

Lectures on Linear Algebra

Dr. rer. nat. Faten Said Abu-Shoga

Islamic University of Gaza

Chapter 1

1.3 Systems of Linear Equations

February 9, 2021

Chapter 1 MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

1.3 SYSTEMS OF LINEAR EQUATIONS

Definition

A linear equation in the variables (unknowns) $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

where $a_1, a_2, \dots, a_n$, and b are real numbers. The scalars $a_1, a_2, \dots, a_n$ are called the coefficients, and b is called the constant term of the equation.

1.3 SYSTEMS OF LINEAR EQUATIONS

Example of linear equation

$$-4x_1 + 8x_2 + 9x_3 + 0x_4 - 12x_5 = 6$$

Example of none linear equation

$$x_1 + 5x_2x_3 = 7$$

$$2x_1 - 7x_2 + x_3^2 = -3$$

$$4\sqrt{x_1} - 3x_2 = 15$$

1.3 SYSTEMS OF LINEAR EQUATIONS

system of linear equations

A system of linear equations is a set of m linear equations in the same n variables, where m and n are positive integers.

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

where a_{ij} denotes the coefficient of x_j in equation i .

A solution of a system of linear equations is a vector $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ in

$\mathcal{R}^n$ such that every equation in the system is satisfied when each x_i is replaced by s_i .

1.3 SYSTEMS OF LINEAR EQUATIONS

Practice Problem

Determine whether (a) $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ and (b) $\mathbf{v} = \begin{bmatrix} 5 \\ 8 \\ 1 \\ 3 \end{bmatrix}$ are solutions of the system of linear equations

$$x_1 + 5x_3 - x_4 = 7$$

$$2x_1 - x_2 + 6x_3 = 8$$

1.3 SYSTEMS OF LINEAR EQUATIONS

SYSTEMS OF 2 LINEAR EQUATIONS IN 2 VARIABLES

A linear equation $ax + by = c$, $a \neq 0$ or $b \neq 0$, is the equation of a line in the xy -plane. Consider the system

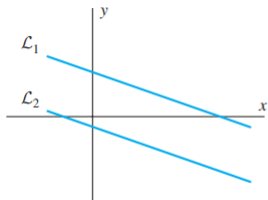
$$a_1x + b_1y = c_1 \quad \text{is the equation of line } \mathcal{L}_1$$

$$a_2x + b_2y = c_2 \quad \text{is the equation of line } \mathcal{L}_2$$

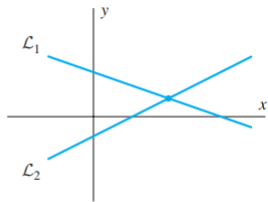
Geometrically, a solution of such a system corresponds to the intersection points of the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

There are three different situations that can arise.

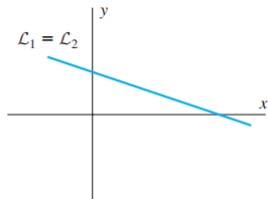
1.3 SYSTEMS OF LINEAR EQUATIONS



$\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel.
No solution



$\mathcal{L}_1$ and $\mathcal{L}_2$ are different but not parallel.
Exactly one solution



$\mathcal{L}_1$ and $\mathcal{L}_2$ are the same.
Infinitely many solutions

1.3 SYSTEMS OF LINEAR EQUATIONS

Remark

Every system of linear equations has no solution, exactly one solution, or infinitely many solutions.

Definition

- 1 A system of linear equations that has one or more solutions is called **consistent**; otherwise, the system is called **inconsistent**.
- 2 Two systems of linear equations that have exactly the same solutions are called **equivalent**.

1.3 SYSTEMS OF LINEAR EQUATIONS

ELEMENTARY ROW OPERATIONS: Example

$$\begin{aligned}x_1 - 2x_2 - x_3 &= 3 \\3x_1 - 6x_2 - 5x_3 &= 3 \\2x_1 - x_2 + x_3 &= 0\end{aligned}$$

Step 1:

$$\begin{aligned}x_1 - 2x_2 - x_3 &= 3 \\-2x_3 &= -6 \\3x_2 + 3x_3 &= -6\end{aligned}$$

Step 2:

$$\begin{aligned}x_1 - 2x_2 - x_3 &= 3 \\3x_2 + 3x_3 &= -6 \\-2x_3 &= -6\end{aligned}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

