 1 - t^2$, $0 \leq t \leq 3$.

Solution. $L = 9\sqrt{2}$.

Example 2. Find the length of the curve given by $x = 1 + \tan^{-1} t$, $y = 1 - \frac{1}{2} \ln(t^2 + 1)$, $0 \leq t \leq 1$.

Solution. $L = \ln(\sqrt{2} + 1)$.

Area of Surfaces of Revolution

Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b,$$

where f' and g' are continuous and not simultaneously zero on $[a, b]$. Assume that C is traversed exactly once as t increases from $t = a$ to $t = b$. Then the area of the surfaces generated by revolving C about the coordinate axes are as follows.

(1) **Revolution about the x-axis** ($y \geq 0$):

$$S = 2\pi \int_a^b y \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

(2) **Revolution about the y-axis** ($x \geq 0$):

$$S = 2\pi \int_a^b x \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

Example 1. Find the area of the surface generated by revolving the curve given by

$$x = 1 + t^2, \quad y = t, \quad 0 \leq t \leq 1$$

about the x -axis.

Solution. $S = 2\pi \int_0^1 t \sqrt{4t^2 + 1} dt = \frac{\pi}{6} (5\sqrt{5} - 1)$.

Example 2. Find the area of the surface generated by revolving the curve given by

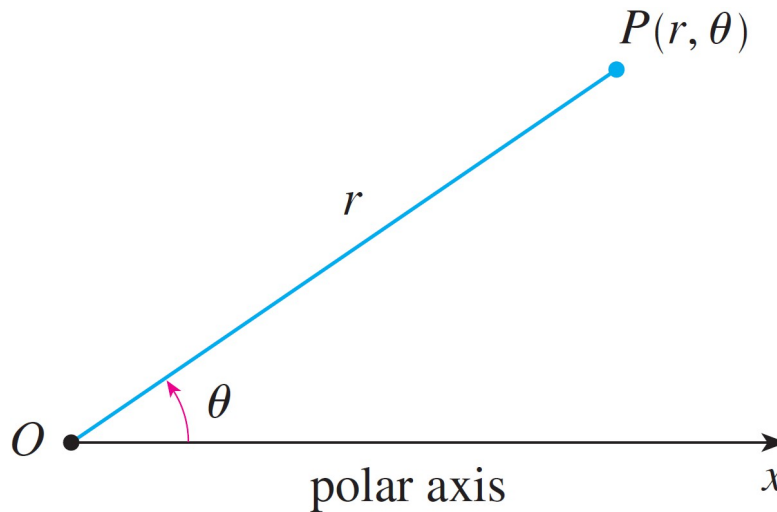
$$x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}, \quad 0 \leq t \leq \sqrt{3}$$

about the y -axis.

Solution. $S = 2\pi \int_0^{\sqrt{3}} \frac{2}{3} t^{3/2} \sqrt{t + t^{-1}} dt = 28\pi/9$.

11.3 Polar Coordinates

In this section, we study polar coordinates and their relations to Cartesian coordinates. While a point in the plane has one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing as we will see in the next section.



Definition of polar coordinates

To define polar coordinates, we first fix an origin O , called the pole, and an initial ray from O . Then each point P in the plane can be located by assigning to it a polar coordinates (r, θ) in which:

- r is the directed distance from O to P ,
- θ is the directed angle from the initial ray to the ray OP .

Remarks.

- (1) θ is positive when it is measured counterclockwise and negative when it is measured clockwise.
- (2) The angle θ associated with a point is not unique.

Example. $P = (2, \frac{\pi}{3}) = (2, \frac{-5\pi}{3}) = (2, \frac{\pi}{3} + 2n\pi), \quad n \in \mathbb{Z}$

Remark. (Negative values of r)

To reach a point (r, θ) , we first turn θ rad from the initial ray. Then if $r > 0$, we go forward r units and if $r < 0$ we go backward $|r|$ units.

Example. Locate the points $(3, \frac{\pi}{4})$ and $(-2, \frac{\pi}{4})$.

Solution.

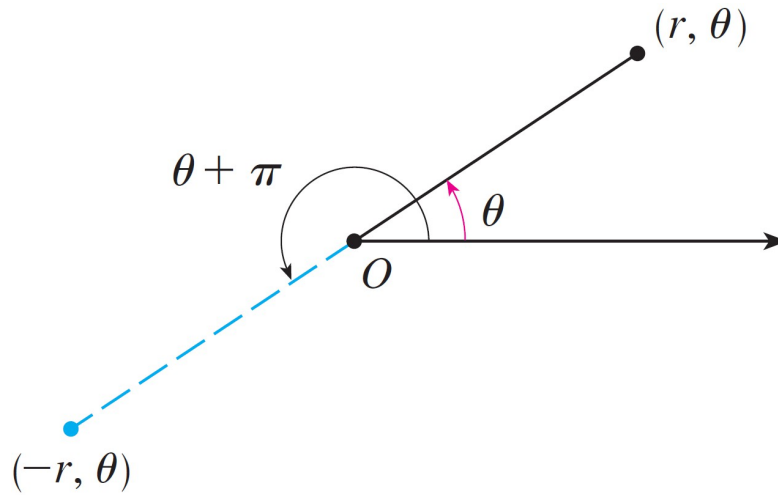
Remark. (All polar coordinates of a point)

If a point P has polar coordinates (r, θ) , then the other polar coordinates are

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + \pi + 2n\pi), \quad n \in \mathbb{Z}.$$

Example. Find all polar coordinates of the point P having coordinates $(3, \frac{\pi}{2})$.

Solution. $P = (3, \frac{\pi}{2}) = (-3, \frac{3\pi}{2}) = (3, \frac{\pi}{2} + 2n\pi) = (-3, \frac{3\pi}{2} + 2n\pi), \quad n \in \mathbb{Z}$



Polar equations and graphs

- (1) The equation $r = a$ is a circle of radius $|a|$ and center O .
- (2) The equation $\theta = \theta_0$ is a line through O making an angle θ_0 with the initial ray.

Example 1. Graph the set of points whose polar coordinates satisfy the conditions

$$2 \leq r \leq 4 \quad \text{and} \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}.$$

Solution.

