

Islamic University Of Gaza
 Faculty of Engineering
 Computer Engineering Department
 Time allowed 85 Min.
 Dr. Musbah Shaat
 ECOM 2311- Discrete Mathematics – 12/11/2013.
 Mid-term Exam



Question No.	Mark
Q1	
Q2	
Q3	
Q4	
TOTAL (100)	

.....Solution key.....الاسم:
الرقم الجامعي:.....

Question one (17 marks):

- (a) Given the proposition

P : Ali is tall and Ahmed is thin.

Write down the negation $\neg P$ (5 points).

We can write the sentence as $P: r \wedge s$
 where r = Ali is tall $\wedge$ s = Ahmed is thin.

$$\text{So, } \neg P = \neg (r \wedge s) \equiv \neg r \vee \neg s$$

$\Rightarrow \neg P$: Ali is not tall or Ahmed is not thin
 (or) Ali is short or Ahmed is fat.

- (b) Let $F(x, y)$ be the statement “ x can fool y ”, where the domain consists of all people in the world. Use quantifiers and logical operators to express each of the statements (12 points).

- (i) Everybody can fool Ali.

$$\forall x F(x, \text{Ali})$$

- (ii) Sara can fool everybody.

$$\forall y F(\text{Sara}, y)$$

- (iii) No one can fool both Ahmed and Hasan.

$$\neg \exists x (F(x, \text{Ahmed}) \wedge F(x, \text{Hasan}))$$

- (iv) No one can fool himself.

$$\neg \exists x F(x, x)$$

Question two (37 marks):

- (a) ~~Given~~ Show that $((p \vee q) \wedge \neg p) \rightarrow q$ is a tautology by developing a series of logical equivalences (12 points).

$$\begin{aligned} & ((p \vee q) \wedge \neg p) \rightarrow q \\ \equiv & \neg((p \vee q) \wedge \neg p) \vee q && \text{using the rule } p \rightarrow q \equiv \neg p \vee q \\ \equiv & (\neg(p \vee q) \vee p) \vee q && \text{applying De Morgan's law} \\ \equiv & ((\neg p \wedge \neg q) \vee p) \vee q \\ \equiv & ((\neg p \vee p) \wedge (\neg q \vee p)) \vee q && (p \wedge q) \vee r = (p \vee r) \wedge (q \vee r) \text{ [distributive law]} \\ \equiv & (T \wedge (\neg q \vee p)) \vee q \\ \equiv & (T \vee q) \wedge (\neg q \vee p \vee q) \\ \equiv & (T \vee q) \wedge (p \vee T) \\ \equiv & T \wedge T \\ \equiv & T \end{aligned}$$

- (b) Let $A \times B = \{(1, 1), (2, 2), (3, 1), (1, 2), (1, 4), (2, 1), (2, 4), (3, 4)\}$, find the power set of B, $P(B)$ (5 points).

From $A \times B$, we can know that $B = \{1, 2, 4\}$.

Hence, the power set

$$P(B) = \{\emptyset, \{1, 2, 4\}, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}\}$$

- (c) Give a proof by contraposition to the statement
 "If the square of an integer m is even, then the integer m must be even" (10 points).

The contrapositive $\neg q \rightarrow \neg p$.

$\Rightarrow$ if m is odd $\rightarrow m^2$ is odd

$\Rightarrow m$ is odd $\Rightarrow m = 2k+1$

$$\begin{aligned} m^2 &= (2k+1)^2 = 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2l + 1 \end{aligned}$$

$\Rightarrow m^2$ is odd.

This completes the proof $\#$.

- (d) Is the function $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $f(x) = 3x^2 + 2$,

(i) One to one.

(ii) Onto function.

Justify your answer (10 points).

(i) $3x^2 + 2$ is one to one as every element in $\mathbb{R}^+$ has a unique pre-image in $\mathbb{R}$

(ii) $f(x) = y = 3x^2 + 2$ is not one to one.

$\Rightarrow$ The domain of x is $\mathbb{R}^+ \Rightarrow$

$$\begin{aligned} x &> 0 \\ x^2 &> 0 \\ 3x^2 &> 0 \\ 3x^2 + 2 &> 2 \\ y &> 2 \end{aligned}$$

Note: any counter example will be ok for the proof.

So, there is no pre-image if $y \leq 2$

$\Rightarrow$ not all y the set is the range $\Rightarrow$ not onto.

Question three (18 marks):

Show that the following argument form is valid by deducing conclusion from the premises step by step through the use base rules of inference or laws of logic.

$$\begin{array}{l} (\neg p \wedge r) \rightarrow u \\ \neg s \rightarrow \neg p \\ \neg s \\ r \\ u \rightarrow q \\ \hline \therefore q \end{array}$$

- | | |
|-------------------------------------|---------------------------|
| (1) $\neg s \rightarrow \neg p$ | premise . |
| (2) $\neg s$ | premise |
| (3) $\neg p$ | (1) & (2) by Modus Ponens |
| (4) r | Premise |
| (5) $\neg p \wedge r$ | conjunction (3) & (4) |
| (6) $\neg p \wedge r \rightarrow u$ | Premise |
| (7) u | modus Ponens (5) & (6) |
| (8) $u \rightarrow q$ | premise . |
| (9) q | Modus Ponens (7) & (8) . |

Question four (28 marks):

- (a) Given $f: \mathbb{N} \rightarrow \mathbb{R}$, $f(x) = 2x^4 + x^3 + 3$. Prove from the definition that $f(x) \in O(x^4)$ (7 points).

assume $x > 1$.

$$\Rightarrow \begin{array}{l} 1 < x \\ x^3 < x^4 \end{array}$$

$$\begin{array}{l} 1 < x \\ 1 < x^4 \\ 3 < 3x^4 \end{array}$$

$$\Rightarrow f(x) = 2x^4 + x^3 + 3 < 2x^4 + x^4 + 3x^4 = 6x^4.$$

$$\Rightarrow \boxed{k=1 \text{ and } C=6}$$

$$\Rightarrow f(x) \in O(x^4).$$

- (b) Give the lowest big-O estimate of

$$f: \mathbb{N} \rightarrow \mathbb{R}, \quad f(n) = \frac{n^4 + \log n}{n^2 + 1}$$

[You should detail all the steps arriving to the answer; direct answer will not be accepted] (7 points)

$$f(n) = \frac{n^4 + \log n}{n^2 + 1} < \frac{n^4 + n}{n^2 + 1} < \frac{n^4 + n}{n^2} < \frac{n^4 + n^4}{n^2} < \frac{2n^4}{n^2} = 2n^2.$$

$$\Rightarrow f(n) \text{ is } O(n^2)$$

$$\boxed{\text{where } k=1 \text{ and } C=2.}$$

(c) Give the lowest big-O estimate for the number of additions used in this algorithm (14 points)

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Procedure exam function (n: positive integer)
i:=1                                     and power of two
t:=0
while i ≤ n
    t:= t + i
    i:=2i
{Output: the variable t}
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* Initially, $i=1$

* In the iteration $k=1$, $i=2$.

$$k=2, \quad i=4=2^2$$

$$k=3, \quad i=8=2^3$$

$$\vdots$$
$$k, \quad i=2^k$$

⇒ the loop will terminate when $i=2^k \geq n$.

and as n is a power of two

$$2^k = n \Rightarrow \log 2^k = \log n.$$

$$\boxed{k = \log n}$$

⇒ In each iteration, we have
single addition

⇒ Total No. of additions = $k = \log n$.

⇒ $O(\log n)$ addition.