

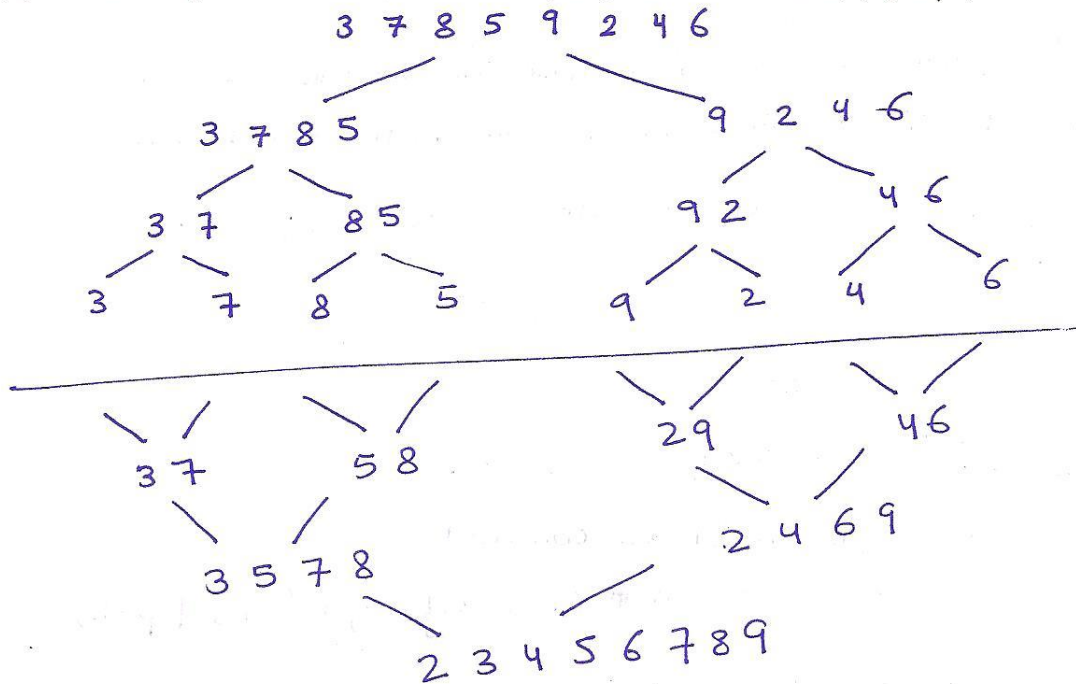


Question No.	Mark
Q1	(17)
Q2	(24)
Q3	(22)
Q4	(12)
TOTAL	(75)

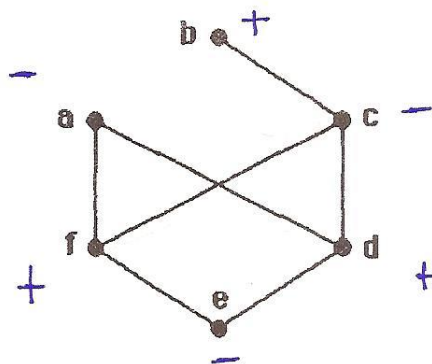
.....Solution key.....الاسم:
الرقم الجامعي:.....

Question One (17 marks):

(a) Use the merge sort to sort 3,7,8,5,9,2,4,6 [Only the tree is needed] (3 points).



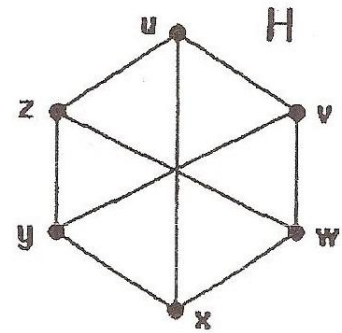
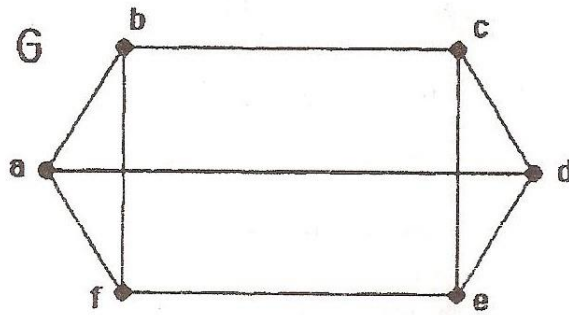
(b) Is the following graph bipartite? Justify your answer (3 points).



