

Lectures on Linear Algebra

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Chapter 1

1.2 Linear Combinations, Matrix-Vector Products, and
Special Matrices

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Chapter 1 MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Definition

A linear combination of vectors $u_1, u_2, \dots, u_k$ is a vector of the form

$$c_1 u_1 + c_2 u_2 + \dots + c_k u_k,$$

where $c_1, c_2, \dots, c_k$ are scalars. These scalars are called the coefficients of the linear combination.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Example

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} = (-3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can also write

$$\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Remark

The set of coefficients that express one vector as a linear combination of the others need not be unique.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

- 1 Determine whether $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
- 2 Determine whether $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
- 3 Determine whether $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Example

Given vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, show that the sum of any two linear combinations of these vectors is also a linear combination of these vectors.

Solution: Suppose that $\mathbf{w}$ and $\mathbf{z}$ are linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Then we may write

$$\mathbf{w} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 \quad \text{and} \quad \mathbf{z} = a'\mathbf{u}_1 + b'\mathbf{u}_2 + c'\mathbf{u}_3$$

where a, b, c, a', b', c' are scalars. So

$$\mathbf{w} + \mathbf{z} = (a + a')\mathbf{u}_1 + (b + b')\mathbf{u}_2 + (c + c')\mathbf{u}_3$$

which is also a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

STANDARD VECTORS IN $\mathcal{R}^2$

We can write any vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathcal{R}^2$ as a linear combination of the two vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as follows:

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

STANDARD VECTORS IN $\mathcal{R}^3$

We can write any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathcal{R}^3$ as a linear combination of the two vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as follows:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

STANDARD VECTORS IN $\mathcal{R}^n$

In general, we define the standard vectors of $\mathcal{R}^n$ by

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Remark

If $\mathbf{u}$ and $\mathbf{v}$ are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Practice Problem

Let $\mathbf{w} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$ and $\mathcal{S} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$

- (a) Without doing any calculations, explain why $\mathbf{w}$ can be written as a linear combination of the vectors in $\mathcal{S}$.
- (b) Express $\mathbf{w}$ as a linear combination of the vectors in $\mathcal{S}$.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

MATRIX-VECTOR PRODUCTS

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{v}$ be an $n \times 1$ vector. We define the matrix-vector product of $\mathbf{A}$ and $\mathbf{v}$, denoted by $\mathbf{Av}$, to be the linear combination of the columns of $\mathbf{A}$ whose coefficients are the corresponding components of $\mathbf{v}$. That is,

$$\mathbf{Av} = v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + \dots + v_n \mathbf{a}_n$$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Then

$$\begin{aligned} A\mathbf{v} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 8 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ 21 \\ 35 \end{bmatrix} + \begin{bmatrix} 16 \\ 32 \\ 48 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix} \end{aligned}$$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Practice Problem

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Compute the

following vectors:

(a) $A\mathbf{v}$

(b) $(A\mathbf{v})^T$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

IDENTITY MATRICES

For each positive integer n , the $n \times n$ identity matrix I_n is the $n \times n$ matrix whose respective columns are the standard vectors $e_1, e_2, \dots, e_n$ in $\mathcal{R}^n$.

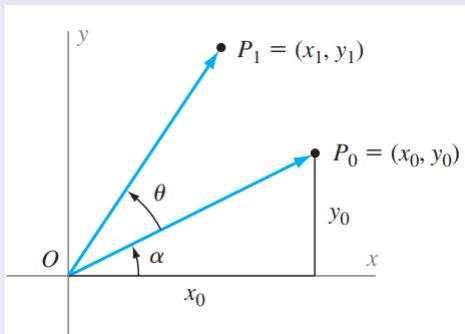
Examples

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

ROTATION MATRICES

Consider a point $P_0 = (x_0, y_0)$ in $\mathcal{R}^n$ with polar coordinates (r, α) , where $r \geq 0$.



Then $(r, \alpha + \theta)$ represents the polar coordinates for P_1 ,

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

$$\begin{aligned}x_1 &= r \cos(\alpha + \theta) \\&= r(\cos \alpha \cos \theta - \sin \alpha \sin \theta) \\&= (r \cos \alpha) \cos \theta - (r \sin \alpha) \sin \theta \\&= x_0 \cos \theta - y_0 \sin \theta\end{aligned}$$

Similarly, $y_1 = x_0 \sin \theta + y_0 \cos \theta$.

We can express these equations as a matrix equation by using a matrix-vector product.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

If we define A_θ by

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

then

$$\begin{aligned} A_\theta \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} x_0 \cos \theta - y_0 \sin \theta \\ x_0 \sin \theta + y_0 \cos \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \end{aligned}$$

We call A_θ the θ -rotation matrix, or more simply, a rotation matrix. For any vector $\mathbf{u}$, the vector $A_\theta \mathbf{u}$ is the vector obtained by rotating $\mathbf{u}$ by an angle θ , where the rotation is counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Example

To rotate the vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ by 30° , we compute $A_{30^\circ} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$; that is

$$\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} \frac{3\sqrt{3}}{2} - \frac{4}{2} \\ \frac{3}{2} + \frac{4\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3\sqrt{3} - 4 \\ 3 + 4\sqrt{3} \end{bmatrix}$$

Remark

The 0° -rotation matrix A_{0° , which leaves a vector unchanged, is given by $A_{0^\circ} = I_2$.

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

PROPERTIES OF MATRIX-VECTOR PRODUCTS

Let A and B be $m \times n$ matrices, and let $\mathbf{u}$ and $\mathbf{v} \in \mathcal{R}^n$. Then

- 1 $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- 2 $A(c\mathbf{u}) = c(A\mathbf{u}) = (cA)\mathbf{u}$ for every scalar c
- 3 $(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$.
- 4 $A\mathbf{e}_j = \mathbf{a}_j$ for $j = 1, 2, \dots, n$, where $\mathbf{e}_j$ is the j th standard vector in $\mathcal{R}^n$
- 5 If B is an $m \times n$ matrix such that $B\mathbf{w} = A\mathbf{w}$ for all $\mathbf{w}$ in $\mathcal{R}^n$, then $B = A$.
- 6 $A\mathbf{0}$ is the $m \times 1$ zero vector.
- 7 If O is the $m \times n$ zero matrix, then $O\mathbf{v}$ is the $m \times 1$ zero vector.
- 8 $I_n\mathbf{v} = \mathbf{v}$

1.2 LINEAR COMBINATIONS, MATRIX VECTOR PRODUCTS, AND SPECIAL MATRICES

Remark

For any $m \times n$ matrix A , any scalars $c_1, c_2, \dots, c_k$, and any vectors $u_1, u_2, \dots, u_k$ in $\mathcal{R}^n$,

$$A(c_1 u_1 + c_2 u_2 + \dots + c_k u_k) = c_1 A u_1 + c_2 A u_2 + \dots + c_k A u_k$$

EXERCISES Pages 25 and 26

1-64