



بسم الله الرحمن الرحيم

تلخيص مادة:

تفاضل وتكامل (ب)

مدرس المساق:

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التلخيص مقدم من:

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تم الرفع بواسطة:

فريق مدونة قسم الرياضيات

لا تنسونا من صالح دعائكم

لكل من ساهم في انجاز هذا العمل

فبراير ٢٠١٤م

زوروا على الرابط التالي

<http://iugmath.blogspot.com>

أو على الفيس بوك

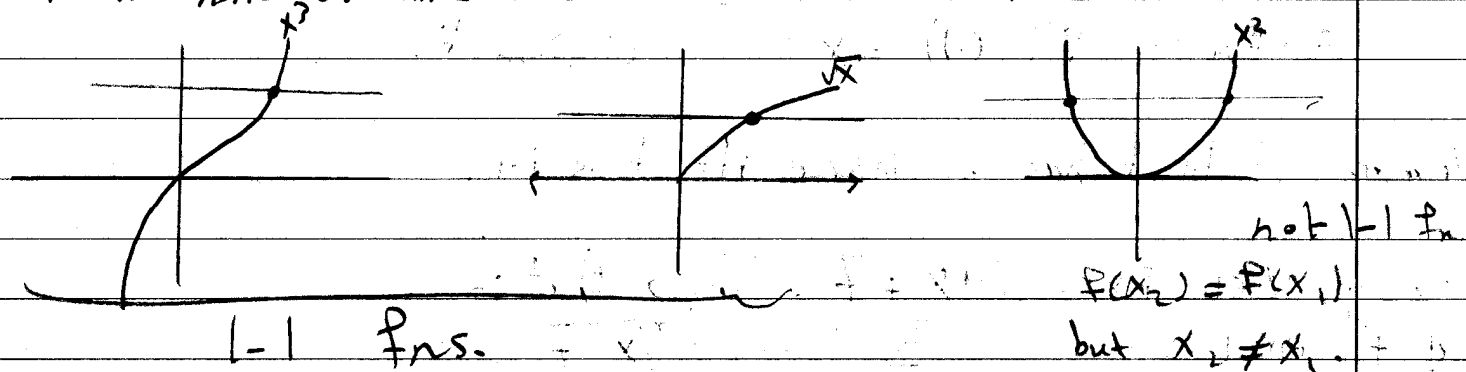
صفحة (قسم الرياضيات)

7.1 / Inverse fns. and their derivatives.

Def. A fn $f: D \rightarrow R$ is called one-to-one if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ $\forall x_1, x_2 \in D$.

* Equivalently, $f: D \rightarrow R$ is called 1-1 if $f(x_1) = f(x_2)$ then $x_1 = x_2$, $\forall x_1, x_2 \in D$

* Graphically, A fn f is 1-1 iff its graph intersects each horizontal line at most once for example, the fn $f(x) = x^3$ and $f(x) = \sqrt{x}$ are 1-1 but the fn $f(x) = x^2$ is not 1-1 because of the horizontal line test.



Example 1 / Determine whether or not the following fns are 1-1.

[1] $f(x) = 9 - x^2$

* $f(2) = 9 - (2)^2 = 5$, $f(-2) = 9 - (-2)^2 = 5$.

$f(2) = f(-2)$ but $2 \neq -2 \Rightarrow f$ is not 1-1.

* Let $f(x_1) = f(x_2)$

$9 - x_1^2 = 9 - x_2^2 \Rightarrow x_1^2 = x_2^2 \Rightarrow |x_1| = |x_2| \Rightarrow$

$x_1 = \pm x_2 \Rightarrow f$ is not 1-1.

[2] $f(x) = (2 - x^3)^{\frac{1}{3}} + 8$

Let $f(x_1) = f(x_2)$

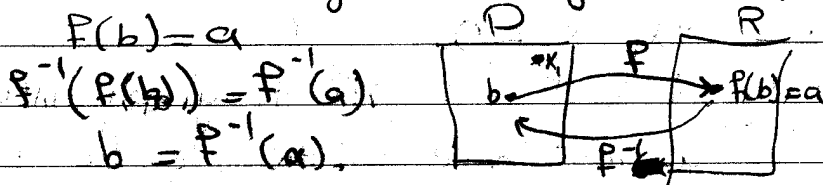
$(2 - x_1^3)^{\frac{1}{3}} + 8 = (2 - x_2^3)^{\frac{1}{3}} + 8$

$2 - x_1^3 = 2 - x_2^3 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

So f is 1-1.

Remark/ Any increasing and decreasing fns are 1-1 because of the horizontal line test.

Def. let $f: D \rightarrow R$ be a one-to-one fn. The inverse of f denoted by $f^{-1}(x)$ is a fn defined on the range of f by $f^{-1}(a) = b$ if $f(b) = a$.



Note/ $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, x \in \text{Dom } f^{-1}$
 $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x, x \in \text{Dom } f$.

Remark/ A fn f has an inverse iff f is 1-1.

Example/ Let $f(x) = 64x^3 + 7$. Show that :-

[a] f^{-1} exists.

[B] $f^{-1}(x) = \frac{1}{4}\sqrt[3]{x-7}$

[a] let $f(x_1) = f(x_2) \Rightarrow 64x_1^3 + 7 = 64x_2^3 + 7$
 $x_1^3 = x_2^3 \Rightarrow x_1 = x_2 \Rightarrow f$ is 1-1 $\Rightarrow f^{-1}$ exists.

[b] $(f^{-1} \circ f)(x) = f^{-1}(64x^3 + 7) = \frac{1}{4}\sqrt[3]{64x^3 + 7 - 7} = \frac{1}{4}(\sqrt[3]{64} \cdot \sqrt[3]{x^3}) = x$
 $(f \circ f^{-1})(x) = f(\frac{1}{4}\sqrt[3]{x-7}) = 64(\frac{1}{4}\sqrt[3]{x-7})^3 + 7 = x$

So $f^{-1}(x) = \frac{1}{4}\sqrt[3]{x-7}$ is the inverse fn of $f(x)$.

Example/ Let $f(x) = \sqrt{x-1}$. Find the value of a if $f^{-1}(a) = 10$

$f^{-1}(a) = 10$

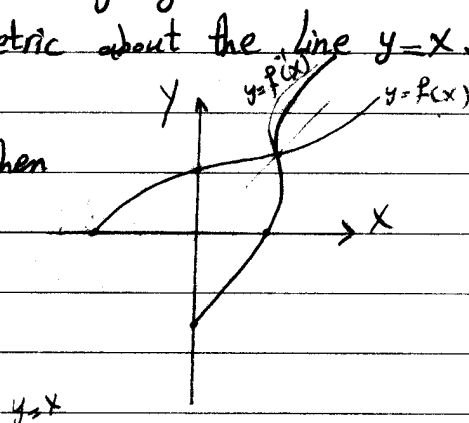
$f(f^{-1}(a)) = f(10) \Rightarrow f(10) = a$

$f(10) = \sqrt{10-1} = ? \Rightarrow \boxed{a=3}$

How to find $f^{-1}(x)$ from the fn $y=f(x)$?

- 1] Solve the eqn $y=f(x)$ to get x as a fn of y .
- 2] To get $f^{-1}(x)$ replace each x by y and y by x .
- 3] The graph of f and f^{-1} are symmetric about the line $y=x$.

* If (a,b) lie in the graph of the fn f , then
 $(b,a) = \dots = f^{-1}$.



Example/ For each of the following fns,

a] Show that f^{-1} exists or not.

b] If f^{-1} exists, find f^{-1} .

1] $f(x) = x^2 - 2$

2] $f(x) = (6-x)^3$

3] $f(x) = (x^3 + 1)^{\frac{1}{3}} + 2$

4] $f(x) = \sin x, x \in [0, 2\pi]$

5] $f(x) = x^2 - 2x, x \leq 1$

6] $f(x) = \frac{\sqrt{x}}{\sqrt{x} - 3}$

1] a] $f(3) = 3^2 - 2 = 7, f(-3) = (-3)^2 - 2 = 7$

$f(3) = f(-3)$ but $3 \neq -3 \Rightarrow f$ is not 1-1 $\Rightarrow f^{-1}$ doesn't exist.

2] a] $f'(x) = 3(6-x)^2(-1) = -3(6-x)^2 \leq 0 \forall x \Rightarrow f$ is decreasing
 $\Rightarrow f$ is 1-1 $\Rightarrow f^{-1}$ exists.

b] $y = (6-x)^3 \Rightarrow \sqrt[3]{y} = 6-x \Rightarrow x = 6 - \sqrt[3]{y}$
 $\Rightarrow f^{-1}(x) = y = 6 - x^{\frac{1}{3}}$

3] a] Let $f(x_1) = f(x_2) \Rightarrow (x_1^3 + 1)^{\frac{1}{3}} + 2 = (x_2^3 + 1)^{\frac{1}{3}} + 2$
 $(x_1^3 + 1)^{\frac{1}{3}} = (x_2^3 + 1)^{\frac{1}{3}} \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$
 $\Rightarrow f$ is 1-1 $\Rightarrow f^{-1}$ exists.

b] $y = (x^3 + 1)^{\frac{1}{3}} + 2 \Rightarrow y - 2 = (x^3 + 1)^{\frac{1}{3}}$
 $\Rightarrow (y-2)^3 = x^3 + 1 \Rightarrow x^3 = (y-2)^3 - 1 \Rightarrow x = \sqrt[3]{(y-2)^3 - 1}$
 $f^{-1}(x) = \sqrt[3]{(x-2)^3 - 1}$

4) a) $f\left(\frac{\pi}{6}\right) = f\left(\frac{5\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}$, $\frac{\pi}{6}, \frac{5\pi}{6} \in [0, 2\pi]$
 But $\frac{\pi}{6} \neq \frac{5\pi}{6} \Rightarrow f$ is not 1-1 in $[0, 2\pi] \Rightarrow f^{-1}$ does not exist.

5) a) $f'(x) = 2x - 2 = 0 \Rightarrow x = 1$ $\leftarrow \text{---} / \text{---} \backslash \text{---} \rightarrow$
 f is decreasing $\forall x \leq 1 \Rightarrow f$ is 1-1 $\Rightarrow f^{-1}$ exists.

b) $y = x^2 - 2x \Rightarrow y = (x^2 - 2x + 1) - 1, x \leq 1$
 $\Rightarrow y + 1 = (x - 1)^2, x \leq 1 \Rightarrow |x - 1| = \sqrt{y + 1} \Rightarrow 1 - x = \sqrt{y + 1}$
 $\Rightarrow x = 1 - \sqrt{y + 1}, x \leq 1$
 $f^{-1}(x) = y = 1 - \sqrt{x + 1}, y \leq 1$
 $1 - \sqrt{x + 1} \leq 1 \Rightarrow \sqrt{x + 1} \geq 0$
 $x + 1 \geq 0 \Rightarrow x \geq -1$

$f^{-1}(x) = 1 - \sqrt{x + 1}, x \geq -1$

6) a) $f(x) = \frac{\sqrt{x}}{\sqrt{x} - 3}, x \in [0, 9) \cup (9, \infty)$

$y = \frac{(\sqrt{x} - 3) + 3}{\sqrt{x} - 3} = 1 + \frac{3}{\sqrt{x} - 3}$

$f'(x) = 3 \cdot \frac{-1}{(\sqrt{x} - 3)^2} \cdot \frac{1}{2\sqrt{x}} = \frac{-3}{2\sqrt{x}(\sqrt{x} - 3)^2} < 0 \quad \forall x \in (0, 9) \cup (9, \infty)$

f is decreasing $\Rightarrow f$ is 1-1 $\Rightarrow f^{-1}$ exists.

b) $y = \frac{\sqrt{x}}{\sqrt{x} - 3} \Rightarrow \sqrt{x}y - 3y = \sqrt{x} \Rightarrow \sqrt{x} - \sqrt{x}y = 3y$

$\sqrt{x}(1 - y) = 3y \Rightarrow \sqrt{x} = \frac{3y}{1 - y} \Rightarrow x = \frac{9y^2}{(1 - y)^2}$

$\Rightarrow f^{-1}(x) = \frac{9x^2}{(1 - x)^2}$

The derivative of inverse fns.

Theorem / if $\text{Dom}(f)$ is the interval I , $f'(x)$ exists and $f'(x) \neq 0$ on I then f^{-1} is diff. at every point in its domain. The value of $(f^{-1})'$ at a point $b = f(a)$ in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$, that is

$$\left. \frac{d}{dx} (f^{-1}(x)) \right|_{x=b=f(a)} = \frac{1}{\left. \frac{d}{dx} (f(x)) \right|_{x=a}}$$

Example / Let $f(x) = x^3 + 2x + 3$. Find $\frac{d}{dx} f^{-1}$ at $x=3$.

$$\text{Put } 3 = a^3 + 2a + 3 \Rightarrow a^3 + 2a = 0$$

$$a(a^2 + 2) = 0 \Rightarrow \underline{a = 0}, a^2 \neq -2$$

$$\left. \frac{d(f^{-1})}{dx} \right|_{x=3=f(0)} = \frac{1}{\left. \frac{df}{dx} \right|_{x=0}} = \frac{1}{3x^2+2} \Big|_{x=0} = \frac{1}{0+2} = \boxed{\frac{1}{2}}$$

Example / Let $f(x) = \frac{2x-1}{x+2}$, $x \neq -2$.

(a) Show that f^{-1} exists.

(b) Find $\left. \frac{d f^{-1}}{dx} \right|_{x=f(0)}$.

$$(a) f'(x) = \frac{(x+2) \cdot 2 - (2x-1) \cdot 1}{(x+2)^2} = \frac{2x+4-2x+1}{(x+2)^2} = \frac{5}{(x+2)^2} > 0 \quad \forall x \neq -2$$

$\Rightarrow f$ is increasing $\Rightarrow f$ is 1-1 $\Rightarrow f^{-1}$ exists.

[b]

$$\left. \frac{d f^{-1}}{dx} \right|_{x=-3} = \frac{1}{\left. \frac{df}{dx} \right|_{x=-1}} = \frac{1}{\frac{5}{(x+2)^2} \Big|_{x=-1}} = \frac{1}{\frac{5}{1}} = \boxed{\frac{1}{5}}$$

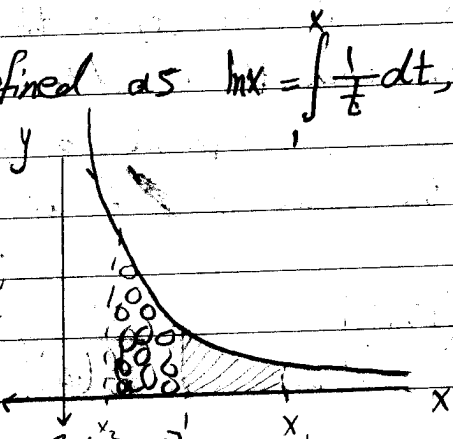
Ex. 1-58
55-60

7.2 ~~Almost~~ Logarithm

Def. The Natural logarithm $\ln x$ is defined as $\ln x = \int_1^x \frac{1}{t} dt, x > 0$

Some properties of $y = \ln x$.

[1] If $x > 1$, then $\ln x = \int_1^x \frac{1}{t} dt$ is the ^{total} area ~~under~~ ^{between} the curve $y = \frac{1}{t}$ and the x -axis on the interval $[1, x]$ $\Rightarrow \ln x > 0 \quad \forall x \in (1, x]$



[2] If $x < 1$, $\ln x = - \int_x^1 \frac{1}{t} dt$ = the negative area between the graph $y = \frac{1}{t}$ and the x -axis on the interval $[x, 1]$

$$\Rightarrow \ln x < 0 \quad \forall x \in [x, 1).$$

$$[3] \text{ If } x = 1 \Rightarrow \ln 1 = \int_1^1 \frac{1}{t} dt = 0 \Rightarrow \boxed{\ln 1 = 0}$$

$$[4] \frac{d}{dx}(\ln x) = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x} > 0 \text{ because } x > 0 \Rightarrow$$

$\ln x$ is increasing $\forall x$

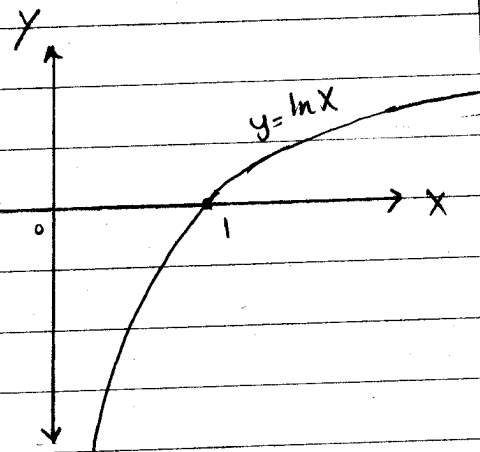
$$[5] \frac{d^2}{dx^2}(\ln x) = \frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2} < 0 \quad \forall x > 0 \Rightarrow$$

the graph of $\ln x$ is concave down.

[6] From the graph of $\ln x$, we have

$$\lim_{x \rightarrow \infty} \ln x = \infty, \quad \lim_{x \rightarrow -\infty} \ln x \text{ undefined}$$

$$\lim_{x \rightarrow 0} \ln x \text{ undefined}$$



[7] If $a, x > 0$ then

(i) $\ln(ax) = \ln a + \ln x$

(ii) $\ln(a/x) = \ln a - \ln x$

(iii) $\ln(1/x) = -\ln x$

(iv) If n is a real num $\Rightarrow \ln x^n = n \ln x$.

[8] $\text{Dom}(\ln x) = (0, \infty)$

[9] $\frac{d}{dx}(\ln x) = \frac{1}{x}$. If u is a diff fn of x , then $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$.

[10] $\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$

$\Rightarrow \int \frac{1}{x} dx = \ln|x| + c$.

$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot -1 = \frac{1}{x}, x < 0$

In general, $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$.

Def. The number e is a num. in the $\text{Dom}(\ln x)$ that satisfy

$\boxed{\ln e = 1}$, $e \approx 2.7$.

Examples:-/

[I] Simplify the following expressions.

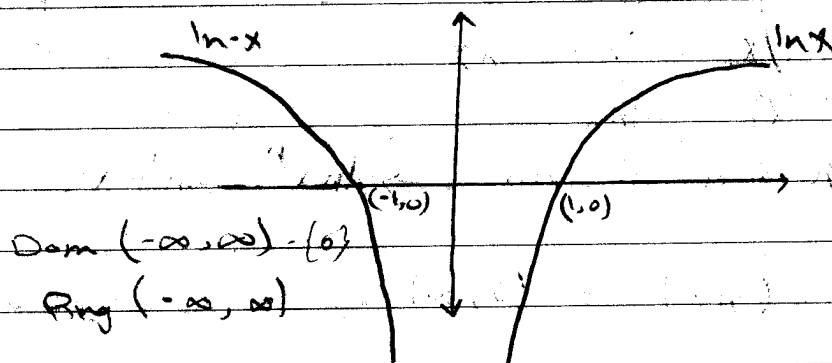
[a] $\frac{\ln 35 + \ln \frac{1}{7}}{\ln 25} = \frac{\ln(5 \cdot 7) - \ln 7}{\ln 5^2} = \frac{\ln 5 + \ln 7 - \ln 7}{2 \ln 5} = \boxed{\frac{1}{2}}$

[b] $2 \ln \sqrt{3} + \frac{1}{2} \ln \left(\frac{4}{9}\right) = \frac{2 \ln 3}{2} + \ln \sqrt{\frac{4}{9}} = \ln 3 + \ln \frac{2}{3} = \ln 3 + \ln 2 - \ln 3 = \boxed{\ln 2}$

Ex 1 Sketch the graph of the following curves:

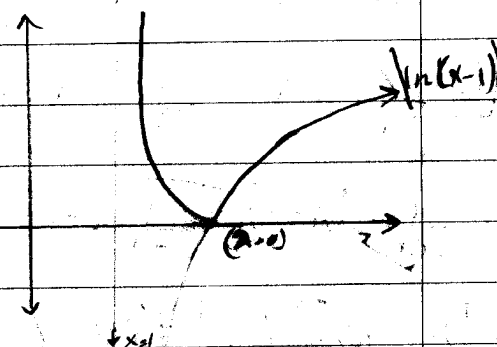
1) $y = \ln|x|$

$$f(x) = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$



Ex 2 $y = \ln(x-1)$

Domain: $(1, \infty)$
Range: $(-\infty, \infty)$



Ex 3 Find the Domain of the function $f(x) = \ln(6x - 2x^2)$

$$\text{Domain} = \{x : 6x - 2x^2 > 0\} = \{x : 2x(3-x) > 0\}$$

$\Rightarrow \text{Domain} = (0, 3)$

Sign $2x$	-	-	+	+	+
Sign $3-x$	+	+	+	-	-
Sign $2x(3-x)$	-	-	+	-	-

Ex 4 Find $\frac{dy}{dx}$ for each of the following:

1) $y = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$

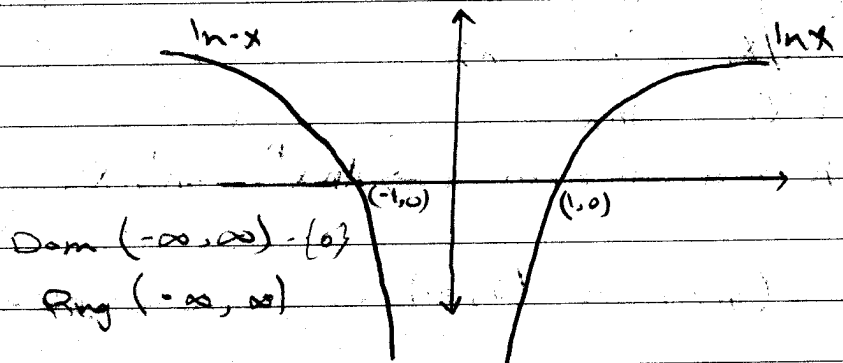
$$\frac{dy}{dx} = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x} \cdot (-1), & x < 0 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

* $\frac{d}{dx}(\ln \sin x) = \frac{1}{\sin x} \cdot \cos x = \cot x$

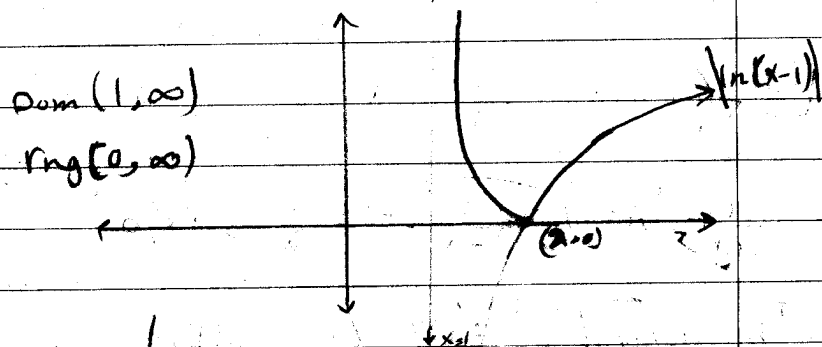
2] Sketch the graph of the following curves:

1] $y = \ln|x|$

$$y = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$



2] $y = |\ln(x-1)|$



3] Find the Dom of the fn $f(x) = \ln(6x - 2x^2)$

$$\text{Dom}(f) = \{x : 6x - 2x^2 > 0\} = \{x : 2x(3-x) > 0\}$$

$$\Rightarrow \text{Dom}(f) = (0, 3)$$

Sign $2x$	-	-	0	+	+	+
Sign $3-x$	+	+	0	-	-	-
Sign $2x(3-x)$	-	-	0	+	-	-

4] Find $\frac{dy}{dx}$ for each of the following:

1] $y = \ln|x| = \begin{cases} \ln x, & x > 1 \\ \ln -x, & x < 1 \end{cases}$

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x}, & x > 1 \\ -\frac{1}{x}, & x < 1 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$\frac{d}{dx} (\ln \sin x) = \frac{1}{\sin x} \cdot \cos x = \cot x$$

$$\boxed{2} \quad y = 10x \ln \sqrt{3+x^2} = 10x \cdot \frac{1}{2} \ln(3+x^2) = 5x \ln(3+x^2)$$

$$\frac{dy}{dx} = 5 \ln(3+x^2) + 5x \cdot \frac{1}{3+x^2} \cdot 2x = 5 \ln(3+x^2) + \frac{10x^2}{3+x^2}$$

$$\boxed{3} \quad y = \ln(\ln(3x+2))$$

$$\frac{dy}{dx} = \frac{1}{\ln(3x+2)} \cdot \frac{1}{3x+2} \cdot 3$$

$$\text{Ex. } y = \ln(\ln(\ln(3x+2))) = \frac{1}{\ln(\ln(3x+2))} \cdot \frac{3}{(3x+2) \ln(3x+2)}$$

$$\boxed{4} \quad y = (\tan(\ln x))^3$$

$$\frac{dy}{dx} = 3 \tan^2(\ln x) \cdot \sec^2(\ln x) \cdot \frac{1}{x}$$

$$\boxed{5} \quad y = \frac{\ln^3 \sqrt{x+1}}{(2-x^2)} = \frac{1}{3} \ln(x+1) - \ln(2-x^2)$$

$$\frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{x+1} - \left(\frac{1}{2-x^2} \cdot -2x \right) = \frac{1}{3(x+1)} + \frac{2x}{2-x^2}$$

$$\boxed{6} \quad y = \int_{\sqrt{x}}^{x^2} \frac{t}{\ln t} dt$$

$$\frac{dy}{dx} = (x^2 \ln x^2) \cdot 2x - (\sqrt{x} \ln \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= 4x^3 \ln x - \frac{1}{4} \ln x$$

$$\boxed{7} \quad y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = \sin x \cdot \ln x$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \frac{\sin x}{x}$$

$$y' = x^{\sin x} \left(\cos x \cdot \ln x + \frac{\sin x}{x} \right)$$

$$\boxed{8} \quad (2y)^x = x^{y^2} \Rightarrow \ln(2y)^x = \ln(x^{y^2}) \Rightarrow x \ln 2y = y^2 \ln x$$

$$\ln 2y + x \cdot \frac{1}{2y} \cdot 2y' = 2y y' \ln x + y^2 \cdot \frac{1}{x}$$

[5] Use logarithmic diff. to find y' for $y = \left(\frac{\sqrt{2x+1} \cdot \sec^3 x}{3(x^2+1)} \right)^3 \Rightarrow \ln y = 3 \ln \left(\frac{\sqrt{2x+1} \cdot \sec^3 x}{3(x^2+1)} \right)$

$$\ln y = 3 \left(\ln \sqrt{2x+1} + \ln \sec^3 x - \ln 3 - \ln(x^2+1) \right)$$

$$= 3 \left(\frac{1}{2} \ln(2x+1) + 2 \ln \sec x - \ln 3 - \ln(x^2+1) \right)$$

$$\frac{y'}{y} = 3 \left(\frac{1}{2} \cdot \frac{1}{2x+1} \cdot 2 + 2 \cdot \frac{1}{\sec x} \cdot \sec x \cdot \tan x - \frac{1}{x^2+1} \cdot 2x \right)$$

$$\frac{y'}{y} = 3 \left(\frac{1}{2x+1} + 2 \tan x - \frac{2x}{x^2+1} \right) \Rightarrow$$

$$y' = 3y \left(\frac{1}{2x+1} + 2 \tan x - \frac{2x}{x^2+1} \right)$$

[6] Evaluate the following integrals:

[1] (i) $\int \cot x \cdot dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$

(ii) $\int \tan x \cdot dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{-\sin x}{\cos x} = -\ln|\cos x| + C$
 $= \ln|\cos x|^{-1} + C = \ln|\sec x| + C$

(iii) $\int \sec x \cdot dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \cdot dx$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + C$$

(iv) $\int \csc x \cdot dx = \int \csc x \cdot \frac{(\csc x - \cot x)}{\csc x - \cot x} dx$

$$= \int \frac{\csc^2 x - \csc x \cdot \cot x}{\csc x - \cot x} dx = \ln|\csc x - \cot x| + C$$

[2] $\int \frac{x^2}{(2x^3+1)} dx = \frac{1}{6} \int \frac{6x^2}{(2x^3+1)} dx = \frac{1}{6} \ln|2x^3+1| + C$

[3] $\int \frac{x+2}{x^2+4x+3} dx = \frac{1}{2} \int \frac{2x+4}{x^2+4x+3} dx = \frac{1}{2} \ln|x^2+4x+3| + C$

$$[4] \int \frac{dx}{x \cot^2(3 + \ln x)}$$

$$\theta = 3 + \ln x$$

$$d\theta = \frac{1}{x} dx$$

$$= \int \frac{1}{\cot^2 \theta} d\theta = \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C$$

$$= \tan(3 + \ln x) - 3 - \ln x + C = \tan(3 + \ln x) - \ln x + C$$

$$C = 3 + C_1$$

$$[5] \int \frac{\ln x}{x(3 + \ln x)} dx$$

$$w = 3 + \ln x$$

$$dw = \frac{1}{x} dx$$

$$= \int \frac{w-3}{w} dw = \int 1 - \frac{3}{w} dw$$

$$= w - 3 \ln|w| + C = 3 + \ln x - 3 \ln|3 + \ln x| + C$$

$$[6] \int \frac{dx}{\sqrt{x} + x} = \int \frac{dx}{\sqrt{x}(1 + \sqrt{x})}$$

$$w = 1 + \sqrt{x}$$

$$dw = \frac{1}{2\sqrt{x}} dx$$

$$2dw = \frac{1}{\sqrt{x}} dx$$

$$= 2 \int \frac{1}{w} dw = 2 \ln|w| + C = 2 \ln|1 + \sqrt{x}| + C$$

$$[7] \int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}} dx$$

$$z = \ln(\sec x + \tan x)$$

$$dz = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) dx$$

$$dz = \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \sec x dx$$

$$= \int \frac{1}{(z)^{\frac{1}{2}}} dz = \int z^{-\frac{1}{2}} dz$$

$$= 2\sqrt{z} + C = 2\sqrt{\ln(\sec x + \tan x)} + C$$

$$[8] \int \tan \cdot \ln(\cos x) \cdot dx$$

$$t = \ln(\cos x)$$

$$dt = \frac{1}{\cos x} \cdot -\sin x \cdot dx$$

$$dt = -\tan x \cdot dx$$

$$-dt = \tan x \cdot dx$$

$$= - \int t \cdot dt = -\frac{t^2}{2} + C$$

$$= -\frac{1}{2} (\ln(\cos x))^2 + C$$

7.3 The exponential fns:-

* Since $\ln x$ is an increasing fn, $\ln x$ is 1-1 $\Rightarrow \ln x$ has an inverse fn denoted by $\ln^{-1} x$.

* From the previous sections, we have
 $\ln e = 1 \Rightarrow e = \ln^{-1} 1 \approx 2.7$

Def. For any real num. x , the exponential fn $\text{Exp}(x)$ is the inverse fn of $\ln x$. That is, $\text{Exp}(x) = e^x = \ln^{-1} x$.

* From the above def, we have

$$\boxed{1} \quad \text{Dom}(e^x) = \text{Rng}(\ln x) = (-\infty, \infty)$$

$$\text{Rng}(e^x) = \text{Dom}(\ln x) = (0, \infty)$$

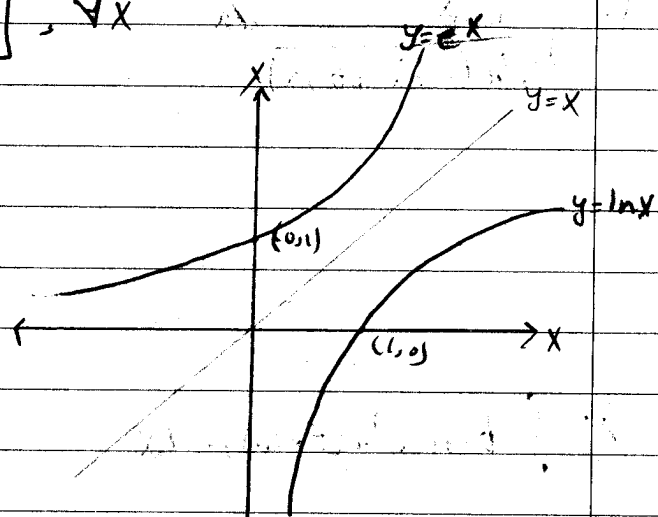
$$\boxed{2} \quad \ln e = \ln(\ln^{-1} 1) = 1 \Rightarrow \underline{\ln e = 1}$$

$$\boxed{3} \quad (i) e^{\ln x} = \ln^{-1}(\ln x) = x, \forall x > 0$$

$$(ii) \ln e^x = x \ln e = x \Rightarrow \boxed{\ln e^x = x}, \forall x$$

$\boxed{4}$ The graph of $y = e^x$ as shown
 $e^x > 0, \forall x$

$$\boxed{5} \quad \lim_{x \rightarrow \infty} e^x = \infty, \lim_{x \rightarrow -\infty} e^x = 0$$



Some information about the fn $y = e^x$.

II $e^x > 0 \quad \forall x$

(i) $e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}$

(ii) $e^{x_1-x_2} = e^{x_1} / e^{x_2}$

(iii) $e^{-x} = 1/e^x$

(iv) $(e^{x_1})^{x_2} = e^{x_1 \cdot x_2}$

III If $y = e^x$ then $\frac{dy}{dx} = e^x$.

Proof/ $\ln y = \ln e^x = x \ln e = x$

$\Rightarrow \frac{y'}{y} = 1 \Rightarrow y' = y = e^x$

* If u is a diff fn of x , then $\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$.

For example/ ~~$\frac{d}{dx}(e^{\sin x}) = \cos x \cdot e^{\sin x}$~~

$\frac{d}{dx}(e^{\sin x}) = \cos x \cdot e^{\sin x}$

IV $\int e^x dx = e^x + C$

$\int e^{ax} dx = \frac{e^{ax}}{a} + C \Rightarrow \int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C$

because $\int e^{ax+b} dx = \int e^{ax} \cdot e^b dx = e^b \int e^{ax} dx = e^b \cdot \frac{e^{ax}}{a} + C = \frac{e^{ax+b}}{a} + C$

* Theorem: $e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = \lim_{y \rightarrow \infty} (1+\frac{1}{y})^y$

Proof/ let $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1 =$

$f'(1) = \lim_{h \rightarrow 0} \frac{f(h+1) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\ln(h+1) - \ln(1)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln(1+h)$

$1 = \lim_{h \rightarrow 0} \ln(1+h)^{\frac{1}{h}} \Rightarrow 1 = \ln(\lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}})$

$e = e^{\ln(\lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}})} \Rightarrow e = \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}}$

Put $h = \frac{1}{y} \Rightarrow y = \frac{1}{h} \Rightarrow e = \lim_{y \rightarrow \infty} (1+\frac{1}{y})^y$

$\left. \begin{array}{l} \frac{1}{y} \rightarrow 0 \\ y \rightarrow \infty \end{array} \right\}$

##

Example 1 / If $f(x) = e^{2x}$. Find $f^{-1}(x)$ if it exists.

$$f'(x) = 2e^{2x} > 0 \quad \forall x \quad \Rightarrow f \text{ is increasing fn} \Rightarrow f \text{ is 1-1}$$

$$\begin{aligned} \ln y &= \ln e^{2x} \Rightarrow \ln y = 2x \ln e \Rightarrow 2x = \ln y \Rightarrow x = \frac{1}{2} \ln y = \ln \sqrt{y} \\ y &= \ln \sqrt{x} = f^{-1}(x) \end{aligned}$$

Example / Find the Dom of the fn $f(x) = \ln(\ln(e^{3x} - 7))$

$$\begin{aligned} \text{Dom}(f) &= \{x : \ln(e^{3x} - 7) > 0\} \\ &= \{x : e^{\ln(e^{3x} - 7)} > e^0\} = \{x : e^{3x} - 7 > 1\} \\ &= \{x : e^{3x} > 8\} = \{x : \ln e^{3x} > \ln 8\} \\ &= \{x : 3x > \ln 8\} = \{x : x > \ln 2\} = (\ln 2, \infty) \end{aligned}$$

Example / Simplify the expression $e^{3 \ln 2} - \ln \sqrt{e^8}$

$$e^{\ln 2^3} - \ln e^3 = 2^3 - 3 \ln e = 8 - 3 = \boxed{5}$$

Example / [A] Solve for x ~~the fn~~ $2e^{\frac{x}{2}} = \ln 2 - \ln 3$

$$2e^{\frac{x}{2}} = \ln(3 \times 4) - \ln 3 = \ln 3 + \ln 4 - \ln 3 = \ln 2^2$$

$$2e^{\frac{x}{2}} = 2 \ln 2$$

$$\frac{x}{2} \ln e = \ln(\ln 2) \Rightarrow x = 2 \ln(\ln 2).$$

[B] Solve y in terms of x for the eqn

$$\ln(y-1) = -\ln 2 = X + \ln X$$

$$\ln\left(\frac{y-1}{2}\right) = X + \ln X \Rightarrow$$

$$e^{\ln\left(\frac{y-1}{2}\right)} = e^{X + \ln X}$$

$$\frac{1}{2}(y-1) = e^X \cdot e^{\ln X} = e^X \cdot X$$

$$y-1 = 2Xe^X \Rightarrow y = 1 + 2Xe^X.$$

Example/ Find y' for each of the following.

a) $y = e^{x^3+2}$ $y' = e^{x^3+2} \cdot 3x^2 = 3x^2 e^{x^3+2}$

b) $y = \sqrt{e^x} \cdot \ln(e^x+1)$ $= e^{\frac{1}{2}x} \cdot \ln(e^x+1)$
 $y' = \frac{1}{2} e^{\frac{1}{2}x} \ln(e^x+1) + e^{\frac{1}{2}x} \cdot \frac{1}{e^x+1} \cdot e^x$

c) $y = \sin(e^{3x}) + e^{\sin(3x)}$
 $y' = \cos(e^{3x}) \cdot 3e^{3x} + e^{\sin(3x)} \cdot \cos(3x) \cdot 3$

d) $y = e^{\sqrt{x}} \cdot x^{\sin x} \rightarrow \ln y = \ln(e^{\sqrt{x}} \cdot x^{\sin x}) = \sqrt{x} \ln e + \sin x \ln x$
 $\frac{1}{y} \cdot y' = \frac{1}{2\sqrt{x}} + \cos x \ln x + \frac{\sin x}{x}$

$y' = y \left(\frac{1}{2\sqrt{x}} + \cos x \ln x + \frac{\sin x}{x} \right) =$

e) $y = \ln(e^{-x} + xe^{-x})$
 $y' = \frac{1}{e^{-x} + xe^{-x}} \cdot (e^{-x} \cdot -1 + e^{-x} - xe^{-x}) = \frac{-xe^{-x}}{e^{-x} + xe^{-x}}$

f) $y = \int_{\ln x}^{2\ln x} \sin e^t dt$

$y' = (\sin e^{2\ln x}) \cdot \frac{2}{x} - (\sin e^{\ln x}) \cdot \frac{1}{x}$

$= \frac{2}{x} \sin e^{2\ln x} - \frac{1}{x} \sin e^{\ln x} = \frac{2}{x} \sin x^2 - \frac{1}{x} \sin x$

g) $e^{\frac{1}{y}} = \ln(xe^{y+1}) = \ln x + \ln e^{y+1} = \ln x + y + 1$

$e^{\frac{1}{y}} \cdot \left(-\frac{1}{y^2}\right) \cdot y' = \frac{1}{x} + y'$

Example / Evaluate the following integrals.

$$[1] \int e^{-5x+8} dx = \frac{e^{-5x+8}}{-5} + C$$

$$[2] \int \frac{\sin e^{-3x}}{e^{3x}} dx$$

$$\rightarrow \frac{-1}{3} \int \sin t dt = \frac{1}{3} \cos t + C$$

$$= \frac{1}{3} \cos e^{-3x} + C$$

$$\begin{cases} t = e^{-3x} \\ dt = e^{-3x} \cdot -3 dx \\ \frac{-1}{3} dt = \frac{1}{e^{3x}} dx \end{cases}$$

$$[3] \int_0^{\sqrt{\pi}} 2x e^{x^2} \cos(e^{x^2}) dx$$

$$\rightarrow \int_1^{\pi} \cos \theta d\theta = \sin \theta \Big|_1^{\pi} = \sin \pi - \sin 1 = \boxed{-\sin 1}$$

$$\begin{cases} \theta = e^{x^2} \\ d\theta = 2x e^{x^2} dx \end{cases}$$

$$[4] \int \frac{e^{\frac{1}{x}}}{x^2} dx$$

$$\Rightarrow \int e^w dw = e^w + C = e^{\frac{1}{x}} + C$$

$$\begin{cases} w = \frac{1}{x} \\ dw = \frac{-1}{x^2} dx \end{cases}$$

$$[5] \int \frac{e^{2x}}{e^x - 1} dx = \int \frac{e^x \cdot e^x}{e^x - 1} dx$$

$$\begin{cases} z = e^x - 1 \\ dz = e^x dx \end{cases}$$

$$\Rightarrow \int \frac{1+z}{z} dz = \int \frac{1}{z} + 1 dz$$

$$= \ln|z| + z + C = \ln|e^x - 1| + e^x - 1 + C$$

$$[6] \int \frac{1}{1+e^x} dx = \int \frac{(1+e^x) - e^x}{1+e^x} dx = \int 1 - \frac{e^x}{1+e^x} dx$$

$$= x - \ln|1+e^x| + C$$

$$= x - \ln(e^x + 1) + C$$

$$\int \frac{e^{-x}}{e^{-x}(1+e^x)} dx = \int \frac{e^{-x}}{(e^{-x}+1)} dx = - \int \frac{-e^{-x}}{e^{-x}+1} dx$$

$$= -\ln|e^{-x}+1| + C$$

$$\boxed{7} \int \frac{e^{\sec x}}{\cos x \cdot \cot x} dx$$

$$t = \sec x$$

$$dt = \sec x \cdot \tan x dx$$

$$dt = \frac{1}{\cos x \cdot \cot x} dx$$

$$\Rightarrow \int e^t dt = e^t + C = e^{\sec x} + C$$

$$\boxed{8} \int \frac{e^x + 1}{e^x - 1} dx = \int 1 + \frac{2}{e^x - 1} dx$$

$$= \int 1 dx + \int \frac{2e^{-x}}{e^{-x}(e^x - 1)} dx$$

$$= x + 2 \int \frac{e^{-x}}{1 - e^{-x}} = x + 2 \ln |1 - e^{-x}| + C$$

$$\frac{e^x - 1}{2} \left[\frac{e^x + 1}{e^x - 1} \right]$$

* The fns a^x and $\log_a x$

Def. If a is a positive real num, then for any real num x
 $a^x = e^{x \ln a}$

* Some informations about $y = a^x$

$\boxed{1}$ If $a > 0$ and any real num n and y , we have

$\boxed{1}$

$$(i) a^x \cdot a^y = a^{x+y}$$

$$(ii) a^{-x} = \frac{1}{a^x}$$

$$(iii) a^x / a^y = a^{x-y}$$

$$(iv) (a^x)^y = a^{x \cdot y}$$

$$\boxed{2} \frac{d}{dx} (a^x) = \frac{d}{dx} (e^{x \ln a}) = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$$

$$\Rightarrow \frac{da^x}{dx} = a^x \cdot \ln a$$

* If u is a diff fn of x , then $\frac{d}{dx} (a^u) = a^u \cdot \ln a \cdot \frac{du}{dx}$

$$\frac{d}{dx}(3^{\sin x}) = 3^{\sin x} \cdot \ln 3 \cdot \cos x$$

$$\frac{d}{dx}(x^x) = \frac{d}{dx}(e^{x \ln x}) = e^{x \ln x} \left(\ln x + \frac{x}{x} \right) = x^x (1 + \ln x)$$

$$\frac{d}{dx}(\sin x^{\cos x}) = \frac{d}{dx}(e^{\cos x \ln \sin x}) = e^{\cos x \ln \sin x} (-\sin x \ln \sin x + \cos x \cdot \frac{1}{\sin x} \cdot \cos x)$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

[5] The graph of $y = a^x$ is the graph of $y = e^x$

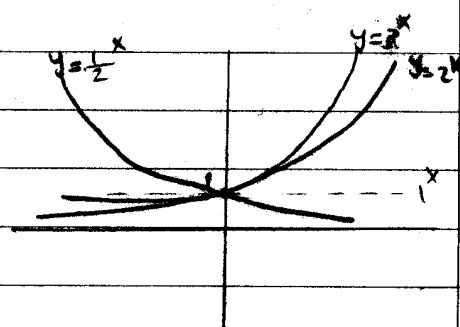
$$\lim_{x \rightarrow \infty} a^x = \infty$$

$$\lim_{x \rightarrow -\infty} a^x = \infty$$

$$\lim_{x \rightarrow \infty} a^{-x} = 0$$

$$\lim_{x \rightarrow -\infty} a^{-x} = 0$$

$a > 1$



positive

Def. For any real num. $a \neq 1$ the fn $\log_a x$ is the inverse fn of a^x , that is, if $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.