Example 2. Graph the set of points whose polar coordinates satisfy the conditions

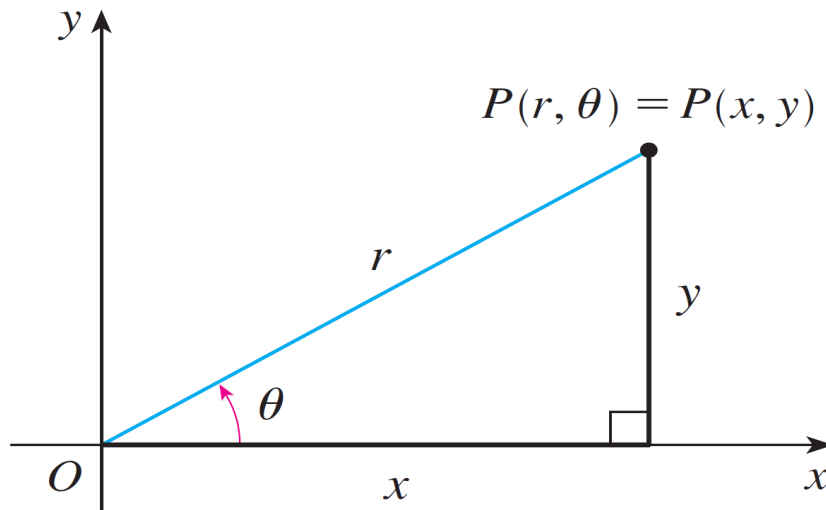
$$r \geq -2 \quad \text{and} \quad \theta = \frac{\pi}{6}.$$

Solution.

Relation between Cartesian and polar coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \frac{\pi}{2}$, $r > 0$ becomes the positive y-axis. The two coordinate systems are then related by the following equations.

$x = r \cos \theta$	$y = r \sin \theta$
$r^2 = x^2 + y^2$	$\tan \theta = \frac{y}{x}$



The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ . On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each selection, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y) . The other polar coordinate representations for the point can be determined from these two, as in the following example.

Example 1. Find the Cartesian coordinates of the point P given in polar coordinates as $P = (-6, -\frac{\pi}{3})$.

Solution. $x = -3, \quad y = 3\sqrt{3}$

Example 2. Find all polar coordinates of the points P_1 and P_2 given in Cartesian coordinates as $P_1 = (\sqrt{3}, 1)$ and $P_2 = (-\sqrt{3}, -1)$.

Solution. For P_1 and P_2 we have $r^2 = 4$ and $\tan \theta = \frac{1}{\sqrt{3}}$. Thus we have for both points $r = \pm 2$. Since P_1 lies in the first quadrant, the corresponding angle is $\theta = \frac{\pi}{6}$. It follows that

$$P_1 = (2, \frac{\pi}{6} + 2n\pi) = (-2, \frac{7\pi}{6} + 2n\pi), \quad n \in \mathbb{Z}.$$

Since P_2 lies in the third quadrant, the corresponding angle is $\theta = \frac{7\pi}{6}$. It follows that

$$P_2 = (2, \frac{7\pi}{6} + 2n\pi) = (-2, \frac{\pi}{6} + 2n\pi), \quad n \in \mathbb{Z}.$$

Example 3. Find polar coordinates, $-\pi \leq \theta \leq \pi$ and $r \leq 0$ of the point P_1 given in Cartesian coordinates as $P_1 = (-\sqrt{3}, 1)$.

Solution. $r^2 = 4$ and $\tan \theta = -1/\sqrt{3}$.

P_1 lies in the second quadrant, thus $\theta = 5\pi/6$.

One polar coordinates of P_1 is $(2, 5\pi/6) = (-2, -\pi/6)$.

Example 4. Find a polar equation for the circle $x^2 + y^2 - 2x = 0$.

Solution. $r^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r \cos \theta = 0$

Example 5. Replace the polar equation $r \cos \theta + 2r \sin \theta = 1$ by equivalent Cartesian equation. Then describe or identify the graph.

Solution. $x + 2y = 1$

Example 6. (Exam)

Replace the polar equation $r = \frac{5}{3 + 2 \cos \theta}$ by equivalent Cartesian equation. Then describe or identify the graph.

Solution. Rewrite the equation as $r(3 + 2 \cos \theta) = 5$. As a result we obtain $3r + 2x = 5$. To eliminate r we write the last equation as $3r = 5 - 2x$ and square both sides. It follows that $9(x^2 + y^2) = 25 - 20x + 4x^2$ or $5x^2 + 9y^2 + 20x = 25$. It is an ellipse.

Example 7. Replace the polar equation $r \cos \left(\theta + \frac{\pi}{3} \right) = 4$ by equivalent Cartesian equation. Then describe or identify the graph.

Solution. It the line $x - \sqrt{3}y = 8$.

11.4 Graphing in Polar Coordinates

In this section we study techniques for graphing equations in polar coordinates.

Symmetry tests for polar graphs

Consider the polar equation $F(r, \theta) = 0$.

1. If $F(r, \theta) = F(r, -\theta)$ or $F(r, \theta) = F(-r, \pi - \theta)$, then the graph of the equation is symmetric about the x -axis.
2. If the points (r, θ) and $(-r, -\theta)$ or $(r, \pi - \theta)$ satisfy the equation, then the graph of the equation is symmetric about the y -axis.
3. If the points (r, θ) and $(-r, \theta)$ or $(r, \pi + \theta)$ satisfy the equation, then the graph of the equation is symmetric about the origin.

Example. Identify the symmetry of the graph of the equation $r = 4 + 3 \cos \theta$.

Solution.

Symmetry about the x -axis:

Replacing (r, θ) by $(r, -\theta)$, we see that the equation does not change. Thus the graph is symmetric about the x -axis.

Symmetry about the y -axis:

Replacing (r, θ) by $(-r, -\theta)$, we see that the equation changes to $r = -4 - 3 \cos \theta$. Replacing (r, θ) by $(r, \pi - \theta)$, we see that the equation changes to $r = 4 - 3 \cos \theta$. Thus the graph fails the symmetry test about the y -axis.