The nodes can be classified into two groups $\{b, d, f\}$ and $\{a, c, e\}$ where the nodes in the same group has no edges between them.

⇒ the graph is bipartite.

(c) Decide whether the graphs G and H are isomorphic. Justify your answer (4 points).

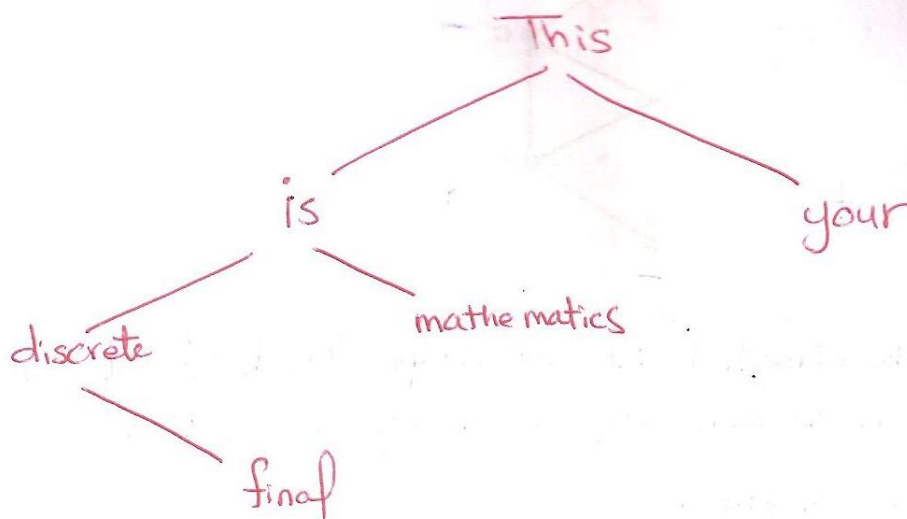


The graphs are not isomorphic as we can construct a cycle with length 3 in G while in H we need min. of 4 to construct any ~~path~~ cycle (closed path).

(d) Any undirected graph is tree if it satisfies two conditions, state them (2 points).

1. the graph should be connected....
2.~ ~ ~ not have any cycle (closed path)

(e) Construct a binary search tree from the words of the sentence This is your discrete mathematics final, using alphabetical order, inserting words in the order they appear in the sentence (5 points).



Question Two (24 marks):

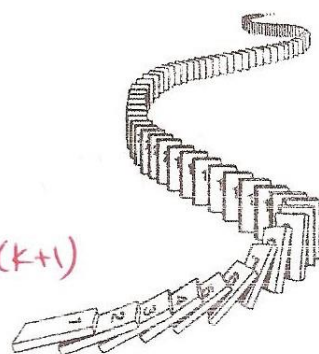
(a) Considering an infinite row of dominoes, illustrate the difference between mathematical induction and strong induction. (2 points).

- Mathematical Induction: $P(k) \rightarrow P(k+1)$

If the k^{th} domino fall, implies that $k+1^{\text{th}}$ domino will fall.

- Strong Induction: $P(1) \wedge P(2) \wedge \dots \wedge P(k) \rightarrow P(k+1)$

If the dominoes from the start to the k^{th} one fall, implies that $k+1^{\text{th}}$ domino will fall.



(b) Using the mathematical induction, prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer (6 points).

Basis step: $n=1$, $P(1) = 4^2 + 5^1 = 16 + 5 = 21$, divides 21
 $\Rightarrow$ True.