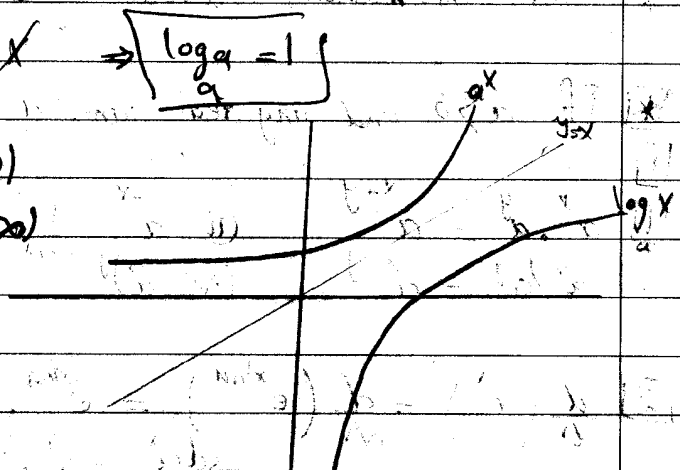
Some information about $y = \log_a x$.

$$(i) a^{\log_a x} = x, x > 0$$

$$(ii) \log_a a^x = x, \forall x \Rightarrow x \log_a a = x \Rightarrow \log_a a = 1$$

$$\text{Dom}(\log_a x) = \text{Rng}(a^x) = (0, \infty)$$

$$\text{Rng}(\log_a x) = \text{Dom}(a^x) = (-\infty, \infty)$$



$$\text{If } a \neq 1 \text{ then } \log_a x = \frac{\ln x}{\ln a}$$

Proof. $a^{\log_a X} = X \Rightarrow \ln a^{\log_a X} = \ln X$
 $= \log_a X \cdot \ln a = \ln X \Rightarrow \log_a X = \frac{\ln X}{\ln a}$

[5] For any positive real num X and y , we have:

(i) $\log_a (X \cdot y) = \log_a X + \log_a y$

(ii) $\log_a (X/y) = \log_a X - \log_a y$

(iii) $\log_a (1/X) = -\log_a X$ (iv) $\log_a X^n = n \log_a X$

[6] $\frac{d}{dx} (\log_a X) = \frac{d}{dx} \left(\frac{\ln X}{\ln a} \right) = \frac{1}{\ln a} \cdot \frac{1}{X} = \frac{1}{X \ln a}$

* If u is a diff fn of X , then $\frac{d}{dx} (\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$

* For example $\frac{d}{dx} (\log_3 (x^2 + \sin x)) = \frac{1}{\ln 3 \cdot (x^2 + \sin x)} \cdot (2x + \cos x)$

Examples:-

[1] Simplify the expression $\frac{1}{2} \log_2 (e^{(\ln 2) \sin x}) = \log_4 2^{\sin x} + 2^{\log_4 x^2}$

$\frac{1}{2} \cdot \frac{\ln(e^{(\ln 2) \sin x})}{\ln 2} = \sin \log_4 2 + (2)^{\frac{2 \log_4 x}{2}}$

$= \frac{1}{2} \cdot \frac{\ln 2 \cdot \sin x}{\ln 2} = \sin x \cdot \frac{\ln 2}{\ln 2^2} + 4^{\frac{\log_4 x}{2}}$

$= \frac{1}{2} \sin x - \frac{1}{2} \sin x + X = \boxed{X}$

[2] Solve for X $\ln e + 4^{-2 \log_4 X} = \frac{1}{X} \log_{10} (100)$

$1 + 4^{\log_4 X^{-2}} = \frac{1}{X} \log_{10} 10^2$

$1 + X^{-2} = \frac{1}{X} \cdot 2 \Rightarrow 1 + \frac{1}{X^2} = \frac{2}{X}$

$X^2 + 1 = 2X \Rightarrow X^2 - 2X + 1 = 0$

$(X-1)^2 = 0 \Rightarrow X-1 = 0 \Rightarrow \boxed{X=1}$

Q3 Find $\frac{dy}{dx}$ for each of the following.

1) $y = 3^{(2x-1)} + 2^{\frac{1}{x}}$

$$\frac{dy}{dx} = 3^{(2x-1)} \ln 3 \cdot 2 + 2^{\frac{1}{x}} \cdot \ln 2 \cdot \left(-\frac{1}{x^2}\right)$$

$$= 2 \ln 3 \cdot 3^{(2x-1)} - \frac{\ln 2}{x^2} \cdot 2^{\frac{1}{x}}$$

2) $y = 2^{\tan x} + \log_2 \sqrt{x^2+1}$

$$\frac{dy}{dx} = 2^{\tan x} \cdot \ln 2 \cdot \sec^2 x + \frac{1}{2} \cdot \frac{1}{(x^2+1) \cdot \ln 2} \cdot 2x$$

$$= \ln 2 \cdot 2^{\tan x} \cdot \sec^2 x + \frac{x}{\ln 2 \cdot (x^2+1)}$$

3) $y = \log_2 \left(\frac{x^2 e^2}{2\sqrt{x+1}} \right)$

$$= \log_2 x^2 + \log_2 e^2 - \log_2 2 - \log_2 \sqrt{x+1}$$

$$= 2 \log_2 x + 2 \log_2 e - 1 - \frac{1}{2} \log_2 (x+1)$$

$$\frac{dy}{dx} = 2 \cdot \frac{1}{x \ln 2} + \frac{1}{2} \cdot \frac{1}{(x+1) \ln 2} \cdot 1$$

4) $y = 3 \log_8 (\log_2 x)$

$$\frac{dy}{dx} = 3 \cdot \frac{1}{\log_2 x \cdot \ln 8} \cdot \frac{1}{x \ln 2} = \frac{3}{\log_2 x \cdot 3 \ln 2 \cdot x \ln 2}$$

$$= \frac{1}{\frac{\ln x}{\ln 2} \cdot \ln 2 \cdot x \ln 2} = \frac{1}{x \ln x \cdot \ln 2}$$

[4] Evaluate the following integrals:

1) $\int_0^1 \left(\frac{1}{3}\right)^{\tan x} \cdot \sec^2 x \, dx$

$$\begin{cases} t = \tan x \\ dt = \sec^2 x \, dx \end{cases}$$

$$\rightarrow \int_0^1 \left(\frac{1}{3}\right)^t dt = \left[\frac{\left(\frac{1}{3}\right)^t}{\ln \frac{1}{3}} \right]_0^1$$

$$= \frac{-1}{\ln 3} \left(\frac{1}{3}^1 - \frac{1}{3}^0 \right) = \frac{-1}{\ln 3} \left(\frac{1}{3} - 1 \right) = \frac{2}{3 \ln 3} = \frac{2}{\ln 27}$$

2) $\int \frac{(3^x + 1)^2}{3^x} dx = \int \frac{3^{2x} + 2 \cdot 3^x + 1}{3^x} dx$

$$= \int 3^x + 2 + 3^{-x} dx$$

$$= \frac{3^x}{\ln 3} + 2x - \frac{3^{-x}}{\ln 3} + C$$

3) $\int \frac{dx}{x(\log x)^2}$

$$w = \log x \Rightarrow dw = \frac{1}{x \ln 8} dx$$

$$\Rightarrow \ln 8 \int \frac{1}{w^2} dw = \ln 8 \int w^{-2} dw$$

$$= \ln 8 \cdot \frac{w^{-1}}{-1} + C = -\frac{\ln 8}{w} + C$$

$$= -\frac{\ln 8}{\log x} + C = \frac{-(\ln 8)^2}{\ln x} + C$$

4) $\int \frac{x \cdot 2^{x^2}}{1 + 2^{x^2}} dx$

$$\begin{aligned} y &= 1 + 2^{x^2} \\ dy &= 2^{x^2} \cdot \ln 2 \cdot 2x \, dx \\ \frac{dy}{2 \ln 2} &= x \cdot 2^{x^2} \, dx \end{aligned}$$

$$\rightarrow \frac{1}{2 \ln 2} \int \frac{1}{y} dy$$

$$= \frac{1}{2 \ln 2} \cdot \ln |y| + C = \frac{1}{2 \ln 2} \ln (1 + 2^{x^2}) + C$$

$$\boxed{5} \int x^{2x} (1 + \ln x) dx$$

$$\Rightarrow \int \frac{1}{2} dy$$

$$= \frac{1}{2} y + C$$

$$= \frac{1}{2} x^{2x} + C$$

$$y = x^{2x} = e^{2x \ln x}$$

$$dy = e^{2x \ln x} \cdot (2 \ln x + \frac{1}{x} \cdot 2x) dx$$

$$dy = 2e^{2x \ln x} (\ln x + 1) dx$$

$$\frac{1}{2} dy = x^{2x} (1 + \ln x) dx$$

7.5 L'Hopital's Rule

① If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example Find the following limits:

$$\boxed{1} \lim_{x \rightarrow 0} \frac{\sin x - x^3}{x^3} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow 0} \frac{-\sin x}{6x} = -\frac{1}{6} \lim_{x \rightarrow 0} \frac{\sin x}{x} = \boxed{-\frac{1}{6}}$$

$$\boxed{2} \lim_{x \rightarrow 0} \frac{x \cdot 2^x}{2^x - 1} = \frac{0}{0} \Rightarrow$$

$$\lim_{x \rightarrow 0} \frac{x \cdot 2^x \cdot \ln 2 + 2^x}{2^x \ln 2} = \lim_{x \rightarrow 0} \frac{x \ln 2 + 1}{\ln 2} = \frac{0 + 1}{\ln 2} = \boxed{\frac{1}{\ln 2}}$$

$$\boxed{3} \lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{x \sin x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{2(e^x - 1) \cdot e^x}{x \cos x + \sin x} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow 0} \frac{2e^{2x} - 2e^x}{x \cos x + \sin x}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{4e^{2x} - 2e^x}{-x \sin x + \cos x + \cos x} = \frac{4 - 2}{0 + 1 + 1} = \boxed{1}$$

$$[4] \lim_{x \rightarrow \infty} \frac{\log_2 x}{\log(x+3)} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x} \ln 2}{(x+3) \ln 3} = \lim_{x \rightarrow \infty} \frac{\ln 3 (x+3)}{x \ln 2} = \frac{\ln 3}{\ln 2} \lim_{x \rightarrow \infty} \frac{x+3}{x} = \frac{\infty}{\infty} \Rightarrow$$

$$\frac{\ln 3}{\ln 2} \lim_{x \rightarrow \infty} \frac{1}{1} = \boxed{\frac{\ln 3}{\ln 2}} = \boxed{\log_2 3}$$

$$[5] \lim_{x \rightarrow \infty} \frac{3^x + 2x}{x^2 + 1} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{3^x \ln 3 + 2}{2x + 1} = \frac{\infty}{\infty} \rightarrow \lim_{x \rightarrow \infty} \frac{3^x (\ln 3)^2}{2} = \frac{(\ln 3)^2}{2} \cdot \infty = \boxed{\infty}$$

[2] ~~Indeterminate~~ Indeterminate product and difference: $\infty - \infty$ and $\infty \cdot \infty$

Example Find the following limits:-

$$[1] \lim_{x \rightarrow 0^+} x (\ln x)^2 = 0 \cdot (-\infty)^2 = 0 \cdot \infty \Rightarrow$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{(\ln x)^2}{\frac{1}{x}} &= \frac{\infty}{\infty} \Rightarrow \lim_{x \rightarrow 0^+} \frac{2 \ln x \cdot \frac{1}{x}}{\frac{-1}{x^2}} = -2 \lim_{x \rightarrow 0^+} \frac{(\ln x)}{\frac{1}{x}} = \frac{\infty}{\infty} \\ &= -2 \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = 2 \lim_{x \rightarrow 0^+} \frac{1}{\frac{1}{x}} = 2 \lim_{x \rightarrow 0^+} x = \boxed{0} \end{aligned}$$

$$[2] \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x \right) \tan x = 0 \cdot \infty \Rightarrow \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x \right) \frac{\sin x}{\cos x} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x \right) \cos x - \sin x}{-\sin x} = \frac{0 - 1}{-1} = \boxed{1}$$

$$[3] \lim_{x \rightarrow 0^+} \ln x - \ln \sin x = -\infty + \infty \Rightarrow \lim_{x \rightarrow 0^+} \ln \frac{x}{\sin x}$$

$$= \ln \left(\lim_{x \rightarrow 0^+} \frac{x}{\sin x} \right) = \ln \left(\lim_{x \rightarrow 0^+} \frac{1}{\cos x} \right) = \ln \left(\frac{1}{1} \right) = \ln 1 = \boxed{0}$$

$$[4] \lim_{x \rightarrow 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \infty - \infty \Rightarrow$$

$$\lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{(x-1) \cdot \frac{1}{x} + \ln x} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x} \Rightarrow \lim_{x \rightarrow 1^+} \frac{-\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{-1}{1+1} = \boxed{-\frac{1}{2}}$$

[5] ~~Indeterminate~~ powers: 1^∞ , 0^∞ and ∞^0

Remark / If $\lim_{x \rightarrow a} \ln f(x) = L$ then $\ln(\lim_{x \rightarrow a} f(x)) = L \Rightarrow$

$$\lim_{x \rightarrow a} f(x) = e^L$$

Example / Find the following limits:-

$$[1] \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^x$$

$$* f(x) = \left(1 + \frac{a}{x} \right)^x$$

$$* \ln f(x) = \ln \left(1 + \frac{a}{x} \right)^x = x \ln \left(1 + \frac{a}{x} \right)$$

$$* \lim_{x \rightarrow \infty} (\ln f(x)) = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{a}{x} \right) = \infty \cdot 0$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{a}{x} \right)}{\frac{1}{x}} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{a}{x}} \cdot \frac{-a}{x^2}}{\frac{-1}{x^2}} = \lim_{x \rightarrow \infty} \frac{a}{1 + \frac{a}{x}} = \frac{a}{1+0} = \underline{a}$$

$$* \text{So } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^x = \boxed{e^a}$$

$$\textcircled{2} \lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1} \right)^x = \lim_{x \rightarrow \infty} \left(\frac{1+\frac{2}{x}}{1-\frac{1}{x}} \right)^x = 1^\infty$$

$$\star \text{ Let } f(x) = \left(\frac{x+2}{x-1} \right)^x$$

$$\star \ln(f(x)) = \ln \left(\frac{x+2}{x-1} \right)^x = x \ln \left(\frac{x+2}{x-1} \right)$$

$$\star \lim_{x \rightarrow \infty} (\ln f(x)) = \lim_{x \rightarrow \infty} x \ln \left(\frac{x+2}{x-1} \right) = \lim_{x \rightarrow \infty} x \ln \left(\frac{1+\frac{2}{x}}{1-\frac{1}{x}} \right) = \infty \cdot 0$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left[x \ln \left(1 + \frac{2}{x} \right) - x \ln \left(1 - \frac{1}{x} \right) \right] = \infty \cdot 0 - \infty \cdot 0$$

$$= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{2}{x} \right)}{\frac{1}{x}} - \lim_{x \rightarrow \infty} \frac{\ln \left(1 - \frac{1}{x} \right)}{\frac{1}{x}} = \frac{0}{0} - \frac{0}{0}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\frac{1}{1+\frac{2}{x}} \cdot \frac{-2}{x^2}}{\frac{-1}{x^2}} - \lim_{x \rightarrow \infty} \frac{\frac{1}{1-\frac{1}{x}} \cdot \frac{1}{x^2}}{\frac{-1}{x^2}} = 2 \lim_{x \rightarrow \infty} \frac{1}{1+\frac{2}{x}} + \lim_{x \rightarrow \infty} \frac{1}{1-\frac{1}{x}} = 2 + 1 = 3$$

$$\text{So } \lim_{x \rightarrow \infty} \left(\frac{x+2}{x-1} \right)^x = \boxed{e^3}$$

$$\textcircled{3} \lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}}$$

$$\star f(x) = (e^x + x)^{\frac{1}{x}}$$

$$\star \ln(f(x)) = \frac{1}{x} \ln(e^x + x)$$

$$\star \lim_{x \rightarrow 0} \ln f(x) = \lim_{x \rightarrow 0} \frac{1}{x} \ln(e^x + x) = \infty \cdot 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow 0} \frac{\frac{1}{e^x + x} \cdot (e^x + 1)}{1}$$

$$= \lim_{x \rightarrow 0} \frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

$$\text{So } \lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} = \boxed{e^2}$$

$$[4] \lim_{x \rightarrow \infty} (x^3 + e)^{\frac{1}{\ln x}} = \infty^0$$

$$\text{let } f(x) = (x^3 + e)^{\frac{1}{\ln x}}$$

$$\ln f(x) = \frac{1}{\ln x} \ln(x^3 + e)$$

$$\lim_{x \rightarrow \infty} \ln f(x) = \lim_{x \rightarrow \infty} \frac{\ln(x^3 + e)}{\ln x} = \frac{\infty}{\infty} \Rightarrow$$

$$\lim_{x \rightarrow \infty} \frac{\frac{3x^2}{x^3 + e}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{3x^2}{x^3 + e} = \frac{\infty}{\infty} \Rightarrow$$

$$\lim_{x \rightarrow \infty} \frac{9x^2}{3x^2} = \underline{\underline{3}}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \ln f(x) = 3 \Rightarrow \boxed{\lim_{x \rightarrow \infty} f(x) = e^3}$$

$$[5] \lim_{x \rightarrow 0^+} x^{\sin x} = 0^0$$

$$\text{let } f(x) = x^{\sin x} \Rightarrow \ln f(x) = \sin x \ln x$$

$$\lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \sin x \cdot \ln x = 0 \cdot \infty \Rightarrow$$

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\sec x} = \frac{-\infty}{\infty} \Rightarrow \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\csc x \cdot \cot x} = -\lim_{x \rightarrow 0^+} \frac{\sin x}{x}$$

$$-\lim_{x \rightarrow 0^+} \frac{\sin^2 x}{\cos x \cdot x} = -\lim_{x \rightarrow 0^+} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0^+} \frac{\sin x}{\cos x} = -1 \cdot 0 = 0$$

$$\Rightarrow \lim_{x \rightarrow 0^+} x^{\sin x} = e^0 = \boxed{1}$$

Example/Find the value of the constant a such that

$$\lim_{x \rightarrow 0} (a^x + x)^{\frac{1}{x}} = 4e$$

$$\lim_{x \rightarrow 0} (a^x + x)^{\frac{1}{x}} = 1^\infty$$

* Let $f(x) = (a^x + x)^{\frac{1}{x}} \Rightarrow \ln f(x) = \frac{1}{x} \ln(a^x + x)$

$$\lim_{x \rightarrow 0} \ln f(x) = \lim_{x \rightarrow 0} \frac{\ln(a^x + x)}{x} = \frac{0}{0} \Rightarrow \text{L'Hôpital's Rule}$$

$$\lim_{x \rightarrow 0} \frac{a^x \ln a + 1}{1} = \lim_{x \rightarrow 0} \frac{a^x \ln a + 1}{a^x + x} = \frac{\ln a + 1}{\ln a + 1} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} (a^x + x)^{\frac{1}{x}} = e^1 = e$$

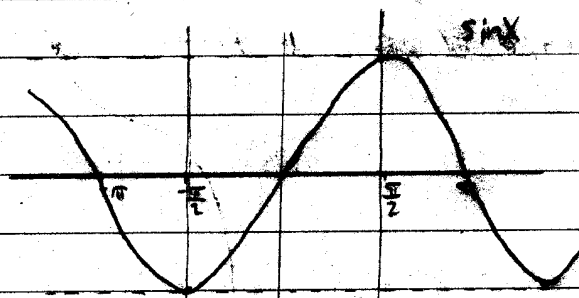
$$\Rightarrow \boxed{a = 4}$$

7.6 Inverse Trigonometric fns.

[I] $y = \sin^{-1} x = \arcsin x$

$$\text{Dom}(\sin x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Rng}(\sin x) = [-1, 1]$$

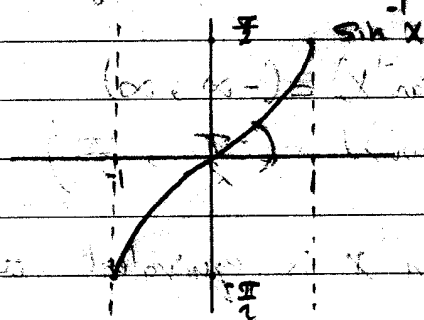


* Some properties of $y = \sin^{-1}(x)$

[1] $\text{Dom}(\sin^{-1} x) = [-1, 1]$

$$\text{Rng}(\sin^{-1} x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

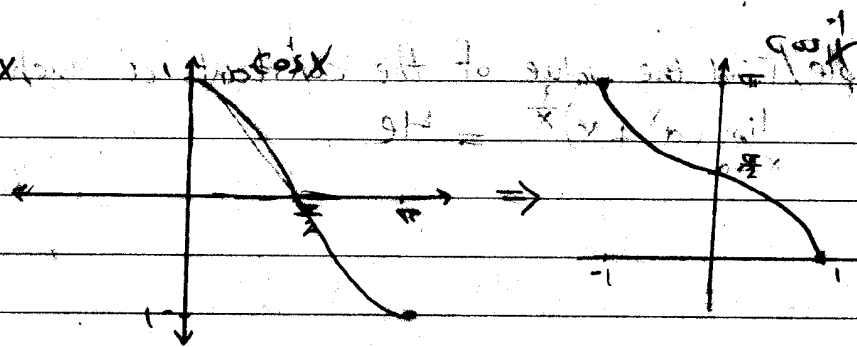
[2] $y = \sin^{-1} x$ is equivalent to $x = \sin y$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



[3] $\sin(\sin^{-1} x) = x$, $x \in [-1, 1]$ and $\sin^{-1}(\sin x) = x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[4] $\lim_{x \rightarrow 1^-} \sin^{-1} x = \frac{\pi}{2}$ and $\lim_{x \rightarrow -1^+} \sin^{-1} x = -\frac{\pi}{2}$

2] $y = \cos^{-1} x = \arccos x$



* Some properties of $y = \cos^{-1} x$

1] $\text{Dom}(\cos^{-1} x) = [-1, 1]$

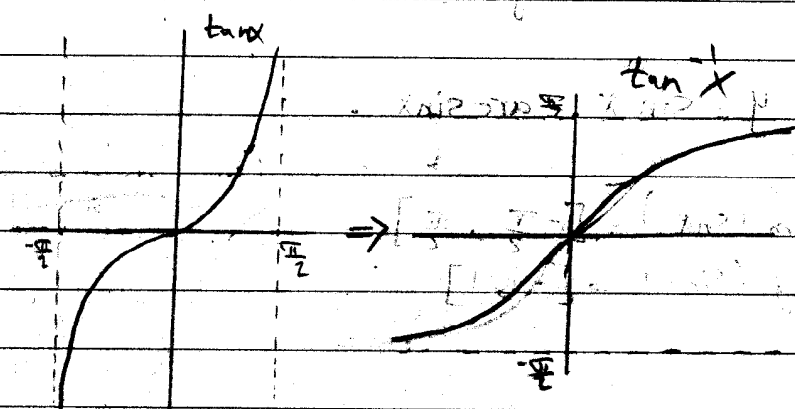
$\text{Rng}(\cos^{-1} x) = [0, \pi]$

2] $y = \cos^{-1} x$ is equivalent to $x = \cos y$.

3] $\cos(\cos^{-1} x) = x$, $x \in [-1, 1]$ and
 $\cos^{-1}(\cos x) = x$, $x \in [0, \pi]$

4] $\lim_{x \rightarrow 1^-} \cos^{-1} x = 0$ and $\lim_{x \rightarrow -1^+} \cos^{-1} x = \pi$

3] $y = \tan^{-1} x = \arctan x$



* Some properties of $y = \tan^{-1} x$

1] $\text{Dom}(\tan^{-1} x) = (-\infty, \infty)$

$\text{Rng}(\tan^{-1} x) = (-\frac{\pi}{2}, \frac{\pi}{2})$

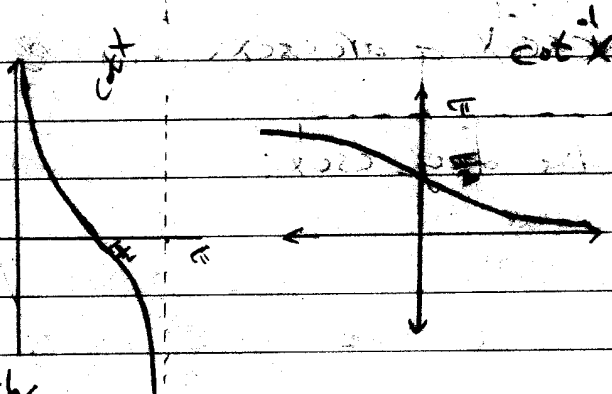
2] $y = \tan^{-1} x$ is equivalent to $x = \tan y$.

3] $\tan(\tan^{-1} x) = x$, $x \in (-\infty, \infty)$

$\tan^{-1}(\tan x) = x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

4] $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$, $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$

4] $y = \cot^{-1} x = \text{arc cot } x$



Some properties of $y = \cot^{-1} x$.

1] $\text{Dom}(\cot^{-1} x) = (-\infty, \infty)$

$\text{Rng}(\cot^{-1} x) = (0, \pi)$

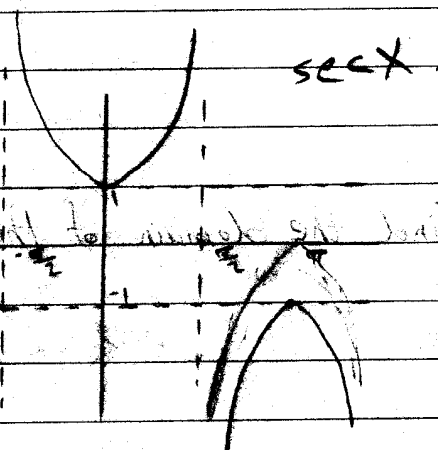
2] $y = \cot^{-1} x$ is equivalent to $x = \cot y$

3] $\cot(\cot^{-1} x) = x, x \in (-\infty, \infty)$

$\cot^{-1}(\cot x) = x, x \in (0, \pi)$

4] $\lim_{x \rightarrow \infty} \cot^{-1} x = 0$ and $\lim_{x \rightarrow -\infty} \cot^{-1} x = \pi$

5] $y = \sec^{-1} x = \text{arc sec } x$



* Some properties of $y = \sec^{-1} x$

1] $\text{Dom}(\sec^{-1} x) = \{x : |x| \geq 1\}$
 $= (-\infty, -1] \cup [1, \infty)$

$\text{Rng}(\sec^{-1} x) = [0, \pi] - \{\pi/2\}$

2] $y = \sec^{-1} x$ is equivalent to $x = \sec y$

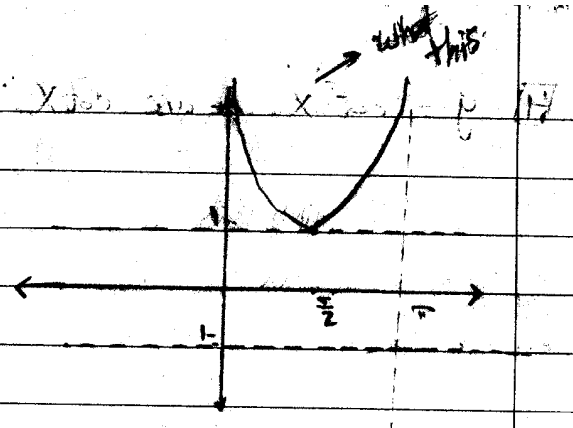
3] $\sec(\sec^{-1} x) = x, x \in (-\infty, -1] \cup [1, \infty)$

$\sec^{-1}(\sec x) = x, x \in [0, \pi] - \{\pi/2\}$

4] $\lim_{x \rightarrow \infty} \sec^{-1} x = \lim_{x \rightarrow -\infty} \sec^{-1} x = \frac{\pi}{2}$

6 $y = \csc^{-1} x = \arccsc x, \quad |x| \geq 1$

Some properties of $y = \csc^{-1} x$:



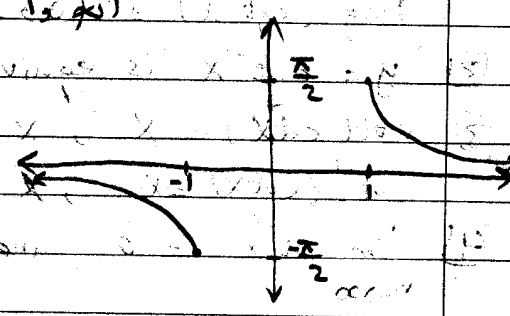
1- $\text{Dom}(\csc^{-1} x) = (-\infty, -1] \cup [1, \infty)$

$\text{Range}(\csc^{-1} x) = [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

2- $\csc^{-1}(\csc^{-1} x) = x \quad \forall x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

$\csc(\csc^{-1} x) = x \quad \forall x \in (-\infty, -1] \cup [1, \infty)$

3- $y = \csc^{-1} x \iff \csc y = x$



14- odd function.

$\Rightarrow \csc^{-1}(-x) = -\csc^{-1} x$

Example/Find the domain of the fn $f(x) = \cos^{-1}(\frac{2^x - 5}{3})$

$\text{Dom}(f) = \{x : -1 \leq \frac{2^x - 5}{3} \leq 1\} = \{x : -2 \leq 2^x - 5 \leq 3\}$

$= \{x : 2 \leq 2^x \leq 8\} = \{x : \ln 2 \leq x \ln 2 \leq \ln 2^3\}$

$= \{x : 1 \leq x \leq 3\} = [1, 3]$

Example/Find the exact value for each of the following expression if it exists

I) $\cos^{-1}(\frac{1}{2})$ II) $\sin^{-1}(\sin \frac{7\pi}{3})$ III) $\sin^{-1}(\sin \frac{9\pi}{5})$ IV) $\sin(\cos^{-1} \frac{\sqrt{2}}{2})$

II) Let $\theta = \cos^{-1}(\frac{1}{2}) \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$

$$[2] \sin^{-1}(\sin(2\pi + \frac{\pi}{3})) = \sin^{-1}(\sin(\frac{\pi}{3})) = \boxed{\frac{\pi}{3}}$$

$$[3] \sin^{-1}(\sin(2\pi - \frac{\pi}{3})) = \sin^{-1}(\sin(-\frac{\pi}{3})) = \boxed{-\frac{\pi}{3}}$$

[4] $\cos(\cos^{-1} \frac{5}{3})$ does not exist, because $\frac{5}{3} \notin \text{Dom}(\cos^{-1}) = [-1, 1]$.

$$[5] \sin(\cos^{-1} \frac{1}{\sqrt{2}}) = \sin(\cos^{-1} \frac{1}{\sqrt{2}})$$

$$\theta = \cos^{-1} \frac{1}{\sqrt{2}} \Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

$$\text{So } \sin(\cos^{-1} \frac{1}{\sqrt{2}}) = \sin(\frac{\pi}{4}) = \boxed{\frac{1}{\sqrt{2}}}$$

[3] Simplify the following expression:

[A] $\tan(\cos^{-1} \frac{x}{3})$

Let $\theta = \cos^{-1} \frac{x}{3}$

$$\cos \theta = \frac{x}{3}$$

$$\tan(\theta) = \frac{\sqrt{9-x^2}}{x}$$



[B] $\sec^2(\cot^{-1} x)$

Let $\theta = \cot^{-1} x$

$$\cot \theta = x$$

$$\sec^2(\theta) = \left(\frac{\sqrt{1+x^2}}{x} \right)^2 = \frac{1+x^2}{x^2}$$

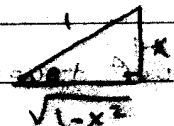


[C] $\cos(2\sin^{-1} x) + 2(4^{\log_2 x})$

Let $\theta = \sin^{-1} x$

$$\sin \theta = x$$

$$\cos(2\theta) + 2(2^{2\log_2 x})$$



$$= (\sqrt{1-x^2})^2 - \left(\frac{x}{1}\right)^2 + 2(x^2) = 1-x^2 - x^2 + 2x^2 = \boxed{1}$$

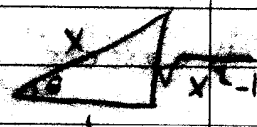
Example/ Solve for x . $\ln(x-e) = e^{-3\ln x} \cdot x^5 - \tan^2(\sec^{-1} x)$

$$\ln(x-e) = e^{\ln x^{-3}} \cdot x^5 - \tan^2 \theta \quad \theta = \sec^{-1} x$$

$$\ln(x-e) = x^{-3} \cdot x^5 - \left(\frac{\sqrt{x^2-1}}{1} \right)^2$$

$$\sec \theta = x$$

$$\ln(x-e) = x^2 - x^2 + 1 = 1$$



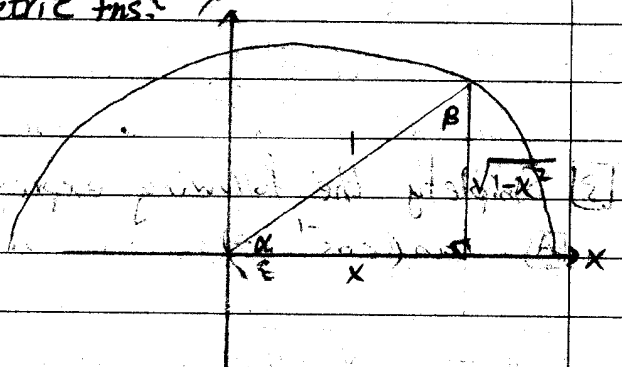
$$e^{\ln(x-e)} = e^1 \Rightarrow x-e = e \Rightarrow \boxed{x=2e}$$

Some identities involving inverse trigonometric fns: x

① (a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad |x| \leq 1$

(b) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad \forall x$

(c) $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}, \quad |x| \geq 1$



Proof (a) $\cos \alpha = x \Rightarrow \alpha = \cos^{-1} x$

$\sin \beta = x \Rightarrow \beta = \sin^{-1} x$

$$\cos^{-1} x + \sin^{-1} x = \alpha + \beta = \frac{\pi}{2} \quad \#$$

(b) $\tan \alpha = \frac{\sqrt{1-x^2}}{x} \Rightarrow \alpha = \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \tan^{-1} \theta$

$\cot \beta = \frac{\sqrt{1-x^2}}{x} \Rightarrow \beta = \cot^{-1} \frac{\sqrt{1-x^2}}{x} = \cot^{-1} \theta$

$$\tan^{-1} \theta + \cot^{-1} \theta = \tan^{-1} x + \cot^{-1} x = \alpha + \beta = \frac{\pi}{2} \quad \#$$

③

[2] $\sin^{-1}(-x) = -\sin^{-1}(x)$ and $\tan^{-1}(-x) = -\tan^{-1}(x)$.

This is true because the graph of $\sin^{-1}x$ and $\tan^{-1}x$ is symmetric about the origin. that is, $\sin^{-1}x$ and $\tan^{-1}x$ are odd fns.

[3] a) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$, $x \geq 1$

b) $\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$, $x \geq 1$

c) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$, $x > 0$

Proof/ [a] Let $\theta = \sec^{-1}x \Rightarrow \sec\theta = x \Rightarrow \frac{1}{\cos\theta} = x$

$\cos\theta = \frac{1}{x} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{x}\right) \Rightarrow \sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \#$

b) Let $\theta = \csc^{-1}x \Rightarrow \csc\theta = x \Rightarrow \frac{1}{\sin\theta} = x \Rightarrow$

$\sin\theta = \frac{1}{x} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{x}\right) \Rightarrow \csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \#$

c) Let $\theta = \cot^{-1}x \Rightarrow \cot\theta = x \Rightarrow \frac{1}{\tan\theta} = x$

$\tan\theta = \frac{1}{x} \Rightarrow \theta = \tan^{-1}\frac{1}{x} \Rightarrow \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) \#$

[4] a) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $|x| \leq 1$

b) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$

Proof/ [a] $\sin^{-1}(+x) + \cos^{-1}(x) = \frac{\pi}{2}$, $|x| \leq 1$

$\sin^{-1}(-x) + \cos^{-1}(-x) = \frac{\pi}{2}$, $|-x| \leq 1 \Rightarrow |x| \leq 1$

$-\sin^{-1}x + \cos^{-1}(-x) = \frac{\pi}{2}$

$-\frac{\pi}{2} + \cos^{-1}x + \cos^{-1}(-x) = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1}x$, $|x| \leq 1$

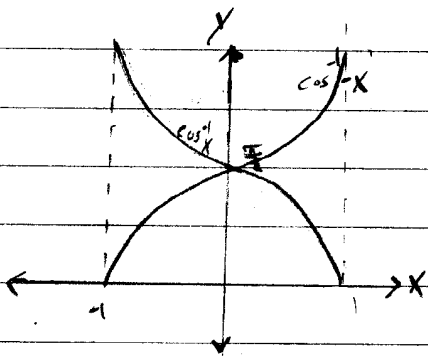
$\boxed{b} \cos^{-1}(-x) = \pi - \cos^{-1}x, \quad |x| \leq 1$ put $\frac{1}{x} = x$
 $\cos^{-1}(-1/x) = \pi - \cos^{-1}(\frac{1}{x}), \quad |\frac{1}{x}| \leq 1$
 $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$ "from 3. a"

#

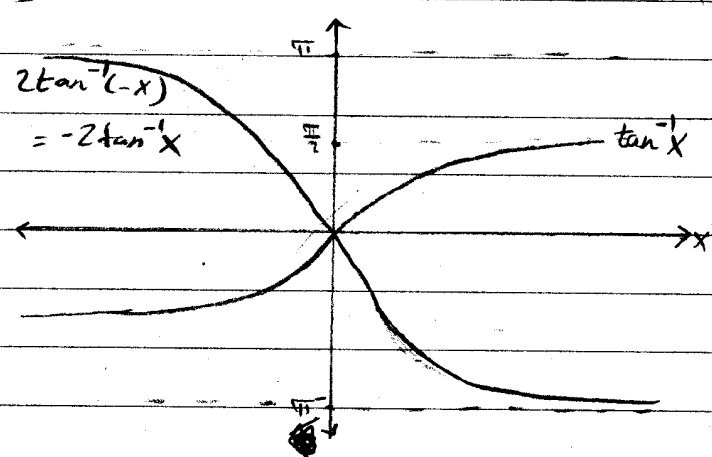
Examples/ II sketch the graph of the following fns.