Symmetry about the origin:

Replacing (r, θ) by $(-r, \theta)$, we see that the equation changes to $r = -4 - 3 \cos \theta$. Replacing (r, θ) by $(r, \pi + \theta)$, we see that the equation changes to $r = 4 - 3 \cos \theta$. Thus the graph fails the symmetry test about the origin.

Remarks.

- (1) It is possible for a graph to be symmetric with respect to the origin or one of the axes without satisfying any of the corresponding conditions. For examples, the graph of $r = \sin\left(\frac{\theta}{2}\right)$ is symmetric about the origin and the two axes but it satisfies the condition of symmetry with respect to the y -axis only.
- (2) This is because a graph can have many polar representations, so many tests are possible. There are many different ways of specifying a point in polar coordinates. You can realize this fact by thinking that every point (r, θ) could also be called $(r, \theta + 2n\pi)$, with n any integer. This fact can affect a symmetry test.
- (3) To test symmetry we must take into consideration all representations of the points mentioned in the symmetry tests above.
- (4) The graph is not symmetric about the origin or one of the axes if all polar coordinates of the symmetric point to (r, θ) does not satisfy the equation of the graph.

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx , not by $r' = df/d\theta$. Using $y = r \sin \theta$, $x = r \cos \theta$, we find $y = f(\theta) \sin \theta$, $x = f(\theta) \cos \theta$, and hence

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + \frac{df}{d\theta} \sin \theta}{-f(\theta) \sin \theta + \frac{df}{d\theta} \cos \theta}$$

Example. Find the slope of the curve $r = 1 + 2 \cos \theta$ at $\theta = \pi/2$.

Solution. Writing x and y as functions of θ we obtain $x = r \cos \theta = \cos \theta + 2 \cos^2 \theta$, $y = r \sin \theta = \sin \theta + 2 \sin \theta \cos \theta$. Thus

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + 2 \cos(2\theta)}{-\sin \theta - 4 \cos \theta \sin \theta}$$

Graphing

The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

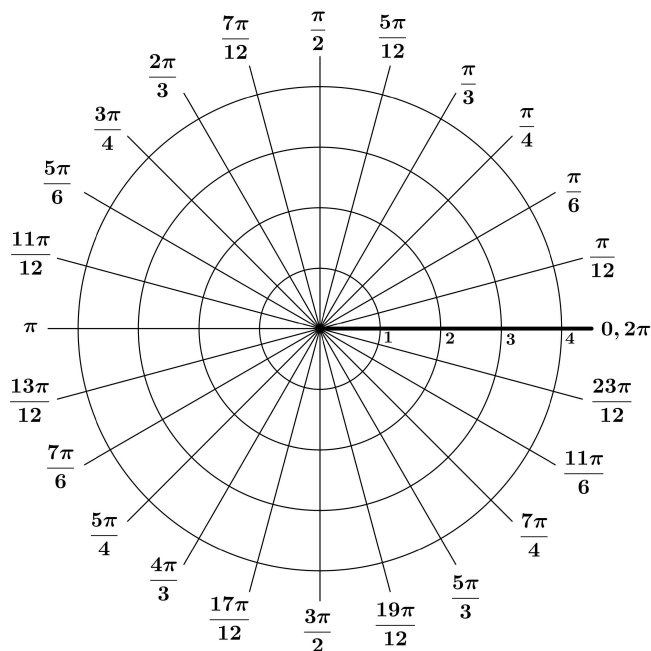
One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ) -values, plot the corresponding points, and connect them in order of increasing θ . This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing that is usually quicker and more reliable is to

1. first graph $r = f(\theta)$ in the Cartesian $r\theta$ -plane
2. then use the Cartesian graph as a table and guide to sketch the polar graph.

Example 1. Graph the cardioid $r = 1 + \cos \theta$.

Solution.

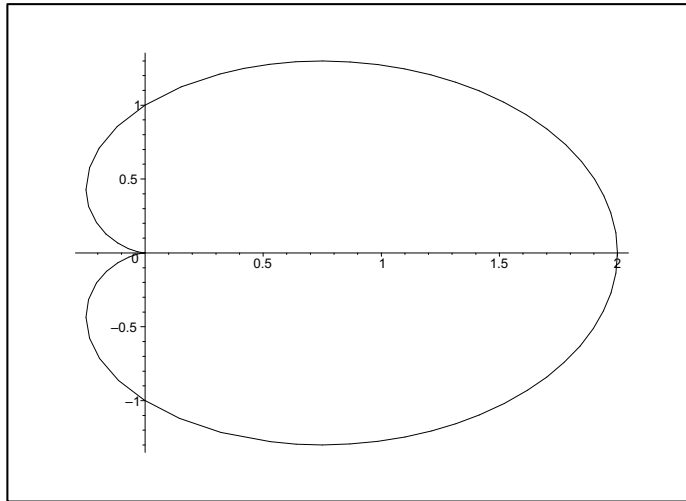
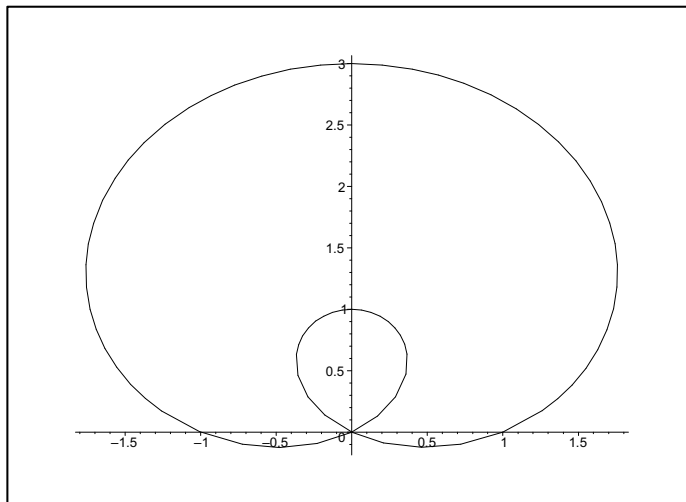
θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$r = 1 + \cos \theta$	2	1.5	1	0.5	0	0.5	1	1.5	1.9	2



Example 2. Identify the symmetry of the Limacon $r = 1 + 2 \sin \theta$ and graph it.