Inductive step: we assume that $P(k) = 4^{k+1} + 5^{2k-1}$ divides 21
we need to show that $P(k+1) = 4^{k+2} + 5^{2k+1}$ is also ~ 21

$$\begin{aligned} \Rightarrow P(k+1) &= 4^{k+2} + 5^{2k+1} \\ &= 4^{k+1} \cdot 4 + 5^{2k-1} \cdot 5^2 \\ &= 4^{k+1} \cdot 4 + 5^{2k-1} \cdot (25) \\ &= 4^{k+1} \cdot 4 + 5^{2k-1} \cdot (21+4) \\ &= 4 \left[4^{k+1} + 5^{2k-1} \right] + \underbrace{21 \cdot 5^{2k-1}}_{\text{divides 21}} \\ &\quad \underbrace{4^{k+1} + 5^{2k-1}}_{\text{divides 21 from the hypothesis}} \end{aligned}$$

$\Rightarrow P(k+1)$ divides 21 $\#$
which completes the proof

(c) The sequence of Lucas numbers is defined recursively by

$$\ell_0 = 2, \ell_1 = 1, \text{ and } \ell_n = \ell_{n-1} + \ell_{n-2}, \text{ for } n \geq 2.$$

i. Find ℓ_6 (2 points).

$$\ell_2 = \ell_1 + \ell_0 = 3, \quad \ell_3 = \ell_2 + \ell_1 = 4, \quad \ell_4 = 7$$

$$\ell_5 = 11, \quad \boxed{\ell_6 = 18}.$$

ii. Show by induction that $\ell_0^2 + \ell_1^2 + \dots + \ell_n^2 = \ell_n \ell_{n+1} + 2$ whenever n is a nonnegative integer (6 points).

Basis Step: $P(0) \Rightarrow \ell_0^2 = \ell_0 \ell_1 + 2 = 2 \times 1 + 2 = 4 = 2^2$
 $\Rightarrow$ True.

Inductive Step: assume $P(k)$ is true.

$$\Rightarrow \ell_0^2 + \ell_1^2 + \dots + \ell_n^2 = \ell_n \ell_{n+1} + 2 \text{ is true.}$$

We need to show that

$$\ell_0^2 + \ell_1^2 + \dots + \ell_{k+1}^2 = \ell_{k+1} \ell_{k+2} + 2.$$

Starting from the hypothesis

$$\ell_0^2 + \ell_1^2 + \dots + \ell_n^2 = \ell_n \ell_{n+1} + 2$$

$$\ell_0^2 + \ell_1^2 + \dots + \ell_n^2 + \ell_{n+1}^2 = \ell_n \ell_{n+1} + 2 + \ell_{n+1}^2$$

$$= \ell_{n+1} \left[\ell_n + \ell_{n+1} \right] + 2$$

$\hookrightarrow$ From the def. of any Lucas num. $\ell_{n+2} = \ell_{n+1} + \ell_n$

$$= \ell_{n+1} \ell_{n+2} + 2$$

and this completes the proof. $\#$

iii. Write a recursive algorithm to evaluate any Lucas number (4 points).

Procedure Lucas (n : nonnegative integer)
If $n=0$ then return 2
else if $n=1$ then return 1
else return Lucas($n-1$) + Lucas($n-2$)
{output is Lucas(n)}

* Steps is also o.k.

(d) Give a big- O estimate of $f(x) = (x+2)\log(x^2+1) + \log(x^3+1)$ (4 points).

Let $y_1(x) = x+2$, $y_2(x) = \log(x^2+1)$, $y_3(x) = \log(x^3+1)$

$\Rightarrow$ For $y_1(x)$, $x+2$ is polynomial of degree 1 $\Rightarrow O(x)$.