$\boxed{a} y = \cos^{-1}(-x)$ $\boxed{b} y = 2\tan^{-1}(-x) = -2\tan^{-1}x$

a)



b)



2 Find the exact value of:

a) $\cot(\sin^{-1}(-1/2) - \sec^{-1}(2)) = \cot(-\sin^{-1}1/2 - \sec^{-1}2)$

$= \cot(-(\csc^{-1}2 + \sec^{-1}2)) = -\cot(\frac{\pi}{2}) = \boxed{0}$

b) $2^{\log_4 \pi^2} - \cos^{-1}(-1/\sqrt{2}) = 2^{2 \log_4 \pi} - (\pi - \cos^{-1}(1/\sqrt{2}))$

$= 4^{\log_4 \pi} - \pi + \cos^{-1}1/\sqrt{2} = \pi - \pi + \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$

$\cos^{-1}1/\sqrt{2} = \theta$
 $\cos \theta = 1/\sqrt{2}$
 $\therefore \theta = \frac{\pi}{4}$

3 Solve for x : $\cos^{-1}(-x) - \ln(xe^{\sin^{-1}x}) = \frac{\pi}{2}$

$\pi - \cos^{-1}(x) - \ln x - \ln e^{\sin^{-1}x} = \frac{\pi}{2}$

$\pi - \cos^{-1}x - \ln x - \sin^{-1}x = \frac{\pi}{2}$

$\pi - (\cos^{-1}x + \sin^{-1}x) - \ln x = \frac{\pi}{2}$

$\pi - \frac{\pi}{2} - \ln x = \frac{\pi}{2}$

$\Rightarrow \ln x = 0$

$\Rightarrow e^{\ln x} = e^0$

$\boxed{x = 1}$

→ Derivatives of inverse trigonometric fns.

If u is a diff fn of x , then

$$[1] \frac{d}{dx} (\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, |u| < 1$$

$$[2] \frac{d}{dx} (\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}, |u| < 1$$

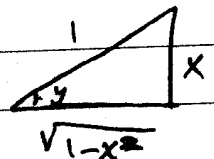
$$[3] \frac{d}{dx} (\tan^{-1} u) = \frac{1}{1+u^2} \frac{du}{dx}$$

$$[4] \frac{d}{dx} (\cot^{-1} u) = \frac{-1}{1+u^2} \frac{du}{dx}$$

$$[5] \frac{d}{dx} (\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, |u| > 1$$

$$[6] \frac{d}{dx} (\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}, |u| > 1$$

Proof / [1] Let $y = \sin^{-1} x \Rightarrow \sin y = x$



$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

[2]

Ww

Some Integrals :-

$$\boxed{1} \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

proof/ put $x=at$
 $dx=adt$

$$\boxed{2} \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\boxed{3} \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C$$

Example/ $\boxed{1}$ Find the following limits.

$$\textcircled{a} \lim_{x \rightarrow \infty} x \tan^{-1}(2/x) = \infty \cdot 0$$

$$\lim_{x \rightarrow \infty} \frac{\tan^{-1}(2/x)}{\frac{1}{x}} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow \infty} \frac{\frac{1}{1+(2/x)^2} \cdot (-2/x^2)}{-1/x^2} = 2 \lim_{x \rightarrow \infty} \frac{1}{1+(2/x)^2}$$

$$= 2 \left(\frac{1}{1+0} \right) = \boxed{2}$$

$$\textcircled{b} \lim_{x \rightarrow 0^+} \frac{\sin^{-1} x^2}{(\sin^{-1} x)^2} = \frac{0}{0} \Rightarrow \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sqrt{1-x^4}} \cdot 2x}{2 \sin^{-1} x \left(\frac{1}{\sqrt{1-x^2}} \right)}$$

$$\lim_{x \rightarrow 0^+} \frac{x}{\left(\sqrt{1-x^2} \cdot \sqrt{1+x^2} \right) \sin^{-1} x} = \lim_{x \rightarrow 0^+} \frac{x}{\sqrt{1+x^2} \sin^{-1} x} = \frac{0}{0}$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{1+x^2} \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x} = \frac{1}{1+0} = \boxed{1}$$

Example/ Find $\frac{dy}{dx}$ for each of the following:-

$$\boxed{1} y = \tan^{-1}(\ln x) \quad \frac{dy}{dx} = \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x}$$

$$\boxed{2} y = \cot^{-1}(\sqrt{x^2 - 1})$$

$$\frac{dy}{dx} = \frac{-1}{1+(\sqrt{x^2-1})^2} \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x = \frac{-x}{1+x^2-1(\sqrt{x^2-1})} = \frac{-1}{x\sqrt{x^2-1}}$$

$$\begin{aligned} \text{[3]} \quad y &= \sqrt{x} (\cos^{-1} \sqrt{x})^4 \\ \frac{dy}{dx} &= \sqrt{x} \left(4(\cos^{-1} \sqrt{x})^3 \cdot \frac{-1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} \right) + (\cos^{-1} \sqrt{x})^4 \cdot \frac{1}{2\sqrt{x}} \\ &= -\frac{2(\cos^{-1} \sqrt{x})^3}{\sqrt{1-x}} + \frac{1}{2\sqrt{x}} (\cos^{-1} \sqrt{x})^4 \end{aligned}$$

$$\begin{aligned} \text{[4]} \quad y &= 3^{\tan x} + \cot^{-1}(3^x) \\ \frac{dy}{dx} &= 3^{\tan x} \cdot \ln 3 \cdot \frac{1}{1+x^2} - \frac{1}{1+(3^x)^2} \cdot 3^x \ln 3 \end{aligned}$$

$$\begin{aligned} \text{[5]} \quad y &= x^{\sec^{-1} x} = e^{\sec^{-1} x \ln x} \\ \frac{dy}{dx} &= e^{\sec^{-1} x \ln x} \cdot \left(\sec^{-1} x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{|x| \sqrt{x^2-1}} \right) \\ &= x^{\sec^{-1} x} \left(\frac{\sec^{-1} x}{x} + \frac{\ln x}{|x| \sqrt{x^2-1}} \right) \end{aligned}$$

$$\begin{aligned} \text{[6]} \quad \sin^{-1}(\ln y) &= \csc^{-1}(e^{xy}) \\ \frac{1}{\sqrt{1-(\ln y)^2}} \cdot \frac{y'}{y} &= \frac{-1}{e^{xy} \sqrt{(e^{xy})^2-1}} \cdot e^{xy} \cdot (y' + xy') \\ \frac{y'}{y \sqrt{1-(\ln y)^2}} &= \frac{-[y + xy']}{\sqrt{e^{2xy}-1}} \end{aligned}$$

Example/Evaluate the following integrals:-

$$\begin{aligned} \text{[1]} \quad \int \frac{dx}{9+3x^2} &= \frac{1}{3} \int \frac{dx}{3+x^2} = \frac{1}{3} \int \frac{dx}{(\sqrt{3})^2+x^2} \\ &= \frac{1}{3} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + C \end{aligned}$$

$$\text{[2]} \quad \int \frac{\sec^2 x}{\sqrt{4-\tan^2 x}} dx$$

$$\begin{aligned} t &= \tan x \\ dt &= \sec^2 x dx \end{aligned}$$

$$\Rightarrow \int \frac{1}{\sqrt{4-t^2}} dt = \sin^{-1} \left(\frac{t}{2} \right) + C$$

$$= \sin^{-1} \left(\frac{\tan x}{2} \right) + C$$

$$\boxed{3} \int \frac{x^3 + 2x}{x^4 + 9} dx = \int \frac{x^3}{x^4 + 9} dx + \int \frac{2x}{x^4 + 9} dx$$

$$= \frac{1}{4} \int \frac{4x^3}{x^4 + 9} dx + 2 \int \frac{x}{(x^2)^2 + 9} dx$$

$$w = x^2$$

$$dw = 2x dx$$

$$\frac{1}{2} dw = x dx$$

$$= \frac{1}{4} \int \frac{4x^3}{x^4 + 9} dx + \frac{2}{2} \int \frac{dw}{w^2 + 9}$$

$$= \frac{1}{4} \ln(x^4 + 9) + \frac{2}{2} \cdot \frac{1}{3} \tan^{-1}\left(\frac{w}{3}\right) + C$$

$$= \frac{1}{4} \ln(x^4 + 9) + \frac{1}{3} \tan^{-1}\left(\frac{x^2}{3}\right) + C$$

$$\boxed{4} \int \frac{x^2 + 2x - 1}{x^2 + 9} dx$$

$$= \int 1 dx + \int \frac{2x - 10}{x^2 + 9} dx$$

$$\frac{x^2 + 9}{x^2 + 9} \cdot \frac{x^2 + 2x - 1}{x^2 + 9} = \frac{x^2 + 2x - 1}{x^2 + 9}$$

$$= \int 1 dx + \int \frac{2x}{x^2 + 9} dx - 10 \int \frac{1}{x^2 + 9} dx$$

$$= x + \ln(x^2 + 9) - \frac{10}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$\boxed{5} \int \frac{1}{\sqrt{e^{2x} - 16}} dx = \int \frac{e^x}{e^x \sqrt{(e^x)^2 - 16}} dx$$

$$= \int \frac{d(e^x)}{e^x \sqrt{(e^x)^2 - 16}} = \frac{1}{4} \sec^{-1}\left(\frac{e^x}{4}\right) + C$$

$$\boxed{6} \int \frac{1}{x \sqrt{x^6 - 16}} dx = \int \frac{x^2}{x^3 \sqrt{(x^3)^2 - 16}} dx \Rightarrow$$

$$t = x^3$$

$$dt = 3x^2 dx$$

$$\frac{1}{3} \int \frac{dt}{t \sqrt{t^2 - 16}} = \frac{1}{3} \cdot \frac{1}{4} \sec^{-1}\left|\frac{t}{4}\right| + C$$

$$\frac{1}{3} dt = x^2 dx$$

$$= \frac{1}{12} \sec^{-1}\left(\frac{x^3}{4}\right) + C$$

$$\begin{aligned} \boxed{7} \int \frac{dx}{\sqrt{2x-x^2}} &= \int \frac{dx}{\sqrt{1-1+2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} \\ &= \int \frac{dw}{\sqrt{1-w^2}} = \sin^{-1} w + C = \sin^{-1}(x-1) + C \end{aligned}$$

$$\begin{aligned} \boxed{8} \int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} &= \int \frac{dx}{(x-2)\sqrt{x^2-4x+4-1}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}} \\ &= \int \frac{d(x-2)}{(x-2)\sqrt{(x-2)^2-1}} = \sec^{-1}|x-2| + C \end{aligned}$$

$$\boxed{9} \int \frac{dx}{\sin^{-1} x \cdot \sqrt{1-x^2}} = \int \frac{d(\sin^{-1} x)}{\sin^{-1} x} = \ln|\sin^{-1} x| + C$$

$$\boxed{10} \int \frac{\cos^3(\sec^{-1} x)}{x\sqrt{x^2-1}} dx, x > 0$$

$$\theta = \sec^{-1} x$$

$$d\theta = \frac{1}{x\sqrt{x^2-1}} dx$$

$$= \int \cos^3 \theta d\theta = \int \cos^2 \theta \cdot \cos \theta d\theta$$

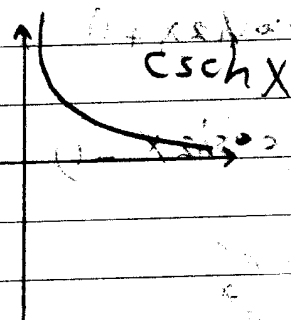
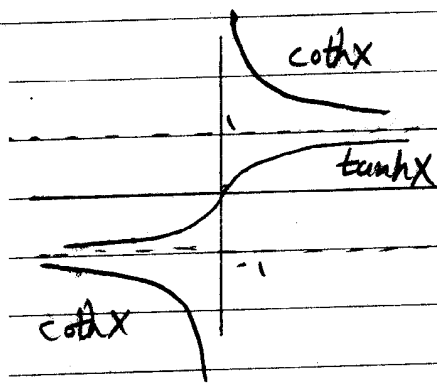
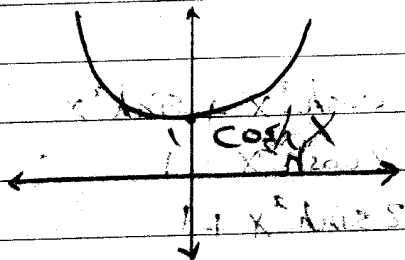
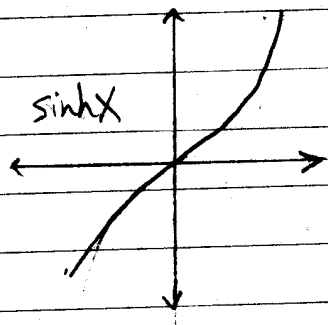
$$= \int (1 - \sin^2 \theta) d(\sin \theta) = \sin \theta - \frac{\sin^3 \theta}{3} + C$$

$$= \sin(\sec^{-1} x) - \frac{1}{3} \sin^3(\sec^{-1} x) + C$$

7.7 Hyperbolic Fns:-

$$\boxed{1} \sinh x = \frac{e^x - e^{-x}}{2} \quad \boxed{2} \cosh x = \frac{e^x + e^{-x}}{2} \quad \boxed{3} \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\boxed{4} \operatorname{csch} x = \frac{2}{e^x - e^{-x}} \quad \boxed{5} \operatorname{sech} x = \frac{2}{e^x + e^{-x}} \quad \boxed{6} \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



- $\operatorname{Dom}(\sinh) = \operatorname{Rng}(\sinh) = (-\infty, \infty)$

- $\operatorname{Dom}(\cosh) = (-\infty, \infty)$

- $\operatorname{Rng}(\cosh) = [1, \infty)$

- $\operatorname{Dom}(\tanh) = (-\infty, \infty)$

- $\operatorname{Rng}(\tanh) = (-1, 1)$

- $\operatorname{Dom}(\coth) = \mathbb{R} - \{0\}$

- $\operatorname{Rng}(\coth) = \mathbb{R} - \{0\} = (-\infty, -1) \cup (1, \infty)$

- $\operatorname{Dom}(\operatorname{sech}) = (-\infty, \infty)$

- $\operatorname{Rng}(\operatorname{sech}) = (0, 1]$

- $\operatorname{Dom}(\operatorname{csch}) = \operatorname{Rng}(\operatorname{csch}) = (-\infty, \infty) - \{0\}$

* $\cosh x$ and $\operatorname{sech} x$ are even fns.
the others are odd fns.

Some Identities of hyperbolic Fns:-

$$\begin{aligned} \text{[1]} \quad \cosh^2 x - \sinh^2 x &= 1 \quad \Rightarrow \\ 1 - \tanh^2 x &= \operatorname{sech}^2 x \\ \coth^2 x - 1 &= \operatorname{csch}^2 x \end{aligned}$$

$$\text{[2]} \quad \sinh 2x = 2 \sinh x \cdot \cosh x.$$

$$\begin{aligned} \text{[3]} \quad \cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 2\cosh^2 x - 1 \\ &= 2\sinh^2 x + 1 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{from 1}$$

$$\text{[4]} \quad \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$$

$$\begin{aligned} \text{[5]} \quad \cosh x + \sinh x &= e^x \\ \cosh x - \sinh x &= e^{-x} \end{aligned}$$

Derivatives of hyperbolic Fns:-

If u is a diff fn of x then

$$\text{[1]} \quad \frac{d}{dx}(\sinh u) = \cosh u \cdot \frac{du}{dx}$$

$$\text{[2]} \quad \frac{d}{dx}(\cosh u) = \sinh u \cdot \frac{du}{dx}$$

$$\text{[3]} \quad \frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

$$\text{[4]} \quad \frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \cdot \frac{du}{dx}$$

$$\text{[5]} \quad \frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \cdot \tanh u \cdot \frac{du}{dx}$$

$$\text{[6]} \quad \frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \cdot \coth u \cdot \frac{du}{dx}$$

$$\boxed{1} \int \sinh ax \cdot dx = \frac{1}{a} \cosh ax + C$$

$$\boxed{2} \int \cosh ax \cdot dx = \frac{1}{a} \sinh ax + C$$

$$\boxed{3} \int \tanh ax \cdot dx = \int \frac{\sinh ax}{\cosh ax} \cdot dx = \frac{1}{a} \ln |\cosh ax| + C$$

$$\boxed{4} \int \coth x \cdot dx = \int \frac{\cosh x}{\sinh x} \cdot dx = \ln |\sinh x| + C$$

$$\boxed{5} \int \operatorname{sech}^2 x \cdot dx = \tanh x + C$$

$$\boxed{6} \int \operatorname{csch}^2 x \cdot dx = -\coth x + C$$

$$\boxed{7} \int \operatorname{sech} x \cdot \tanh x \cdot dx = -\operatorname{sech} x + C$$

$$\boxed{8} \int \operatorname{csch} x \cdot \coth x \cdot dx = -\operatorname{csch} x + C$$

Examples:- $\boxed{1}$ If $\cosh x = \frac{13}{5}$, $x > 0$. Find:

$\boxed{a}$ $\tanh x$ and $\operatorname{sech} x$. $\boxed{b}$ the value of x .

$$\boxed{a} \cosh^2 x - \sinh^2 x = 1$$

$$\left(\frac{13}{5}\right)^2 - 1 = \sinh^2 x \Rightarrow \sinh x = \sqrt{\frac{144}{25}} = \frac{12}{5}$$

$$\star \tanh x = \frac{\frac{12}{5}}{\frac{13}{5}} = \frac{12}{13}$$

$$\star \operatorname{sech} x = \frac{1}{\cosh x} = \frac{5}{13}$$

$$\boxed{b} \cosh x + \sinh x = e^x$$

$$\frac{13}{5} + \frac{12}{5} = e^x \Rightarrow 5 = e^x$$

$$\boxed{x = \ln 5}$$

[2] Simplify the following expressions:

[A] $\sinh(2\ln x)$

[B] $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

[A] $\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{x^2 - x^{-2}}{2} = \frac{x^4 - 1}{2x^2}$

[B] $\ln(\cosh x + \sinh x)(\cosh x - \sinh x) = \ln(\cosh^2 x - \sinh^2 x) = \ln 1 = 0$

or/ $\ln(e^x) + \ln(e^{-x}) = x - x = 0$

[3] Find y' for each of the following:

[I] $y = 2\sqrt{x} \tanh \sqrt{x}$

$y' = 2\sqrt{x} \cdot \operatorname{sech}^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} + \frac{2}{2\sqrt{x}} \tanh \sqrt{x}$
 $= \operatorname{sech}^2 \sqrt{x} + \frac{\tanh \sqrt{x}}{\sqrt{x}}$

[2] $y = \ln(\sinh x) + \operatorname{csch}(\ln 2x)$

$y' = \frac{1}{\sinh x} \cdot \cosh x - \operatorname{csch}(\ln 2x) \cdot \coth(\ln 2x) \cdot \frac{1}{2x} \cdot 2$
 $= \coth x - \frac{1}{x} \operatorname{csch}(\ln 2x) \coth(\ln 2x)$

[3] $y = \cosh \sqrt[5]{e^{2x} + 2}$

$y' = 5 \cosh^4 \sqrt[5]{e^{2x} + 2} \cdot \sinh \sqrt[5]{e^{2x} + 2} \cdot \frac{1}{2\sqrt[5]{e^{2x} + 2}} \cdot e^{2x} \cdot 2$

[4] $y = \sinh(\tan x) + \cot^{-1}(\sinh x)$

$y' = \cosh(\tan x) \cdot \sec^2 x - \frac{1}{1 + \sinh^2 x} \cdot \cosh x$

$= \sec^2 x \cdot \cosh(\tan x) - \frac{1}{\cosh^2 x} \cdot \cosh x$

$= \sec^2 x \cdot \cosh(\tan x) - \operatorname{sech} x$

$$[5] y = \cosh x \cdot \tan^{-1}(\sinh x)$$

$$y' = \cosh x \cdot \frac{1}{1 + \sinh^2 x} \cdot \cosh x + \sinh x \cdot \tan^{-1}(\sinh x)$$

$$= 1 + \sinh x \tan^{-1}(\sinh x)$$

$$[6] y = x^{\cosh x} = e^{\cosh x \ln x}$$

$$y' = e^{\cosh x \ln x} \left(\cosh x \cdot \frac{1}{x} + \sinh x \cdot \ln x \right)$$

[4] Evaluate the following integrals:-

$$[1] \int \operatorname{sech} x \, dx = \int \frac{2}{e^x + e^{-x}} \, dx \cdot \frac{e^x}{e^x} = \int \frac{2e^x \, dx}{e^{2x} + 1} \quad \begin{matrix} t = e^x \\ dt = e^x dx \end{matrix}$$

$$= \int \frac{2 \, dt}{t^2 + 1} = 2 \tan^{-1}(t) + C = 2 \tan^{-1}(e^x) + C$$

$$[2] \int e^{-x} \cosh x \, dx = \int e^{-x} \frac{(e^x + e^{-x})}{2} \, dx = \frac{1}{2} \int (1 + e^{-2x}) \, dx$$

$$= \frac{1}{2} \left(x + \frac{e^{-2x}}{-2} \right) + C = \frac{1}{2} x - \frac{1}{4} e^{-2x} + C$$

$$[3] \int \operatorname{sech}^3 x \tanh x \, dx = \int \operatorname{sech}^2 x \cdot \operatorname{sech} x \tanh x \, dx$$

$$= \int \operatorname{sech}^2 x \, d(\operatorname{sech} x) = -\frac{1}{3} \operatorname{sech}^3 x + C$$

$$[4] \int \frac{\sinh 2x}{1 + \cosh x} \, dx = \int \frac{2 \sinh x \cosh x}{1 + \cosh x} \, dx \quad \begin{matrix} w = 1 + \cosh x \\ dw = \sinh x \, dx \end{matrix}$$

$$\Rightarrow 2 \int \frac{(w-1)}{w} \, dw = 2 \int 1 - \frac{1}{w} \, dw = 2(w - \ln|w|) + C$$

$$= 2(1 + \cosh x) - 2 \ln(1 + \cosh x) + C$$

$$\boxed{5} \int \frac{1 + \tanh x}{\cosh^2 x} dx = \int (\operatorname{sech}^2 x + \operatorname{sech}^2 x \tanh x) dx$$

$$= \int \operatorname{sech}^2 x dx = \int \operatorname{sech} x d(\operatorname{sech} x)$$

$$= \tanh x - \frac{\operatorname{sech}^2 x}{2} + C$$

$$\boxed{6} \int \tanh x \ln(\cosh x) dx$$

$$t = \ln(\cosh x)$$

$$dt = \frac{1}{\cosh x} \cdot \sinh x dx = \tanh x dx$$

$$= \int t dt = \frac{1}{2} t^2 + C = \frac{1}{2} \ln^2(\cosh x) + C$$

$$\text{Ex. } \int \frac{\cos^{-1}(\tanh x) \operatorname{sech}^2 x}{\sqrt{1 - \tanh^2 x}} dx$$

~~$$\int \cos^{-1}(\tanh x) dx$$~~

$$= - \int t dt$$

$$= -\frac{1}{2} t^2 + C$$

$$= -\frac{1}{2} (\cos^{-1}(\tanh x))^2 + C$$

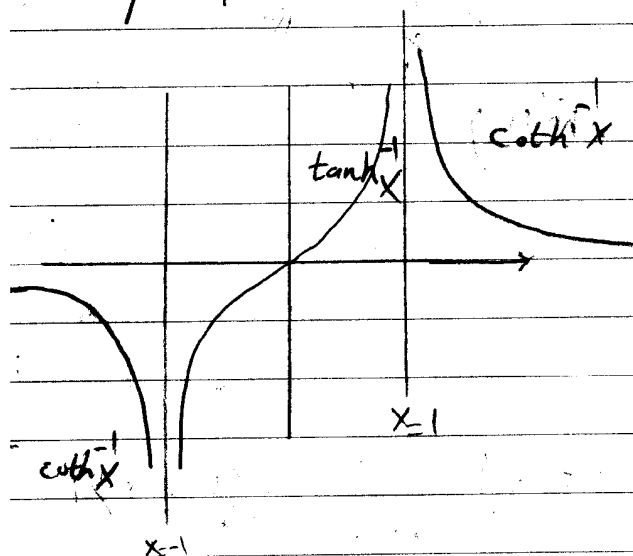
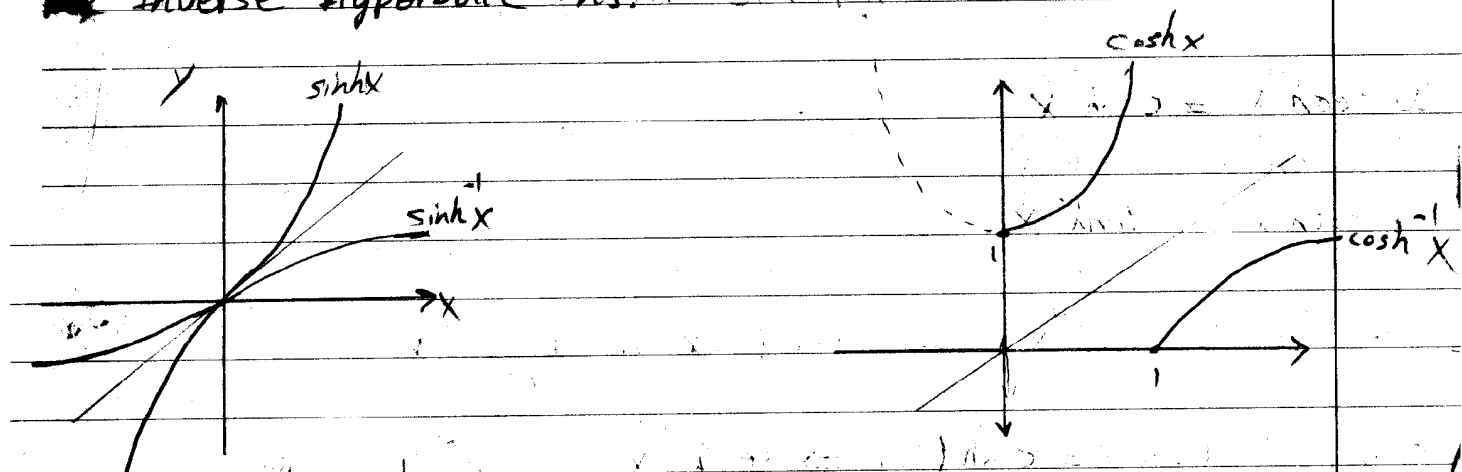
$$t = \cos^{-1}(\tanh x)$$

$$dt = \frac{-1}{\sqrt{1 - \tanh^2 x}} \cdot \operatorname{sech}^2 x dx$$

$$t = \cos^{-1}(\tanh x)$$

~~$$\int \cos^{-1}(\tanh x) dx$$~~

Inverse Hyperbolic Fns.



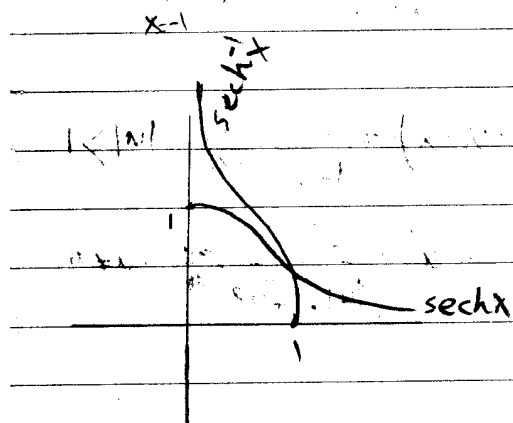
$$\text{Dom}(\tanh^{-1} x) = (-1, 1)$$

$$\text{Rng} = (-\infty, \infty)$$

$$\text{Dom}(\coth^{-1} x) = |x| < 1$$

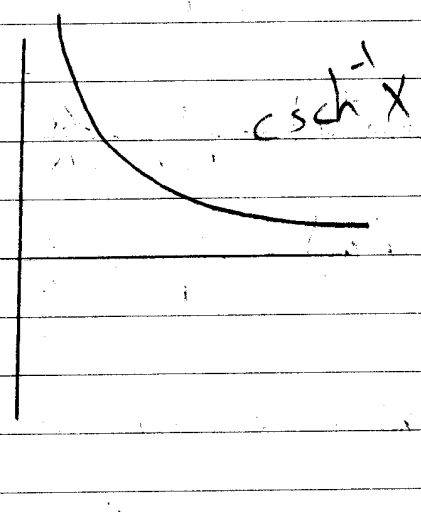
$$= (-1, 1) \cup (1, \infty)$$

$$\text{Rng} = \mathbb{R} - \{0\}$$



$$\text{Dom}(\text{sech}^{-1} x) = (0, 1]$$

$$\text{Rng} = [0, \infty)$$



Some rules about inverse hyperbolic fns:-

$$\boxed{1} \quad \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\boxed{2} \quad \operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\boxed{3} \quad \operatorname{coth}^{-1} x = \tanh^{-1} \frac{1}{x}$$

Proof 1 / let $y = \operatorname{sech}^{-1} x \Rightarrow \operatorname{sech} y = x \Rightarrow \frac{1}{\cosh y} = x$

$$\cosh y = \frac{1}{x} \Rightarrow y = \cosh^{-1} \left(\frac{1}{x} \right) \Rightarrow \operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \quad \#$$

✓ Example / Find the exact value of $\operatorname{sech}(\cosh^{-1} \frac{4}{3})$
 $\operatorname{sech}(\cosh^{-1} \frac{4}{3}) = \operatorname{sech}(\operatorname{sech}^{-1} \frac{3}{4}) = \frac{3}{4}$

Derivatives of inverse hyperbolic fns.

If u is a diff fn of x then

$$\boxed{1} \quad \frac{d}{dx} (\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}} \cdot \frac{du}{dx}$$

$$\boxed{2} \quad \frac{d}{dx} (\cosh^{-1} x) = \frac{1}{\sqrt{u^2-1}} \cdot \frac{du}{dx}, \quad |u| > 1$$

$$\boxed{3} \quad \frac{d}{dx} (\tanh^{-1} u) = \frac{1}{1-u^2} \cdot \frac{du}{dx}, \quad |u| < 1$$

$$\boxed{4} \quad \frac{d}{dx} (\coth^{-1} x) = \frac{1}{1-u^2} \cdot \frac{du}{dx}, \quad |u| > 1$$

$$\boxed{5} \quad \frac{d}{dx} (\operatorname{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}} \cdot \frac{du}{dx}, \quad 0 < u < 1$$

$$\boxed{6} \quad \frac{d}{dx} (\operatorname{csch}^{-1} x) = \frac{-1}{|u|\sqrt{1+u^2}} \cdot \frac{du}{dx}, \quad u \neq 0$$

From the above derivatives we have:-

$$\boxed{1} \quad \int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1} \left(\frac{x}{a} \right) + c$$

$$\boxed{2} \quad \int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c$$

$$\boxed{3} \quad \int \frac{1}{a^2-x^2} dx = \int \frac{1}{a^2-x^2} dx = \frac{1}{a} \tanh^{-1} \left(\frac{x}{a} \right) + c, \quad |x| < a$$

$$\frac{1}{a} \coth^{-1} \left(\frac{x}{a} \right) + c, \quad |x| > a$$

$$4) \int \frac{1}{x\sqrt{a^2-x^2}} dx = -\frac{1}{a} \operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C, \quad 0 < x < a$$

$$5) \int \frac{1}{x\sqrt{a^2+x^2}} dx = \frac{-1}{a} \operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C, \quad x \neq 0$$

Example/ [1] Solve for x : $e^{\cosh^{-1}(2x)} + \frac{d}{dx}(\cos^{-1}(\sin x)) = 0, \quad 0 \leq x < 1$

$$e^{\cosh^{-1}(2x)} + \frac{-1}{\sqrt{1-\sin^2 x}} \cdot \cos x = 0$$

$$e^{\cosh^{-1}(2x)} + \frac{-\cos x}{\cos x} = 0 \Rightarrow e^{\cosh^{-1}(2x)} = 1 = e^0$$

$$1 - \sin^2 x = \cos^2 x$$

$$\Rightarrow \cosh^{-1}(2x) = 0 \Rightarrow 2x = \cosh(0) = 1 \Rightarrow \boxed{x = \frac{1}{2}}$$

[2] Find $\frac{dy}{dx}$ for each of the following:

[1] $y = (1-x)\tanh^{-1}x$
 $y' = (1-x) \cdot \frac{1}{1-x^2} + (-1) \cdot \tanh^{-1}x$
 $= \frac{1}{1+x} - \tanh^{-1}x$

[2] $y = \cosh^{-1}(2\sqrt{x+1})$

$$y' = \frac{1}{\sqrt{4(x+1)-1}} \cdot 2 \cdot \frac{1}{2\sqrt{x+1}} = \frac{1}{\sqrt{4x+3}} - \frac{1}{\sqrt{x+1}}$$

[3] $y = \cosh x \cdot \tan(\sinh^{-1}x)$

$$y' = \cosh x \cdot \sec^2(\sinh^{-1}x) \cdot \frac{1}{\sqrt{1+x^2}} + \tan(\sinh^{-1}x) \cdot \sinh x$$

[4] $y = \coth(e^x) + \csc^{-1}(2^x)$

$$y' = \frac{1}{1-\coth^2 e^x} \cdot -\csc^2(e^x) \cdot e^x + \frac{-1}{2^x \sqrt{1+(2^x)^2}} \cdot 2^x \ln 2$$

$$[5] y = \sqrt{\operatorname{sech}^{-1} x} + e^{\tanh^{-1}(2x)}$$

$$y' = \frac{1}{2\sqrt{\operatorname{sech}^{-1} x}} \cdot \frac{-1}{x\sqrt{1-x^2}} + e^{\tanh^{-1}(2x)} \cdot \frac{1}{1-4x^2} \cdot 2$$

Example/ Evaluate the following integrals:-

$$[1] \int \frac{e^x}{\sqrt{e^{2x}-9}} dx = \int \frac{d(e^x)}{(e^x)^2-9} = \cosh^{-1}\left(\frac{e^x}{3}\right) + C$$

$$[2] \int \frac{dx}{\sqrt{x^2-10x+34}} = \int \frac{dx}{\sqrt{x^2-10x+25+9}} = \int \frac{dx}{\sqrt{(x-5)^2+9}} = \int \frac{d(x-5)}{\sqrt{(x-5)^2+9}} \\ = \sinh^{-1}\left(\frac{x-5}{3}\right) + C$$

$$[3] \int \frac{\cot x}{\sqrt{5-\sin^2 x}} dx = \int \frac{\cos x}{\sin x \sqrt{5-\sin^2 x}} dx = \int \frac{d(\sin x)}{\sin x \sqrt{5-\sin^2 x}} = \frac{1}{\sqrt{5}} \operatorname{sech}^{-1}\left(\frac{\sin x}{\sqrt{5}}\right) + C$$

$$[4] \int \frac{1}{x\sqrt{49+9x^4}} dx \Rightarrow \int \frac{x dx}{x^2\sqrt{49+(3x^2)^2}} \quad \begin{aligned} y &= 3x^2 \\ dy &= 6x dx \\ \frac{1}{6} dy &= x dx \end{aligned} \\ \Rightarrow \frac{1}{6} \int \frac{dy}{\frac{y}{3}\sqrt{49+y^2}} = \frac{1}{2} \int \frac{dy}{y\sqrt{49+y^2}} \\ = \frac{1}{2} \cdot \frac{1}{7} \cdot \operatorname{csch}^{-1}\left|\frac{y}{7}\right| + C = \frac{1}{14} \operatorname{csch}^{-1}\left(\frac{3x^2}{7}\right) + C$$

$$[5] \int_{-1/2}^{5/2} \frac{dx}{2-x^2+4x+5} = \int_{-1/2}^{5/2} \frac{dx}{2-(x^2-4x)+5} = \int_{-1/2}^{5/2} \frac{dx}{2-(x^2-4x+4)+4+5} = \int_{-1/2}^{5/2} \frac{dx}{9-(x-2)^2} \\ = \int_0^{1/2} \frac{dw}{9-w^2} = \frac{1}{3} \tanh^{-1}\left(\frac{w}{3}\right) \Big|_0^{1/2} = \frac{1}{3} \tanh^{-1}\left(\frac{1}{6}\right)$$

Logarithm Formulas for inverse hyperbolic fns.

$$\textcircled{1} \sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \quad -\infty < x < \infty$$

$$\textcircled{2} \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$$

$$\textcircled{3} \tanh^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{1-x}\right), \quad |x| < 1$$

$$\textcircled{4} \coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \quad |x| > 1$$

$$\textcircled{5} \operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right), \quad 0 < x < 1$$

$$\textcircled{6} \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \quad x \neq 0$$

Proof / $\textcircled{1}$ Let $y = \sinh^{-1} x \Rightarrow x = \sinh y$

$$x = \frac{e^y - e^{-y}}{2} \Rightarrow e^y - e^{-y} = 2x \Rightarrow$$

$$e^{2y} - 1 = 2xe^y \Rightarrow (e^y)^2 - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = x \pm \sqrt{x^2 + 1}$$

$$e^y = x + \sqrt{x^2 + 1}$$

$$e^y = x - \sqrt{x^2 + 1} \quad \text{impossible}$$

$$y = \ln(x + \sqrt{x^2 + 1}) \Rightarrow \sinh^{-1} x = \ln(x + \sqrt{1+x^2})$$

Ex. $\textcircled{1}$ Solve for x: $\frac{d}{dx} \left[\int_0^x \frac{2 \cosh^{-1}(t)}{t} dt \right] = 3^{\log_3 2}$

$$\Rightarrow \frac{2 \cosh^{-1}(\ln x) \cdot e^x}{e^x} = 0 = 3^{\log_3 2} = 3^{2 \log_3 2}$$

$$2 \cosh^{-1}(x) = 9^{\log_3 2} = 2 \Rightarrow$$

$$\cosh^{-1}(x) = 1 \Rightarrow x = \cosh(1)$$

$$x = \frac{e + e^{-1}}{2} \Rightarrow$$

$\textcircled{2}$ Show that $\frac{d}{dx} \int_{e^{-\sin x}}^{e^{\sin x}} \ln t \cdot dt = \sin 2x \cdot \sinh(\sin x)$

$$\ln e^{\sin x} \cdot e^{\sin x} \cdot \cos x - \ln e^{-\sin x} \cdot e^{-\sin x} \cdot (-\cos x)$$

$$\sin x \cdot \cos x \cdot e^{\sin x} - \sin x \cdot \cos x \cdot e^{-\sin x} \Rightarrow$$

$$\frac{2 \sin x \cos x \cdot e^{\sin x}}{2} - \frac{2 \sin x \cos x \cdot e^{-\sin x}}{2}$$

$$\Rightarrow \sin 2x = \frac{e^{\sin x} - e^{-\sin x}}{2} = \sin 2x \cdot \frac{(e^{\sin x} - e^{-\sin x})}{2} = \sin 2x \cdot \sinh(\sin x)$$

[3] Find $\frac{dy}{dx}$ for:-

a) $y = x^{\cosh x} \Rightarrow \ln y = \cosh x \cdot \ln x \Rightarrow$

$$\frac{y'}{y} = \sinh x \cdot \ln x + \cosh x \cdot \frac{1}{x} \Rightarrow$$

b) $y = \sinh x \Rightarrow \ln y = x \ln \sinh x$

$$\frac{y'}{y} = \ln \sinh x + x \cdot \frac{1}{\sinh x} \cdot \cosh x \Rightarrow$$

c) $y = \coth^{-1} \sqrt{x^2 + 1} \Rightarrow y' = \frac{1}{\sqrt{x^2 + 1}} \cdot \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x$

$$= \frac{x}{x^2 + 1}$$

d) $y = \tanh^{-1} \left(\frac{1}{\coth x} \right) = \tanh^{-1} (\tanh x) = x$

$$y' = 1$$

e) $y = \cosh^{-1} (\sinh 3x)$

$$y' = \frac{1}{\sqrt{\sinh^2 3x - 1}} \cdot \cosh 3x \cdot 3$$

[4] Evaluate:-

a) $\int \frac{1}{e^x \sqrt{e^{-2x} - 16}} dx = \int \frac{1}{e^x \sqrt{(e^{-x})^2 - 16}} dx$

7.8 Relative Rates of Growth.

Def Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

[1] f grows faster than g as $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$

[2] g grows slower than f as $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0$.