Solution.

θ	$-\pi/2$	$-\pi/4$	$-\pi/6$	$-\pi/12$	0	$\pi/6$	$\pi/4$	$\pi/2$
$r = 1 + 2 \sin \theta$	-1	-0.4	0	0.4	1	2	2.4	3

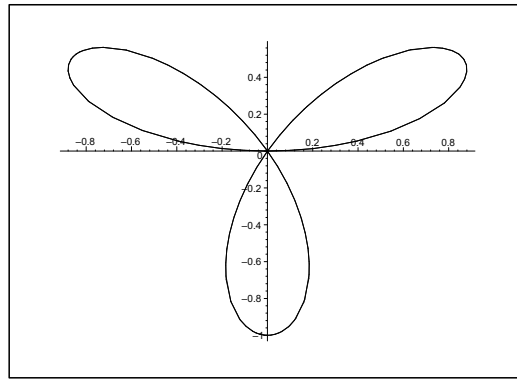
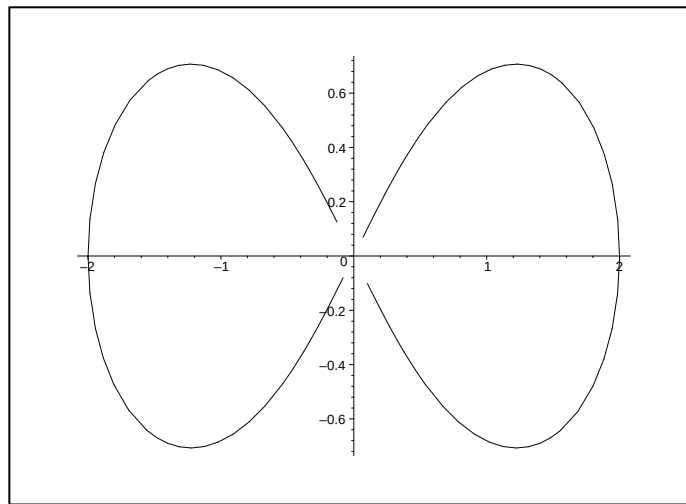
Figure 1: $r = 1 + \cos \theta$.Figure 2: $r = 1 + 2 \sin \theta$

Example 3. Graph the curve $r = \sin(3\theta)$.

Solution.	θ	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
	$r = \sin(3\theta)$	1	0	-0.7	-1	0	1	0.7	0	-1

Example 4. Identify the symmetry of the Limniscate $r^2 = \cos(2\theta)$ and sketch its graph.

Solution.	θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
	$r = \pm\sqrt{\cos(2\theta)}$	± 1	± 0.7	0	Not defined	Not defined	Not defined	Not defined	± 0.7

Figure 3: $r = \sin(3\theta)$ Figure 4: $r^2 = \cos(2\theta)$

Example 5. Graph the curve $r^2 = 9 \sin \theta$.

Solution.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$r = \pm 3\sqrt{\sin \theta}$	0	± 2.12	± 2.52	± 2.79	± 3	± 2.79	± 2.52	± 2.12	0

Intersection points

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. One sure way to identify all the points of intersection is to graph the equations.

Remarks.

(1) The curves $r = f(\theta)$ and $r = g(\theta)$ intersect at the origin if and only if

$$f(\alpha) = g(\beta)$$

for suitable α and β .

(2) The curves $r = f(\theta)$ and $r = g(\theta)$ intersect at a point $(f(\alpha), \alpha)$ other than the origin if and only if

$$f(\alpha) = g(\alpha + 2n\pi) \quad \text{or} \quad f(\alpha) = g(\alpha + \pi + 2n\pi)$$

for some integer n .

Example 1. Show that the point $(\frac{1}{2}, \frac{3\pi}{2})$ lie on the curve $r = -\sin \frac{\theta}{3}$.

Solution.

Example 2. Find the points of intersections of the curves $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

Solution. $(1, \pi/2)$, $(1, 3\pi/2)$, $(0, 0) = (0, \pi)$.

11.5 Areas and Lengths in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

Area in the plane

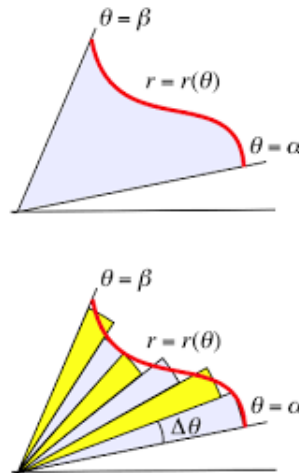


Fig. 7: Area with polar coordinates

Consider the region bounded by the rays $\theta = \alpha$, $\theta = \beta$, and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors. The typical sector has radius $r_k = f(\theta_k)$ and center angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k (πr_k^2) $= \frac{1}{2}[f(\theta_k)]^2 \Delta\theta_k$. Thus the area of this region is approximated by

$$A \approx \sum_{k=1}^n \frac{1}{2} [f(\theta_k)]^2 \Delta\theta_k.$$

Thus the area of the region bounded by the rays $\theta = \alpha$, $\theta = \beta$, and the curve $r = f(\theta)$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 d\theta.$$

Example 1. Find the area of the following circles:

(a) $r = 2$.

(b) $r = 4 \sin \theta$.

(c) $r = -4 \cos \theta$.

Solution. (a)

$$A = \frac{1}{2} \int_0^{2\pi} r^2 d\theta = \int_0^{2\pi} 2 d\theta = 4\pi.$$

(b)

$$A = \frac{1}{2} \int_0^{\pi} r^2 d\theta = \int_0^{\pi} 8 \sin^2 \theta d\theta = 4\pi.$$

(c)

$$A = \frac{1}{2} \int_{\pi/2}^{3\pi/2} r^2 d\theta = \int_{\pi/2}^{3\pi/2} 8 \cos^2 \theta d\theta = 4\pi.$$

Example 2. Find the area of the region enclosed by the curve $r = 1 - \cos \theta$.