$\Rightarrow$ For $y_2(x)$: $\log(x^2+1) < \log(x^2+x^2)$ as the log is +ve increasing fn.
 $= \log(2x^2)$

$= \log 2 + \log x^2$

$= \log 2 + 2\log x \Rightarrow O(\log x)$

$\Rightarrow$ For $y_3(x)$, $\log(x^3+1) < \log(x^3+1)$

$= \log(2x^3)$

$= \log 2 + 3\log x \Rightarrow O(\log x)$

$\Rightarrow O(f(x)) = [O(x) \cdot O(\log x)] + O(\log x)$

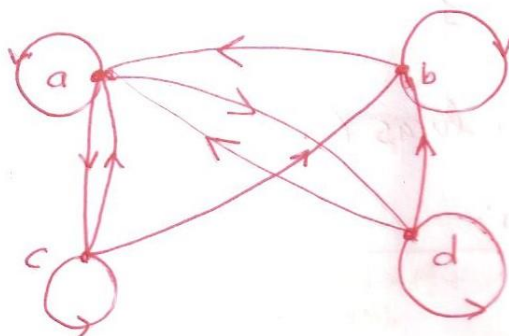
$O(f(x)) = O(x \log x)$

Question Three (22 marks):

(a) Consider a relation R defined by the following the adjacency matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(i) Draw the corresponding directed graph (2 points).



Any other counter example will be o.k

(ii) Is R reflexive? Justify your answer (2 points).

The relation is reflexive as $(a,a), (b,b), (c,c), (d,d) \in R$.

(iii) Is R symmetric? Justify your answer (2 points).

It is not symmetric as $(c,b) \in R$ while $(b,c) \notin R$.

(iv) Is R anti-symmetric? Justify your answer (2 points).

It is not anti-symmetric $(a,c) \in R \nmid (c,a) \in R$ while $a \neq c$.

(v) Is R transitive? Justify your answer (2 points).

It is not transitive, $(b,a) \in R \nmid (a,d) \in R$ while $(b,d) \notin R$.

(vi) Find the symmetric closure on the same set (2 points).

Symmetric closure = $R \cup R^{-1}$ where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

$\Rightarrow$ Element of $R = \{(a,a), (a,c), (b,a), (b,b), (b,d), (c,a), (c,b), (c,c), (d,a), (d,b), (d,d)\}$

$\Rightarrow$ Symmetric closure = $R \cup R^{-1} = R \cup \{(a,b), (a,d), (b,c)\}$

(b) Give a recursive definition of the sequence $\{a_n\}, n = 0, 1, 2, 3, \dots$ if

(i) $a_n = 2^n$ (2 points).

Basis : $a_0 = 2^0 = 1$

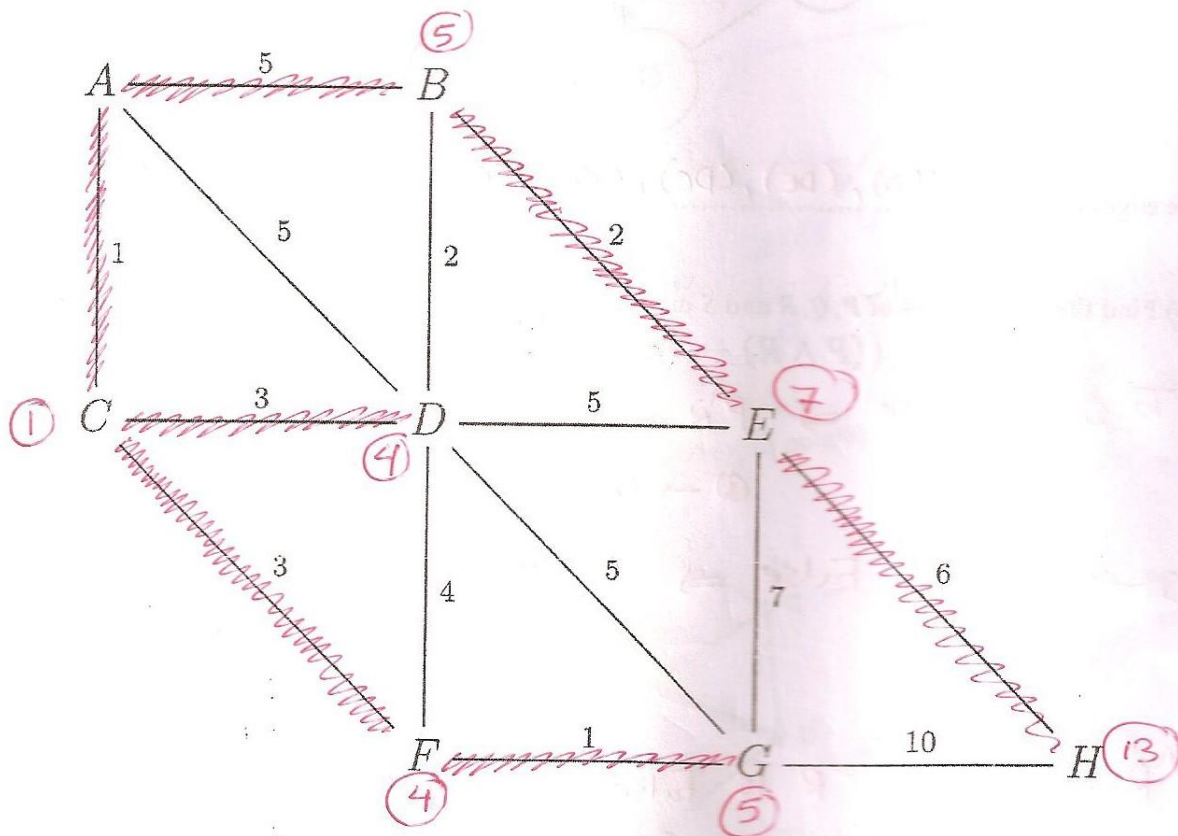
Recursive : $a_{n+1} = 2^{n+1} = 2^n \cdot 2^1 = a_n \cdot 2$