[3] f and g has the same rate as $x \rightarrow \infty$ if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$, $0 < L < \infty$

Example / Discuss the growth of the following pairs of fns.:

[a] 3^x and e^x [b] $\log_{10} X$ and X^2 [c] X^2 and $\sqrt{4X^4 + 3X^2 + 1}$

[a] $\lim_{x \rightarrow \infty} \frac{3^x}{e^x} = \frac{\infty}{\infty} \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{3}{e}\right)^x = \infty$ because $\frac{3}{e} > 1$

So 3^x grows faster than e^x as $x \rightarrow \infty$.

[b] $\lim_{x \rightarrow \infty} \frac{\log_{10} X}{X^2} = \frac{\infty}{\infty} \Rightarrow \lim_{x \rightarrow \infty} \frac{\frac{1}{X \ln 10}}{2X} = \lim_{x \rightarrow \infty} \frac{1}{(2 \ln 10) X^2} = \frac{1}{\infty} = 0$

So $\log_{10} X$ grows slower than X^2 as $x \rightarrow \infty$.

[c] $\lim_{x \rightarrow \infty} \frac{\sqrt{4X^4 + 3X^2 + 1}}{X^2} = \frac{\infty}{\infty} \Rightarrow \lim_{x \rightarrow \infty} \sqrt{4 + \frac{3}{X^2} + \frac{1}{X^2}} = 2$

So X^2 and $\sqrt{4X^4 + 3X^2 + 1}$ grows at the same rate as $x \rightarrow \infty$

Note / If f grows at the same rate as g as $x \rightarrow \infty$, g grows at the rate of h as $x \rightarrow \infty$, then f grows at the same rate as h as $x \rightarrow \infty$, that is

if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L_1$, $\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = L_2$, $0 < L_1, L_2 < \infty$

then $\lim_{x \rightarrow \infty} \frac{f(x)}{h(x)} = \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} = L_1 \cdot L_2$

Show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \rightarrow \infty$.

Let $g(x) = x$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+5}}{x} = 1$$

$$\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = \lim_{x \rightarrow \infty} \frac{x}{(2\sqrt{x}-1)^2} = \lim_{x \rightarrow \infty} \frac{1}{\left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2} = \frac{1}{4}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+5}}{(2\sqrt{x}-1)^2} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+5}}{x} \cdot \frac{x}{(2\sqrt{x}-1)^2} =$$

Ch.8 Techniques of Integration

Some integral formulas:-

If a is a real num. then:-

$$\textcircled{1} \int a \, dx = ax + C$$

$$\textcircled{2} \int \frac{a}{x} \, dx = a \ln|x| + C$$

$$\textcircled{3} \int e^{ax} \, dx = \frac{e^{ax}}{a} + C$$

$$\textcircled{4} \int ax^n \, dx = a \frac{x^{n+1}}{n+1} + C$$

$$\textcircled{5} \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

$$\textcircled{6} \int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\textcircled{7} \int \cos(ax) = \frac{1}{a} \sin ax + C$$

$$\textcircled{8} \int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C$$

$$\textcircled{9} \int \tan$$

||

||

}

8.1 Integration by parts.

If u and v are diff. fns of x then: $\int u dv = uv - \int v du$

Example / Evaluate the following integrals:

$$\textcircled{1} \int x^5 \ln x \, dx$$

$$u = \ln x \quad dv = x^5 \, dx$$
$$du = \frac{1}{x} \, dx \quad v = \frac{1}{6} x^6$$

$$= \frac{1}{6} x^6 \ln x - \int \frac{1}{6} x^6 \cdot \frac{1}{x} \, dx$$
$$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 \, dx =$$

$$\textcircled{2} \int \ln x \, dx$$

$$u = \ln x \quad dv = dx$$
$$du = \frac{1}{x} \, dx \quad v = x$$

$$= x \ln x - \int \frac{1}{x} \cdot x \, dx$$
$$= x \ln x - x + C$$

$$\textcircled{3} \int \frac{\ln x}{\sqrt{x}} \, dx$$

$$u = \ln x \quad dv = x^{\frac{1}{2}} \, dx$$
$$du = \frac{1}{x} \, dx \quad v = 2 x^{\frac{1}{2}}$$

$$= 2\sqrt{x} \ln x - 2 \int \frac{1}{\sqrt{x}} \, dx =$$

$$\textcircled{4} \int x e^{3x} \, dx =$$

$$u = x \quad dv = e^{3x} \, dx$$
$$du = dx \quad v = \frac{e^{3x}}{3}$$

$$x \cdot \frac{e^{3x}}{3} - \int \frac{1}{3} e^{3x} \, dx =$$

$$\textcircled{5} \int 3x^2 \tan^{-1}(x) \, dx.$$

$$u = \tan^{-1} x \quad dv = 3x^2 \, dx$$
$$du = \frac{1}{1+x^2} \, dx \quad v = x^3$$

$$= x^3 \tan^{-1} x - \int \frac{x^3}{1+x^2} \, dx$$

$$= x^3 \tan^{-1} x - \int \left(x - \frac{x}{1+x^2} \right) \, dx$$

$$= x^3 \tan^{-1} x - \frac{1}{2} x^2 + \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$\begin{array}{r} x \\ x^2+1 \overline{) x^3} \\ \underline{x^3+x} \\ -x \end{array}$$

$$\boxed{6} \int \sin^{-1}(x) dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int w^{-\frac{1}{2}} dw$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

$$u = \sin^{-1} x \quad dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = x$$

$$w = 1-x^2$$

$$dw = -2x dx$$

$$\frac{1}{2} dw = -x dx$$

$$\boxed{7} \int x^2 \cosh(2x) dx$$

$$= \frac{1}{2} x^2 \sinh 2x - \int x \sinh 2x dx$$

$$= \frac{1}{2} x^2 \sinh 2x - \frac{1}{2} x \cosh 2x + \frac{1}{2} \int \cosh 2x dx$$

$$= \frac{1}{2} x^2 \sinh 2x - \frac{1}{2} x \cosh 2x + \frac{1}{4} \sinh 2x + C$$

$$u = x^2 \quad dv = \cosh 2x dx$$

$$du = 2x dx \quad v = \frac{\sinh 2x}{2}$$

$$u = x \quad dv = \sinh 2x dx$$

$$du = dx \quad v = \frac{1}{2} \cosh 2x$$

Deriv		Integrals
x^2	+	$\cosh 2x$
$2x$	=	$\frac{1}{2} \sinh 2x$
2	+	$\frac{1}{4} \cosh 2x$
0	=	$\frac{1}{8} \sinh 2x$

Remark/ If n is a positive integer, we use the above method in the following cases:-

$$\boxed{1} \int x^n \cosh ax dx$$

$$\boxed{2} \int x^n \sinh ax dx$$

$$\boxed{3} \int x^n \cos ax dx$$

$$\boxed{4} \int x^n \sin ax dx$$

$$\boxed{5} \int x^n e^{ax} dx$$

$$\boxed{6} \int x^n a^x dx$$

For example / $\int x^2 \cdot 3^x dx$

Deriv		Integrals
x^2	+	3^x
$2x$	=	$\frac{3^x}{\ln 3}$
2	+	$\frac{3^x}{(\ln 3)^2}$
0	=	$\frac{3^x}{(\ln 3)^3}$

Remark / Sometimes we use substitution and then we use integration by parts.

Example / Evaluate the following Integrals

① $\int x (\ln x)^2 dx$

$$= \int e^y (y)^2 \cdot e^y dy$$

$$= \int y^2 \cdot e^{2y} dy$$

$$= \frac{1}{2} e^{2y} y^2 - \int y e^{2y} dy$$

$$= \frac{1}{2} e^{2y} y^2 - \frac{1}{2} e^{2y} y + \frac{1}{4} e^{2y} + c$$

$$= \frac{1}{2} (\ln x)^2 e^{2 \ln x} - \frac{1}{2} \ln x \cdot (e)^{2 \ln x} + \frac{1}{4} e^{2 \ln x} + c$$

$$y = \ln x \Rightarrow e^y = x$$

$$dy = \frac{1}{x} dx$$

$$dy = \frac{1}{e^y} dx$$

$$u = y^2 \quad dv = e^{2y} dy$$

$$du = 2y dy \quad v = \frac{1}{2} e^{2y}$$

$$u = y \quad dv = e^{2y} dy$$

$$du = dy \quad v = \frac{1}{2} e^{2y}$$

② $\int \cos(\sqrt[3]{x}) dx$

$$= 3 \int t^2 \cos t dt$$

$$= 3 t^2 \sin t - 6 t \cos t - 6 \sin t + c$$

$$= 3 x^{2/3} \sin \sqrt[3]{x} - 6$$

$$t = \sqrt[3]{x} \Rightarrow x = t^3$$

$$dx = 3t^2 dt$$

Deriv

Integ.

t^2	$\rightarrow$	$\cos t$
$2t$	$\rightarrow$	$\sin t$
2	$\rightarrow$	$-\cos t$
0	$\rightarrow$	$-\sin t$

③ $\int (\sin^{-1} x)^2 dx$

$$= \int y^2 \cos y dy$$

$$y = \sin^{-1} x \Rightarrow x = \sin y$$

$$dx = \cos y dy$$

$$\begin{aligned}
 [4] \int x^5 e^{x^2} dx \\
 = \int x \cdot (x^2)^2 e^{x^2} dx \\
 = \frac{1}{2} \int t^2 e^t dt
 \end{aligned}$$

$$t = x^2 \rightarrow dt = 2x dx$$

$$\begin{array}{cc}
 \text{deriv} & \text{Integ} \\
 t^2 & e^t
 \end{array}$$

$$[5] \int_0^{\sqrt{1/2}} 2x \sin^{-1}(x^2) dx.$$

$$\theta = x^2$$

$$d\theta = 2x dx$$

$$u = \sin^{-1} \theta \quad du = d\theta$$

$$du = \frac{1}{\sqrt{1-\theta^2}} d\theta \quad u = \theta$$

$$w = 1 - \theta^2$$

$$dw = -2\theta d\theta$$

$$\begin{aligned}
 &= \int_{1/2}^1 \sin^{-1}(\theta) d\theta \\
 &= \theta \sin^{-1} \theta \Big|_0^{1/2} - \int_0^{1/2} \frac{\theta}{\sqrt{1-\theta^2}} d\theta \\
 &= \theta \sin^{-1} \theta \Big|_0^{1/2} + \frac{1}{2} \int_1^{3/4} w^{-1/2} dw
 \end{aligned}$$

=

$$[6] \int \sec^3 x \cdot dx$$

$$u = \sec x \quad du = \sec^2 x dx$$

$$du = \sec x \tan x dx \quad v = \tan x$$

$$= \int \sec^2 x \cdot \sec x dx$$

$$= \tan x \sec x - \int \sec x \tan x \cdot \tan x dx$$

$$= \tan x \sec x - \int (\sec^2 x - 1) \sec x dx$$

$$\int \sec^3 x dx = \tan x \sec x - \int \sec^3 x dx + \int \sec x dx$$

$$2 \int \sec^3 x dx = \tan x \sec x + \ln |\sec x + \tan x| + C$$

$$\int \sec^3 x dx = \frac{1}{2} \left(\tan x \sec x + \ln |\sec x + \tan x| \right) + C_1$$

$$C_1 = \frac{1}{2} C$$

Ex $\int \csc^3 x \cdot dx$

7 $\int e^{2x} \cos 3x \, dx$

$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x \, dx$

$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \left(\frac{1}{3} e^{2x} \cos 3x \right) + \frac{2}{3} \left(-\frac{2}{3} \int e^{2x} \cos 3x \, dx \right)$

$\int e^{2x} \cos 3x \, dx = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x \, dx$

$\frac{13}{9} \int e^{2x} \cos 3x \, dx = \dots =$

$u = e^{2x} \quad dv = \cos 3x \, dx$
 $du = 2e^{2x} \, dx \quad v = \frac{1}{3} \sin 3x$

$u = e^{2x} \quad dv = \sin 3x \, dx$
 $du = 2e^{2x} \, dx \quad v = -\frac{1}{3} \cos 3x$

8 $\int \cos(\ln x) \, dx$

$= \int \cos(y) e^y \, dy$

$y = \ln x \Rightarrow x = e^y$
 $dx = e^y \, dy$

Example/ Use integration by parts to establish the following reduction formula:

$$\int (\ln x)^n dx = x (\ln x)^n - n \int (\ln x)^{n-1} dx, \quad n \text{ is a positive integer}$$

$$u = (\ln x)^n, \quad dv = dx$$
$$du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx, \quad v = x$$

$$\Rightarrow x (\ln x)^n - \int n(\ln x)^{n-1} dx$$

$$= x (\ln x)^n - n \int (\ln x)^{n-1} dx.$$

Example/ $\int (\ln x)^3 dx$.

$$= x (\ln x)^3 - 3 \int (\ln x)^2 dx$$

$$= x (\ln x)^3 - 3 \left[x (\ln x)^2 - 2 \int (\ln x) dx \right]$$

$$= x \ln^3 x - 3x \ln^2 x + 6 \int \ln x dx$$

$$= x \ln^3 x - 3x \ln^2 x + 6 \left(x \ln x - \int 1 dx \right)$$

$$= x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C.$$

8.2 Trigonometric integrals:-

[1] Integrals in the form $\int \sin^m x \cos^n x dx$, where m and n are nonnegative integers.

We have three cases:-

case 1:- If m is ~~an~~ odd, we write $m=2k+1$ and use $\sin^2 x = 1 - \cos^2 x$.

case 2:- n is odd, $n=2k+1$ use $\cos^2 x = 1 - \sin^2 x$.

case 3:- both m and n are even, we ~~use~~ use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

Example/ Evaluate the following integrals.

$$[1] \int \sin x \cos^4 x dx = -\int \cos^4 x d(\cos x) = -\frac{1}{5} \cos^5 x + C.$$

$$\begin{aligned} [2] \int \sin^3 x \cos^4 x dx &= \int \sin^2 x \cdot \sin x \cos^4 x dx \\ &= -\int (1 - \cos^2 x) \cos^4 x d(\cos x) = +\int \cos^6 x - \cos^4 x d(\cos x) \\ &= -\frac{1}{7} \cos^7 x + \frac{1}{5} \cos^5 x + C \end{aligned}$$

$$\begin{aligned} [3] \int \sin^6 x \cos^3 x dx &= \int \sin^5 x \cdot \cos^2 x \cos x dx \\ &= \int \sin^5 x (1 - \sin^2 x) d(\sin x) \\ &= \int \sin^6 x - \sin^8 x d(\sin x) = \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C \end{aligned}$$

$$\begin{aligned} [4] \int \sin^3 x \cos^3 x dx &= \int \sin^2 x \cdot \cos^2 x \cos x dx \\ &= \int \sin^2 x (1 - \sin^2 x) d(\sin x) = \int \sin^2 x - \sin^4 x d(\sin x) \\ &= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C \end{aligned}$$

$$\begin{aligned} [5] \int \sin^4 x \cos^2 x dx &= \int \sin^2 x \cdot \sin^2 x \cos^2 x dx \\ &= \int \sin^2 x (\sin x \cos x)^2 dx = \int \frac{1}{2} (1 - \cos 2x) \left(\frac{1}{2} \sin 2x \right)^2 dx \end{aligned}$$

$$= \frac{1}{8} \int (\sin^2 2x - \sin^2 2x \cos 2x) dx = \frac{1}{8} \int \frac{1}{2} (1 - \cos 4x) dx - \frac{1}{8} \int \frac{1}{2} \sin^2 2x d(\sin 2x)$$

$$= \frac{1}{16} \left(x - \frac{\sin 4x}{4} \right) - \frac{1}{16} \left(\frac{\sin^3 2x}{3} \right) + C$$

$$= \frac{1}{16} x - \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C.$$

[2] Eliminating square roots.

Example/ Evaluate the following integrals:

$$\begin{aligned}
 \text{Q1} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2 x}{\sqrt{1-\cos x}} dx &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{(2\sin \frac{x}{2} \cos \frac{x}{2})^2}{\sqrt{2\sin^2(\frac{x}{2})}} dx = \frac{4}{\sqrt{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}{\sin \frac{x}{2}} dx \\
 &= \frac{4}{\sqrt{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin \frac{x}{2} \cos^2 \frac{x}{2} dx = 2 \cdot \frac{4}{\sqrt{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2 \frac{x}{2} d(\cos \frac{x}{2}) \\
 &= \frac{-8}{\sqrt{2}} \cdot \frac{\cos^3 \frac{x}{2}}{3} \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{-8}{\sqrt{2}} \left(\cos^3 \frac{3\pi}{4} - \cos^3 \frac{\pi}{4} \right) \\
 &= \frac{-8}{\sqrt{2}} \left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{2}}{8} \right) = \frac{\sqrt{2}}{2} - \frac{2}{3}
 \end{aligned}$$

Another solution:

$$\begin{aligned}
 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2 x}{\sqrt{1-\cos x}} \cdot \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}} dx &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2 x \cdot \sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}} dx \\
 &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2 x \sqrt{1+\cos x}}{\sin x} dx = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x \cdot \sqrt{1+\cos x} dx \\
 &= - \int_{\frac{3}{2}}^1 w^{\frac{1}{2}} dw = \int_1^{\frac{3}{2}} w^{\frac{1}{2}} dw = \frac{2}{3} w^{\frac{3}{2}} \Big|_1^{\frac{3}{2}} = \frac{2}{3} \left(\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1 \right)
 \end{aligned}$$

$u = 1 + \cos x$
 $du = -\sin x dx$

$$\boxed{6} \int_0^{\pi/2} x \sqrt{1 - \cos 2x} \, dx = \int_0^{\pi/2} x \sqrt{2 \sin^2 x} \, dx = \sqrt{2} \int_0^{\pi/2} x \sin x \, dx$$

$$\sqrt{2} (-x \cos x + \sin x) \Big|_0^{\pi/2}$$

$$= \boxed{\sqrt{2}}$$

Deriv.	Integ.
x	$\sin x$
$\downarrow$	$\uparrow$
1	$-\cos x$
$\downarrow$	$-\sin x$

3 Integrals of powers of $\tan x$ and $\sec x$

Example / Evaluate the following integrals:

$$\begin{aligned} \boxed{1} \int \sec^6 x \, dx &= \int (\sec^4 x)^2 \sec^2 x \, dx \\ &= \int (1 + \tan^2 x)^2 d(\tan x) = \int (1 + 2\tan^2 x + \tan^4 x) d(\tan x) \\ &= \tan x + \frac{2}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C \end{aligned}$$

$$\begin{aligned} \boxed{2} \int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int (\sec^2 x - 1) \tan^2 x \, dx \\ &= \int \sec^2 x \tan^2 x \, dx - \int \tan^2 x \, dx = \int \tan^2 x \, d(\tan x) - \int (\sec^2 x - 1) \, dx \\ &= \frac{1}{3} \tan^3 x - \tan x - x + C \end{aligned}$$

$$\begin{aligned} \boxed{3} \int \tan^5 x \, dx &= \int \tan^2 x \cdot \tan^3 x \, dx = \int (\sec^2 x - 1) \tan^3 x \, dx \\ &= \int \sec^2 x \tan^3 x \, dx - \int \tan^3 x \, dx \\ &= \int \tan^3 x \, d(\tan x) + \int \tan^2 x \cdot \tan x \, dx \\ &= \frac{1}{4} \tan^4 x - \int (\sec^2 x - 1) \tan x \, dx \\ &= \frac{\tan^4 x}{4} - \int \sec^2 x \tan x \, dx + \int \tan x \, dx \\ &= \frac{\tan^4 x}{4} - \frac{1}{2} \tan^2 x + \ln |\cos x| + C \end{aligned}$$

$$\boxed{4} \int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \cdot \sec x \tan x \, dx$$

$$= \int \sec^2 x (\sec^2 x - 1) \, d(\sec x)$$

$$= \int (\sec^4 x - \sec^2 x) \, d(\sec x) = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

$$\boxed{5} \int \sec^4 x \tan^5 x \, dx$$

$$= \int \sec^2 x \cdot \sec^2 x \cdot \tan^5 x \, dx = \int (1 + \tan^2 x) \tan^5 x \, d(\tan x)$$

$$= \int (\tan^5 x + \tan^7 x) \, d(\tan x) = \frac{1}{6} \tan^6 x + \frac{1}{8} \tan^8 x + C$$

$$\boxed{6} \int \sec^4 x \tan^4 x \, dx$$

$$= \int \sec^2 x \cdot \sec^2 x \cdot \tan^4 x \, dx = \int (1 + \tan^2 x) \tan^4 x \, d(\tan x)$$

$$= \int \tan^4 x + \tan^6 x \, d(\tan x) = \frac{1}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C$$

4] Integrals of Products of sines and cosines :-

we use the following formulas :-

$$\textcircled{1} \sin mx \cdot \cos nx = \frac{1}{2} [\sin(m-n)x + \cos(m+n)x]$$

$$\textcircled{2} \cos mx \cdot \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

$$\textcircled{3} \sin mx \cdot \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$$

Example. Evaluate the following integrals:

$$[a] \int \sin 2x \cos 3x \, dx = \frac{1}{2} \int [\sin(2-3)x + \sin(2+3)x] \, dx$$

$$= \frac{1}{2} \int (\sin -x + \sin 5x) \, dx = \frac{1}{2} \int (-\sin x + \sin 5x) \, dx$$

$$= \frac{1}{2} \left(\cos x - \frac{\cos 5x}{5} \right) + C$$

$$[b] \int \cos 3x \cdot \cos 4x \, dx = \frac{1}{2} \int (\cos(3-4)x + \cos(3+4)x) \, dx$$

$$= \frac{1}{2} \int \cos -x \, dx + \cos 7x \, dx = \frac{1}{2} \int (\cos x + \cos 7x) \, dx$$

$$= \frac{1}{2} \sin x + \frac{1}{14} \sin 7x + C$$

$$[c] \int \sin \theta \cos \theta \cos 3\theta \, d\theta = \frac{1}{2} \int \sin 2\theta \cos 3\theta \, d\theta$$

$$= \frac{1}{4} \int (\sin(2-3)\theta + \sin(2+3)\theta) \, d\theta = \frac{1}{4} \int (\sin -\theta + \sin 5\theta) \, d\theta$$

$$= \frac{1}{4} \left(\cos \theta - \frac{1}{5} \cos 5\theta \right) + C$$

$$[d] \int \frac{\sec^3 x}{\tan x} \, dx = \int \frac{\sec^2 x \cdot \sec x}{\tan x} \, dx = \int \frac{(1+\tan^2 x) \sec x}{\tan x} \, dx$$

$$= \int (\cot x + \tan x) \sec x \, dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} \, dx + \int \tan x \sec x \, dx$$

$$= \int \csc x \, dx + \int \tan x \sec x \, dx = \ln |\csc x - \cot x| + \sec x + C$$

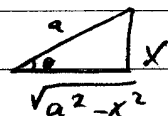
8.3 Trigonometric ~~substitution~~ Substitution

or $(a^2 - x^2)^n$

[1] If the integrand contains $\sqrt{a^2 - x^2}$, Put $x = a \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\sin \theta = \frac{x}{a}$$

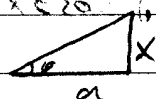
$$\theta = \sin^{-1} \frac{x}{a}, \quad dx = a \cos \theta d\theta$$



[2] If the integrand contains $\sqrt{a^2 + x^2}$ or $(x^2 + a^2)^n$, Put $x = a \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\tan \theta = \frac{x}{a} \Rightarrow dx = a \sec^2 \theta d\theta$$

$$\theta = \tan^{-1} \frac{x}{a}$$

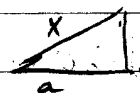


[3] If the integrand contains $\sqrt{x^2 - a^2}$ or $(x^2 - a^2)^n$, Put $x = a \sec \theta$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\sec \theta = \frac{x}{a}$$

$$\Rightarrow dx = a \sec \theta \tan \theta d\theta$$

$$\theta = \sec^{-1} \frac{x}{a}$$



Example / Evaluate the following integrals:-

[1] $\int \frac{\sqrt{9-x^2}}{x^2} dx$

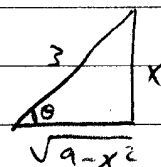
put $x = 3 \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$dx = 3 \cos \theta d\theta$

$$\Rightarrow \int \frac{\sqrt{9-9\sin^2 \theta}}{9\sin^2 \theta} \cdot 3 \cos \theta d\theta = \frac{3 \cdot 3}{9} \int \frac{\cos \theta \cdot \cos \theta}{\sin^2 \theta} d\theta = \int \cot^2 \theta d\theta$$

$$= \int (\csc^2 \theta - 1) d\theta = -\cot \theta - \theta + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$



[2] $\int \frac{1}{x^3(x^2+4)} dx$

$x = 2 \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2 \sec^2 \theta d\theta$

$$= \int \frac{2 \sec^2 \theta d\theta}{8 \tan^3 \theta (4 \tan^2 \theta + 4)} = \frac{2}{8 \cdot 4} \int \frac{\sec^2 \theta d\theta}{\tan^3 \theta (\sec^2 \theta)}$$

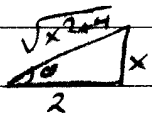
$$= \frac{1}{16} \int \cot^3 \theta \, d\theta = \frac{1}{16} \int \cot^2 \theta \cdot \cot \theta \, d\theta$$

$$= \frac{1}{16} \int (\csc^2 \theta - 1) \cot \theta \, d\theta = \frac{1}{16} \int \csc^2 \theta \cot \theta \, d\theta - \frac{1}{16} \int \cot \theta \, d\theta$$

$$= -\frac{1}{16} \int \cot \theta \, d(\cot \theta) - \frac{1}{16} \int \frac{\cos \theta}{\sin \theta} \, d\theta$$

$$= -\frac{1}{16} \cdot \frac{\cot^2 \theta}{2} - \frac{1}{16} \ln |\sin \theta| + C$$

$$= -\frac{1}{32} \left(\frac{2}{x} \right)^2 - \frac{1}{16} \ln \left| \frac{x}{\sqrt{x^2+4}} \right| + C$$

$$\tan \theta = \frac{x}{2}$$


$$= -\frac{1}{8x^2} - \frac{1}{16} \ln \left| \frac{x}{\sqrt{x^2+4}} \right| + C$$

$$\text{[3]} \int \frac{\sqrt{5x^2-9}}{x} \, dx = \int \frac{\sqrt{5(x^2-\frac{9}{5})}}{x} \, dx = \sqrt{5} \int \frac{\sqrt{x^2-\frac{9}{5}}}{x} \, dx$$

$$\sqrt{5} \int \frac{\sqrt{\frac{9}{5} \sec^2 \theta - \frac{9}{5}}}{\frac{3}{\sqrt{5}} \sec \theta} \cdot \frac{3}{\sqrt{5}} \sec \theta \tan \theta \, d\theta$$

$$x = \frac{3}{\sqrt{5}} \sec \theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$dx = \frac{3}{\sqrt{5}} \sec \theta \tan \theta \, d\theta$$

$$\sqrt{5} \int \frac{\frac{3}{\sqrt{5}} \sqrt{\sec^2 \theta - 1}}{\frac{3}{\sqrt{5}} \sec \theta} \cdot \frac{3}{\sqrt{5}} \sec \theta \tan \theta \, d\theta = 3 \int \sqrt{\tan^2 \theta} \tan \theta \, d\theta$$

$$\sec \theta = \frac{\sqrt{5}x}{3}$$

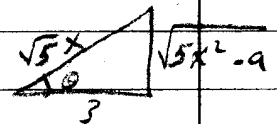
$$= 3 \int \tan^2 \theta \, d\theta = 3 \int (\sec^2 \theta - 1) \, d\theta$$

$$\cos \theta = \frac{3}{\sqrt{5}x}$$

$$= 3(\tan \theta - \theta) + C$$

$$= 3 \tan \theta - 3\theta + C$$

$$= \frac{3}{\sqrt{5}} \frac{\sqrt{5x^2-9}}{3} - 3 \sec^{-1} \frac{\sqrt{5}x}{3} + C$$



$$[4] \int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx.$$

$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

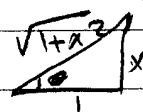
$$\int \frac{\tan \theta \cdot \theta}{(1+\tan^2 \theta)^{3/2}} \sec^2 \theta d\theta.$$

$$= \int \frac{\tan \theta \cdot \theta \cdot \sec^2 \theta}{(\sec^2 \theta)^{3/2}} d\theta = \int \frac{\theta \tan \theta}{\sec \theta} d\theta = \int \theta \sin \theta d\theta.$$

$$= -\theta \cos \theta + \sin \theta + C$$

$$= -\tan^{-1} x \left(\frac{1}{\sqrt{1+x^2}} \right) + \frac{x}{\sqrt{1+x^2}} + C$$

Deriv.	Integ.
θ	$\sin \theta$
1	$-\cos \theta$
0	$-\sin \theta$



$$[5] \int \frac{dx}{e^x \sqrt{e^{2x} - 16}}$$

$$e^x = 4 \sec \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

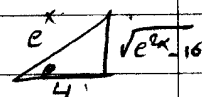
$$e^x dx = 4 \sec \theta \tan \theta d\theta$$

$$dx = \tan \theta d\theta$$

$$= \int \frac{\tan \theta d\theta}{4 \sec \theta \sqrt{16 \sec^2 \theta - 16}} = \frac{1}{4 \cdot 4} \int \frac{\tan \theta}{\sec \theta \sqrt{\tan^2 \theta}} d\theta$$

$$= \frac{1}{16} \int \frac{\tan \theta}{\sec \theta \tan \theta} d\theta = \left(\frac{1}{16} \right) \int \frac{1}{\sec \theta} d\theta = \left(\frac{1}{16} \right) \int \cos \theta d\theta = \frac{1}{16} \sin \theta + C$$

$$= \frac{1}{16} \left(\frac{\sqrt{e^{2x} - 16}}{e^x} \right) + C$$



$$[6] \int \frac{(\ln x)^3}{x \sqrt{1+\ln^2 x}} dx.$$

$$\ln x = \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{1}{x} dx = \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} \frac{\tan^3 \theta \sec^2 \theta d\theta}{\sqrt{1+\tan^2 \theta}} = \int_0^{\pi/4} \frac{\tan^3 \theta \cdot \sec^2 \theta}{\sec \theta} d\theta$$

$$= \int_0^{\pi/4} \tan^2 \theta \sec \theta d\theta$$

$$= \int_0^{\pi/4} \tan^2 \theta \cdot (\tan \theta \sec \theta) d\theta$$

$$= \int_0^{\pi/4} (\sec^3 \theta - \sec \theta) d(\sec \theta)$$

$$= \left[\frac{1}{3} \sec^3 \theta - \sec \theta \right]_0^{\pi/4}$$

=

$$[7] \int \sqrt{\frac{4-x}{x}} dx$$

$$x = y^2 \\ dx = 2y dy$$

$$= \int \sqrt{\frac{4-y^2}{y^2}} \cdot 2y dy$$

$$2 \int \frac{\sqrt{4-y^2}}{y} dy = 2 \int \sqrt{4-y^2} dy$$

$$= 2 \int \sqrt{4-4\sin^2 \theta} \cdot 2 \cos \theta d\theta \quad y = 2\sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ dy = 2 \cos \theta d\theta$$

$$= 2 \cdot 2 \cdot 2 \int \sqrt{\cos^2 \theta} \cdot \cos \theta d\theta$$

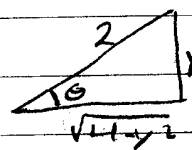
$$= 8 \int \cos \theta \cdot \cos \theta d\theta = 8 \int \cos^2 \theta d\theta$$

$$= \frac{1}{2} \cdot 8 \int (1 + \cos 2\theta) d\theta$$

$$= 4 \left(\theta + \frac{\sin 2\theta}{2} \right) + C = 4\theta + 4 \sin \theta \cos \theta + C$$

$$= 4 \cdot \sin^{-1} \left(\frac{y}{2} \right) + 4 \left(\frac{y}{2} \right) \left(\frac{\sqrt{4-y^2}}{2} \right) + C$$

$$= 4 \sin^{-1} \left(\frac{y}{2} \right) + y \sqrt{4-y^2} + C$$



$$\boxed{8} \int \frac{x}{\sqrt{x^2-2x-3}} dx.$$

$$\int \frac{x dx}{\sqrt{(x^2-2x+1)-4}} = \int \frac{x dx}{\sqrt{(x-1)^2-4}}$$

$$x-1 = 2\sec \theta$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

$$\rightarrow \int \frac{(1+2\sec \theta) \cdot 2\sec \theta \tan \theta d\theta}{\sqrt{4\sec^2 \theta - 4}}$$

$$dx = 2\sec \theta \tan \theta d\theta$$

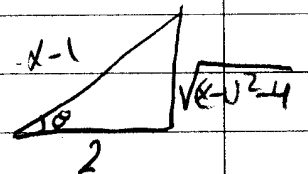
$$= \int \frac{(1+2\sec \theta) \sec \theta \cancel{\tan \theta} d\theta}{\sqrt{\cancel{\tan^2 \theta}}}$$

$$= \int (1+2\sec \theta) \sec \theta d\theta$$

$$= \int \sec \theta + 2\sec^2 \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + 2\tan \theta + C.$$

$$= \ln \left| \frac{x-1}{2} + \frac{\sqrt{(x-1)^2-4}}{2} \right| + 2 \left(\frac{\sqrt{(x-1)^2-4}}{2} \right) + C$$



$$\ln |x-1 + \sqrt{x^2-2x-3}| = \ln 2 + \sqrt{x^2-2x-3} + C.$$

8.4 Integration of rational fns by partial fractions.

Def. A rational fn $R(x)$ is defined as $R(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials with no common factors.

II If $\deg(f) < \deg(g)$ and $g(x)$ is written as product of distinct first degree factors. عوامل منفصلة من الدرجة الأولى غير متكررة

Example. Evaluate the following integrals:-

$$[1] \int \frac{x+3}{2x^3-8x} dx = \int \frac{x+3}{2x(x^2-4)} dx = \int \frac{x+3}{2x(x-2)(x+2)} dx$$

$$\frac{x+3}{2x(x-2)(x+2)} = \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$x+3 = A(x-2)(x+2) + B(2x)(x+2) + C(2x)(x-2) \quad \text{نضرب الطرفين بـ } 2x(x-2)(x+2)$$

$$\text{If } x=0, \text{ then } \rightarrow 3 = A(-4) \Rightarrow A = \frac{-3}{4}$$

$$\text{If } x=2, \text{ then } 2+3 = 16B \Rightarrow B = \frac{5}{16}$$

$$\text{If } x=-2, \text{ then } -2+3 = 16C \Rightarrow C = \frac{1}{16}$$

$$\int \frac{x+3}{2x(x+2)(x-2)} dx = \int \frac{\frac{-3}{4}}{2x} dx + \int \frac{\frac{5}{16}}{x-2} dx + \int \frac{\frac{1}{16}}{x+2} dx$$

$$= \frac{-3}{4} \cdot \frac{1}{2} \ln|x| + \frac{5}{16} \ln|x-2| + \frac{1}{16} \ln|x+2| + C$$

$$[2] \int \frac{x^2+3}{x^2-3x+2} dx$$

$$= \int 1 + \frac{3x+1}{(x-1)(x-2)} dx$$

$$= \int 1 dx + \int \frac{3x+1}{(x-1)(x-2)} dx$$

$$\frac{3x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\frac{x^2-3x+2}{x^2-3x+2} \cdot \frac{x^2+3}{x^2+3}$$

$$\frac{x^2+3}{x^2-3x+2} = \frac{3x+1}{x^2-3x+2}$$

$$3x+1 = A(x-2) + B(x-1)$$

If $x = 1$, then $4 = -A \Rightarrow A = -4$

If $x = 2$, then $7 = B \Rightarrow B = 7$

$$\begin{aligned} \int \frac{x^2+3}{x^2-3x+2} dx &= \int 1 dx + \int \frac{-4}{x-1} dx + \int \frac{7}{x-2} dx \\ &= x - 4 \ln|x-1| + 7 \ln|x-2| + C \end{aligned}$$

[2] If $\deg(f) < \deg(g)$ and $g(x)$ is written as ~~repeated~~ First degree factor.

Example/ [a] $\int \frac{2x^2+3x+3}{(x+1)^3} dx$

$$\frac{2x^2+3x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\begin{aligned} 2x^2+3x+3 &= A(x+1)^2 + B(x+1) + C \\ &= Ax^2 + 2Ax + A + Bx + B + C \\ &= Ax^2 + (2A+B)x + (A+B+C) \end{aligned}$$

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$$\therefore \underline{A=2}, \quad 2A+B=3 \Rightarrow \underline{B=-1}, \quad A+B+C=3 \Rightarrow \underline{C=2}$$

$$\begin{aligned} \int \frac{2x^2+3x+3}{(x+1)^3} dx &= \int \frac{2}{x+1} dx + \int \frac{-1}{(x+1)^2} dx + \int \frac{2}{(x+1)^3} dx \\ &= 2 \ln|x+1| + \frac{1}{x+1} - \frac{1}{(x+1)^2} + C \end{aligned}$$

[b] $\int \frac{x^2}{(x-1)(x^2+2x+1)} dx = \int \frac{x^2}{(x-1)(x+1)^2} dx$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

If $x=1$ then $A = \frac{1}{4}$

If $x=-1$ then $C = -\frac{1}{2}$

If $x=0$ then $0 = A - B - C$

$$B = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$\begin{aligned} \int \frac{x^2 dx}{(x-1)(x^2+2x+1)} &= \int \frac{\frac{1}{4}}{x-1} dx + \int \frac{\frac{3}{4}}{x+1} dx + \int \frac{-\frac{1}{2}}{(x+1)^2} dx \\ &= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2(x+1)} + C \end{aligned}$$

□ $\int \frac{16x^3}{4x^2-4x+1} dx$

$$= \int 4x+4 dx + \int \frac{12x-4}{4x^2-4x+1} dx$$

$$\begin{array}{r} 4x+4 \overline{) 16x^3-16x^2+4x} \\ \underline{16x^3-16x^2+4x} \\ 0 \end{array}$$

$$\frac{12x-4}{(2x-1)^2} = \frac{A}{(2x-1)} + \frac{B}{(2x-1)^2}$$

$$12x-4 = A(2x-1) + B$$

$$12x-4 = 2Ax + (A+B)$$

$$12 = 2A \Rightarrow \underline{A=6}, \quad A+B=-4 \Rightarrow \underline{B=-10}$$

$$\int \frac{16x^3}{4x^2-4x+1} dx = \int 4x+4 dx + \int \frac{6}{2x-1} dx + \int \frac{-10}{(2x-1)^2} dx$$

$$= 2x^2 + 4x + 3 \ln|2x-1| - \frac{1}{(2x-1)} + C$$

3] If $\deg(f) < \deg(g)$ and $g(x)$ is written as product of ~~the~~ distinct second degree factor.