Solution. We graph the curve and determine that the radius r sweeps out the region exactly once as θ runs from 0 to 2π . Thus the area is $A = 3\pi/2$

Example 3. Find the area of the region enclosed by the cardioid $r = 2(1 + \sin \theta)$.

Solution. We graph the curve and determine that the radius r sweeps out the region exactly once as θ runs from 0 to 2π . Thus the area is

$$A = \frac{1}{2} \int_0^{2\pi} r^2 d\theta = \int_0^{2\pi} [3 + 4 \sin \theta - \cos(2\theta)] d\theta = 6\pi.$$

Example 4. Find the area of the region enclosed by the lemniscate $r^2 = \sin(2\theta)$.

Solution. We graph the curve and determine that the radius r sweeps out the region in the first quadrant exactly once as θ runs from 0 to $\pi/2$. Thus the area is

$$A = 2 \left(\frac{1}{2} \int_0^{\pi/2} r^2 d\theta \right) = \int_0^{\pi/2} \sin(2\theta) d\theta = -\frac{1}{2} \cos(2\theta) \Big|_0^{\pi/2} = 1.$$

Example 5. Find the area of the region enclosed by one loop of the lemniscate $r^2 = 16 \cos(2\theta)$.

Solution. From the graph of the curve we see that the area is

$$A = \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 d\theta = 8 \int_{-\pi/4}^{\pi/4} \cos(2\theta) d\theta = 4 \sin(2\theta) \Big|_{-\pi/4}^{\pi/4} = 8.$$

Example 6. (Not clear and omitted) Find the area of the smaller loop of the limaçon $r = 1 + 2 \sin \theta$.

Solution. After sketching the curve, we see that the left hand side of smaller loop is traced out by the point as increases from $\theta = -\pi/2$ to $\theta = -\pi/6$. Thus

$$A = 2 \int_{-\pi/2}^{-\pi/6} \frac{1}{2} r^2 d\theta = \int_{-\pi/2}^{-\pi/6} [1 + 4 \sin \theta + 4 \sin^2 \theta] d\theta = .$$

Area between two curves

The area of the region

$$0 \leq r_1(\theta) \leq r \leq r_2(\theta), \quad \alpha \leq \theta \leq \beta$$

is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)d\theta.$$

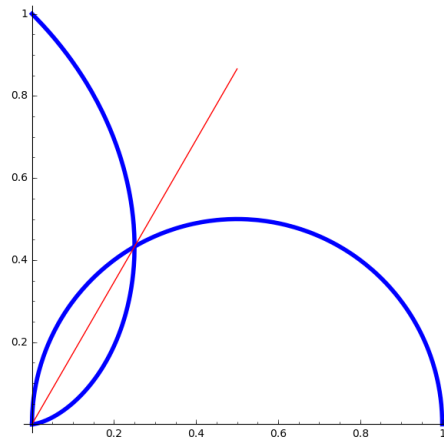
Example 1. Find the area of the region inside the circle $r = 2 \cos \theta$ and outside the circle $r = 1$.

Solution. $A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} [4 \cos^2 \theta - 1] d\theta = \frac{1}{2} [\theta + \sin(2\theta)]_{-\pi/3}^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$

Example 2. Find the area of the region inside $r = 5$ and outside $r = 1 + \cos \theta$.

Solution. $47\pi/2$

Example 3. Find the area of the region shared by the circle $r = \cos \theta$ and the cardioid $r = 1 - \cos \theta$.



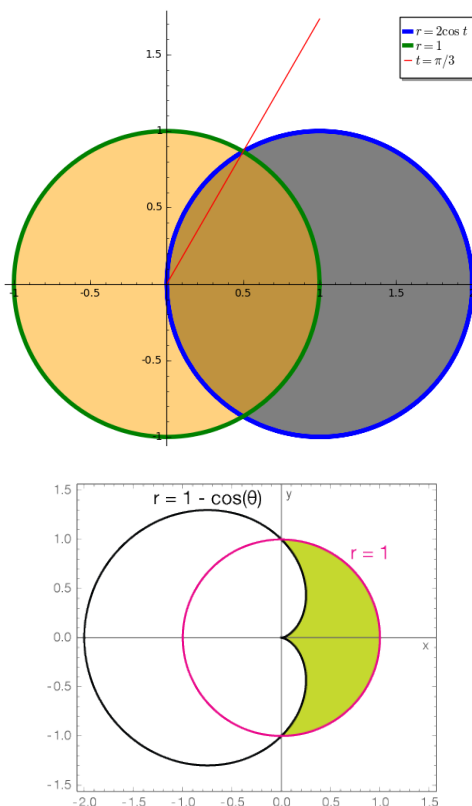
Solution. $A = 2 \left[\frac{1}{2} \int_0^{\pi/3} (1 - \cos \theta)^2 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} \cos^2 \theta d\theta \right] = \frac{7\pi}{12} - \sqrt{3}.$

Example 4. Find the area of the region shared by the curve $r = 1$ and $r = 2 \cos \theta$.

Solution. $A = 2 \left[\frac{1}{2} \int_0^{\pi/3} d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} 4 \cos^2 \theta \right] = \frac{4\pi}{3} - \sqrt{3}$

Example 5. Find the area inside of the function $r = 1$ and outside of the cardioid $r = 1 - \cos \theta$.

Solution. $A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} [1 - (1 - \cos \theta)^2] d\theta = 2 - \frac{\pi}{4}$



Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$, and if the point traces the curve exactly once as runs from α to β , then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Example 1. Find the length of the curve $r = 4$.

Solution.

Example 2. Find the length of the curve $r = \sqrt{1 + \sin(2\theta)}$, $0 \leq \theta \leq \pi\sqrt{2}$.

Solution. $L = 2\pi$

11.6 Conic Sections

In this section we define and review parabolas, ellipses, and hyperbolas geometrically and derive their standard Cartesian equations. These curves are called conic sections or conics because they are formed by cutting a double cone with a plane.

Parabolas

Definition. (Parabola, Focus, Directrix)

A parabola is the set of all points in the plane that are equidistant from a given fixed point and a given fixed line. The fixed point is called the focus of the parabola and the fixed line is called the directrix.

Definition. (Axis and vertex)

- The line with respect to which the parabola is symmetric is called the axis of the parabola.
- The point where the parabola crosses its axis is called the vertex.