(ii) $a_n = \{1, 0, 1, 0, 1, 0, \dots\}$ (2 points).

Basis : $a_0 = 1, a_1 = 0$

Recursive : $a_n = a_{n-2}, n \geq 2$

(c) Use the shortest path algorithm to find the length of the shortest path from A to H in the following weighted graph (6 points).

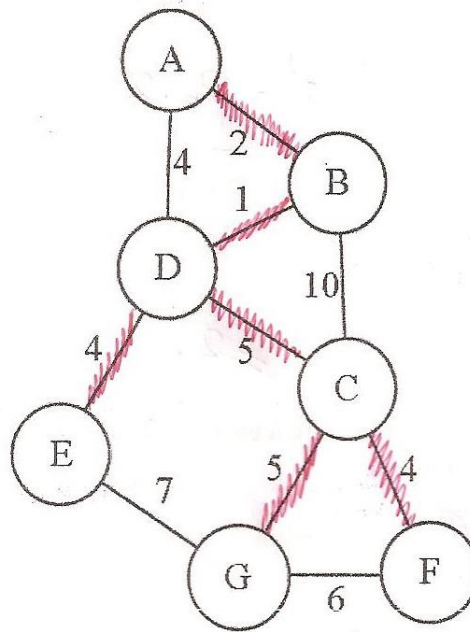


The shortest path is: $A \rightarrow B \rightarrow E \rightarrow H$

The length is: 13

Question Four (12 marks):

- (a) Using Kruskal's algorithm, find the length of the minimum spanning tree on this graph (6 points).



The edges : (A→B), (BD), (DE), (DC), (CG), (CF) The length is: 21

- (b) Find the truth values of P, Q, R and S making the following propositional function FALSE,
 $((P \wedge R) \rightarrow \neg Q) \rightarrow (Q \rightarrow (R \vee S))$ (6 points)

① To get it false $\Rightarrow P \wedge R \rightarrow \neg Q$ should T
 $Q \rightarrow (R \vee S)$ should F

② $Q \rightarrow R \vee S$ is False $\Rightarrow Q$ is True
 $R \vee S$ is False $\Rightarrow R$ is False
 S is False

③ $P \wedge R \rightarrow \neg Q$, R is False from step 2.
 $\Rightarrow P \wedge R$ is False.
 $\Rightarrow$ The sentence is True.

$\Rightarrow$ The overall sentence is False when:
 Q is True, R is False, S is False, P (T or F) don't care