Example / Evaluate the following integrals:-

$$a) \int \frac{x^2 + x}{x^4 + 5x^2 + 6} dx = \int \frac{x^2 + x}{(x^2 + 2)(x^2 + 3)} dx$$

$$\frac{x^2 + x}{(x^2 + 2)(x^2 + 3)} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{x^2 + 3}$$

$$\begin{aligned} x^2 + x &= (Ax + B)(x^2 + 3) + (Cx + D)(x^2 + 2) \\ &= Ax^3 + 3B + 3Ax + Bx^2 + Cx^3 + 2Cx + Dx^2 + 2D \\ &= (A + C)x^3 + (B + D)x^2 + (3A + 2C)x + (3B + 2D) \end{aligned}$$

$$\begin{aligned} A + C &= 0 & \Rightarrow -2A - 2C &= 0 \\ B + D &= 1 & \Rightarrow -2B - 2D &= -2 \\ 3A + 2C &= 1 & \Rightarrow 3A + 2C &= 1 \\ 3B + 2D &= 0 & \Rightarrow 3B + 2D &= 0 \end{aligned} \quad \Rightarrow \begin{aligned} A &= 1, C = -1 \\ B &= -2, D = 3 \end{aligned}$$

$$\begin{aligned} \int \frac{x^2 + x}{x^4 + 5x^2 + 6} dx &= \int \frac{x - 2}{x^2 + 2} dx + \int \frac{-x + 3}{x^2 + 3} dx \\ &= \int \frac{x}{x^2 + 2} dx - 2 \int \frac{1}{x^2 + 2} dx - \int \frac{x}{x^2 + 3} dx + \int \frac{3}{x^2 + 3} dx \\ &= \frac{1}{2} \ln|x^2 + 2| - \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2} \ln|x^2 + 3| + \frac{3}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C \end{aligned}$$

$$B) \int \frac{2y^4}{y^3 - y^2 + y - 1} dy$$

$$= \int (2y + 2) dy + \int \frac{2}{y^2(y-1) + (y-1)} dy$$

$$= \int 2y + 2 dy + \int \frac{2}{(y-1)(y^2+1)} dy$$

$$\frac{2}{(y-1)(y^2+1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$$

$$\begin{aligned} &\frac{2y+2}{y^3 - y^2 + y - 1} \sqrt{2y^4} \\ &\frac{2y^4 - 2y^3 + 2y^2 - 2y}{2y^3 - 2y^2 + 2y - 2} \\ &\frac{2y^4 - 2y^3 + 2y^2 - 2y - 2}{2} \end{aligned}$$

4) If $\deg(f) < \deg(g)$ and $g(x)$ is written as repeated second degree factors

Example/ $\int \frac{x^3 + 4x^2 - 4x - 1}{(x^2 + 1)^2} dx$

$$\frac{x^3 + 4x^2 - 4x - 1}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

$$\begin{aligned} x^3 + 4x^2 - 4x - 1 &= (Ax + B)(x^2 + 1) + (Cx + D) \\ &= Ax^3 + Ax + Bx^2 + B + Cx + D \\ &= Ax^3 + (A + C)x + Bx^2 + (B + D) \end{aligned}$$

مقابل

$A = 1$, $B = +4$

$A + C = -4 \Rightarrow \underline{C = -5}$, $B + D = -1 \Rightarrow \underline{D = -5}$

$$\int \frac{x^3 + 4x^2 - 4x - 1}{(x^2 + 1)^2} dx = \int \frac{x + 4}{x^2 + 1} dx + \int \frac{-5x - 5}{(x^2 + 1)^2} dx$$

$$= \frac{1}{2} \int \frac{2x}{x^2 + 1} dx + 4 \int \frac{1}{x^2 + 1} dx + \frac{-5}{2} \int \frac{2x}{(x^2 + 1)^2} dx - 5 \int \frac{1}{(x^2 + 1)^2} dx$$

$$= \frac{1}{2} \ln(x^2 + 1) + 4 \tan^{-1} x - \frac{5}{2} \int \frac{dt}{t^2} - 5 \int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^2} \quad \left| \begin{array}{l} x = \tan \theta \\ t = x^2 + 1 \end{array} \right.$$

$$= \frac{1}{2} \ln(x^2 + 1) + 4 \tan^{-1} x - \frac{5}{2} \left(\frac{-1}{t} \right) - 5 \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta$$

$$\begin{aligned}
 &= \frac{1}{2} \ln(x^2+1) + 4 \tan^{-1} x + \frac{5}{t^2} - 5 \int \cos^2 \theta \, d\theta \\
 &= \quad \quad \quad - \frac{5}{2} \int (1 + \cos 2\theta) \, d\theta \\
 &= \quad \quad \quad \left(\theta + \frac{\sin 2\theta}{2} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 \text{Q9} \int \frac{\sin \theta \, d\theta}{\cos^2 \theta + \cos \theta - 2} \\
 = - \int \frac{-\sin \theta \, d\theta}{\cos^2 \theta + \cos \theta - 2}
 \end{aligned}$$

$$\begin{aligned}
 \text{put } t &= \cos \theta \\
 dt &= -\sin \theta \, d\theta
 \end{aligned}$$

$$= - \int \frac{dt}{t^2 + t - 2} = - \int \frac{1}{(t+2)(t-1)} \, dt$$

$$\frac{1}{(t+2)(t-1)} = \frac{A}{t+2} + \frac{B}{t-1}$$

$$1 = A(t-1) + B(t+2)$$

$$\text{if } t=1 \Rightarrow 1 = B(3) \Rightarrow \underline{B = 1/3}$$

$$\text{if } t=2 \Rightarrow 1 = A(-3) \Rightarrow \underline{A = -1/3}$$

$$\frac{1}{(t+2)(t-1)} = \frac{-1/3}{t+2} + \frac{1/3}{t-1}$$

$$- \int \frac{1}{(t+2)(t-1)} = \int \frac{1/3}{t+2} \, dt + \int \frac{-1/3}{t-1} \, dt$$

=

$$= \frac{1}{3} \ln|t+2| - \frac{1}{3} \ln|t-1| + C$$

$$= \frac{1}{3} \ln|\cos\theta+2| - \frac{1}{3} \ln|\cos\theta-1| + C$$

$$= \frac{1}{3} \ln \left| \frac{\cos\theta+2}{\cos\theta-1} \right| + C$$

$$[B] \int \frac{1}{x(x^{20}+1)} dx$$

$$\int \frac{x^{19}}{x^{20}(x^{20}+1)} dx$$

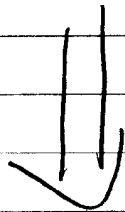
$$t = x^{20} + 1$$

$$dt = 20x^{19} dx$$

$$\frac{1}{20} dt = x^{19} dx$$

$$\Rightarrow \frac{1}{20} \int \frac{dt}{(t-1)t}$$

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1}$$



نفس الطريقة

8.5 / Integral Tables:

Example Evaluate the following Integrals:

$$\text{I} \int \frac{dx}{x\sqrt{x+4}}$$

$$x = y^2 \\ dx = 2y dy$$

$$\rightarrow \int \frac{2y dy}{y^2 \sqrt{y^2+4}} = 2 \int \frac{dy}{y \sqrt{y^2+4}}$$

$$= 2 \cdot -\frac{1}{2} \operatorname{csch}^{-1} \left| \frac{y}{2} \right| + C$$

$$= -\operatorname{csch}^{-1} \left| \frac{\sqrt{x}}{2} \right| + C$$

another soln:

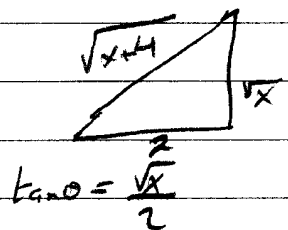
$$x = 4 \tan^2 \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ dx = 8 \tan \theta \sec^2 \theta d\theta$$

$$= \int \frac{8 \tan \theta \sec^2 \theta d\theta}{4 \tan^2 \theta \sqrt{4 \tan^2 \theta + 4}} = 2 \int \frac{\sec^2 \theta d\theta}{2 \tan \theta \sec \theta} = \int \frac{\sec \theta d\theta}{\tan \theta}$$

$$= \int \csc \theta d\theta = \ln |\csc \theta + \cot \theta| + C$$

$$= \ln \left| \frac{\sqrt{x+4}}{\sqrt{x}} + \frac{2}{\sqrt{x}} \right| + C$$

$$= \ln \left| \frac{\sqrt{x+4} + 2}{\sqrt{x}} \right| + C$$



$$[2] \int \frac{\sqrt{9-4x}}{x^2} dx$$

$$\text{put } x = \frac{9}{4} \sin^2 \theta$$

$$dx = \frac{9}{2} \sin \theta \cos \theta d\theta$$

$$\Rightarrow \int \frac{\sqrt{9-9\sin^2 \theta}}{\frac{81}{16} \sin^4 \theta} \cdot \frac{9}{2} \sin \theta \cos \theta d\theta$$

$$= \frac{8}{3} \int \frac{\cos \theta \cdot \sin \theta \cdot \cos \theta}{\sin^4 \theta} d\theta$$

$$= \frac{8}{3} \int \frac{\cos^2 \theta}{\sin^3 \theta} d\theta$$

$$= \frac{8}{3} \int \frac{(1-\sin^2 \theta)}{\sin^3 \theta} d\theta$$

$$= \frac{8}{3} \int \left(\frac{1}{\sin^3 \theta} - \frac{1}{\sin \theta} \right) d\theta$$

$$= \frac{8}{3} \int (\csc^3 \theta - \csc \theta) d\theta$$

$$= \frac{8}{3} \int \csc^3 \theta \cdot \csc \theta d\theta - \frac{8}{3} \int \csc \theta d\theta$$

$$= \frac{8}{3} \int (1 + \cot^2 \theta) \csc \theta d\theta - \frac{8}{3} \int \csc \theta d\theta = \frac{8}{3} \int \csc \theta d\theta + \frac{8}{3} \int \csc \theta \cot^2 \theta d\theta - \frac{8}{3} \int \csc \theta d\theta$$

$$\int \frac{\cos^2 \theta}{\sin^3 \theta} \cdot \frac{1}{\sin \theta} d\theta = \int \cot^2 \theta \cdot \csc \theta d\theta$$

$$u = \csc \theta \quad dv = \cot^2 \theta d\theta$$

$$du = -\csc \theta d\theta \quad v = -\csc \theta$$

$$- \cot \theta \cdot \csc \theta - \int \csc \theta \cot^2 \theta d\theta$$

$$u = \csc \theta \quad dv = \csc^2 \theta$$

$$[3] \int x \sqrt{4x - x^2} dx$$

$$\int x \sqrt{4 - 4 + 4x - x^2} dx = \int x \sqrt{4 - (x^2 - 4x + 4)} dx$$

$$= \int x \sqrt{4 - (x-2)^2} dx$$

$$y = x - 2$$

$$dy = dx$$

$$= \int (y+2) \sqrt{4 - y^2} dy$$

$$= \int y \sqrt{4 - y^2} dy + \int 2 \sqrt{4 - y^2} dy$$

$$= -\frac{1}{2} \int \sqrt{4 - y^2} d(4 - y^2) + 2 \int \sqrt{4 - y^2} dy$$

$$\rightarrow \frac{-1}{\frac{2}{3}} \left(\frac{4-y^2}{\frac{2}{2}} \right)^{\frac{3}{2}} + 2 \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta d\theta \quad \begin{cases} y = 2\sin\theta \\ dy = 2\cos\theta d\theta \end{cases}$$

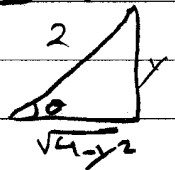
$$= -\frac{1}{3} (4-y^2)^{3/2} + 8 \int \cos^2\theta d\theta$$

$$= -\frac{1}{3} (4-y^2)^{3/2} + \frac{8}{2} \int (1 + \cos 2\theta) d\theta$$

$$= -\frac{1}{3} (4-y^2)^{3/2} + 4 \left(\theta + \frac{\sin 2\theta}{2} \right) + C$$

$$= -\frac{1}{3} (4-(x-2)^2)^{3/2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + 4 \cdot \frac{x-2}{2} \cdot \frac{\sqrt{4-(x-2)^2}}{2} + C$$

$$= -\frac{1}{3} (4x-x^2)^{3/2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) + (x-2)\sqrt{4x-x^2} + C$$



$$\boxed{4} \int x \cos^{-1} x dx$$

$$u = \cos^{-1} x \quad dv = x dx$$

$$du = \frac{-1}{\sqrt{1-x^2}} dx \quad v = \frac{1}{2} x^2$$

$$= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \frac{\sin^2\theta \cdot \cos\theta d\theta}{\sqrt{1-\sin^2\theta}}$$

$$x = \sin\theta$$

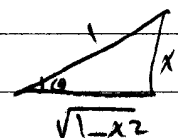
$$dx = \cos\theta d\theta$$

$$= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{2} \int \sin^2\theta d\theta$$

$$= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \int (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \left(\theta - \frac{\sin 2\theta}{2} \right) + C$$

$$= \frac{1}{2} x^2 \cos^{-1} x + \frac{1}{4} \sin^{-1} x - \frac{1}{4} \cdot x \cdot \sqrt{1-x^2} + C$$



$$5) \int \frac{\tan^{-1} x}{x^2} dx$$

$$u = \tan^{-1} x \quad v = \frac{1}{x^2} \quad du = \frac{1}{1+x^2} dx \quad dv = -\frac{1}{x^3} dx$$

$$\Rightarrow \frac{-\tan^{-1} x}{x} + \int \frac{1}{x(1+x^2)} dx$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)(x) = Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A$$

$$A = 1, C = 0, B = -1$$

$$\Rightarrow \int \frac{1}{x(1+x^2)} dx = \int \frac{1}{x} dx + \int \frac{-x}{x^2+1} dx = \ln|x| - \frac{1}{2} \ln|x^2+1| + C$$

$$\Rightarrow \int \frac{\tan^{-1} x}{x^2} dx = \frac{-\tan^{-1} x}{x} + \ln|x| - \frac{1}{2} \ln|x^2+1| + C$$

$$6) \int \frac{\sqrt{x}}{\sqrt{1-x}} dx$$

$$y = \sqrt{x} \Rightarrow x = y^2 \quad dx = 2y dy$$

$$\Rightarrow \int \frac{y}{\sqrt{1-y^2}} \cdot 2y dy = \int \frac{2y^2}{\sqrt{1-y^2}} dy$$

$$= 2 \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta$$

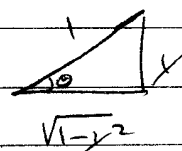
$$= 2 \int \sin^2 \theta d\theta$$

$$= \frac{2}{2} \int (1 - \cos 2\theta) d\theta = \theta - \frac{\sin 2\theta}{2} + C$$

$$y = \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad dy = \cos \theta d\theta$$

$$= \sin^{-1} y - y\sqrt{1-y^2} + C$$

$$= \sin^{-1} \sqrt{x} - \sqrt{x}\sqrt{1-x} + C.$$



$$\boxed{7} \int \frac{dx}{\tan x \sqrt{4 - \sin^2 x}}$$

$$y = \sin x$$

$$dy = \cos x dx$$

$$\Rightarrow \int \frac{dy}{\cos x \left(\frac{\sin x}{\cos x} \right) \sqrt{4 - y^2}} = \int \frac{dy}{y \sqrt{4 - y^2}}$$

$$= -\frac{1}{2} \operatorname{sech}^{-1} \left(\frac{y}{2} \right) + C$$

$$= -\frac{1}{2} \operatorname{sech}^{-1} \left(\frac{\sin x}{2} \right) + C$$

$$\boxed{8} \int \frac{dx}{x \sqrt{3 + (\ln x)^2}}$$

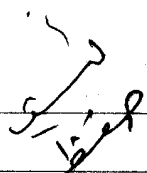
$$y = \ln x$$

$$dy = \frac{1}{x} dx$$

$$\Rightarrow \int \frac{dy}{\sqrt{3 + y^2}} = \sinh^{-1} \left(\frac{y}{\sqrt{3}} \right) + C$$

$$= \sinh^{-1} \left(\frac{\ln x}{\sqrt{3}} \right) + C.$$

Some reduction formulas



If n is a positive ~~integer~~ integer then

$$\boxed{1} \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\boxed{2} \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\boxed{3} \int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

$$\boxed{4} \int \sin^n x \cos^m x \, dx = \frac{-\sin^{n-1} x \cos^{m+1} x}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} x \cos^m x \, dx$$

, $n \neq -m$.

Example / Evaluate :-

$$\boxed{1} \int \tan^6 x \, dx$$

$$= \frac{1}{5} \tan^5 x - \int \tan^4 x \, dx$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \int \tan^2 x \, dx$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - \int 1 \, dx$$

$$= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x + C$$

$$[2] \int \sin^2(2\theta) \cos^3(2\theta) d\theta.$$

$$= \frac{1}{2} \int \sin^2 X \cos^3 X dX.$$

$$X = 2\theta$$

$$dX = 2d\theta$$

$$= \frac{1}{2} \times \frac{\sin X \cos^4 X}{5} + \frac{1}{5} \int 1 \cdot \cos^2 X dX$$

$$= \frac{1}{10} \sin X \cos^4 X + \frac{1}{5} \times \frac{1}{3} \cos^2 X \sin X + \frac{2}{3} \int \cos X dX$$

$$= \frac{1}{10} \sin X \cos^4 X + \frac{1}{15} \cos^2 X \sin X + \frac{2}{3} \sin X + C$$

$$= \frac{1}{10} \sin 2\theta \cos^4 2\theta + \frac{1}{15} \cos^2 2\theta \sin 2\theta + \frac{2}{3} \sin 2\theta + C$$

8.7 / Improper Integrals.

Def/ The definite integrals with infinite limits of integration are called improper integrals of type I.

① If f is continuous on $[a, \infty)$ then $\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$

② $= = = =$ on $(-\infty, b]$ then $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$

③ $= = = =$ on $(-\infty, \infty)$ then $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$

Def/ In each of the above cases, if the limit is infinite we say that the improper integral converges and the value of the limit is the value of the integral.

If the limit fails to exist, the improper integral is called diverges.

Remark/ Conv. $\pm$ Conv. = Conv.

Conv. $\pm$ div = div

div $\pm$ div = may be ^{تعاى} Conv. or may be ^{تباعى} div.

Example/ Determine whether the following improper integrals, Conv. or div?

II) $\int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2} dx$

$= \int_0^b \frac{16 \tan^{-1} x}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b 16 \tan^{-1} x d(\tan^{-1} x)$

$= \lim_{b \rightarrow \infty} 16 \cdot \frac{(\tan^{-1} x)^2}{2} \Big|_0^b = \cancel{16 \cdot \frac{(\tan^{-1} x)^2}{2}} \cdot 8 \cdot \frac{(\frac{\pi}{2})^2}{2} \neq 0$

$= 8 \left(\lim_{b \rightarrow \infty} (\tan^{-1} b)^2 - \lim_{b \rightarrow \infty} (\tan^{-1} 0)^2 \right) \neq \cancel{16 \cdot \frac{\pi^2}{2}}$

$= 8 \left(\left(\frac{\pi}{2} \right)^2 - 0 \right) = 2\pi^2$

$\Rightarrow \int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2} dx$ conv.

$$\boxed{2} \int_2^{\infty} \frac{2}{x^2-x} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{2}{x^2(1-\frac{1}{x})} dx$$

$$y = 1 - \frac{1}{x}$$

$$dy = \frac{1}{x^2} dx$$

$$= 2 \lim_{b \rightarrow \infty} \int \frac{1}{y} dy = 2 \lim_{b \rightarrow \infty} \ln|y| = 2 \lim_{b \rightarrow \infty} \ln \left| 1 - \frac{1}{x} \right| \Big|_2^b$$

$$= 2 \lim_{b \rightarrow \infty} \left(\ln \left(1 - \frac{1}{b} \right) - \ln \left| 1 - \frac{1}{2} \right| \right)$$

$$= 2 \lim_{b \rightarrow \infty} \left(\ln(1) - \ln \frac{1}{2} \right) = \underline{\underline{2 \ln 2}}$$

$$\text{So } \int_2^{\infty} \frac{2}{x^2-x} dx \text{ conv.}$$

$$\boxed{3} \int_{-\infty}^{-1} \frac{1}{\sqrt{x-1}} dx = \lim_{a \rightarrow -\infty} \int_a^{-1} (x-1)^{-\frac{1}{2}} dx$$

$$= \lim_{a \rightarrow -\infty} \frac{(x-1)^{\frac{1}{2}}}{\frac{3}{2}} \Big|_a^{-1} = \frac{3}{2} \lim_{a \rightarrow -\infty} \left(\sqrt{4} - (a-1)^{\frac{1}{2}} \right)$$

$$= \frac{3}{2} (\sqrt{4} - \infty) = -\infty$$

$$\text{So } \int_{-\infty}^{-1} \frac{1}{\sqrt{x-1}} dx \text{ div}$$

$$\boxed{4} \int_{-\infty}^{\infty} \operatorname{sech} x dx = \int_{-\infty}^{\infty} \frac{2}{e^x + e^{-x}} dx = 2 \int_0^{\infty} \frac{2}{e^x + e^{-x}} dx$$

$$= 2 \lim_{b \rightarrow \infty} \int_0^b \frac{2e^x}{(e^x)^2 + 1} dx = 2 \cdot 2 \lim_{b \rightarrow \infty} \tan^{-1}(e^x) \Big|_0^b$$

$$= 2 \cdot 2 \lim_{b \rightarrow \infty} \left(\tan^{-1} e^b - \tan^{-1} 1 \right) = 4 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \underline{\underline{\pi}}$$

$$\text{So } \int_{-\infty}^{\infty} \operatorname{sech} x dx \text{ conv.}$$

Theorem:- If $a > 0$ then $\int_a^{\infty} \frac{dx}{x^p}$ conv. if $p > 1$ and div if $p \leq 1$.

Def. Integrals of fns that become infinite at a point within the interval of integration ^{are} called improper integrals of type II.

① If f is continuous on $(a, b]$ and discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

② = = = on $[a, b)$ = = at b , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

③ If f is discontinuous at $c \in (a, b)$ and continuous on $[a, c) \cup (c, b]$ then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$= \lim_{t \rightarrow c^-} \int_a^t f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

Def. In each of the above cases, if the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral.

If the limit does not exist, the improper integral is called diverges.

Example/Determine whether the following integrals conv. or div?

II) $\int_0^1 \frac{4x \cdot dx}{\sqrt{1-x^4}} = \lim_{c \rightarrow 1^-} \int_0^c \frac{4x \cdot dx}{\sqrt{1-(x^2)^2}}$ $y = x^2$
 $dy = 2x \cdot dx$

$$= 2 \lim_{c \rightarrow 1^-} \int_0^c \frac{dy}{\sqrt{1-y^2}} = 2 \lim_{c \rightarrow 1^-} \sin^{-1} y$$

$$= 2 \lim_{c \rightarrow 1^-} \sin^{-1} x^2 \Big|_0^c = 2 \left(\lim_{c \rightarrow 1^-} \sin^{-1} c^2 - 0 \right)$$

$$= 2 \left(\frac{\pi}{2} \right) = \underline{\underline{\pi}} \Rightarrow \int_0^1 \frac{4x \cdot dx}{\sqrt{1-x^4}} \text{ conv.}$$

$$\textcircled{2} \int_2^4 \frac{dx}{(x-2)^2} = \lim_{c \rightarrow 2^+} \int_c^4 \frac{1}{(x-2)^2} dx = \lim_{c \rightarrow 2^+} \left[\frac{-1}{x-2} \right]_c^4$$

$$= \lim_{x \rightarrow 2} \left(\frac{-1}{2} + \frac{1}{x-2} \right) = \frac{-1}{2} + \frac{1}{\text{V.S.P.Q}} = \infty$$

So $\int_2^4 \frac{dx}{(x-2)^2}$ div.

$$\textcircled{2} \int_0^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{dx}{(x-1)^2} + \int_1^2 \frac{dx}{(x-1)^2}$$

$$= \lim_{c \rightarrow 1^-} \int_c^2 \frac{1}{(x-1)^2} dx + \lim_{c \rightarrow 1^+} \int_2^c \frac{1}{(x-1)^2} dx$$

$$= \lim_{c \rightarrow 1^-} \frac{-1}{x-1} \Big|_0^c + \lim_{c \rightarrow 1^+} \frac{-1}{x-1} \Big|_c^2$$

$$\lim_{C \rightarrow 1} \frac{-1}{C-1} \div \frac{-1}{-1} = \frac{-1}{-1} = 1 \quad \therefore 1' = \infty$$

$$= \frac{-1}{1} + \lim_{c \rightarrow 1^+} \frac{1}{c-1} = -1 + \infty = \infty$$

$$\Rightarrow \int_1^2 \frac{dx}{(x-1)^2} \quad \text{--- div}$$

$$\Rightarrow \int_0^2 \frac{dx}{(x-1)^2} \quad \underline{\underline{\text{div.}}}$$

may be convex
and may be
concave.

$$\boxed{4} \int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_{-1}^{+1} \frac{dx}{\sqrt{|y|}} = 2 \int_0^1 \frac{dy}{\sqrt{|y|}} = 2 \int_0^1 \frac{dy}{\sqrt{y}}$$

$$= 2 \lim_{c \rightarrow 0^+} \int_c^1 \frac{dy}{\sqrt{y}} = 2 \lim_{c \rightarrow 0^+} 2\sqrt{y} \Big|_c^1 = 4 \lim_{c \rightarrow 0^+} (\sqrt{1} - \sqrt{c}) = \boxed{4}$$

5. $\int_0^2 \frac{1}{\sqrt{|x-1|}} dx$ conv.

Test of convergence of the improper integrals:-

II Direct comparison Test:-

Let $f(x)$ and $g(x)$ be continuous on $[a, \infty)$ with $0 < f(x) \leq g(x) \forall x \geq a$

Then:-

① If $\int_a^\infty g(x) dx$ converge then $\int_a^\infty f(x) dx$ converge

② If $\int_a^\infty f(x) dx$ diverge then $\int_a^\infty g(x) dx$ diverge

Remark/ Let f and g be continuous fns on $(a, b]$. If f and g are discontinuous at $x=a$ and $0 < f(x) \leq g(x) \forall x \in (a, b]$ then

① If $\int_a^b g(x) dx$ conv., then $\int_a^b f(x) dx$ conv.

② If $\int_a^b f(x) dx$ div, then $\int_a^b g(x) dx$ div.

$[a, b]$ مائتمة الفترة $(a, b]$ مائتمة الفترة

Note/ The above remark is true on the interval $[a, b]$

Example/ Test the convergence of the following improper integrals:-

① $\int_1^\infty e^{-x^2} dx$

$$e^{-x^2} = \frac{1}{e^{x^2}} \leq \frac{1}{e^x}$$

$$\int_1^\infty \frac{1}{e^x} dx = \lim_{b \rightarrow \infty} \int_1^b e^{-x} dx = \lim_{b \rightarrow \infty} \left. \frac{e^{-x}}{-1} \right|_1^b$$

$$= -\lim_{b \rightarrow \infty} \left. \frac{1}{e^x} \right|_1^b = -1 \left(\frac{1}{\infty} - \frac{1}{e} \right) = \frac{1}{e}$$

$$S = \int_1^\infty \frac{1}{e^x} dx \text{ conv.} \Rightarrow \int_1^\infty e^{-x^2} dx \text{ conv.}$$

② $\int_1^{\infty} e^{-x} \cos^2 x \, dx$

$$e^{-x} \cos^2 x = \frac{\cos^2 x}{e^x} < \frac{1}{e^x}$$

But $\int_1^{\infty} \frac{1}{e^x} \, dx$ conv. from example 1.

By the Direct comparison test, $\int_1^{\infty} e^{-x} \cos^2 x \, dx$ conv.

③ $\int_1^{\infty} \frac{x}{\sqrt{4x^5+1}} \, dx$

لدينا $\frac{x}{\sqrt{4x^5+1}} < \frac{x}{\sqrt{4x^5}} = \frac{x}{2x^{5/2}} = \frac{1}{2x^{3/2}}$
 لأن $\frac{1}{2x^{3/2}}$ متناهي في الصغر
 فإن $\int_1^{\infty} \frac{x}{\sqrt{4x^5+1}} \, dx$ متناهي في الصغر

$$\frac{x}{\sqrt{4x^5+1}} < \frac{x}{\sqrt{4x^5}} = \frac{x}{2x^{5/2}} = \frac{1}{2x^{3/2}}$$

But $\int_1^{\infty} \frac{1}{x^{3/2}} \, dx$ conv. because $p=3/2 > 1$

By the D.C.T, the integral $\int_1^{\infty} \frac{x}{\sqrt{4x^5+1}} \, dx$ conv.

④ $\int_1^{\infty} \frac{2+\cos x}{x} \, dx$

$$\frac{2+\cos x}{x^2} \geq \frac{1}{x}$$

But $\int_1^{\infty} \frac{1}{x} \, dx$ div because $p=1$

By the D.C.T, the integral $\int_1^{\infty} \frac{2+\cos x}{x} \, dx$ div.

* $\int_1^{\infty} \frac{2+\cos x}{x^2} \, dx$

$$\frac{2+\cos x}{x^2} < \frac{2}{x^2} \quad \text{But } \int_1^{\infty} \frac{1}{x^2} \, dx \text{ conv because } p > 1$$

$$⑤ \int_2^{\infty} \frac{x}{\sqrt{x^4-1}} dx$$

$$\frac{x}{\sqrt{x^4-1}} > \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x}$$

But $\int_2^{\infty} \frac{1}{x} dx$ div because $p=1$

By D.C.T, $\int_2^{\infty} \frac{x}{\sqrt{x^4-1}} dx$ div.

$$⑥ \int_3^{\infty} \frac{\sqrt{2x^2+7}}{x^3} dx \quad \frac{\sqrt{2x^2+7}}{x^3} < \frac{\sqrt{2x^2+7x^2}}{x^3} = \frac{3}{x^2}$$

But $\int_3^{\infty} \frac{1}{x^2} dx$ conv. because $p=2 > 1$

By the D.C.T, $\int_3^{\infty} \frac{\sqrt{2x^2+7}}{x^3} dx$ conv.

$$⑦ \int_0^{\pi} \frac{1}{\sqrt{x} + \sin x} dx \quad \frac{1}{\sqrt{x} + \sin x} < \frac{1}{\sqrt{x}} \quad \forall x \in [0, \pi]$$

$$\int_0^{\pi} \frac{1}{x^{1/2}} dx = \lim_{c \rightarrow 0^+} \int_c^{\pi} x^{-1/2} dx = 2 \lim_{c \rightarrow 0^+} x^{1/2} \Big|_c^{\pi} = 2 \lim_{c \rightarrow 0^+} \sqrt{\pi} - \sqrt{c} = 2\sqrt{\pi}$$

So $\int_0^{\pi} \frac{1}{\sqrt{x}} dx$ conv.

By the D.C.T, $\int_0^{\pi} \frac{dx}{\sqrt{x} + \sin x}$ conv.

$$⑧ \int_0^{\pi} \frac{\sin x}{\sqrt{\pi-x}} dx \quad \frac{\sin x}{\sqrt{\pi-x}} \leq \frac{1}{\sqrt{\pi-x}} \quad \forall x \in [0, \pi]$$

$$\int_0^{\pi} \frac{1}{\sqrt{\pi-x}} dx = \lim_{c \rightarrow \pi^-} \int_0^c (\pi-x)^{-1/2} dx = -2 \lim_{c \rightarrow \pi^-} (\pi-x)^{1/2} \Big|_0^c = -2 \lim_{c \rightarrow \pi^-} (\pi-c)^{1/2} - \pi^{1/2} = -2(\sqrt{0} - \sqrt{\pi}) = 2\sqrt{\pi}$$

So $\int_0^{\pi} \frac{1}{\sqrt{\pi-x}} dx$ conv.

By the D.C.T, $\int_0^{\pi} \frac{\sin x}{\sqrt{\pi-x}} dx$ conv.

$$\boxed{9} \int_{100}^{\infty} \frac{x-99}{\sqrt{x^5+4x+1}} dx \quad \frac{x-99}{\sqrt{x^5+4x+1}} < \frac{x}{\sqrt{x^5+4x+1}} < \frac{x}{\sqrt{x^5}} = \frac{1}{x^{3/2}}$$

But $\int_{100}^{\infty} \frac{1}{x^{3/2}} dx$ conv. because $p=3/2 > 1$

By D.C.T, $\int_{100}^{\infty} \frac{x-99}{\sqrt{x^5+4x+1}} dx$ conv.

2 Limit Comparison Test -

If f and g are positive continuous fns on $[a, \infty)$ and if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$, $0 < L < \infty$, then $\int_a^{\infty} f(x) dx$ and $\int_a^{\infty} g(x) dx$ are both conv. or both div.

Example Test the convergence of the following improper integrals:-

$$\boxed{1} \int_4^{\infty} \frac{2 dx}{x^{3/2}-1}$$

$$\int_4^{\infty} \frac{1}{x^{3/2}} dx \text{ conv. because } p=\frac{3}{2} > 1$$

$$\lim_{x \rightarrow \infty} \frac{2}{x^{3/2}-1} = \lim_{x \rightarrow \infty} \frac{2x^{3/2}}{x^{3/2}-1} = \lim_{x \rightarrow \infty} \frac{2}{1-\frac{1}{x^{3/2}}} = \frac{2}{1-0} = \underline{\underline{2}}$$

By the L.C.T, $\int_4^{\infty} \frac{2}{x^{3/2}-1} dx$ conv.

$$\boxed{2} \int_1^{\infty} \frac{1}{\sqrt{3x+1}} dx$$

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx \text{ div because } p=\frac{1}{2} < 1$$

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{3x+1}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{3x+1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x}{3x+1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{1}{3+\frac{1}{x}}} = \frac{1}{\sqrt{3}}$$

By the L.C.T, $\int_1^{\infty} \frac{1}{\sqrt{3x+1}} dx$ div.

$$\frac{1}{\sqrt{3x+1}} \geq \frac{1}{\sqrt{3x+x}} = \frac{1}{2\sqrt{x}}$$

$$\text{div} \quad \int_1^{\infty} \frac{1}{\sqrt{3x+1}} dx \text{ div.}$$

والمثل، لا

$$\boxed{3} \int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx.$$

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ conv. because } p=2 > 1$$

$$\lim_{x \rightarrow \infty} \frac{\tan^{-1} x}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\text{By the L.C.T. } \int_1^{\infty} \frac{\tan^{-1} x}{x^2} dx \text{ conv.}$$

$$\frac{\tan^{-1} x}{x^2} < \frac{\pi/2}{x^2} \leftarrow \text{conv}$$

$$\boxed{4} \int_1^{\infty} \frac{dx}{\sqrt{x^2-x+1}}$$

$$\int_1^{\infty} \frac{1}{x} dx \text{ div because } p=1$$

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2-x+1}} / \frac{1}{x} = \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2-x+1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^2}{x^2-x+1}} =$$

$$\lim_{x \rightarrow \infty} \sqrt{\frac{1}{1-\frac{1}{x}+\frac{1}{x^2}}} = 1$$

$$\text{By I.C.T. } \int_1^{\infty} \frac{1}{\sqrt{x^2-x+1}} dx \text{ div.}$$

$$\frac{1}{\sqrt{x^2-x+1}} > \frac{1}{\sqrt{x^2+1}} > \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}x} \text{ div.}$$

$$[5] \int_1^{\infty} \frac{1}{e^x - 2^x} dx$$

$\int_1^{\infty} \frac{1}{e^x} dx$ conv. (previous example)

$$\lim_{x \rightarrow \infty} \frac{1}{e^x - 2^x} \bigg/ \frac{1}{e^x} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x - 2^x} \Rightarrow \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{2^x}{e^x}} \quad e^x > 2^x$$

$$= \frac{1}{1 - \lim_{x \rightarrow \infty} \left(\frac{2}{e}\right)^x} = \frac{1}{1 - 0} = 1$$

By L.C.T, $\int_1^{\infty} \frac{1}{e^x - 2^x} dx$ conv.

$$[6] \int_0^{\infty} \frac{dx}{\sqrt{x^6 + 1}} = \int_0^1 \frac{1}{\sqrt{x^6 + 1}} dx + \int_1^{\infty} \frac{1}{\sqrt{x^6 + 1}} dx$$

$$[0,1] \forall x \text{ ... } = \leq \int_0^1 1 dx + \int_1^{\infty} \frac{1}{\sqrt{x^6 + 1}} dx = 1 + \int_1^{\infty} \frac{1}{\sqrt{x^6 + 1}} dx$$

But $\int_1^{\infty} \frac{1}{x^3} dx$ conv because $p=3 > 1$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x^6 + 1}}}{\frac{1}{x^3}} = \lim_{x \rightarrow \infty} \frac{x^3}{\sqrt{x^6 + 1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^6}{x^6 + 1}} = \lim_{x \rightarrow \infty} \sqrt{\frac{1}{1 + \frac{1}{x^6}}} = 1$$

By L.C.T, $\int_1^{\infty} \frac{1}{\sqrt{x^6 + 1}} dx$ conv.

$\Rightarrow 1 + \text{conv.} = \text{conv.}$

By the D.C.T, $\int_0^{\infty} \frac{1}{\sqrt{x^6 + 1}} dx$ conv.

Ex. 8.7

1-63 odd 65 و 66

ch. 10 / Infinite sequences and infinite series

10.1 Infinite sequences.

Def. An infinite sequence of numbers is an expression in the form
 $a_1, a_2, \dots, a_n, \dots = \{a_n\}_{n=1}^{\infty}$

a_1 is the first term, $\dots$, a_n is the n -th term, $\dots$

Example (1) The seq $\{2n\} = 2, 4, 6, \dots, 2n, \dots$ (increasing seq)

(2) The seq $\{\frac{1}{n}\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$ (decreasing seq)

(3) The seq $\{(-1)^{n+1} \frac{1}{n}\} = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots$ (oscillation seq)

(4) The seq $\{\frac{n-1}{n}\} = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}, \dots$ (increasing seq)

(5) The seq $\{7\} = 7, 7, 7, \dots, 7, \dots$ (constant seq)

(6) The seq with $a_1 = 1$, $a_{n+1} = \frac{a_n}{n+1}$ is completed as follows:

$$a_1 = 1, a_2 = a_{1+1} = \frac{a_1}{1+1} = \frac{1}{2} = \frac{1}{2!}$$

$$a_3 = \frac{a_2}{2+1} = \frac{\frac{1}{2}}{3} = \frac{1}{2 \times 3} = \frac{1}{3!}$$

$$a_4 = \frac{a_3}{3+1} = \frac{\frac{1}{3!}}{4} = \frac{1}{2 \times 3 \times 4} = \frac{1}{4!}$$

↓

The seq is $\{\frac{1}{n!}\} = 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \dots, \frac{1}{n!}, \dots$

a_{n+1} recurrence relation

Def. For the seq $\{a_n\}$, if $\lim_{n \rightarrow \infty} a_n$ exists, we say that the

seq. conv. and if it does not exist, we say that the seq. ~~div.~~ div.

Example/

① The seq $\{2n\}$ $\lim_{n \rightarrow \infty} 2n = \infty \Rightarrow \text{div.}$

② The seq $\{\frac{1}{n}\}$ $\lim_{n \rightarrow \infty} \frac{1}{n} = \text{Zero} \Rightarrow \text{Conv.}$

③ The seq $\{\frac{n-1}{n}\}$ $\lim_{n \rightarrow \infty} \frac{n-1}{n} = 1 \Rightarrow \text{Conv.}$

④ The seq $\{(-1)^{n+1}\} = 1, -1, 1, -1, \dots \Rightarrow \text{div}$ because it oscillate between two different limits -1 and 1 .

Theorem/ Let $\{a_n\}$ and $\{b_n\}$ be seq. of real num. and let A and B be real num.

If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$ then

① $\lim_{n \rightarrow \infty} (a_n \pm b_n) = A \pm B$

② $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$

③ $\lim_{n \rightarrow \infty} (K \cdot b_n) = K \cdot B$, K is any real num.

④ $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n}\right) = \frac{A}{B}$, $B \neq \text{Zero}$.

Theorem / Sandwich theorem for sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be seq of real num. If $a_n \leq b_n \leq c_n$ for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L \Rightarrow \lim_{n \rightarrow \infty} b_n = L$

Example / Find the formula for the n -th term of the following sequences and determine if seq conv or div.

① $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots$

② $0, 3, 8, 15, 24, \dots$

③ $1, 3, 1, 3, \dots$

④ $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots$

⑤ $-1, 2, -3, 4, -5, \dots$

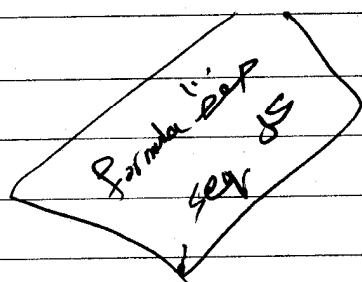
① $\left\{\frac{1}{n^2}\right\}$ conv. because $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$

② $\{n^2 - 1\}$ div $= \lim_{n \rightarrow \infty} n^2 - 1 = \infty$

③ $\{2 + (-1)^n\}$ div $=$ it oscillate between two different limits 1 and 3

④ $\left\{\frac{2^n - 1}{2^n}\right\} = \left\{1 - \frac{1}{2^n}\right\}$ conv. because $\lim_{n \rightarrow \infty} 1 - \frac{1}{2^n} = 1$

⑤ $\{(-1)^n \cdot n\}$ div because it oscillate between -ve value and +ve value.



Example 2/ Find the limits of the following seq.s and determine their convergence?

[1] $a_n = \frac{2n+1}{1-3\sqrt{n}}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n+1}{1-3\sqrt{n}} = \frac{\infty}{-\infty} \Rightarrow \lim_{n \rightarrow \infty} \frac{2}{\frac{-3}{2\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{4\sqrt{n}}{-3} = \underline{\underline{-\infty}}$$

So the seq $\left\{ \frac{2n+1}{1-3\sqrt{n}} \right\}$ div.

[2] $a_n = \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right)$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(2 - \frac{1}{2^n}\right) \cdot \lim_{n \rightarrow \infty} \left(3 + \frac{1}{2^n}\right) = 2 \cdot 3 = \underline{\underline{6}}$$

So the seq $\left\{ \left(2 - \frac{1}{2^n}\right) \left(3 + \frac{1}{2^n}\right) \right\}$ conv.

[3] $a_n = \ln n - \ln(n+1)$

$$\lim_{n \rightarrow \infty} a_n = \ln n - \ln(n+1) = \infty - \infty \Rightarrow \lim_{n \rightarrow \infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln\left(\lim_{n \rightarrow \infty} \frac{n}{n+1}\right) = \ln\left(\lim_{n \rightarrow \infty} 1\right) = \ln(1) = \underline{\underline{zero}}$$

So the seq $\{\ln n - \ln(n+1)\}$ conv.

[4] $a_n = \frac{1}{\sqrt{n}} \tan^{-1} n$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \tan^{-1} n = \lim_{n \rightarrow \infty} \frac{\tan^{-1} n}{\sqrt{n}} = \frac{\pi/2}{\infty} = \underline{\underline{zero}}$$

So seq $\left\{ \frac{1}{\sqrt{n}} \tan^{-1} n \right\}$ conv.