Remark. If the focus F lies on the directrix L , the parabola is the line through F perpendicular to L . We consider this to be a degenerate case and assume henceforth that F does not lie on L .

Standard-form equations for parabolas with vertices at the origin

A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0, p)$ on the positive y -axis and that the directrix is the line $L: y = -p$. A point $P = (x, y)$ lies on the parabola if and only if

$$PF = PQ,$$

where Q is the point at which the perpendicular line from P meets L . As a result we obtain

$$y = \frac{x^2}{4p}.$$

A similar derivation, with x and y interchanged, yields the following equation for a parabola with focus $F = (p, 0)$ and directrix $x = -p$

$$x = \frac{y^2}{4p}.$$

In summary, we have

Equation	Focus	Directrix	Axis	Opens
$y = ax^2$	$(0, \frac{1}{4a})$	$y = \frac{-1}{4a}$	y -axis	up if $a > 0$ and down if $a < 0$
$x = ay^2$	$(\frac{1}{4a}, 0)$	$x = \frac{-1}{4a}$	x -axis	To the right if $a > 0$ and to the left if $a < 0$

Example 1. Find the vertex, focus, and directrix of the parabola $y = \frac{1}{2}x^2$.

Solution.

Example 2. Find an equation for the parabola whose focus is $(0, 2)$ and directrix is $y = -2$. Then sketch the parabola.

Solution.

Example 3. Find the vertex, focus, and directrix of the parabola $x = -3y^2$.

Solution.

The parabolas $y = ax^2 + bx + c$ and $x = ay^2 + by + c$

Vertical and horizontal shifting transform the equation $y = ax^2$ into

$$(y - h) = a(x - k)^2$$

or

$$y = ax^2 + bx + c.$$

Similarly $x = ay^2$ is transformed into

$$x = ay^2 + by + c.$$

Example 1. Find an equation for the parabola obtained by shifting $y = 4x^2$ to the left 2 units and up 1 unit.

Solution.

Example 2. (Exam)

Find the vertex, focus, and directrix of the parabola $x^2 - 4x - 8y = 12$.

Solution.

Focus: $(2, 0)$

Directrix: $y = -4$

Vertex: $(2, -2)$

Ellipses

Definition. (Ellipse, Foci)

An ellipse is the set of all points in the plane whose distances from two fixed points have a constant sum. The two fixed points are called the foci of the ellipse.

Definition. (Focal Axis, Center, Vertices)

- The line through the foci of an ellipse is the focal axis of the ellipse.
- The midpoint between the foci is the center.
- The points where the focal axis and the ellipse cross are the vertices of the ellipse.

Standard-form equations for ellipses with center at the origin

An ellipse has its simplest equation when its center and foci are on one of the coordinate axes. For example, suppose that the foci are $F_1(-c, 0)$ and $F_2(c, 0)$. A point $P = (x, y)$ lies on the ellipse if and only if $PF_1 + PF_2 = 2a$. As a result we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b = \sqrt{a^2 - c^2}$. A similar derivation, with x and y interchanged, yields the equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

for an ellipse with foci $F_{1,2} = (0, \pm c)$.

Definition. (Major and Minor Axes)

- 1) The line segment of length $2a$ joining the points $(\pm a, 0)$ is called the major axes of the ellipse.
- 2) The line segment of length $2b$ joining the points $(0, \pm b)$ is called the minor axes of the ellipse.
- 3) The number a is called the semimajor axis and the number b is called the semiminor axis.

As a summary we have

- **Foci on the x-axis:**

Standard-form-equation:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$
Center:	$(0, 0)$
Center-to-focus distance:	$c = \sqrt{a^2 - b^2}.$
Foci:	$(\pm c, 0)$
Vertices:	$(\pm a, 0)$
Semimajor axis:	a
Semiminor axis:	b

- **Foci on the y-axis:**

Standard-form-equation:	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b$
Center:	$(0, 0)$
Center-to-focus distance:	$c = \sqrt{a^2 - b^2}.$
Foci:	$(0, \pm c)$
Vertices:	$(0, \pm a)$
Semimajor axis:	a
Semiminor axis:	b

Example 1. Consider the ellipse $4x^2 + 25y^2 = 100$. Put the equation in standard form and then sketch the ellipse and find the foci.

Solution.

Standard form:	$\frac{x^2}{25} + \frac{y^2}{4} = 1$
Center:	$(0, 0)$
Center-to-focus distance:	$c = \sqrt{a^2 - b^2} = \sqrt{21}.$
Foci:	$(\pm\sqrt{21}, 0)$
Vertices:	$(\pm 5, 0)$

Example 2. Find an equation for the ellipse whose foci are $(0, \pm 5)$ and whose vertices are $(0, \pm 7)$.

Solution. We have $a = 7$, $c = 5$. Thus $b^2 = 24$ and the equation reads

$$\frac{x^2}{24} + \frac{y^2}{49} = 1.$$

Example 3. Find an equation for the ellipse obtained by shifting $x^2 + 4y^2 = 36$ up 3 units and to the right 1 unit. Then find the new center, foci, and vertices.

Solution. Equation $(x - 1)^2 + 4(y - 3)^2 = 36$.

Example 4. Find the vertices, foci, and center of the ellipse $x^2 + 3y^2 - 14x + 24y + 49 = 0$.

Solution. $\frac{(x - 2)^2}{9} - \frac{(y + 3)^2}{4} = 1$

	$\frac{x^2}{48} + \frac{y^2}{16} = 1$	$\frac{(x-1)^2}{48} + \frac{(y-4)^2}{16} = 1$
Center	$(0, 0)$	$(1, 4)$
Foci	$(\pm\sqrt{32}, 0)$	$(1 \pm \sqrt{32}, 4)$
Vertices	$(\pm\sqrt{48}, 0)$	$(1 \pm \sqrt{13}, 4)$

Hyperbolas

Definition. (Hyperbola, Foci)

A hyperbola is the set of all points in the plane whose distances from two fixed points have a constant difference. The two fixed points are called the foci of the hyperbola.

Definition. (Focal Axis, Center, Vertices)

- The line through the foci of a hyperbola is the focal axis of the hyperbola.
- The midpoint between the foci is the center.
- The points where the focal axis and the hyperbola cross are the vertices of the hyperbola.

Standard-form equations for hyperbolas with center at the origin

A hyperbola has its simplest equation when its center and foci are on one of the coordinate axes. For example, suppose that the foci are $F_1(-c, 0)$ and $F_2(c, 0)$. A point $P = (x, y)$ lies on the hyperbola if and only if $PF_1 - PF_2 = 2a$. As a result we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $b = \sqrt{c^2 - a^2}$. A similar derivation, with x and y interchanged, yields the following equation for a hyperbola with foci $F_{1,2} = (0, \pm c)$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Asymptotes of Hyperbolas and Graphing

If we solve

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for y we obtain

$$y^2 = b^2 \left(\frac{x^2}{a^2} - 1 \right)$$

or taking square roots,

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}$$

As $x \rightarrow \pm\infty$, the factor approaches 1, and the factor is dominant. Thus the lines

$$y = \pm \frac{b}{a}x$$

are the two asymptotes of the hyperbola. The asymptotes give the guidance we need to graph hyperbolas quickly. The fastest way to find the equations of the asymptotes is to replace the 1 in the equation by 0 and solve the new equation for y . **As a summary we have**

• **Foci on the x-axis:**

Standard-form-equation:	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Center:	$(0, 0)$
Center-to-focus distance:	$c = \sqrt{a^2 + b^2}$
Foci:	$(\pm c, 0)$
Vertices:	$(\pm a, 0)$
Asymptotes:	$y = \pm \frac{b}{a}x$

• **Foci on the y-axis:**

Standard-form-equation:	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Center:	$(0, 0)$
Center-to-focus distance:	$c = \sqrt{a^2 + b^2}$
Foci:	$(0, \pm c)$
Vertices:	$(0, \pm a)$
Asymptotes:	$y = \pm \frac{a}{b}x$

Example 1. Consider the hyperbola $y^2 - 3x^2 = 9$. Put the equation in standard form and then sketch the hyperbola and find the foci, vertices, and asymptotes.

Solution. Writing the equation in standard form gives $\frac{y^2}{9} - \frac{x^2}{3} = 1$. Thus we have $a = 3$, $b = \sqrt{3}$ and $c = \sqrt{12}$.

Foci: $(0, \pm c) = (0, \pm\sqrt{12})$

Vertices: $(0, \pm a) = (0, \pm 3)$

Asymptotes: $y = \pm \frac{a}{b}x = \pm\sqrt{3}x$.

Example 2. Find an equation for the hyperbola with center on the origin, one focus is $(-4, 0)$, and asymptotes $y = \pm \frac{1}{2}x$.

Solution. $c = 4$ and $\frac{b}{a} = \frac{1}{2} \Rightarrow a = 2b$. Using $c = \sqrt{a^2 + b^2} \Rightarrow 16 = 5b^2$. Thus $b = \frac{4}{\sqrt{5}}$, $a = \frac{8}{\sqrt{5}}$ and the equation reads $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Example 3. (Exam)

Find the center, vertices, foci, and asymptotes of the hyperbola $4x^2 - 16x - 9y^2 - 54y = 101$.

Solution. $\frac{(x-2)^2}{9} - \frac{(y+3)^2}{4} = 1$

	$\frac{x^2}{9} - \frac{y^2}{4} = 1$	$\frac{(x-2)^2}{9} - \frac{(y+3)^2}{4} = 1$
Center	$(0, 0)$	$(2, -3)$
Foci	$(\pm\sqrt{13}, 0)$	$(2 \pm \sqrt{13}, -3)$
Vertices	$(\pm 3, 0)$	$(2 \pm 3, -3)$
Asymptotes	$y = \pm \frac{2}{3}x$	$y + 3 = \pm \frac{2}{3}(x - 2)$

11.7 Conics in Polar Coordinates

In this section we study the equations of lines, circles, and conic sections in polar coordinates. We develop that equations here after introducing the idea of eccentricity (deviation of a curve or orbit from circularity).

Eccentricity

Definition. (Eccentricity)

(1) The eccentricity of an ellipse or a hyperbola is

$$e = \frac{\text{distance between foci}}{\text{distance between vertices}} = \frac{c}{a}.$$

(2) The eccentricity of a parabola is $e = 1$.

Example 1. Find the eccentricity of the ellipse $4x^2 + 9y^2 = 36$.

Solution. Foci on the x -axis.

$$a = 3, \quad b = 2, \quad c = \sqrt{5}. \quad \text{Thus } e = \frac{c}{a} = \frac{\sqrt{5}}{3}.$$

Example 2. Find the eccentricity of the hyperbola $25y^2 - 16x^2 = 400$.

Solution. Foci on the y -axis.

$$a = 4, \quad b = 5, \quad c = \sqrt{41}. \quad \text{Thus } e = \frac{c}{a} = \frac{\sqrt{41}}{4}.$$

Directrices

Definition. (Directrices)

(1) The directrices of an ellipse or a hyperbola with foci on the x -axis and center $(0, 0)$ are

$$x = \pm \frac{a}{e} = \pm \frac{a^2}{c}.$$

(2) The directrices of an ellipse or a hyperbola with foci on the y -axis and center $(0, 0)$ are

$$y = \pm \frac{a}{e} = \pm \frac{a^2}{c}.$$

Example. (Exam)

Find the standard-form equation for the hyperbola with center at the origin, focus are $(-6, 0)$ and directrix $x = -2$.

Solution. Equation is $\frac{x^2}{12} - \frac{y^2}{24} = 1$.

$c = 6$, directrix $x = \pm a^2/c \Rightarrow a = \sqrt{12}$. Thus $b = \sqrt{c^2 - a^2} = \sqrt{24}$.

Focus-directrix equation

The the equation of the conic section with eccentricity e , focus F and corresponding directrix D is given by

$$PF = ePD,$$

where PF is the distance between any point $P(x, y)$ and F and PD is the distance between $P(x, y)$ and the directrix D .

(1) If $e = 1$, the conic section is a parabola.