[5] $a_n = \frac{\ln n}{\ln(2n)}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{\ln 2n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{2}{2n}} = \underline{\underline{1}}$$

So the seq $\left\{ \frac{\ln n}{\ln 2n} \right\}$ conv.

$$[6] a_n = n(1 - \cos \frac{1}{n})$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n \times \lim_{n \rightarrow \infty} (1 - \cos \frac{1}{n}) \quad \text{① } \infty \cdot 0$$

$$\lim_{n \rightarrow \infty} 1 - \cos \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin(\frac{1}{n}) \cdot \frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \sin \frac{1}{n} = \sin 0 = 0$$

So the seq $\{n(1 - \cos \frac{1}{n})\}$ conv.

$$[7] a_n = n - \sqrt{n^2 - 1}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (n - \sqrt{n^2 - 1}) = \lim_{n \rightarrow \infty} \frac{n^2 - n^2 + 1}{n + \sqrt{n^2 - 1}} = \lim_{n \rightarrow \infty} \frac{1}{n + \sqrt{n^2 - 1}} = 0$$

So the seq $\{n - \sqrt{n^2 - 1}\}$ conv.

$$[8] a_n = \frac{\sin^2 n}{2^n}$$

$$0 \leq \sin^2 n \leq 1$$

$$\lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{\sin^2 n}{2^n} \leq \lim_{n \rightarrow \infty} \frac{1}{2^n}$$

By Sandwich theorem,

$$\lim_{n \rightarrow \infty} \frac{\sin^2 n}{2^n} = \text{zero}$$

$$0 \leq \lim_{n \rightarrow \infty} \frac{\sin^2 n}{2^n} \leq 0$$

So the seq $\{\frac{\sin^2 n}{2^n}\}$ conv.

$$[9] a_n = \frac{n!}{n^n}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdot n \dots n}$$

$$0 < \frac{n(n-1)(n-2) \dots 3 \cdot 2}{n \cdot n \cdot n \dots n} < 1$$

$$0 < \frac{n(n-1) \dots 3 \cdot 2}{n \cdot n \dots n} = \frac{1}{n} < \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} 0 < \lim_{n \rightarrow \infty} \frac{n!}{n^n} < \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

By Sandwich Theorem, $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = \text{zero}$

So the seq $\left\{ \frac{n!}{n^n} \right\}$ conv.

Remark /

If $\{a_n\}$ is conv. (or div) seq, then $\{c a_n\}$ is conv. (or div) seq, where c is a non zero constant.

Some basic limits:

$$[1] \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$[2] \lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

$$[3] \lim_{n \rightarrow \infty} \sqrt[n]{x} = \lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1, x > 0$$

$$[4] \lim_{n \rightarrow \infty} x^n = \text{zero}, \text{ if } |x| < 1$$

$$[5] \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \text{ (any } x)$$

$$[6] \lim_{n \rightarrow \infty} \frac{x^n}{n!} = \text{zero} \text{ (any } x)$$

Example / Find the limits of the following sequences and determine its convergence?

$$[1] a_n = \sqrt[n]{3n^3}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} (3n^3)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 3^{\frac{1}{n}} \cdot \lim_{n \rightarrow \infty} (n^3)^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} 3^{\frac{1}{n}} \cdot \lim_{n \rightarrow \infty} (n^{\frac{1}{n}})^3 = 3^0 \cdot (1)^3 = \underline{\underline{1}} \end{aligned}$$

So the seq $\left\{ \sqrt[n]{3n^3} \right\}$ conv.

$$[2] a_n = (n+3)^{\frac{1}{n+3}}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} (n+3)^{\frac{1}{n+3}} = \lim_{m \rightarrow \infty} (m)^{\frac{1}{m}} \quad \text{put } m = n+3 \\ &= \lim_{m \rightarrow \infty} m^{\frac{1}{m}} = \underline{\underline{1}} \end{aligned}$$

So the seq $\left\{ (n+3)^{\frac{1}{n+3}} \right\}$ conv.

$$[5] a_n = \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{3}\right)^n + \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2^n}}$$

$$= \text{Zero} + \lim_{n \rightarrow \infty} \frac{1}{(2^n)^{1/2}}$$

$$= \frac{1}{\infty} = \underline{\underline{\text{Zero}}}$$

So the seq $\left\{\left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}\right\}$ conv.

$$[4] a_n = \frac{3^n \cdot 6^n}{2^n \cdot n!}$$

$$\lim_{n \rightarrow \infty} \frac{3^n \cdot 6^n}{2^n \cdot n!} = \lim_{n \rightarrow \infty} \frac{18^n \cdot 2^n}{n!} = \lim_{n \rightarrow \infty} \frac{(36)^n}{n!} = \text{Zero}$$

So the seq $\left\{\frac{3^n \cdot 6^n}{2^n \cdot n!}\right\}$ conv.

$$[5] a_n = \frac{n!}{(10^6)^n}$$

$$\lim_{n \rightarrow \infty} \frac{n!}{(10^6)^n} = \infty$$

So the seq $\left\{\frac{n!}{(10^6)^n}\right\}$ div.

$$[6] a_n = \frac{(3n+1)^n}{3^{n-1}}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{3n(1 + \frac{1}{3n})}{3^{n-1}} \right)^n$$

$$= \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{3n})^n}{(1 + \frac{1}{3n})^n} = \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{3n})^n}{(1 + \frac{1}{3n})^n} = \lim_{n \rightarrow \infty} 1 = 1$$

$$= e^{1/3} = e^{1/3}$$

So the seq $\left\{\left(\frac{3n+1}{3}\right)^n\right\}$ conv.

$$[7] a_n = \left(1 - \frac{1}{n^2}\right)^n$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e^{-1} \cdot e^1 = e^0 = \underline{\underline{1}}$$

So the seq $\left\{\left(1 - \frac{1}{n^2}\right)^n\right\}$ conv.

$$[8] a_n = \sqrt[n]{n^2 + n}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} (n^2 + n)^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \left[n^2 \left(1 + \frac{1}{n}\right)\right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left(n^{\frac{2}{n}}\right)^2 \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) \\ &= (1)^2 \cdot 1 = \underline{\underline{1}} \end{aligned}$$

So the seq $\{\sqrt[n]{n^2 + n}\}$ conv.

Def. A seq $\{a_n\}$ with property that $a_n \leq a_{n+1}, \forall n$ is called a nondecreasing seq.

$$\begin{aligned} a_{n+1} - a_n &\geq 0 \\ \frac{a_{n+1}}{a_n} &\geq 1 \end{aligned}$$

Example/ The following sequences are nondecreasing.

$$[1] 3, 6, 9, \dots = \{3n\}$$

$$[2] \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots = \left\{\frac{n}{n+1}\right\}$$

$$[3] 3, 3, 3, \dots = \{3\}$$

Def/ [1] A seq $\{a_n\}$ is bounded from above if there exists a number M such that, $a_n \leq M \forall n$. The num. M is called an upper bound for the seq $\{a_n\}$.

[2] If M is an upper bound for the seq $\{a_n\}$ but no number less than M is an upper bound of $\{a_n\}$, then M is the least upper bound for the seq $\{a_n\}$.

[3] A seq $\{a_n\}$ is bounded from below if $\exists$ a num m such that $a_n \geq m$ for all n .
 m is called a lower bound of the seq $\{a_n\}$.

[4] If m is a lower bound for the the seq $\{a_n\}$ but no num greater than m is a lower ~~bound~~ bound for $\{a_n\}$ then m is called the greatest lower bound for the seq $\{a_n\}$.

[5] If the seq $\{a_n\}$ bounded from above and below the seq $\{a_n\}$ is called bounded. If $\{a_n\}$ is not bounded we say that the seq $\{a_n\}$ is an unbounded seq.

Theorem / A nondecreasing seq. of real num. conv. iff it's bounded from above

* If a nondecreasing seq. conv. it conv. to its least upper bound.

Example /

* The seq $\{3n\} = 3, 6, \dots, 3n, \dots$ has no upper bound, that is, $\{3n\}$ is unbounded $\Rightarrow \{3n\}$ div.

* The seq $\frac{1}{2}, \frac{2}{3}, \dots, \frac{n}{n+1}, \dots$ has upper bound 1, 2, 3 (say) but 1 is the least upper bound because the seq $\{\frac{n}{n+1}\}$ is nondecreasing and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \Rightarrow$ The seq. conv.

* The seq $\{\frac{4n+1}{n+1}\} = \frac{5}{2}, \frac{9}{3}, \dots, \frac{4n+1}{n+1}, \dots$ is nondecreasing

and $\frac{4n+1}{n+1} = 4 - \frac{3}{n+1} \Rightarrow \lim_{n \rightarrow \infty} 4 - \frac{3}{n+1} = 4 \Rightarrow$ the seq

bounded from above by 4 and 4 is the least upper bound $\Rightarrow$ the seq $\{\frac{4n+1}{n+1}\}$ conv.

Def. The seq $\{a_n\}$ is called monotonic if it either increasing or decreasing.

Example. The seq $\left\{\frac{3}{n+1}\right\}$ is decreasing $\Rightarrow$ The seq $\left\{\frac{3}{n+1}\right\}$ is monotonic.

Theorem If a seq $\{a_n\}$ is both bounded and monotonic, then the seq conv.

Example / The seq $\left\{\frac{1}{n^2}\right\} = 1, \frac{1}{4}, \frac{1}{9}, \dots, \frac{1}{n^2}, \dots$

The seq is decreasing $\Rightarrow$ The seq $\left\{\frac{1}{n^2}\right\}$ is monotonic.

This seq is bounded above by 1 and below by zero $\Rightarrow$ the seq $\left\{\frac{1}{n^2}\right\}$ is bounded.

$\Rightarrow$ The seq $\left\{\frac{1}{n^2}\right\}$ conv.

Ex 10.1

1-99 odd
102

10.2 / Infinite series

Def. An infinite series is the sum of an infinite seq. of numbers, that is, Infinite series is an expression in the form:

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$$

convergence of the series.

$$S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, \dots$$

⊛ If the seq. of partial sum is $\{S_1, S_2, \dots, S_n, \dots\} = \{S_n\}$

⊛ If the seq. of partial sums $|S_n|$ conv. to a limit L , then the series $\sum_{n=1}^{\infty} a_n$ conv. and its sum is L that is $\sum_{n=1}^{\infty} a_n = L$.

⊛ If the seq. of partial sums is not conv. we say that the series div.

Example. Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n-1}$

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 + (-1) + 1 + (-1) + \dots$$

$$S_1 = 1, S_2 = 1 + (-1) = 0, S_3 = 1 + (-1) + 1 = 1$$

the seq. of partial sum $|S_n| = 1, 0, 1, 0, \dots$ div. seq. because it oscillate between two different limits 1 and 0.

So the series $\sum_{n=1}^{\infty} (-1)^{n-1}$ div.

Geometric Series:-

Def. the geometric series is an expression in the form ~~the~~
 $a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$

where a is the first term and r is the ratio

* The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ conv. if $|r| < 1$ and div if $|r| \geq 1$

* If the geometric series conv. then its sum:

$$S = \frac{a}{1-r}$$

Example, which of the following series conv. and which div. If the series conv. find its sum.

$$[1] \sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n} = \sum_{n=0}^{\infty} 5 \left(-\frac{1}{4}\right)^n$$

conv. geom. series because $|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$

$$\text{it's sum is } S = \frac{5}{1 - (-\frac{1}{4})} = \underline{\underline{\frac{4}{5}}}$$

$$[2] \sum_{n=0}^{\infty} \frac{2^{n+3}}{3^n} = \sum_{n=0}^{\infty} \frac{2^n \cdot 2^3}{3^n} = \sum_{n=0}^{\infty} 8 \left(\frac{2}{3}\right)^n$$

conv. geom. series because $|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1$

$$S = \frac{8}{1 - \frac{2}{3}} = \underline{\underline{24}}$$

$$[3] \sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \dots + \frac{(-1)^n}{5^n}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$$

conv. geom. series because $|r| = \left|-\frac{1}{5}\right| = \frac{1}{5} < 1$

$$S = \frac{1}{1 - \frac{1}{5}} = \underline{\underline{\frac{5}{4}}}$$

$$[4] \left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \dots = \sum_{h=0}^{\infty} \left(\frac{-2}{3}\right)^{h+2} = \sum_{h=0}^{\infty} \frac{4}{9} \left(\frac{-2}{3}\right)^h$$

Conv. geom series because $|r| = \left|\frac{-2}{3}\right| = \frac{2}{3} < 1$

$$S = \frac{\frac{4}{9}}{1 - \frac{-2}{3}} = \frac{4}{15}$$

$$[5] \sum_{h=0}^{\infty} \left(\frac{\pi}{e}\right)^h$$

div. geom. series because $|r| = \frac{\pi}{e} > 1$

Example / Express the num $1.\overline{28}$ as a ratio of two integers.

$$1.\overline{28} = 1.282828\dots$$

$$= 1 + \frac{28}{100} + \frac{28}{10000} + \frac{28}{1000000} + \dots$$

$$= 1 + \frac{28}{100} \left(1 + \frac{1}{100} + \frac{1}{10000} + \dots\right) = 1 + \frac{28}{100} \sum_{h=0}^{\infty} \left(\frac{1}{100}\right)^h$$

Conv. geom series because $|r| = \frac{1}{100} < 1$

$$S = 1 + \frac{28}{100} \left(\frac{1}{1 - \frac{1}{100}}\right) = \frac{127}{99}$$

Example / Find the value of x for which the series.

$\sum_{h=0}^{\infty} \left(\frac{-1}{2}\right)^h (x-3)^h$ conv and find the sum of the series

$$\sum_{h=0}^{\infty} \left(\frac{-1}{2} (x-3)\right)^h = \sum_{h=0}^{\infty} \left(\frac{3}{2} - \frac{3}{x}\right)^h$$

• Since the series conv. then

$$r = \left|\frac{3}{2} - \frac{3}{x}\right| < 1$$

$$-1 < \frac{3}{2} - \frac{3}{x} < 1 \Rightarrow -2 < 3 - x < 2 \Rightarrow x > 1$$

$$\Rightarrow x \in (1, 5)$$

$$S = \frac{1}{1 - (\frac{2-x}{2})} = \frac{2}{x-1}$$

Telescoping Series:-

Example/ Discuss the convergence of the following series If the series conv. find its sum:-

$$[a] \sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$$

$$[b] \sum_{n=1}^{\infty} \tan^{-1} n - \tan^{-1}(n+1)$$

[a] The series is $\frac{4}{1 \times 5} + \frac{4}{5 \times 9} + \frac{4}{9 \times 13} + \dots$

$$\frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$$

$$\begin{aligned} 4 &= A(4n+1) + B(4n-3) \\ &= 4An + A + 4Bn - 3B \\ &= (4A + 4B)n + (A - 3B) \end{aligned}$$

By Equating

$$\left. \begin{aligned} 4A + 4B &= 0 \Rightarrow A + B = 0 \\ A - 3B &= 4 \end{aligned} \right\} \Rightarrow B = -1, A = 1$$

$$\begin{aligned} \text{So } \sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} &= \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right) \\ &= \left(1 - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \left(\frac{1}{9} - \frac{1}{13} \right) + \dots + \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right) + \dots \end{aligned}$$

The n^{th} term of the seq of partial sums is

$$\begin{aligned} S_n &= \left(1 - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{9} \right) + \left(\frac{1}{9} - \frac{1}{13} \right) + \dots + \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right) \\ &= \left(1 - \frac{1}{4n+1} \right) \end{aligned}$$

$$|S_n| = \left\{ 1 - \frac{1}{4n+1} \right\}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{4n+1} = 1$$

So the seq of partial sums $|S_n|$ conv.

The Series $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ conv.

and its Sum = 1

[b] $\sum_{n=1}^{\infty} \tan^{-1} n - \tan^{-1}(n+1)$

The n^{th} term of the seq of partial sum's is

$$S_n = (\tan^{-1} 1 - \tan^{-1} 2) + (\tan^{-1} 2 - \tan^{-1} 3) + \dots + (\tan^{-1} n - \tan^{-1}(n+1)) + \dots$$

$$= \frac{\pi}{4} - \tan^{-1}(n+1)$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{\pi}{4} - \tan^{-1}(n+1) \right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

So the seq of partial sums $|S_n|$ conv $\Rightarrow$

The series $\sum_{n=1}^{\infty} \tan^{-1} n - \tan^{-1}(n+1)$ conv.

and its sum is $-\frac{\pi}{4}$

Theorem / If the series $\sum_{n=1}^{\infty} a_n$ conv. then
 $\lim_{n \rightarrow \infty} a_n = \text{zero}$

The n^{th} term test for divergence:

The Series $\sum_{n=1}^{\infty} a_n$ div if $\lim_{n \rightarrow \infty} a_n \neq 0$ or fails to exist.

Example / Test the convergence of the following series:

$$\boxed{1} \sum_{n=0}^{\infty} \frac{n!}{(1000)^n}$$

$$\lim_{n \rightarrow \infty} \frac{n!}{(1000)^n} = \infty \quad \text{--- periodic example}$$

By the n^{th} term test, the series $\sum_{n=1}^{\infty} \frac{n!}{(1000)^n}$ div.

$$\boxed{2} \sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n} \Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{2^n}{4^n} + \frac{4^n}{4^n}}{\frac{2^n}{4^n} + \frac{4^n}{4^n}} = \lim_{n \rightarrow \infty} \frac{(\frac{1}{2})^n + 1}{(\frac{1}{2})^n + 1} = 1 \neq 0$$

By the n^{th} term test the series div.

$$\boxed{3} \sum_{n=1}^{\infty} \frac{n+1}{\sqrt{4n^2+3}}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{\sqrt{4n^2+3}} = \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{\sqrt{\frac{4n^2}{n^2} + \frac{3}{n^2}}} = \lim_{n \rightarrow \infty} \frac{1+\frac{1}{n}}{\sqrt{4+\frac{3}{n^2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2} \neq 0$$

By the n^{th} term test, the series div.

$$\boxed{4} \sum_{n=1}^{\infty} \left(\frac{n}{n-1} \right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{1-\frac{1}{n}} \right)^n = \sum_{n=1}^{\infty} \frac{1}{(1-\frac{1}{n})^n}$$

$$\lim_{n \rightarrow \infty} \frac{1}{(1-\frac{1}{n})^n} = \frac{1}{e^{-1}} = e \neq 0$$

By the n^{th} term test, the series div.

$$\boxed{5} \sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$$

$$\lim_{n \rightarrow \infty} \frac{e^n}{e^n + n} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{n}{e^n}} = \frac{1}{1 + \lim_{n \rightarrow \infty} \frac{n}{e^n}}$$

$$= \frac{1}{1 + \lim_{n \rightarrow \infty} \frac{1}{e^n}} = 1 \neq 0$$

By the n^{th} term test, the series div.

$$\boxed{6} \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

$$\lim_{n \rightarrow \infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n \rightarrow \infty} \frac{n}{n+1}\right) = \ln\left(\lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}}\right) \Rightarrow$$

$$= \ln(1) = 0$$

The n^{th} term test is inconclusive. نقير

$$\text{But } \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} (\ln n) - (\ln(n+1))$$

the n^{th} term of the seq of partial sums is

$$S_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \dots + (\ln n - \ln(n+1))$$

$$= \ln 1 - \ln(n+1) = -\ln(n+1)$$

$$S_n = \lim_{n \rightarrow \infty} -\ln(n+1) = -\infty$$

So the seq of partial sum div

$\Rightarrow$ The series $\sum \ln\left(\frac{n}{n+1}\right)$ div.

Theorem. If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then:

① $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$ is conv.

② If K is a nonzero constant then $\sum K a_n = K \sum a_n = KA$ is conv.

Corollaries ① If K is a nonzero constant and $\sum a_n$ div, then $\sum K a_n$ div.

② Conv series $\pm$ Conv. series = Conv.

Conv series $\pm$ div series = div.

div series $\pm$ div series = maybe conv or maybe div.

Example If $\sum_{n=1}^{\infty} (-1)^{2n} = 1 + 1 + 1 + \dots$

and $\sum_{n=1}^{\infty} (-1)^{2n-1} = -1 + (-1) + (-1) + \dots$

then The two series are diverge.

But $\sum_{n=1}^{\infty} (-1)^{2n} + \sum_{n=1}^{\infty} (-1)^{2n-1} = 0 + 0 + 0 + \dots = 0$

So the Sum of the two series Conv.

Example 2/ Discuss the convergence of the following series. If the series conv., find its sum.

$$\text{[1]} \sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right) = \sum_{n=0}^{\infty} 5 \left(\frac{1}{2} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^n$$

= conv. geom. series because $|r| = \frac{1}{2} < 1$

conv. : : : $|r| = \frac{1}{3} < 1$

= conv. - conv. = conv.

$$S = S_1 - S_2 = \frac{5}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{3}} = \underline{\underline{\frac{17}{2}}}$$

$$\text{[2]} \sum_{n=0}^{\infty} \left(\frac{e}{\pi} \right)^n + \left(\frac{\pi}{e} \right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{e}{\pi} \right)^n + \sum_{n=0}^{\infty} \left(\frac{\pi}{e} \right)^n$$

conv. geom. series because $|r| = \frac{e}{\pi} < 1$

div. geom. series because $|r| = \frac{\pi}{e} > 1$

= conv. + div = div.

$$S_0 \sum_{n=0}^{\infty} \left(\frac{e}{\pi} \right)^n + \left(\frac{\pi}{e} \right)^n \text{ div.}$$

Remark/ Adding or deleting a finite num. of terms to a series does not alternate the convergence or divergence of the series. That is,

[1] If $\sum_{n=1}^{\infty} a_n$ conv. (div) then $\sum_{n=100}^{\infty} a_n$ conv. (div) and

If $\sum_{n=49}^{\infty} a_n$ conv. (div) then $\sum_{n=0}^{\infty} a_n$ conv. (div)

Remark / (Reindexing)

$$\left. \begin{aligned} \sum_{n=1}^{\infty} ar^{n-1} &= \sum_{n+1=1}^{\infty} ar^{(n+1)-1} = \sum_{n=0}^{\infty} ar^n \\ \text{and } \sum_{n=1}^{\infty} ar^{n-1} &= \sum_{n-3=1}^{\infty} ar^{(n-3)-1} = \sum_{n=4}^{\infty} ar^{n-4} \end{aligned} \right\} \Rightarrow$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \sum_{n=4}^{\infty} ar^{n-4}$$

Ex 10.2
(1-81) ~~old~~
(82-92)

10.3 Integral Test :-

Def. The series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} + \dots$

is called harmonic series.

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ div.

Def. The P-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots$

is called the p-series or hyper harmonic series.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv. if $p > 1$ and div if $p \leq 1$

Example / Determine the convergence of the following series

① $\sum_{n=1}^{\infty} \frac{-8}{n} = -8 \sum_{n=1}^{\infty} \frac{1}{n}$ div harmonic series.

② $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3 \sum_{n=1}^{\infty} \frac{1}{(n)^{1/2}}$ div (p-series because $p = \frac{1}{2} < 1$)

③ $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{(n)^{3/2}}$ conv. p-series because $p = \frac{3}{2} > 1$

$$\textcircled{4} \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$= \sum_{k+\frac{1}{2}=1}^{\infty} \frac{1}{m}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m}$$

div. harmonic series.

$$m=2n-1$$

$$2n=m+1$$

$$n=\frac{1}{2}m+\frac{1}{2}$$

The Integral Test.

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms. Let $f(x)$ be a f_n obtained by replacing n by x in the formula of a_n . If $f(x)$ is positive, continuous and decreasing for all $x \geq N$, then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ are both conv. or div.

Example / Test the convergency of the following series:

$$\textcircled{II} \sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$$

$$m=\sqrt{n} \Rightarrow n=m^2$$

$$\sum_{m=2}^{\infty} \frac{\ln m^2}{m} = 2 \sum_{m=2}^{\infty} \frac{\ln m}{m}$$

$$f(x) = \frac{\ln x}{x} \quad \text{positive and continuous } \forall x \geq 2.$$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} < 0 \quad \forall x \geq 3$$

So f is +ve, continuous and decreasing $\forall x \geq 3$

$$2 \int_3^{\infty} \frac{\ln x}{x} dx = 2 \lim_{b \rightarrow \infty} \int_3^b \frac{\ln x}{x} d(\ln x)$$

$$= 2 \lim_{b \rightarrow \infty} \frac{(\ln x)^2}{2} \Big|_3^b = \lim_{b \rightarrow \infty} (\ln b)^2 - (\ln 3)^2 = \infty$$

$$\Rightarrow \therefore 2 \int_3^{\infty} \frac{\ln x}{x} dx \text{ div}$$

By the Integral Test, $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ div.

$$\boxed{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$$

$$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)} \quad \text{positive and continuous } \forall x \geq 1$$

$$a_n = \frac{1}{\sqrt{n}(\sqrt{n}+1)} > a_{n+1} = \frac{1}{\sqrt{n+1}(\sqrt{n+1}+1)} \Rightarrow \text{decreasing}$$

$$\begin{aligned} \int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx = \lim_{b \rightarrow \infty} 2 \int_1^b \frac{d(\sqrt{x}+1)}{\sqrt{x}+1} \\ &= 2 \lim_{b \rightarrow \infty} \ln|\sqrt{x}+1| \Big|_1^b = 2 \lim_{b \rightarrow \infty} (\ln(\sqrt{b}+1) - \ln(2)) \\ &= \infty \end{aligned}$$

$$\text{So } \int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx \quad \text{div.}$$

$$\text{By the Integral test, } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)} \quad \text{div.}$$

$$\boxed{3} \sum_{n=3}^{\infty} \frac{1}{n \ln x \sqrt{(\ln x)^2 - 1}}$$

$$f(x) = \frac{1}{x \ln x \sqrt{(\ln x)^2 - 1}} \quad \text{+ve and continuous } \forall x \geq 3$$

$$a_n = \frac{1}{n \ln x \sqrt{(\ln x)^2 - 1}} > a_{n+1} = \frac{1}{(n+1) \ln(n+1) \sqrt{(\ln(n+1))^2 - 1}} \Rightarrow \text{decreasing}$$

$$\int_3^{\infty} \frac{1}{x \ln x \sqrt{(\ln x)^2 - 1}} dx = \lim_{b \rightarrow \infty} \int_3^b \frac{d(\ln x)}{\ln x \sqrt{(\ln x)^2 - 1}} = \lim_{b \rightarrow \infty} \sec^{-1} \ln x \Big|_3^b$$

$$= \lim_{b \rightarrow \infty} (\sec^{-1} \ln b - \sec^{-1} \ln 3) = \frac{\pi}{2} - \sec^{-1} \ln 3$$

$$\text{So } \int_3^{\infty} \frac{1}{x \ln x \sqrt{(\ln x)^2 - 1}} dx \quad \text{conv.}$$

$$\text{By the Integral Test, } \sum_{n=3}^{\infty} \frac{1}{n \ln x \sqrt{(\ln x)^2 - 1}} \quad \text{conv.}$$

$$[4] \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$f(x) = \frac{1}{x(\ln x)^2} \quad \text{+ve and continuous } \forall x \geq 2$$

$$a_n = \frac{1}{n(\ln n)^2} > a_{n+1} = \frac{1}{(n+1)(\ln(n+1))^2} \Rightarrow \text{decreasing}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{d(\ln x)}{(\ln x)^2} = \lim_{b \rightarrow \infty} \left. \frac{-1}{\ln x} \right|_2^b$$

$$= \lim_{b \rightarrow \infty} \frac{-1}{\ln b} + \frac{1}{\ln 2} = \frac{1}{\ln 2}$$

$$\text{So } \int_2^{\infty} \frac{1}{x(\ln x)^2} dx \text{ conv.}$$

$$\text{By the Integral Test, } \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \text{ conv.}$$

$$[5] \sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$$

$$f(x) = \frac{e^x}{1+e^{2x}} \quad \text{+ve and continuous } \forall x \geq 1$$

$$f'(x) = \frac{(1+e^{2x})e^x - e^x(2e^{2x})}{(1+e^{2x})^2} = \frac{e^x + e^{3x} - 2e^{3x}}{(1+e^{2x})^2} = \frac{e^x - e^{2x}}{(1+e^{2x})^2}$$

$$= \frac{e^x(1-e^{2x})}{(1+e^{2x})^2} < 0 \quad \forall x \geq 1 \Rightarrow \text{decreasing}$$

$$\int_1^{\infty} \frac{e^x}{1+e^{2x}} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{d(e^x)}{1+(e^x)^2} = \lim_{b \rightarrow \infty} \left. \tan^{-1}(e^x) \right|_1^b$$

$$= \lim_{b \rightarrow \infty} \tan^{-1} e^b - \tan^{-1} e^1 = \frac{\pi}{2} - \tan^{-1} e$$

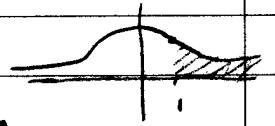
$$\text{So } \int_1^{\infty} \frac{e^x}{1+e^{2x}} dx \text{ conv.}$$

$$\text{By the Integral test, } \sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} \text{ conv.}$$

$$\boxed{\text{Q}} \sum_{n=1}^{\infty} \operatorname{sech}^2 n.$$

$f(x) = \operatorname{sech}^2 x$ +ve, continuous and decreasing $\forall x \geq 1$

(from the graph of $\operatorname{sech} x$).



$$\int_1^{\infty} \operatorname{sech}^2 x \, dx = \lim_{b \rightarrow \infty} \int_1^b \operatorname{sech}^2 x \, dx = \lim_{b \rightarrow \infty} \tanh x \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} \tanh b - \tanh 1 = \underline{1 - \tanh 1}$$

$\therefore \int_1^{\infty} \operatorname{sech}^2 x \, dx$ conv.

By the Integral Test, $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$ conv.

Ex 10.3

141, 42, 44,
55, 56, 58

10.4/ Comparison Tests.

Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with ~~non~~ nonnegative terms. Suppose that for some integer N , $d_n \leq a_n \leq c_n$ for all $n \geq N$.

(A) If $\sum c_n$ conv. then $\sum a_n$ also conv.

(B) If $\sum d_n$ div then $\sum a_n$ also div.

Example/ Test the convergence of the following series.

$$[1] \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}} \quad \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

But $\sum \frac{1}{n^{3/2}}$ conv. p-series because $p = \frac{3}{2} > 1$

By the D.C.T, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ conv.

$$[2] \sum_{n=1}^{\infty} \frac{2+\cos n}{\sqrt{n+8}}$$

$$\frac{2+\cos n}{\sqrt{n+8}} \geq \frac{1}{\sqrt{n+8}} \geq \frac{1}{\sqrt{n+8n}} = \frac{1}{\sqrt{9n^2}}$$

By $\sum \frac{1}{\sqrt{9n^2}}$ div p-series because $p = \frac{1}{2} < 1$

By D.C.T, $\sum_{n=1}^{\infty} \frac{2+\cos n}{\sqrt{n+8}}$ div.

$$[3] \sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$$

$$\frac{n+2}{n^2-n} > \frac{n}{n^2-n} > \frac{n}{n^2} = \frac{1}{n}$$

But $\sum \frac{1}{n}$ div harmonic series.

By D.C.T, $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ div.

$$[4] \sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$$

$$\frac{10n+1}{n(n+1)(n+2)} < \frac{10n+1}{n(n+1)(n+2)} = \frac{11n}{n(n+1)(n+2)} < \frac{11n}{n \cdot n \cdot n} = \frac{11}{n^2}$$

But $\sum \frac{11}{n^2}$ conv. p-series because $p=2 > 1$

By D.C.T, $\sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)}$ conv.

$$[5] \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2+\ln n}$$

$$\frac{\sqrt{n}}{2+\ln n} > \frac{\sqrt{n}}{2+n} > \frac{\sqrt{n}}{2+n} = \frac{\sqrt{n}}{3n} = \frac{1}{3n^{\frac{1}{2}}}$$

But $\sum \frac{1}{3n^{\frac{1}{2}}}$ div p-series because $p=\frac{1}{2} < 1$

By D.C.T, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2+\ln n}$ div.

$$[6] \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1} \quad \frac{\ln(n+1)}{n+1} > \frac{1}{n+1} > \frac{1}{n+n} = \frac{1}{2n} \quad \forall n \geq 2$$

But $\sum \frac{1}{2n}$ div harmonic series

By D.C.T, $\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n+1}$ div.

$$[7] \sum_{n=1}^{\infty} \frac{\tanh n}{n^2} \quad \frac{\tanh n}{n^2} < \frac{1}{n^2}$$

But $\sum \frac{1}{n^2}$ conv p-series because $p=2 > 1$

By the D.C.T, $\sum_{n=1}^{\infty} \frac{\tanh n}{n^2}$ conv.

10.5 The Ratio and n -th Root tests.

1 The Ratio Test:

Let $\sum a_n$ be a series with positive terms and suppose that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$. Then:

1 The series $\sum a_n$ conv. if $\rho < 1$

2 $\sum a_n$ = div if $\rho > 1$ or ρ is infinite.

3 $\sum a_n$ test inconclusive (fail) if $\rho = 1$

2 The n th root test:

Let $\sum a_n$ be a series with $a_n > 0 \forall n > N$ and suppose that $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$ then

Example/Test the convergence of the following series:

1 $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{10^{n+1}} \bigg/ \frac{n!}{10^n}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)n!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n \rightarrow \infty} \frac{n+1}{10} = \frac{1}{10} < 1$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ div.

OR $\lim_{n \rightarrow \infty} \frac{n!}{10^n} = \frac{1}{\text{zero}} = \infty \Rightarrow \text{div.}$

$$② \sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{n}}}{\ln n} = \frac{1}{\infty} = \text{zero} < 1$$

By the n -th root test, the series $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ conv.

$$③ \sum_{n=1}^{\infty} n! \left(\frac{3}{n}\right)^n$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{(n+1)! \left(\frac{3}{n+1}\right)^{n+1}}{n! \left(\frac{3}{n}\right)^n} = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot 3^{n+1} \cdot n^n}{n! \cdot (n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{3^{n+1} n^n}{(n+1)^{n+1}} = 3 \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = 3 \lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{3}{e} > 1 \end{aligned}$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} n! \left(\frac{3}{n}\right)^n$ div.

$$④ \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot n^n}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \lim_{n \rightarrow \infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1 \end{aligned}$$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ conv.

$$⑤ \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(1 - \frac{1}{n}\right)^{n^2}} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} = \frac{1}{e} < 1$$

By the n -th root test, the series $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

$$⑥ \sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$$

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\left(1 - \frac{1}{3n}\right)^n} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n}\right) = 1 - \frac{1}{\infty} = 1$$

The n -th root test fails.

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3^n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{-1}{3^n}\right)^n = e^{-\frac{1}{3}} = \frac{1}{e^{\frac{1}{3}}} \neq 0$$

By the n -th term test, the series $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3^n}\right)^n$ diverges.

$$[7] \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)(3^n+1)}$$

$$P = \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)(3^{n+1}+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)(3^{n+1}+1)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)(3^n+1)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$= \lim_{n \rightarrow \infty} \frac{(2n+1)}{(2n+2)} \cdot \lim_{n \rightarrow \infty} \frac{3^n+1}{3^{n+1}+1}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{2} \cdot \lim_{n \rightarrow \infty} \frac{\frac{1}{3} + \frac{1}{2^{n+1}}}{1 + \frac{1}{2^{n+1}}} = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$$

By the Ratio Test, the series conv.

$$\text{نظرًا إلى } \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)(3^n+1)} < \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{2 \cdot 4 \cdots (2n)(3^n+1)} = \frac{1}{3^n+1} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

But $\sum \left(\frac{1}{3}\right)^n$ conv. geom. series because $|r| = \frac{1}{3} < 1$

By D.C.T, the series conv.

$$[8] \sum_{n=1}^{\infty} \frac{n! \ln n}{n \cdot (n+2)!}$$

$$P = \lim_{n \rightarrow \infty} \frac{(n+1)! \ln(n+1)}{(n+1)(n+3)!} \cdot \frac{n! \ln n}{n \cdot (n+2)!}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)! \ln(n+1)}{(n+1)(n+3)(n+2)!} \cdot \frac{n! \ln n}{n! \ln n} = \lim_{n \rightarrow \infty} \frac{n}{n+3} \cdot \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n)}$$

$$= 1 \cdot \lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

So the ratio test is inconclusive (Fail).

$$\sum_{n=1}^{\infty} \frac{n! \ln(n)}{n(n+2)(n+1)n!} = \sum_{n=1}^{\infty} \frac{\ln n}{n(n+2)(n+1)}$$

$$\frac{\ln n}{n(n+1)(n+2)} < \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < \frac{1}{n \cdot n} = \frac{1}{n^2}$$

But $\sum \frac{1}{n^2}$ conv. p-series because $p=2 > 1$

By the D.C.T, $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$ conv.

9) $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n)^{n^2}}$

$$\rho = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n!)^n}{(n)^{n^2}}} = \lim_{n \rightarrow \infty} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2) \dots (n-2+1)}{n \cdot n \cdot n \dots n}$$

$$= \lim_{n \rightarrow \infty} \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \dots \cdot \frac{2}{n} \cdot \frac{1}{n} < \lim_{n \rightarrow \infty} \frac{n-1}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{n-1}{n} \cdot \frac{n-1}{n}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n-1}{n} \right)^{n-1} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^{n-1} = \frac{1 - \frac{1}{n}}{1 - \frac{1}{n}} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n} \right)^n \cdot \frac{1}{1 - \frac{1}{n}}$$

$$= e^{-1} \cdot \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{n}} = e^{-1} \cdot 1 = \frac{1}{e} < 1$$

By the n-th root test, the series $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n)^{n^2}}$ conv

10) $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$

$$\rho = \lim_{n \rightarrow \infty} \frac{4^{n+1} (n+1)!^2}{(2n+2)!} \div \frac{4^n (n!)^2}{(2n)!} = \lim_{n \rightarrow \infty} \frac{4(n+1)(n+1)!^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n (n!)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{4(n+1)^2}{2(n+1)(2n+1)} = \lim_{n \rightarrow \infty} \frac{2(n+1)}{(2n+1)} = \lim_{n \rightarrow \infty} \frac{2}{2} = 1$$

So the ratio test fails.

$$\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} = \frac{2n+2}{2n+1} > 1 \Rightarrow \frac{a_{n+1}}{a_n} > 1$$

$$a_{n+1} > a_n \quad \forall n$$

$$a_1 = \frac{4^1 (1!)^2}{(2)!} = 2 \Rightarrow a_n > 2 \Rightarrow \lim_{n \rightarrow \infty} a_n > 2 \neq 0$$

By the n -th term test, the series $\sum \frac{4^n (n!)^2}{(2n)!}$ div.

$$[N] a_1 = 3, a_{n+1} = \frac{n}{n+1} a_n$$

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \Rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

So the ratio test fails.

$$a_1 = 3, a_2 = a_{1+1} = \frac{1}{2} a_1 = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

$$a_3 = \frac{2}{2+1} a_2 = \frac{2}{3} \cdot \frac{3}{2} = 1, a_4 = \frac{3}{3+1} a_3 = \frac{3}{4} \cdot 1 = \frac{3}{4}$$

$$a_5 = \frac{4}{4} \cdot \frac{3}{4} = \frac{3}{4}$$

$$\text{The series is } 3 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \dots = \sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$$

div harmonic series.

Ex 10.5) 1-62 multiple of 3.

10.6 Alternating Series

Def. The alternating series is an expression in the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n+1} a_n + \dots$$

$$\text{or } \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \dots + (-1)^n a_n + \dots$$

Theorem: Convergence of alternating series.

The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ conv. if the following three conditions are satisfied:

- ① The a_n 's are all positives
- ② the positive a_n 's are nonincreasing, that is, $a_n \geq a_{n+1} \forall n \geq N$ for some integer N .
- ③ $\lim_{n \rightarrow \infty} a_n = \text{Zero}$.

Example / Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv. or div.

- ① The a_n 's $= \frac{1}{n}$ are all positive.
- ② $a_n = \frac{1}{n} > a_{n+1} = \frac{1}{n+1} \Rightarrow$ decreasing $\forall n \geq 1$
- ③ $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = \text{Zero}$

By the above theorem, the alternating harmonic series conv.

Def. If the series of absolute series $\sum |a_n|$ conv. then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is called conv. absolutely

Def. If $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is not conv. absolutely but satisfies the ~~the~~ three conditions of the ~~the~~ previous theorem, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is called conv. conditionally.

Note ① $|(-1)^{n+1}| = |(-1)^n| = 1$

② conv. absolutely means conv. and conv. conditionally means conv.

③ If the third condition of the previous theorem is not satisfied, that is, $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ div.

Example Which of the following series conv., which conv. conditionally, and which div ??

① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{3^{n+1}}$

The series of abs values is $\sum \frac{4^n}{3^{n+1}} = \sum \frac{1}{3} \left(\frac{4}{3}\right)^n$

div geom series because $|r| = \frac{4}{3} > 1$

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{3} \left(\frac{4}{3}\right)^n = \infty$

Since $(\frac{4}{3})^n > 1$

This means that the third condition of the theorem isn't satisfied, that is, $\lim_{n \rightarrow \infty} a_n \neq \text{Zero}$.

So the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{3^{n+1}}$ div.

$$\boxed{2} \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$$

The series of abs. values is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$f(x) = \frac{1}{x \ln x}$ positive and continuous $\forall x \geq 2$

$$a_n = \frac{1}{n \ln n} > a_{n+1} = \frac{1}{(n+1) \ln(n+1)} \Rightarrow \text{decreasing } \forall x \geq 2$$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{d(\ln x)}{\ln x} = \lim_{b \rightarrow \infty} \ln |\ln x| \Big|_2^b$$

$$= \lim_{b \rightarrow \infty} \ln |\ln b| - \ln |\ln 2| = \infty - \text{---} = \infty$$

$$\text{So } \int_2^{\infty} \frac{1}{x \ln x} dx \text{ div}$$

By the Integral test, the series of absolute values div.