(2) If $e < 1$, the conic section is an ellipse.

(3) If $e > 1$, the conic section is a hyperbola.

Example 1. Consider the ellipse centered at the origin whose focus is $(-3, 0)$ with corresponding directrix $x = -5$. Find the eccentricity of the ellipse and its standard-form equation.

Solution. Directrix $x = -a/e = -a^2/c = -5$ and $c = 3 \Rightarrow a^2 = 15 \Rightarrow a = \sqrt{15}$. Thus $e = a/5 = \sqrt{15}/5$.

Equation

$$\begin{aligned} PF &= ePD. \\ \Rightarrow \sqrt{(x+3)^2 + y^2} &= \frac{\sqrt{15}}{5} \sqrt{(x+5)^2}. \\ \Rightarrow 2x^2 + 5y^2 &= 30. \end{aligned}$$

Example 2. Find an equation for the ellipse whose eccentricity is $e = 1/3$ and that has a focus $(2, 0)$ with corresponding directrix $x = 4$.

Solution. Equation

$$\begin{aligned} PF &= ePD. \\ \Rightarrow \sqrt{(x-2)^2 + y^2} &= \frac{1}{3} \sqrt{(x-4)^2}. \\ \Rightarrow 8x^2 - 28x + 9y^2 + 20 &= 0. \end{aligned}$$

Polar Equations

The polar equations for conic section are simple when one focus is on $(0, 0)$ and the corresponding directrix is a line parallel to a coordinate axis.

Let us place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$, $k > 0$. Using focus-directrix equation $PF = ePD$, we obtain

$$r = e(k - r \cos \theta)$$

or

$$r = \frac{ek}{1 + e \cos \theta}.$$

If the corresponding directrix is the vertical line $x = -k$, $k > 0$, to the left of the origin, then the equation is

$$r = \frac{ek}{1 - e \cos \theta}.$$

If the corresponding directrix is the horizontal line $y = k$, $k > 0$, above the origin, then the equation is

$$r = \frac{ek}{1 + e \sin \theta}.$$

If the corresponding directrix is the horizontal line $y = -k$, $k > 0$, below the origin, then the equation is

$$r = \frac{ek}{1 - e \sin \theta}.$$

Example 1. Find an equation for the conic section with one focus at the origin and corresponding directrix $x = 4$ and eccentricity $e = 2$.

Solution.

$$r = \frac{ek}{1 + e \cos \theta} = \frac{8}{1 + 2 \cos \theta}.$$

Example 2. Sketch the ellipse $r = \frac{4}{2 - \cos \theta}$. Include the directrix corresponding to the focus at the origin, label the vertices and center with appropriate polar coordinates.

Solution.

- $e = 1/2$, $k = 4$ and the directrix corresponding to the focus at the origin is $x = -4$.
- Vertices: $\theta = 0 \Rightarrow r = 4$ and $\theta = \pi \Rightarrow r = 4/3$. Thus the vertices are $(4, 0)$ and $(4/3, \pi)$.
- Center: The distance between the vertices is $4 + (4/3) = 16/3$. Thus the distance between the vertices and the center is $8/3$. It follows that the center is $(4/3, 0)$.

Example 3. Sketch the ellipse $r = \frac{5}{3 + 2 \sin \theta}$. Include the directrix corresponding to the focus at the origin, label the vertices, center and the other focus with appropriate polar coordinates.

Solution.

- $e = 2/3$, $k = 5/2$ and the directrix corresponding to the focus at the origin is $y = 5/2$.
- Vertices: $\theta = \pi/2 \Rightarrow r = 1$ and $\theta = -\pi/2 \Rightarrow r = 5$. Thus the vertices are $(1, \pi/2)$ and $(5, -\pi/2)$.
- Center: The distance between the vertices is $1 + 5 = 6$. Thus the distance between the vertices and the center is 3 . It follows that the center is $(2, -\pi/2)$.

Example 4. Sketch the parabola $r = \frac{5}{2 - 2 \sin \theta}$. Include the directrix corresponding to the focus at the origin, label the vertex.

Solution.

- $e = 1$, $k = 5/2$ and the directrix corresponding to the focus at the origin is $y = -5/2$.
- Vertex: $\theta = \pi/2 \Rightarrow r$ is not defined and $\theta = -\pi/2 \Rightarrow r = 5/4$. Thus the vertex is $(5/4, -\pi/2)$.

Lines

Suppose the perpendicular from the origin to line L meets L at a point $P_0(r_0, \theta_0)$, $r_0 \geq 0$. Then, if $P(r, \theta)$ is any point on L , the points P_0 , P , and O are the vertices of a right triangle. Thus we have

$$r_0 = r \cos(\theta - \theta_0).$$

This is the standard polar equation for the line L which meets the perpendicular from the origin at $P_0(r_0, \theta_0)$.

Example 1. Find polar and Cartesian equations for the line which meets the perpendicular from the origin at $(2, \pi/3)$.

Solution. $r \cos(\theta - \frac{\pi}{3}) = 2 \Rightarrow x + \sqrt{3}y = 2$.

Example 2. Find polar equation in the form $r_0 = r \cos(\theta - \theta_0)$ for the line $\sqrt{3}x - y = 1$.

Solution. $r(\sqrt{3} \cos \theta - \sin \theta) = 1 = r[\cos \theta \cos \theta_0 + \sin \theta \sin \theta_0]$.

$$\Rightarrow \cos \theta_0 = \sqrt{3}, \quad \sin \theta_0 = -1.$$

$$\Rightarrow \tan \theta_0 = \frac{-1}{\sqrt{3}} \Rightarrow \theta_0 = -\pi/6.$$

It follows that the equation is

$$1 = r \cos(\theta + \pi/6).$$

Circles

The polar equation for circle with radius a and center at (r_0, θ_0) is given by

$$a^2 = r_0^2 + r^2 - 2r_0r \cos(\theta - \theta_0).$$

Example. Find polar equation for the circle through the origin with radius 3 and center at the negative y -axis.

Solution. Center is $(3, -\pi/2)$. Thus equation is

$$9 = 9 + r^2 - 6r \cos(\theta + \pi/2).$$

or

$$r = -6 \sin \theta.$$