1 The a_n 's are all positive $\forall n \geq 2$.

$$\boxed{2} a_n = \frac{1}{n \ln n} > a_{n+1} = \frac{1}{(n+1) \ln(n+1)} \quad \forall n \geq 2 \quad \text{"decreasing"}$$

$$\boxed{3} \lim_{n \rightarrow \infty} \frac{1}{n \ln n} = \frac{1}{\infty} = \text{Zero.}$$

The series $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ conv. conditionally.

$$\boxed{3} \sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$$

absolute

The series of abs values is $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

$$\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$$

But $\sum \frac{1}{n^2}$ conv. p-series because $p=2 > 1$

By D.C.T, the series of abs values $\sum \frac{|\sin n|}{n^2}$ conv.

So $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$ conv. absolutely.

$$[4] \sum_{n=1}^{\infty} (-2)^{n+1} / (n+5)^n \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^{n+1}}{n+5^n}$$

the series of abs values is $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} = \sum_{n=1}^{\infty} \frac{2 \cdot 2^n}{n+5^n}$

$$\frac{2 \cdot 2^n}{n+5^n} < \frac{2 \cdot 2^n}{5^n} = 2 \left(\frac{2}{5}\right)^n$$

But $\sum_{n=1}^{\infty} 2 \left(\frac{2}{5}\right)^n$ conv. geom. series because $|r| = \frac{2}{5} < 1$

By D.C.T, the series of abs values $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n}$ conv.

So $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n}$ conv. absolutely.

$$[5] \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{2n^3+3}$$

the series of abs values is $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+3}$ $\left\{ \frac{(3n)^3 (1)}{(1+5)} \right\}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^3}{2n^3+3} = \frac{1}{2} \neq 0$$

By the n-th term test, the series of abs. values div.

But the third condition of the theorem doesn't satisfied.

$$\text{that is, } \lim_{n \rightarrow \infty} a_n = \frac{1}{2} \neq 0 \Rightarrow$$

the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{2n^3+3}$ div.

$$[6] \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n n!}{n^n}$$

the series of abs. values is $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$\rho = \lim_{n \rightarrow \infty} \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = \lim_{n \rightarrow \infty} \frac{2(2)^n (n+1)n!}{(n+1)(n+1)^n 2^n n!} \cdot \frac{n^n}{2^n n!}$$

$$= \lim_{n \rightarrow \infty} \frac{2 n^n}{(n+1)^n} = 2 \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = 2 \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

By the ratio test, the series of abs values conv.
 So the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n n!}{n^n}$ conv absolutely.

$$[7] \sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n\sqrt{n}+1}$$

the series of abs values is $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}+1}$

$$\frac{\tan^{-1} n}{n\sqrt{n}+1} < \frac{\frac{\pi}{2}}{n\sqrt{n}+1} < \frac{\frac{\pi}{2}}{n\sqrt{n}} = \frac{\frac{\pi}{2}}{(n)^{3/2}}$$

but $\sum \frac{\pi}{n^{3/2}}$ conv. p-series because $p = \frac{3}{2} > 1$

By the D.C.T, the series of abs values conv.

So the series $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n\sqrt{n}+1}$ conv. absolutely.

$$[8] \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$$

the series of abs. values is $\sum_{n=1}^{\infty} \frac{(n!)^2 3^n}{(2n+1)!}$

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \frac{(n+1)!^2 3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{3^n n!^2} = \lim_{n \rightarrow \infty} \frac{(n+1)^2 (n!)^2 3 \cdot 3^n}{(2n+3)(2n+2)(2n+1)!} \cdot \frac{(2n+1)!}{3^n n!^2} \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^2 3}{2(2n+3)} = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} < 1 \end{aligned}$$

By the Ratio Test, the series of abs values conv.

So the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 3^n}{(2n+1)!}$ conv. absolutely.

10.7] Power Series.

Def. The Power series is a series in the form

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots + C_n(x-a)^n + \dots$$

where $a, C_0, C_1, \dots, C_n, \dots$ are constants and a is the center of the power series.

* If $a=0$, then power series is in the form

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n + \dots$$

* Convergence of the power series

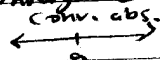
For any power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ or $\sum_{n=0}^{\infty} C_n x^n$, ~~conv~~ exactly

one of the following is true:-

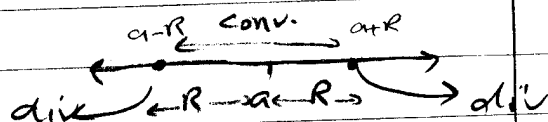
[1] The series conv. only at $x=a$ and div. elsewhere. The div. radius of convergence is $R=0$.



[2] The series conv. absolutely $\forall x$. The radius of convergence is $R=\infty$.



[3] There's a +ve real num. R (the radius of convergence of the power series) such that the series conv. abs for $|x-a| < R$ and div for $|x-a| > R$



Notes/

[1] The series may or may not conv at the endpoints $x = a-R$ and $x = a+R$.

[2] For the convergence of the power series, we use the ratio test or the n -th root test.

Examples:- The interval and radius of convergence of the following power series:

$$\boxed{1} \sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^n (x-3)^n$$

$$\rho = \lim_{n \rightarrow \infty} \left| \sqrt[n]{\left(\frac{n}{2}\right)^n (x-3)^n} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{n}{2}\right) (x-3) \right|$$

$$= |x-3| \lim_{n \rightarrow \infty} \frac{n}{2} = 0 < 1 \text{ only at } x=3$$

The series conv. at $x=3$

The Radius of convergence is $R = \text{zero}$.

$$\boxed{2} \sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{n! x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1) n! x \cdot x^n}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{n! x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x}{2(2n+1)} \right| = \left| \frac{x}{2} \right| \lim_{n \rightarrow \infty} \frac{1}{2n+1} = \frac{|x|}{2} \times 0 = \text{zero} < 1$$

$\forall x$

The series conv. abs $\forall x$.

The radius of convergence is $R = \infty$.

$$\boxed{3} \sum_{n=0}^{\infty} \frac{n x^n}{4^n (n^2 + 1)}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1) x^{n+1}}{4^{n+1} ((n+1)^2 + 1)} \cdot \frac{4^n (n^2 + 1)}{n x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)x \cdot x^n}{4^n \cdot 4(n^2+2n+2)} \cdot \frac{4^n (n^2+1)}{1 \cdot x^n} \right|$$

$$= \left| \frac{x}{4} \right| \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{(n^2+1)}{(n^2+2n+2)} = \frac{|x|}{4} \cdot 1 \cdot 1 = \frac{|x|}{4}$$

The series convs abs. if $\frac{|x|}{4} < 1$ and div if $\frac{|x|}{4} > 1$

* If $x = -4$, then series is $\sum_{n=0}^{\infty} \frac{n(-4)^n}{4^n(n^2+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n n}{4^n(n^2+1)}$

div conv. div.
-4 4
 $|x| < 4$
 $-4 < x < 4$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{n}{(n^2+1)}$$

The series of abs values is $\sum \frac{n}{n^2+1}$

$$\frac{n}{n^2+1} > \frac{n}{n^2+n^2} = \frac{n}{2n^2} = \frac{1}{2n}$$

But $\sum \frac{1}{2n}$ div because it harmonic series.

By the D.C.T, the series of abs value $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ div $\forall x > 2$

div
أقل من 2
بالتالي
المتسلسلة
تتبدل

(i) a_n 's are +ve (ii) $f(x) = \frac{x}{x^2+1} \Rightarrow f'(x) = \frac{x^2+1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$

أقل من
2
بالتالي
Zero

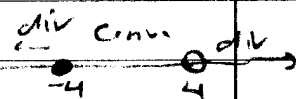
So $f'(x)$ decreasing

(iii) $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n^2+1} = \text{Zero}$

So the series $\sum (-1)^n \frac{n}{n^2+1}$ convs conditionally.

* If $x = 4$, the series $\sum \frac{n(4)^n}{4^n(n^2+1)} = \sum \frac{n}{n^2+1}$ div - from above

The interval of convergence $[-4, 4)$



The ratio of convergence $R = 4$

$$\boxed{4} \sum_{n=1}^{\infty} \frac{(-1)^n (2x-3)^n}{2n-1}$$

$(-1)^n$ القيمة المطلقة له 1 فلا يؤثر

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{(2x-3)^{n+1}}{2(n+1)-1} \div \frac{(2x-3)^n}{2n-1} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(2x-3)(2x-3)^n}{2n+1} \cdot \frac{2n-1}{(2x-3)^n} \right| = |2x-3| \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} = |2x-3|$$

The series conv. abs if $|2x-3| < 1$ and div if $|2x-3| > 1$

$$-1 < 2x-3 < 1$$

$$2 < 2x < 4$$

$$1 < x < 2$$

* If $x=1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{2n-1} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ ليست after لأنه دائماً موجب $\frac{1}{2n}$

$\frac{1}{2n-1} > \frac{1}{2n}$ But $\sum \frac{1}{2n}$ div. harmonic series.

By the D.C.T, $\sum \frac{1}{2n-1}$ div.

If $x=2$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$

The series of abs values is $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ div from above

(i) a_n 's are +ve $\forall n \geq 1$

(ii) $a_n = \frac{1}{2n-1} > a_{n+1} = \frac{1}{2(n+1)-1} \Rightarrow$ decreasing

(iii) $\lim_{n \rightarrow \infty} \frac{1}{2n-1} = \text{zero}$

So the series conv. conditionally at $x=2$.

The interval of convergence is $(1, 2]$

The radius of convergence is $\frac{2-1}{2} = \underline{\underline{\frac{1}{2}}}$

Example / Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ and find the sum of the series as f_n .

The series is geom. series that conv. if $|r| = \left|\frac{x^2+1}{3}\right| < 1$

$$\Rightarrow \frac{|x^2+1|}{3} < 1 \Rightarrow |x^2+1| < 3 \Rightarrow x^2+1 < 3$$

$$|x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$$

The interval of convergence is $(-\sqrt{2}, \sqrt{2})$

The sum of the series

$$S = \frac{1}{1 - \frac{x^2+1}{3}} = \boxed{\frac{3}{2-x^2}}$$

Term-by-term diff theorem:

If $\sum_{n=0}^{\infty} C_n(x-a)^n$ conv. for $\alpha-R < x < \alpha+R$, it defines as

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n, \quad \alpha-R < x < \alpha+R$$

This f_n has derivatives of all orders ~~inside~~ inside the interval of convergence as follows

$$f'(x) = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2} \dots$$

Each of these derived series conv. at every interior point of the interval of convergence of the original series.

Term-by-Term Integration theorem:-

Suppose that $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$ conv. for $a-R < x < a+R$.

Then $\sum_{n=0}^{\infty} \frac{C_n(x-a)^{n+1}}{n+1}$ conv. for $a-R < x < a+R$ and

$$\int f(x) dx = \sum_{n=0}^{\infty} \frac{C_n(x-a)^{n+1}}{n+1} + C \text{ for } a-R < x < a+R$$

Example / Find a fn $f(x)$ that represented by the power series $\sum_{n=0}^{\infty} (-1)^n x^n$. Then use the result to find a power series that represent the following fns.

① $g(x) = \frac{1}{(1+x)^2}$ ② $g(x) = \ln(1+x)$

$$\sum_{n=0}^{\infty} (-1)^n (x)^n = 1 - x + x^2 - x^3 + x^4 - \dots + (-1)^n x^n + \dots$$

conv. geom. series if $|r| = |-x| = |x| < 1 \rightarrow -1 < x < 1$

The given series conv. on $(-1, 1)$

The Sum is $S = \frac{1}{1 - (-x)} = \frac{1}{1+x}$

$$S. \sum_{n=0}^{\infty} (-1)^n (x)^n = \frac{1}{1+x}$$

② $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots + (-1)^n x^n + \dots$

$$\frac{-1}{(1+x)^2} = -1 + 2x - 2x^2 + 4x^3 - \dots + (-1)^n n x^{n-1} + \dots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \dots + (-1)^{n+1} n x^{n-1} + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

$$\boxed{b} \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots + (-1)^n x^n + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^n x^{n+1}}{n+1} + \dots + C$$

If $x=0$ then $\ln(1) = \text{Zero} + C \Rightarrow C = \text{Zero}.$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^n x^{n+1}}{n+1} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

I [S]

10.8 Taylor and Maclaurin series

Def. [1] Let f be a fn with derivatives of all orders through out some interval containing a as an interior point. The Taylor series generated by the fn f at the point a as

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!} = \underbrace{f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}}_{\text{Taylor polynomial}}$$

[2] If $a=0$, we get the Maclaurin series that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} = \underbrace{f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n)}(0)x^n}{n!}}_{\text{Maclaurin polynomial}}$$

Example /

[1] Find the Taylor series for the fn $f(x) = e^x$ at $x=1$.

$$f(x) = e^x \Rightarrow f(1) = e^1 = e$$

$$f'(x) = e^x \Rightarrow f'(1) = e^1 = e$$

$$f''(x) = e^x \Rightarrow f''(1) = e^1 = e$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e^1 = e$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)(x-1)^k}{k!} = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \dots + \frac{f^{(n)}(1)(x-1)^n}{n!} + \dots$$

$$e^x = e + \frac{e(x-1)}{1!} + \frac{e(x-1)^2}{2!} + \dots + \frac{e(x-1)^n}{n!} + \dots$$

[2] Find the Taylor series for the fn $f(x) = x \ln x$ at $x=e$.

$$f(x) = x \ln x$$

$$f'(x) = x \left(\frac{1}{x}\right) + \ln x$$

$$f''(x) =$$

$$f(e) = e \ln e = e$$

$$f'(e) = 1 + \ln e = 2$$

$$f(x) = x \ln x$$

$$f'(x) = 1 + \ln x$$

$$f''(x) = \frac{1}{x} = \frac{0!}{x^1}$$

$$f'''(x) = -\frac{1}{x^2} = -\frac{1!}{x^2}$$

$$f^{(4)}(x) = \frac{2}{x^3} = \frac{2!}{x^3}$$

$$f^{(5)}(x) = -\frac{6}{x^4} = -\frac{3!}{x^4}$$

$$f^{(6)}(x) = \frac{24}{x^5} = \frac{4!}{x^5}$$

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-2)!}{x^{n-1}}$$

$$f(e) = e$$

$$f'(e) = 2$$

$$f''(e) = \frac{1}{e} = \frac{0!}{e^1}$$

$$f'''(e) = -\frac{1}{e^2} = -\frac{1!}{e^2}$$

$$f^{(4)}(e) = \frac{2}{e^3} = \frac{2!}{e^3}$$

$$f^{(5)}(e) = -\frac{6}{e^4} = -\frac{3!}{e^4}$$

$$f^{(6)}(e) = \frac{24}{e^5} = \frac{4!}{e^5}$$

$$f^{(n)}(e) = \frac{(-1)^{n-1} (n-2)!}{e^{n-1}}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(e)(x-e)^k}{k!}$$

$$= f(e) + \frac{f'(e)(x-e)}{1!} + \frac{f''(e)(x-e)^2}{2!} + \dots + \frac{f^{(n)}(e)(x-e)^n}{n!} + \dots$$

$$x \ln x = e + 2(x-e) + \frac{(x-e)^2}{e \cdot 2!} - \frac{1}{e^2} \frac{(x-e)^3}{3!} + \dots + \frac{(-1)^{n-1} (n-2)!}{e^{n-1}} \frac{(x-e)^n}{n!} + \dots$$

$$= e + 2(x-e) + \frac{(x-e)^2}{e \cdot 2!} - \frac{1}{e^2} \frac{(x-e)^3}{3!} + \frac{2!}{e^3} \frac{(x-e)^4}{4 \cdot 3 \cdot 2!} - \frac{3!}{e^4} \frac{(x-e)^5}{5 \cdot 4 \cdot 3!} + \dots$$

$$+ \frac{(-1)^{n-1} (n-2)!}{e^{n-1}} \frac{(x-e)^n}{n(n-1)(n-2)!} + \dots$$

$$= e + 2(x-e) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (x-e)^n}{e^{n-1} n(n-1)}$$

[3] Find the Maclaurin series for the following fns.

(a) $f(x) = \sin x$

(b) $f(x) = \sin 3x$

(c) $f(x) = \cos x$

(d) $f(x) = \cos 3x$

(a) $f(x) = \sin x$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(2n)}(x) = (-1)^n \sin x$$

$$f^{(2n+1)}(x) = (-1)^n \cos x$$

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -1$$

$$f^{(2n)}(0) = (-1)^n \sin 0 = 0$$

$$f^{(2n+1)}(0) = (-1)^n \cos 0 = (-1)^n$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0) x^k}{k!} = \cancel{f(0)} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{(2n+1)}}{(2n+1)!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

x²ⁿ by 5
→ exponential

[b] Replace each x by 3x in part (a) we get:-

$$\sin 3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \dots + \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

[c] Differentiate the series (a) we get.

$$\cos x = \frac{d}{dx}(\sin x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots + \frac{(-1)^n (2n+1)x^{2n}}{(2n+1)!} - \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{2n!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

$$[d] \cos^2 x = \frac{1}{2} (1 + \cos 2x)$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots + \frac{(-1)^n (2x)^{2n}}{2n!} - \dots$$

$$\cos^2 x = \frac{1}{2} \left((1+1) - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots + \frac{(-1)^n (2x)^{2n}}{2n!} + \dots \right)$$

$$= 1 - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} - \frac{2^5 x^6}{6!} + \dots + \frac{(-1)^n 2^{2n-1} x^{2n}}{2n!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{2n!}$$

$$\text{Ex 10.8} = (1-35) \text{ odd}$$

$$37, 39, 41, 43, 45.$$

10.10 Applications of power series:- (Binomial series)

Def. The binomial series is the Maclaurin series of $f(x) = (1+x)^m$, where m is constant.

To get the general form of the binomial series we make the following steps:-

$$f(x) = (1+x)^m \quad f(0) = 1^m = 1$$

$$f'(x) = m(1+x)^{m-1} \quad f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2} \quad f''(0) = m(m-1)$$

$$f^{(n)}(x) = m(m-1)(m-2)\dots(m-(n-1))(1+x)^{m-n} \rightarrow f^{(n)}(0) = m(m-1)(m-2)\dots(m-(n-1))$$

$$\begin{aligned} (1+x)^m &= f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots \\ &= 1 + mx + \frac{m(m-1)x^2}{2!} + \dots + \frac{m(m-1)(m-2)\dots(m-(n-1))x^n}{n!} + \dots \end{aligned}$$

$\Rightarrow$ Conv. on $|x| < 1$

Example/ Find the binomial series for the fn $f(x) = \frac{1}{\sqrt{1-x}}$

$$f(x) = \frac{1}{(1-x)^{1/2}} = (1-x)^{-1/2} = (1+(-x))^{-1/2}$$

$$= 1 + \left(\frac{-1}{2}\right)(-x) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)(-x)^2}{2!} + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)(-x)^3}{3!} + \dots$$

$$+ \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\dots\left(\frac{-1}{2}-n+1\right)(-x)^n}{n!} + \dots$$

$$= 1 + \frac{1 \cdot x}{2} + \frac{\frac{1}{2} \cdot \frac{3}{2} x^2}{2!} + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} x^3}{3!} + \dots + \frac{1 \cdot 3 \cdot 5 \dots (2n-1) x^n}{2^n \cdot n!}$$

$$= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2n-1) x^n}{2^n \cdot n!}$$

$$\begin{aligned} \frac{-1}{2} - n + 1 &= \frac{1-n}{2} \\ &= \frac{-2n+1}{2} = -\frac{(2n-1)}{2} \end{aligned}$$

Ex = (1-13) (29-35) odd

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Ch. 11 Parametric eqns and polar coordinates.

11.1 parametrizations of plane curves.

Def. If $x = f(t)$ and $y = g(t)$ are given fns of t and $t \in I$ then the set of points $(x, y) = (f(t), g(t))$ is a parametric curve and the eqns $x = f(t), y = g(t), t \in I$ are called the parametric eqns of curve and t is called the parameter of the curve.

Note // A given curve can be represent by different sets of parametric eqns.

Examples - The following parametric eqns describe the motion of a particle in the x - y plane.

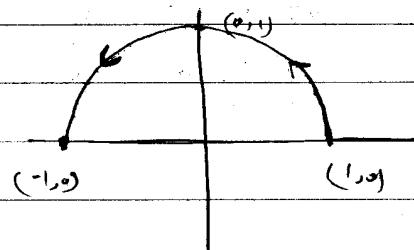
- a) Identify the particle's path.
- b) Graph its cartesian eqn on the given interval.
- c) Show the direction of motion.

1) $x = \cos t, y = \sin t, 0 \leq t \leq \pi$

a) $x^2 + y^2 = \cos^2 t + \sin^2 t$
 $x^2 + y^2 = 1$

circle with center at the origin and radius 1.

b) If $t = 0 \Rightarrow x = 1, y = 0 \Rightarrow (1, 0)$
 $t = \frac{\pi}{2} \Rightarrow x = 0, y = 1 \Rightarrow (0, 1)$
 $t = \pi \Rightarrow x = -1, y = 0 \Rightarrow (-1, 0)$



- c) The graph is the upper half of the circle in the counter clock wise direction.

[2] $x = t, y = \sqrt{1-t^2}, -1 \leq t \leq 1$

[a] $y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$
 $x^2 + y^2 = 1$

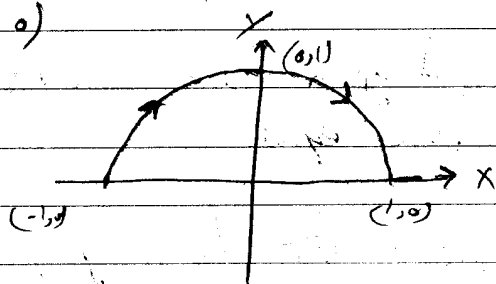
eqn of a circle with center at the origin and radius 1.

[b] If $t = -1, x = -1, y = 0 \Rightarrow (-1, 0)$

$t = 0, x = 0, y = 1 \Rightarrow (0, 1)$

$t = 1, x = 1, y = 0 \Rightarrow (1, 0)$

[c] The graph is the upper half of the circle in the clockwise direction.



[3] $x = 3t, y = 9t^2, -\infty < t < \infty$

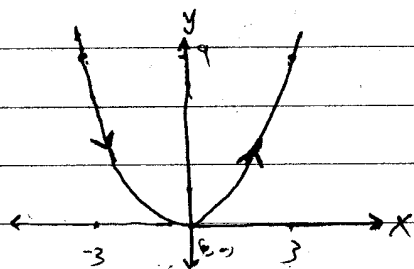
[a] $y = (3t)^2 = x^2$

eqn of a parabola open up with vertex at the origin.

[b] If $t = -1, x = -3, y = 9 \Rightarrow (-3, 9)$

$t = 0, x = 0, y = 0 \Rightarrow (0, 0)$

$t = 1, x = 3, y = 9 \Rightarrow (3, 9)$



[c] The graph is a parabola with vertex at the origin. the direction from up to down on the left until the origin (0,0) and from down to up on the Right.

4) $x = 4\sin t$, $y = 5\cos t$, $0 \leq t \leq 2\pi$

a) $\frac{x}{4} = \sin t$, $\frac{y}{5} = \cos t$

$$\frac{x^2}{16} + \frac{y^2}{25} = \sin^2 t + \cos^2 t = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$$

eqn of an ellipse with major axis along the y-axis and center at the origin.

b) If $t = 0$, $x = 0$, $y = 5 \Rightarrow (0, 5)$

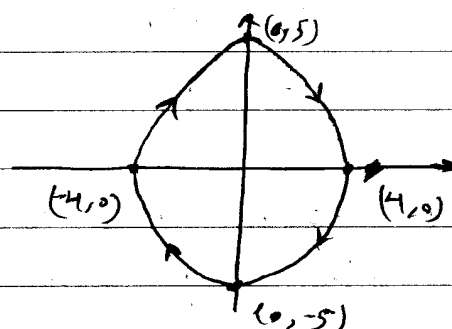
$t = \frac{\pi}{2}$, $x = 4$, $y = 0 \Rightarrow (4, 0)$

$t = \pi$, $x = 0$, $y = -5 \Rightarrow (0, -5)$

$t = \frac{3\pi}{2}$, $x = -4$, $y = 0 \Rightarrow (-4, 0)$

$t = 2\pi$, $x = 0$, $y = 5 \Rightarrow (0, 5)$

c) The direction of motion is on the clockwise direction



5) $x = -\sec t$, $y = \tan t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

a) $x^2 - y^2 = \sec^2 t - \tan^2 t = 1 \Rightarrow x^2 - y^2 = 1$

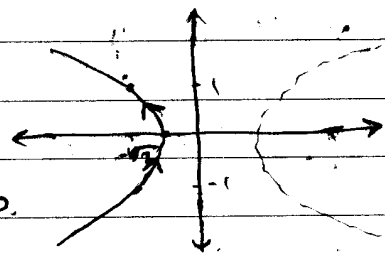
eqn of hyperbola with ~~hyper~~ focal axis along the x-axis and center at the origin.

b) If $t = -\frac{\pi}{4}$, $x = -\sqrt{2}$, $y = -1 \Rightarrow (-\sqrt{2}, -1)$

$t = 0$, $x = -1$, $y = 0 \Rightarrow (-1, 0)$

$t = \frac{\pi}{4}$, $x = -\sqrt{2}$, $y = 1 \Rightarrow (-\sqrt{2}, 1)$

The graph is the left ~~and~~ branch of the hyperbola.



(c) The direction of motion from down to up.

⇒ Remark

The parametric eqns of a line passes through (x_0, y_0) and (x_1, y_1) are $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$, $-\infty < t < \infty$

Example/ Find the parametrization for the line segment with end's points $(-1, 3)$ and $(3, -2)$

(x_0, y_0) (x_1, y_1)

$$x = -1 + (3 - (-1))t = -1 + 4t$$

$$y = 3 + (-2 - 3)t = 3 - 5t$$

If $t=0$, then $(x, y) = (-1, 3)$

$t=1$, then $(x, y) = (3, -2)$

$$\begin{cases} -1 + 4t = -1 & \Rightarrow t=0 \\ 3 - 5t = 3 & \Rightarrow t=0 \\ -1 + 4t = 3 & \Rightarrow t=1 \end{cases}$$

The parametric eqn of the line segment are

$$\begin{cases} x = -1 + 4t \\ y = 3 - 5t \end{cases} \quad 0 \leq t \leq 1$$

$$\frac{x+1}{4} = \frac{-y+3}{5}$$

Ex 11.1, 1-21 mod.

$$y - 3 = -\frac{5}{4}(x + 1)$$

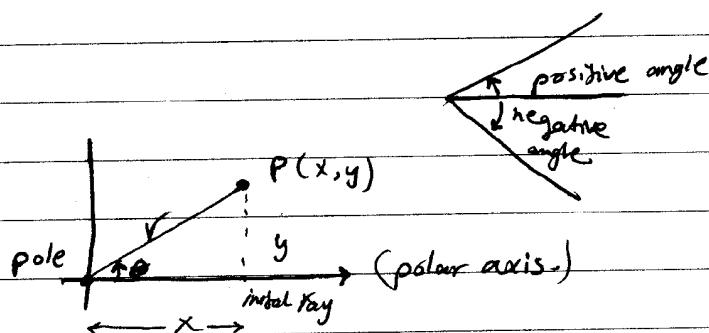
$$y = -\frac{5}{4}x + \frac{7}{4}$$

$$x=t \Rightarrow y = -\frac{5}{4}t + \frac{7}{4}$$

11.3 Polar Coordinates:

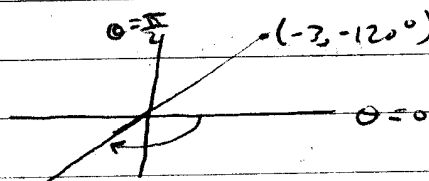
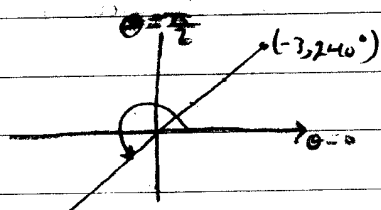
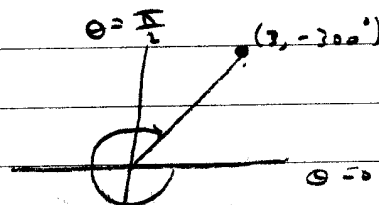
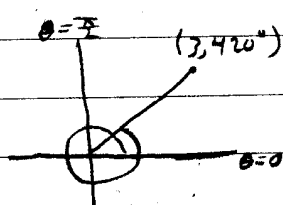
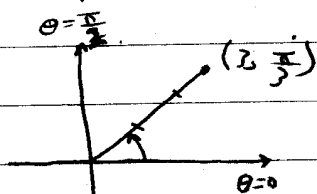
Cartesian

$(x, y) \rightarrow (r, \theta)$



* The representation of a point in polar coordinates is not unique.

Example/ Sketch the point $(3, \frac{\pi}{3})$ in polar coordinates.



In general, any point (r, θ) is equivalent to $(r, \theta + 2n\pi)$, $n = 0, 1, 2, \dots$

Remark | If r is negative we can write it positive by adding (π) to the angle θ , that is, $(-r, \theta)$ is equivalent to $(r, \theta + \pi)$

Remark | [1] The graph of the curve $r = a$ is a circle with center at the origin and radius $|a|$.

[2] The graph of $\theta = \theta_0$ is a line through the origin making an angle θ_0 with the polar axis.

Example/ Is the curve $r = 2 \sin 2\theta$ passes through the point $(2, \frac{3\pi}{4})$?
 $r = 2 \sin 2(\frac{3\pi}{4})$

$$= 2 \sin \frac{3\pi}{2} = \underline{\underline{-2}}$$

But $(2, \frac{3\pi}{4})$ is equivalent to $(-2, \frac{3\pi}{4} + \pi) = (-2, \frac{7\pi}{4})$

$$r = 2 \sin 2(\frac{7\pi}{4}) = 2 \sin \frac{7\pi}{2} = 2 \sin(2\pi + \frac{3\pi}{2})$$

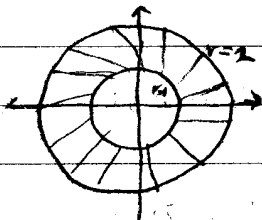
$$= 2 \sin(\frac{3\pi}{2}) = \underline{\underline{-2}}$$

So $(-2, \frac{7\pi}{4})$ lie on the curve $r = 2 \sin 2\theta \Rightarrow$

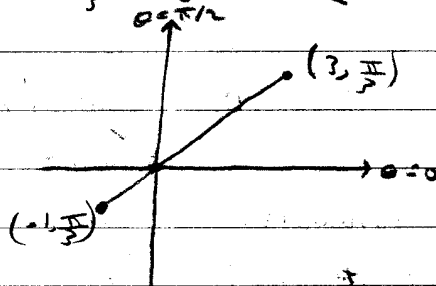
The equivalent point $(2, \frac{3\pi}{4})$ lie on the curve.

Example/ Graph the set of points whose polar coordinates satisfy the following eqns and inequalities.

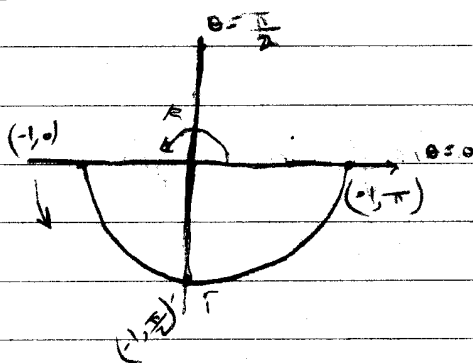
1) $1 \leq r \leq 2$



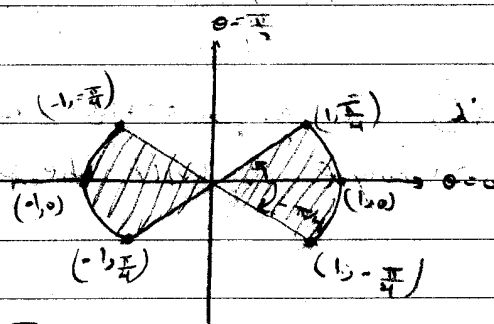
2) $\theta = \frac{\pi}{3}, -1 \leq r \leq 3$



3) $0 \leq \theta \leq \pi, r = -1$



4) $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, -1 \leq r \leq 1$

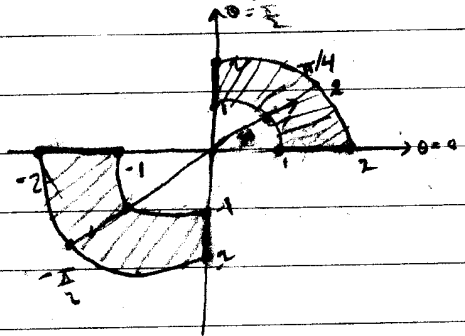


$\theta = -\frac{\pi}{4}$
 $-1 \leq r \leq 1$

5) $0 \leq \theta \leq \frac{\pi}{2}$, $1 \leq r \leq 2$

$r > 1 \Rightarrow r > 1$ or $r < -1$

$r \leq 2 \Rightarrow -2 \leq r \leq 2$



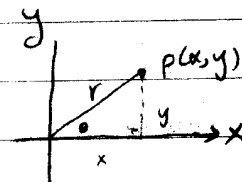
$r > 1, r < -1$ $0 \leq \theta \leq \frac{\pi}{2}$
 $-2 \leq r \leq 2$

* The point $(0, \theta_0)$ is the origin (pole).

The relation between polar and Cartesian Coordinates

$y = r \sin \theta$
 $x = r \cos \theta$

$\Rightarrow x^2 + y^2 = r^2$



$\Rightarrow \frac{y}{x} = \tan \theta$

Example / [1] Find the cartesian coordinates of the point $(-\sqrt{2}, \frac{3\pi}{4})$

$x = r \cos \theta = -\sqrt{2} \cos \frac{3\pi}{4} = -1$

$y = r \sin \theta = -\sqrt{2} \sin \frac{3\pi}{4} = -1$

The point $(-\sqrt{2}, \frac{3\pi}{4})$ in polar coordinates is equivalent to $(-1, -1)$ in cartesian coordinates.

[2] Find all possible polar coordinates of the point $(4\sqrt{3}, 4)$

$r = \sqrt{x^2 + y^2} = \sqrt{(4\sqrt{3})^2 + 4^2} = 8$

$\tan \theta = \frac{y}{x} \Rightarrow \theta = \pi/6$

The possible polar coordinates of the point are

$(8, \frac{\pi}{6} \pm 2n\pi)$, $n = 0, 1, 2, \dots$

and

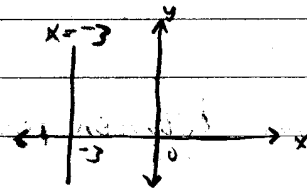
$(-8, \frac{\pi}{6} + \pi \pm 2n\pi) = (-8, \frac{7\pi}{6} \pm 2n\pi)$, $n = 0, 1, 2, \dots$

Ex 3) Replace the following polar eqns by the equivalent Cartesian eqns and describe the graph?

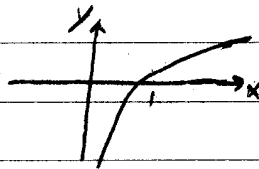
[a] $r = -3 \sec \theta$ [b] $r \sin \theta = \ln r + \ln \cos \theta$ [c] $r = 4 \tan \theta \sec \theta$
 [d] $r^2 = -6r \sin \theta$ [e] $r^2 \sin \theta = 2$ [f] $r \sin(\theta + \frac{\pi}{8}) = 2$

[a] $r = -3 \sec \theta \Rightarrow r = \frac{-3}{\cos \theta} \Rightarrow r \cos \theta = -3$
 $\Rightarrow x = -3$

line to the left of the y-axis.



[b] $r \sin \theta = \ln r \cos \theta$
 $y = \ln x$
 (natural logarithm)

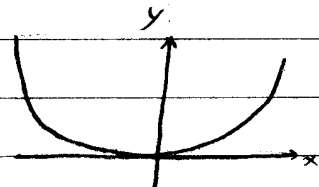


[c] $r = 4 \tan \theta \sec \theta = 4 \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta}$

$\Rightarrow r \cos^2 \theta = 4 \sin \theta \Rightarrow r^2 \cos^2 \theta = 4 r \sin \theta$

$x^2 = 4y$

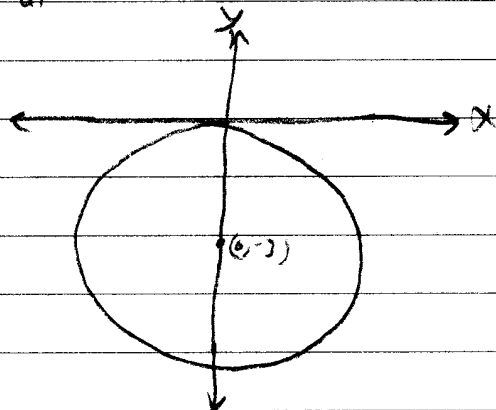
$y = \frac{1}{4}x^2$



[d] $r^2 = -6r \sin \theta \Rightarrow x^2 + y^2 = -6y$

$x^2 + y^2 + 6y = 0 \Rightarrow x^2 + y^2 + 6y + 9 = 9$

$x^2 + (y+3)^2 = 9$ circle with center at the point $(0, -3)$ and radius 3.



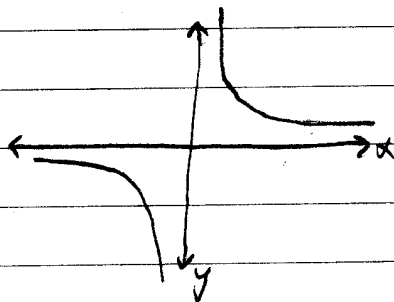
$$\textcircled{e} \quad r^2 \sin 2\theta = 2$$

$$r^2 (2 \sin \theta \cos \theta) = 2$$

$$r \sin \theta \cdot r \cos \theta = 1$$

$$y \cdot x = 1 \Rightarrow y = \frac{1}{x}$$

eqn of hyperbola



$$\textcircled{f} \quad r \sin \left(\theta + \frac{\pi}{2} \right) = 2$$

$$r \left(\sin \theta \cos \frac{\pi}{2} + \cos \theta \sin \frac{\pi}{2} \right) = 2$$

$$r \sin \theta \cdot \frac{\sqrt{2}}{2} + r \cos \theta \cdot \frac{1}{2} = 2$$

$$\sqrt{3} y + x = 4$$

eqn of a line

4) Replace the following cartesian eqns by it equivalent polar eqn

$$\textcircled{a} \quad y^2 = 4x \quad \textcircled{b} \quad (x-5)^2 + y^2 = 25$$

$$\textcircled{c} \quad 4xy = 9 \quad \textcircled{d} \quad x^2 - y^2 = 4$$

$$\textcircled{a} \quad r^2 \sin^2 \theta = 4r \cos \theta \Rightarrow r^2 \sin^2 \theta - 4r \cos \theta = 0$$

$$r(r \sin^2 \theta - 4 \cos \theta) = 0 \Rightarrow \underline{r=0}, \quad r \sin^2 \theta = 4 \cos \theta$$

$$r = \frac{4 \cos \theta}{\sin^2 \theta} = 4 \csc \theta \cot \theta$$

$$\boxed{r=0, r=4 \csc \theta \cot \theta}$$

$$\textcircled{b} \quad x^2 - 10x + 25 + y^2 = 25$$

$$(x^2 + y^2) - 10x = 0$$

$$r^2 - 10r \cos \theta = 0$$

$$r(r - 10 \cos \theta) = 0$$

$$\underline{r=0}, \quad \underline{r=10 \cos \theta}$$

Ⓔ $4xy = 9$

$4r \cos \theta \cdot r \sin \theta = 9 \Rightarrow 4r^2 \sin \theta \cos \theta = 9$

$2r^2 (\sin 2\theta) = 9 \Rightarrow r^2 = \frac{9}{2 \sin 2\theta} = \frac{9}{2} \csc 2\theta$

$r = \pm \frac{3}{\sqrt{2}} \sqrt{\csc 2\theta}$

Ⓕ $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 4$

$r^2 (\cos^2 \theta - \sin^2 \theta) = 4 \Rightarrow r^2 \cos 2\theta = 4$

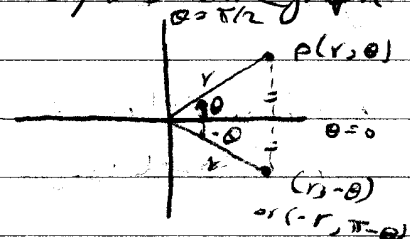
$r = \frac{\pm 2}{\sqrt{\cos 2\theta}} = \pm 2 \sqrt{\sec 2\theta}$

Ex (11.3)
1-67 odd.

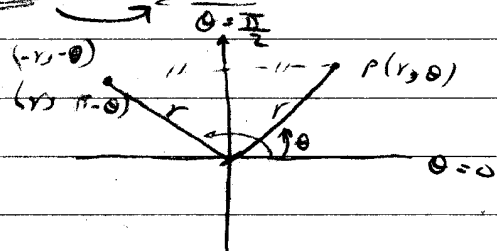
11.4 Graphing in polar Coordinates

(I) Symmetry /

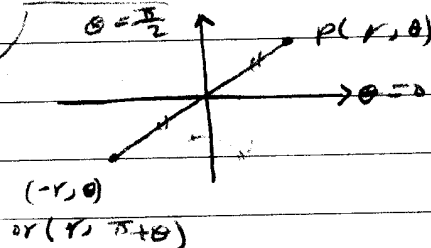
Ⓐ Symmetry about the x-axis: means that if the point (r, θ) lie on the graph then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.



Ⓑ Symmetry about the y-axis: means that if the point (r, θ) lie on the graph then the point $(-r, -\theta)$ or $(r, \pi - \theta)$ lies on the graph.



[C] Symmetry about the origin: means that, If the point (r, θ) lie on the graph then the point $(-r, \theta)$ or $(r, \pi + \theta)$ lies on the graph.



[2] Slope: If $r = f(\theta)$ and $x = r \cos \theta = f(\theta) \cos \theta$.
 $y = r \sin \theta = f(\theta) \sin \theta$.

and if f is diff. fn of θ and $\frac{dx}{d\theta} \neq 0$, then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \text{slope at any points}$$

Example/ Find the slope of the curve $r = 1 - \cos \theta$ at the point $\theta = -\frac{\pi}{2}$.

$$\frac{dr}{d\theta} = \sin \theta \quad \text{slope} = \frac{\sin \theta \sin \theta + (1 - \cos \theta) \cos \theta}{\sin \theta \cos \theta - (1 - \cos \theta) \sin \theta}$$

$$\text{slope} \Big|_{\theta = -\frac{\pi}{2}} = \frac{\sin^2(-\frac{\pi}{2}) + (1 - \cos(-\frac{\pi}{2})) \cos(-\frac{\pi}{2})}{\frac{1}{2} \sin \cdot \pi - (1 - \cos(-\frac{\pi}{2})) \sin(-\frac{\pi}{2})} = \frac{(-1)^2 + (1-0) \cdot 0}{0 - (1-0)(-1)} = [1]$$

[2] Sketch the graph of $r = 2 + 4 \cos \theta$ (limaçon with inner loop).

* Symmetry about x-axis:

$$2 + 4 \cos(-\theta) = 2 + 4 \cos \theta = r \Rightarrow \text{The graph is symmetric about x-axis.}$$

* Symmetry about y-axis:

$$2 + 4 \cos(\pi - \theta) = 2 + 4(\cos \pi \cos \theta + \sin \pi \sin \theta) = 2 - 4 \cos \theta \neq r$$

But $2 + 4 \cos(-\theta) = 2 + 4 \cos \theta = r \neq -r \Rightarrow$ no symmetry about the y-axis.

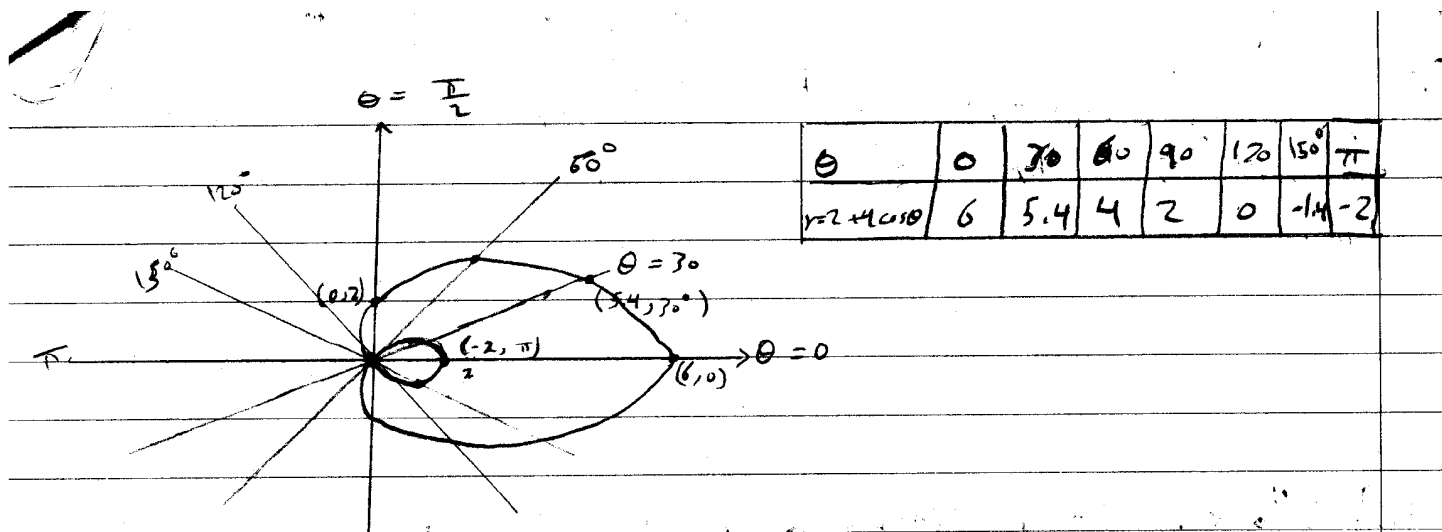
* Symmetry about the origin:

$$2 + 4 \cos(\pi + \theta) = 2 + 4(\cos \pi \cos \theta - \sin \pi \sin \theta) = 2 - 4 \cos \theta \neq r$$

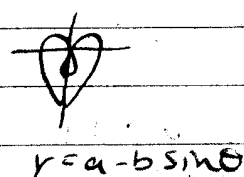
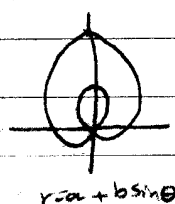
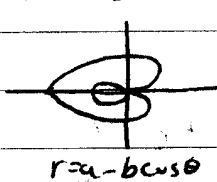
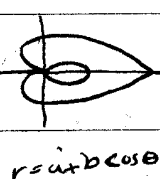
But $-r = -2 - 4 \cos \theta \neq r$.

$\therefore$ symmetric about the origin.

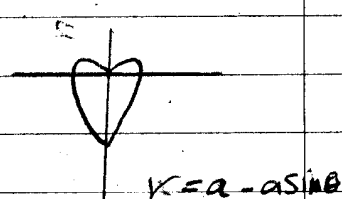
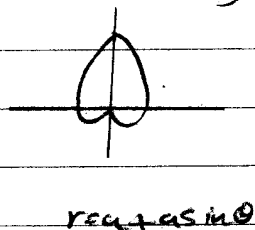
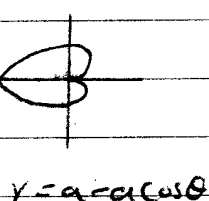
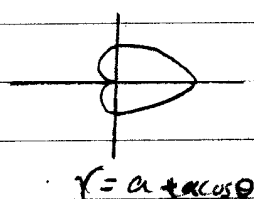
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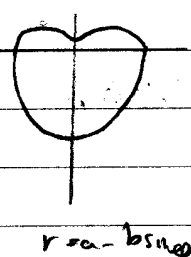
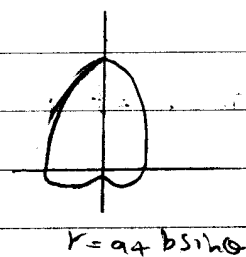
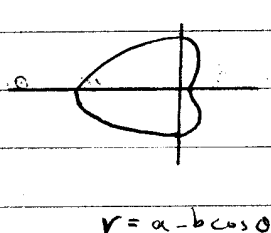
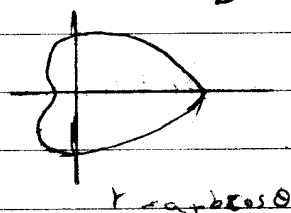
[1] $r = a + b \cos \theta$, with $\frac{a}{b} < 1$



[2] If $a = b \Rightarrow r = a + a \cos \theta$ (cardioid)

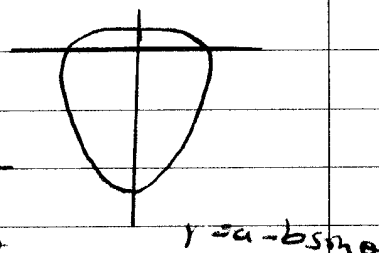
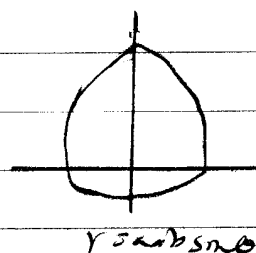
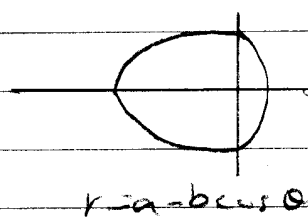
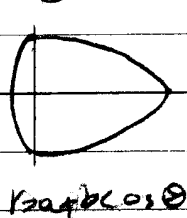


[3] If $1 < \frac{a}{b} < 2$, $r = a + b \cos \theta$



"Limaçon with a dent"

[4] $\frac{a}{b} \geq 2$ (convex limaçon).



3) Sketch the graph of the curves. $\begin{cases} \text{(a)} r^2 = 4 \sin 2\theta \\ \text{(b)} r^2 = 4 \cos 2\theta \end{cases}$

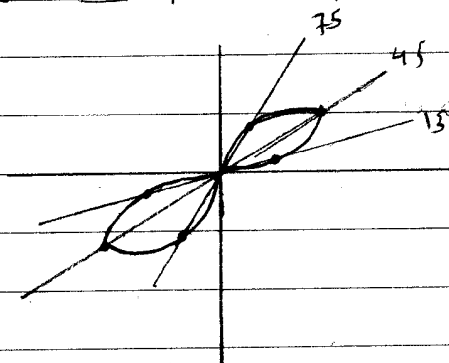
(a) X-axis: $4 \sin 2(0) = 4 \sin(-2\theta) = -4 \sin 2\theta = -r^2 \neq r^2$
 But $4 \sin 2(\pi - \theta) = 4 \sin(2\pi - 2\theta) = 4 \sin(-2\theta) = -4 \sin 2\theta = -r^2 \neq (r)^2$

no symmetry about the x-axis

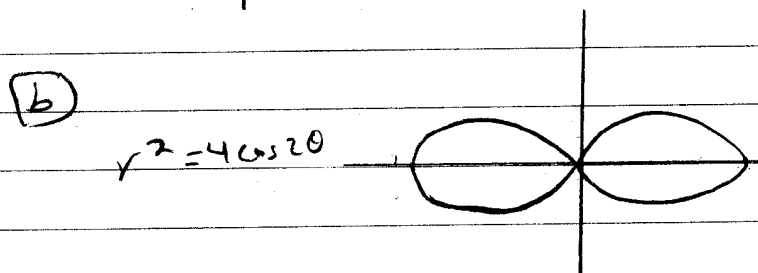
(y) $4 \sin 2(\pi - \theta) = 4 \sin(2\pi - 2\theta) = 4 \sin(-2\theta) = -r^2 \neq r^2$
 But $4 \sin 2(-\theta) = -4 \sin 2\theta = -r^2 \neq (-r)^2$
 no symmetry about the y-axis.

origin $4 \sin 2(\pi + \theta) = 4 \sin(2\pi + 2\theta) = 4 \sin 2\theta = r^2$
 The graph is symmetric about the origin.

θ	0	15°	45°	75°	90°
$r = \pm 2\sqrt{\sin 2\theta}$	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0



(a)



4) Sketch the graph of the following curves:-

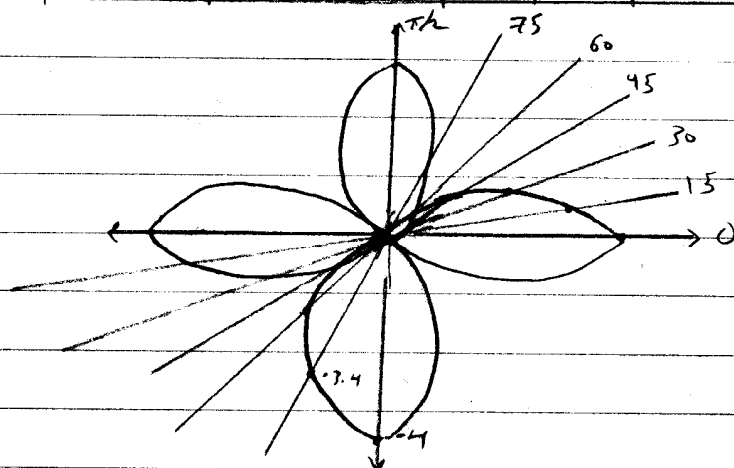
(a) $r = 4 \cos 2\theta$ (b) $r = 4 \sin 2\theta$ (c) $r = \sin 3\theta$ (d) $r = \cos 3\theta$.

(a) $4 \cos 2(-\theta) = 4 \cos(-2\theta) = 4 \cos 2\theta = r$
Symmetric about the x-axis

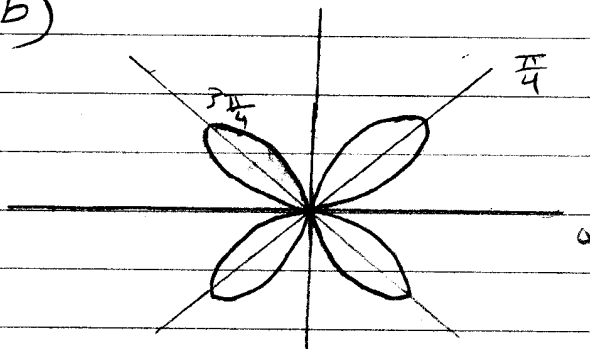
(b) $4 \cos 2(\pi - \theta) = 4 \cos 2\theta = r$
Symmetric about y-axis

origin Symmetric about the origin because two symmetries give the other symmetry.

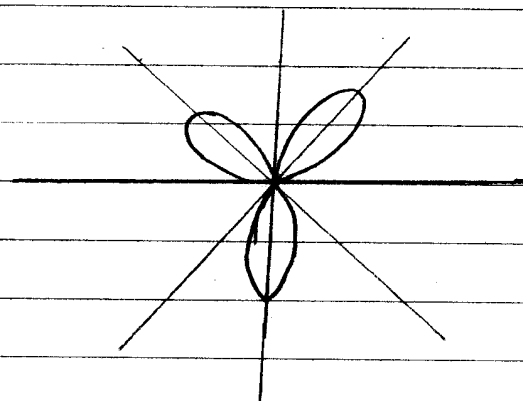
θ	0	15°	30°	45°	60°	75°	90°
$r = 4 \cos 2\theta$	4	3.4	2	0	-2	-3.4	-4



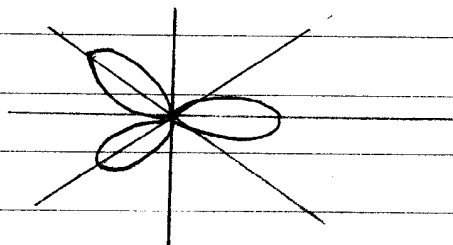
(b)



(c)



(d)

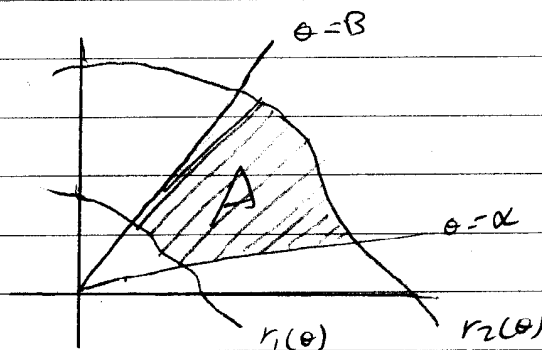
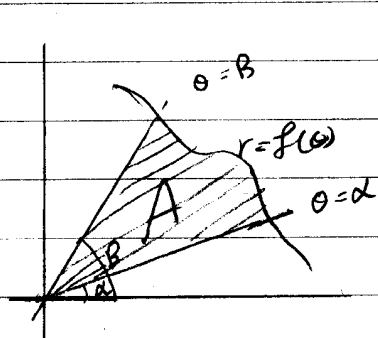


11.5 / Areas and lengths in polar coordinates

(I) Area in the plane

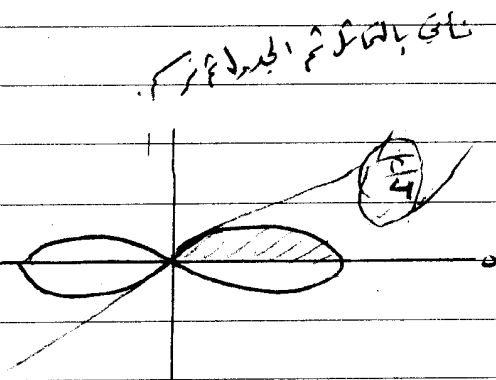
Def. (I) The area of the region between the origin and the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ is $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

(2) If $0 \leq r_1(\theta) \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$ then the area of the region A is $A = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta$



Example: Find the area of the region inside the lemniscate $r^2 = 2a^2 \cos 2\theta$, $a > 0$.

$$\begin{aligned}
 A &= 4 \int_0^{\pi/4} \frac{1}{2} r^2 d\theta \\
 &= 2 \int_0^{\pi/4} (2a^2 \cos 2\theta) d\theta \\
 &= 4a^2 \left(\frac{\sin 2\theta}{2} \right) \Big|_0^{\pi/4} \\
 &= 2a^2 (1 - 0) = \boxed{2a^2}
 \end{aligned}$$



11.7 Conics in polar Coordinates-

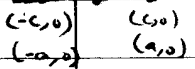
II Eccentricity - الإختلاف المركز

* For an ellipse, we have

(a) If $c = 0$ then ~~the~~ $F_1 \equiv F_2$ (foci) $\equiv$ center $\rightarrow$ The ellipse becomes a circle.



(b) If $c = a$ then the ellipse becomes a line segment.



Def. For the ellipse and hyperbola, the eccentricity e is defined as $e = \frac{\text{distance between foci}}{\text{distance between vertices}} \rightarrow e = \frac{2c}{2a} = \frac{c}{a}$

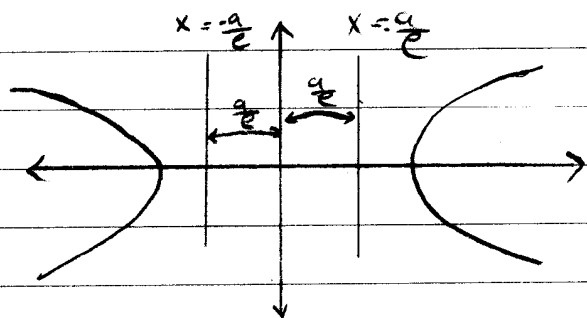
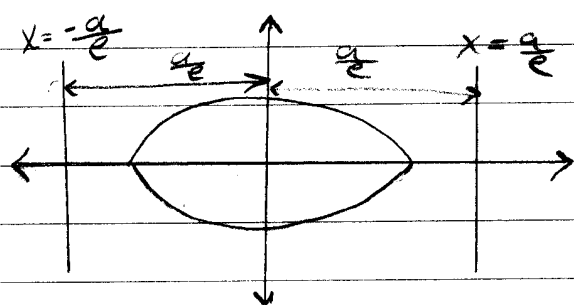
Remark:-

1] The Conic section is parabola if $e = 1$.

2] $=$ is an ellipse if $0 < e < 1$ because $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} < 1$.

3] $=$ is hyperbola if $e > 1$ because $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} > 1$.

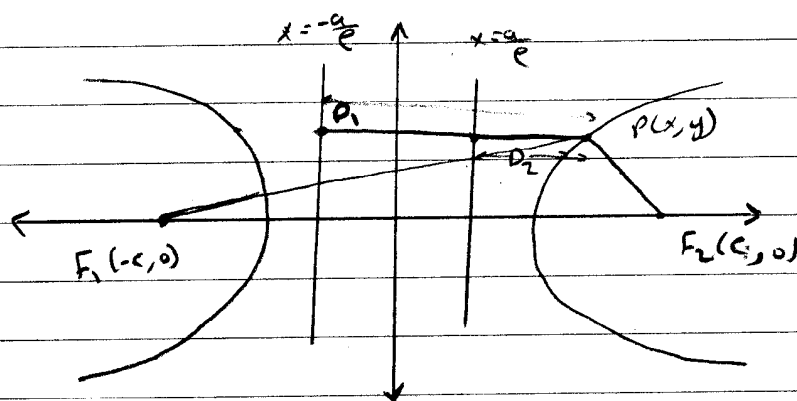
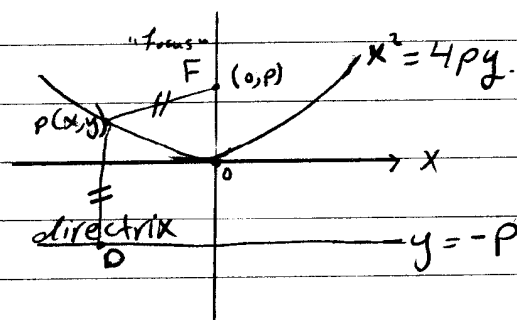
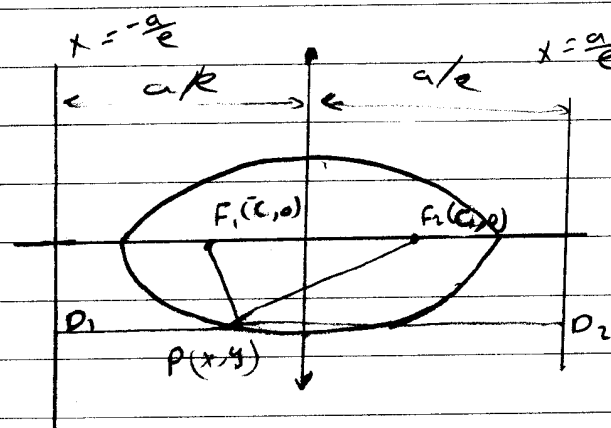
Def. The directrix of an ellipse (hyperbola) is the line perpendicular to the focal axis and at distance $\pm \frac{a}{e}$ from the center.



2] Focus directrix eqn:

1] The focus directrix of a parabola $PF = 1 \cdot PD$

2] $\frac{PF}{PD} = e$ of an ellipse and hyperbola are $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.



Example 1]

Find the standard form eqn of the conic section with $e=3$ and foci $(0, \pm 3)$. Also, find its vertices and directrix

$e=3 > 1 \Rightarrow$ The conic section is hyperbola.

$$c=3, \quad a = \frac{c}{e} = \frac{3}{3} = 1$$

$$b = \sqrt{c^2 - a^2} = \sqrt{9 - 1} = \sqrt{8}$$

the eqn is $\boxed{\frac{y^2}{1} - \frac{x^2}{8} = 1}$

* The vertices are $(0, \pm 1)$

* The directrix are $y = \pm \frac{1}{3}$.

Ex [2]

Find the eccentricity, vertices and the standard form eqn of an ellipse with center at the origin, focus (4, 0) and the corresponding directrix $X = \frac{16}{3}$.

$$c = 4, \quad x = \frac{a}{e} = \frac{16}{3} \Rightarrow \underline{3a = 16e}$$

$$e = \frac{c}{a} = \frac{4}{a} \Rightarrow \underline{e = \frac{4}{a}}$$

$$\text{So } 3a = 16 \left(\frac{4}{a} \right) \Rightarrow \underline{a = \frac{8}{\sqrt{3}}}$$

$$b = \sqrt{a^2 - c^2} = \sqrt{\frac{64}{3} - 16} = \underline{\frac{4}{\sqrt{3}}}$$

$$\text{The eqn: } \frac{x^2}{\left(\frac{8}{\sqrt{3}}\right)^2} + \frac{y^2}{\left(\frac{4}{\sqrt{3}}\right)^2} = 1 \Rightarrow \boxed{\frac{3x^2}{64} + \frac{3y^2}{16} = 1}$$

The vertices are $\left(\pm \frac{8}{\sqrt{3}}, 0\right)$

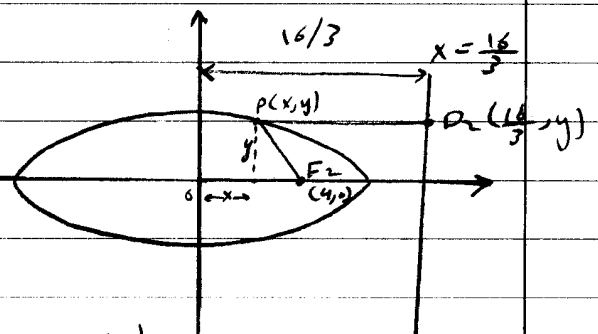
$$e = \frac{4}{\frac{8}{\sqrt{3}}} = \frac{4\sqrt{3}}{8} = \underline{\frac{\sqrt{3}}{2}}$$

* Another soln.

$$PF_2 = e \cdot PD_2$$

$$\sqrt{(x-4)^2 + (y-0)^2} = \frac{\sqrt{3}}{2} \sqrt{\left(x - \frac{16}{3}\right)^2 + 0}$$

$$x^2 - 8x + 16 + y^2 = \frac{3}{4} \left(x^2 - \frac{32}{3}x + \frac{256}{9} \right)$$



$$\frac{3x^2}{64} + \frac{3y^2}{16} = 1$$

شوية تجميل

[3] Find the eccentricity, vertices and the standard form eqn of the hyperbola with center at the origin, one focus $(-6, 0)$ and the corresponding directrix $x = -2$.

$$c = 6, \quad x = -\frac{a}{e} = -2 \Rightarrow \underline{a = 2e}$$

$$e = \frac{c}{a} = \frac{6}{2e} \Rightarrow \underline{e = \sqrt{3}}$$

$$\underline{a = 2\sqrt{3}} \Rightarrow \text{The vertices are } (\pm 2\sqrt{3}, 0).$$

$$b = \sqrt{c^2 - a^2} = \underline{\sqrt{24}}$$

$$\text{the eqn: } \boxed{\frac{x^2}{12} - \frac{y^2}{24} = 1}$$

[3] The Polar eqns for conic sections:-

If we place one focus at the origin and the corresponding directrix $x = k$ is to the right of the origin, we have

$$PF = e PD$$

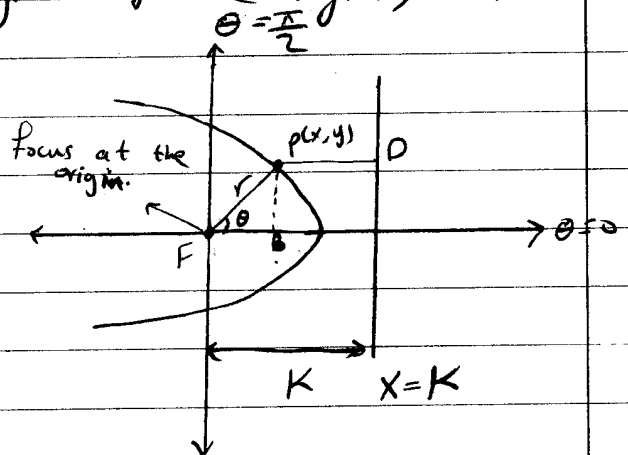
$$r = e(k - FB)$$

$$r = e(k - r \cos \theta)$$

$$= ke - re \cos \theta$$

$$r(1 + e \cos \theta) = ke$$

$$\boxed{r = \frac{ke}{1 + e \cos \theta}}$$



If the directrix is to the left of the origin:-

$$\boxed{r = \frac{ke}{1 - e \cos \theta}}$$

If the directrix up the origin.

$$r = \frac{ke}{1 + e \sin \theta}$$

If the directrix down the origin.

$$r = \frac{ke}{1 - e \sin \theta}$$

Example 1 /

Find the polar eqn for the following conic sections with one focus at the origin and.

a) $e = 1$, vertex $(5, \pi/2)$

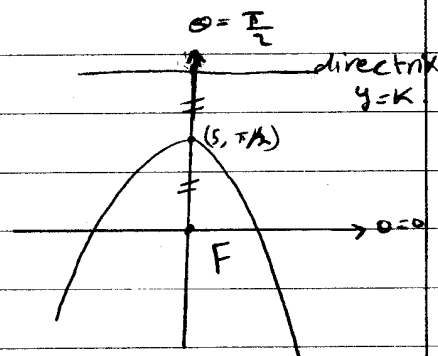
b) $e = 2$, directrix $r = 4 \sec \theta$

c) $e = \frac{1}{5}$, $y = -10$.

a) $e = 1 \Rightarrow$ the conic section is parabola.

$$k = 5 + 5 = 10$$

$$r = \frac{ke}{1 + e \sin \theta} = \frac{10}{1 + \sin \theta}$$



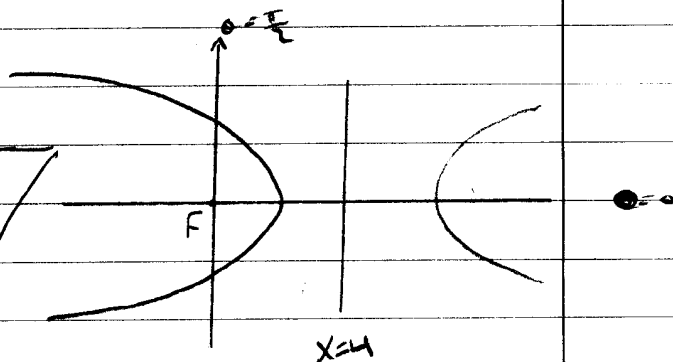
b) The directrix $r = 4 \sec \theta \Rightarrow r = \frac{4}{\cos \theta} \Rightarrow r \cos \theta = 4$

$$\Rightarrow \underline{X = 4}$$

$e = 2 > 1 \Rightarrow$ The conic section is hyperbola.

$$k = 4,$$

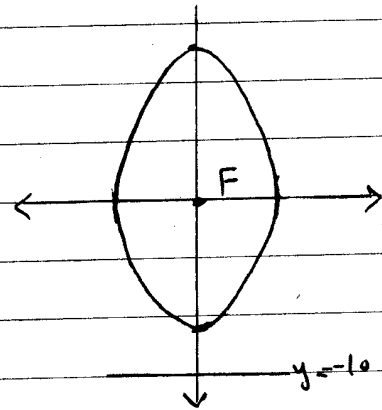
$$r = \frac{4(2)}{1 + (2) \cos \theta} \Rightarrow r = \frac{8}{1 + 2 \cos \theta}$$



Ex 1] $e = \frac{1}{5} < 1 \Rightarrow$ The conic section is ellipse.

$$K = 10,$$

$$r = \frac{10(\frac{1}{5})}{1 - \frac{1}{5}\sin\theta} \Rightarrow r = \frac{10}{5 - \sin\theta}$$



Ex 2] Sketch and classify the following conic sections. Label the vertices, centers (as well) and sketch the directrix that corresponds to the focus at the origin. Find the other focus if it exists.

[a] $r = \frac{8}{2 - 2\sin\theta}$

[b] $r = \frac{30}{10 - 5\cos\theta}$

[c] $r = \frac{4}{2 - \sin\theta}$

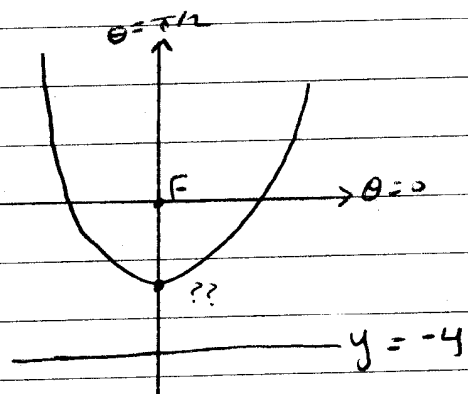
[a] $r = \frac{4}{1 - \sin\theta} \Rightarrow e = 1$, the conic section is parabola.

$$Ke = 4 \Rightarrow K = 4$$

* To get the vertex put $\theta = -\frac{\pi}{2}$

$$r = \frac{4}{1 - \sin(-\frac{\pi}{2})} = 2$$

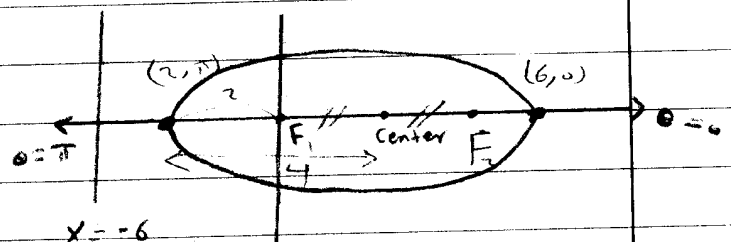
The vertex is $(2, -\frac{\pi}{2})$



[b] $r = \frac{3}{1 - \frac{1}{2}\cos\theta} \Rightarrow e = \frac{1}{2}$, the conic section is ellipse.

$$Ke = 3 \Rightarrow K = 6$$

The directrix is $x = -6$



To get the vertices, put $\theta = 0, \theta = \pi$

$$\text{If } \theta = 0 \Rightarrow r = \frac{3}{1 - \frac{1}{2}} = 6.$$

$$\text{If } \theta = \pi \Rightarrow r = \frac{3}{1 + \frac{1}{2}} = 2.$$

The Vertices are $(6, 0)$ and $(2, \pi)$

$$2a = 6 + 2 \Rightarrow a = 4$$

The center is $(2, 0)$

The other focus is $(4, 0)$

$$\boxed{c} \quad r = \frac{2}{1 - \frac{1}{2} \sin \theta} \Rightarrow e = \frac{1}{2} < 1 \Rightarrow \text{The Conic section is ellipse}$$

$$ke = 2 \Rightarrow \underline{k = 4}$$

The directrix is $y = -4$

To get the vertices put $\theta = -\frac{\pi}{2}, \theta = \frac{\pi}{2}$

$$\text{If } \theta = -\frac{\pi}{2} \Rightarrow r = \frac{2}{1 - \frac{1}{2} \sin(-\frac{\pi}{2})} = \frac{4}{3}$$

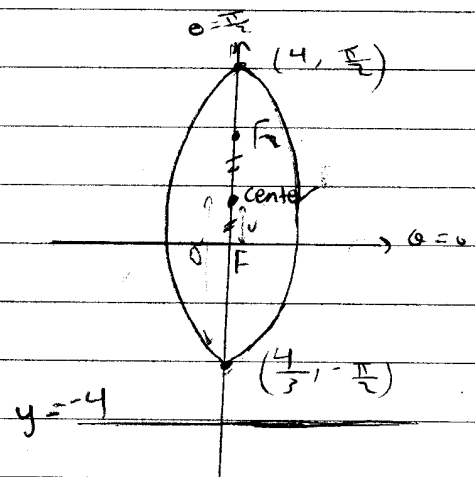
$$\text{If } \theta = \frac{\pi}{2} \Rightarrow r = 4$$

The vertices are $(\frac{4}{3}, -\frac{\pi}{2})$ and $(4, \frac{\pi}{2})$

$$2a = \frac{4}{3} + 4 = \frac{16}{3} \Rightarrow a = \frac{8}{3}$$

$$c = \frac{8}{3} - \frac{4}{3} = \frac{4}{3} \Rightarrow \text{The center is } (\frac{4}{3}, \frac{\pi}{2})$$

The other focus is $(\frac{8}{3}, \frac{\pi}{2})$

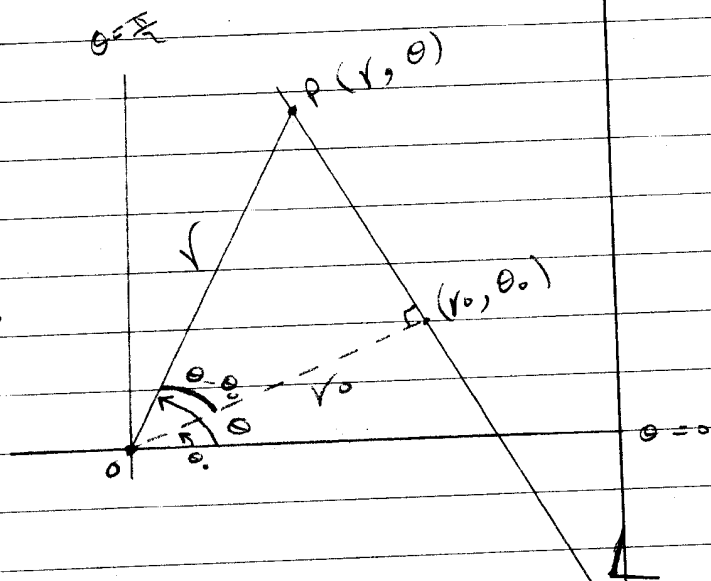


H] Lines:-

$$\cos(\theta - \theta_0) = \frac{r_0}{r}$$

$$\boxed{r \cos(\theta - \theta_0) = r_0}$$

the eqn of a line L in polar coordinates.



Example / Sketch the line $r \cos(\theta - \frac{2\pi}{3}) = 3$ and find its cartesian eqn.

To get the intersection with x-axis ($\theta = 0$) Put $\theta = 0 \Rightarrow$

$$r \cos(0 - \frac{2\pi}{3}) = 3 \Rightarrow r = \frac{3}{-\frac{1}{2}} = -6 \Rightarrow \text{The point } (-6, 0)$$

The point of intersection with y-axis ($\theta = \frac{\pi}{2}$) is given as follow $r \cos(\frac{\pi}{2} - \frac{2\pi}{3}) = 3 \Rightarrow r \cos(-\frac{\pi}{6}) = 3 \Rightarrow$

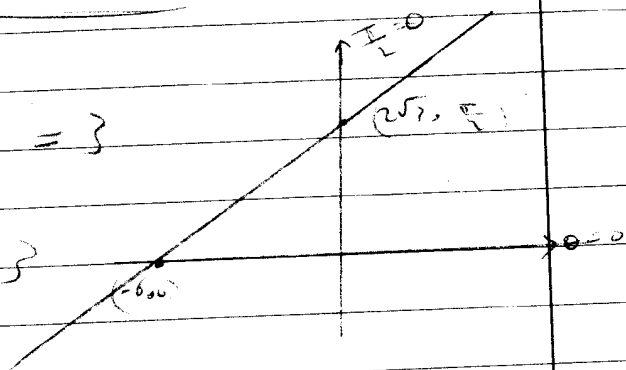
$$r = 2\sqrt{3} \Rightarrow \text{The point is } (2\sqrt{3}, \frac{\pi}{2})$$

$$r \left(\cos \theta \cos \frac{2\pi}{3} + \sin \theta \sin \frac{2\pi}{3} \right) = 3$$

$$r \left(\cos \theta \left(-\frac{1}{2}\right) + \sin \theta \left(\frac{\sqrt{3}}{2}\right) \right) = 3$$

$$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3$$

$$\boxed{\sqrt{3}y - x = 6}$$

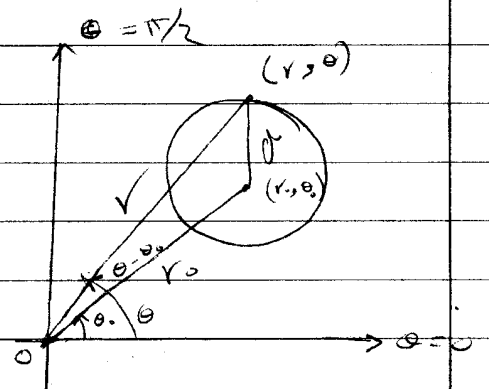


5 Circles :- From the law of cosine, we have.

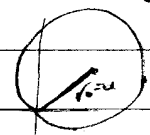
$$\cos(\theta - \theta_0) = \frac{r_0^2 + r^2 - a^2}{2rr_0}$$

$$2rr_0 \cos(\theta - \theta_0) = r_0^2 + r^2 - a^2$$

$$r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) = a^2$$



* If the circle passes through the origin, then $r_0 = a \Rightarrow$
 $r^2 + a^2 - 2ra \cos(\theta - \theta_0) = a^2$
 $\Rightarrow r = 2a \cos(\theta - \theta_0)$



* If the circle passes through origin and its center on the positive x-axis then $\theta_0 = 0 \Rightarrow r = 2a \cos \theta$

* If the center ~~is~~^{on} the negative x-axis then $\theta_0 = \pi \Rightarrow$
 $r = -2a \cos \theta$

* If the circle has a center on the positive y-axis then
 $\theta_0 = \pi/2 \Rightarrow r = 2a \cos(\theta - \pi/2)$
 $= 2a(\cos \theta \cos \pi/2 + \sin \theta \sin \pi/2)$
 $= 2a \sin \theta$

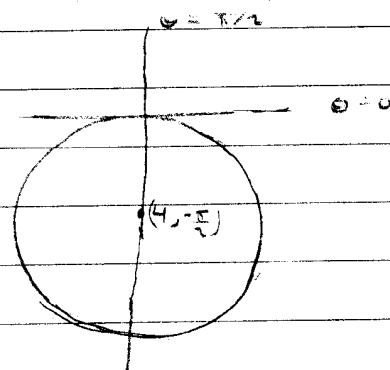
* If the center is on the down y-axis then $\theta_0 = -\pi/2$
 and the eqn $\Rightarrow r = -2a \sin \theta$

Example :-

Sketch the circle $r = -8 \sin \theta$. Find its center in polar coordinates and Find its radius.

$$r = -8 \sin \theta = -2a \sin \theta \Rightarrow \underline{a = 4} = \text{radius of the circle}$$

The center of the circle is $(4, -\frac{\pi}{2})$ or $(4, \frac{3\pi}{2})$



Ex :- 1-74 odd