ELEMENTARY ROW OPERATIONS: Example

Step 3:

$$x_1 - 2x_2 - x_3 = 3$$

$$3x_2 + 3x_3 = -6$$

$$x_3 = 3$$

Step 4:

$$x_1 - 2x_2 = 6$$

$$3x_2 = -15$$

$$x_3 = 3$$

Step 5:

$$x_1 - 2x_2 = 6$$

$$x_2 = -5$$

$$x_3 = 3$$

1.3 SYSTEMS OF LINEAR EQUATIONS

ELEMENTARY ROW OPERATIONS: Example

Step 6:

$$\begin{aligned}x_1 &= -4 \\x_2 &= -5 \\x_3 &= 3\end{aligned}$$

so that $\begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix}$ is a solution of the system. Indeed, it is the only solution.

1.3 SYSTEMS OF LINEAR EQUATIONS

ELEMENTARY ROW OPERATIONS: The matrix equation

A system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

We can write the system in matrix form $AX = b$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

A is called the coefficient matrix (or the matrix of coefficients) of the system. If we write the matrix $[A \ b]$, then we call this matrix the augmented matrix of the system.

1.3 SYSTEMS OF LINEAR EQUATIONS

ELEMENTARY ROW OPERATIONS: The matrix equation

A system of linear equations

$$\begin{aligned}x_1 - 2x_2 - x_3 &= 3 \\ 3x_1 - 6x_2 - 5x_3 &= 3 \\ 2x_1 - x_2 + x_3 &= 0\end{aligned}$$

We can write the system in matrix form $AX = b$, where

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -5 \\ 2 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad [Ab] = \begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

Definition

Any one of the following three operations performed on a matrix is called an elementary row operation:

- 1 *Interchange any two rows of the matrix. (interchange operation)*
- 2 *Multiply every entry of some row of the matrix by the same nonzero scalar.(scaling operation)*
- 3 *Add a multiple of one row of the matrix to another row. (row addition operation)*

1.3 SYSTEMS OF LINEAR EQUATIONS

Let

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -3 & -7 \end{bmatrix}$$

The following sequence of elementary row operations transforms A into B :

$$\begin{aligned} A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} &\xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{-2r_1 + r_2 \rightarrow r_2} \\ \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -3 \\ 3 & 1 & 0 & 2 \end{bmatrix} &\xrightarrow{-3r_1 + r_3 \rightarrow r_3} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -3 \\ 0 & -5 & -3 & -7 \end{bmatrix} \\ \xrightarrow{-\frac{1}{3}r_2 \rightarrow r_2} &\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -3 & -7 \end{bmatrix} = B \end{aligned}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

We may perform several elementary row operations in succession, indicating the operations by stacking the individual labels above a single arrow.

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 3 & 1 & 0 & 2 \end{bmatrix} \xrightarrow[\begin{smallmatrix} -3\mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \end{smallmatrix}]{\begin{smallmatrix} -2\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2 \end{smallmatrix}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -3 & -3 \\ 0 & -5 & -3 & -7 \end{bmatrix}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

REDUCED ROW ECHELON FORM

Definition

we call a row of a matrix **a zero row** if all its entries are 0 and **a nonzero row** otherwise. We call the leftmost nonzero entry of a nonzero row its **leading entry**.

$$\begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & * & * & 0 & 0 & * \\ 0 & 1 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

Definitions

A matrix is said to be in row echelon form if it satisfies the following three conditions:

- 1 Each nonzero row lies above every zero row.
- 2 The leading entry of a nonzero row lies in a column to the right of the column containing the leading entry of any preceding row.
- 3 If a column contains the leading entry of some row, then all entries of that column below the leading entry are 0.

If a matrix also satisfies the following two additional conditions, we say that it is in reduced row echelon form.

- 4 If a column contains the leading entry of some row, then all the other entries of that column are 0.
- 5 The leading entry of each nonzero row is 1.

1.3 SYSTEMS OF LINEAR EQUATIONS

Example

The following matrices are *not* in reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

Solution of linear system

A system of linear equations with unique solution

$$x_1 = -4$$

$$x_2 = -5$$

$$x_3 = 3$$

1.3 SYSTEMS OF LINEAR EQUATIONS

Solution of linear system

A system of equations has infinitely many solutions

$$x_1 - 3x_2 + 2x_4 = 7$$

$$x_3 + 6x_4 = 9$$

$$x_5 = 2$$

$$0 = 0.$$

The augmented matrix of this system is

$$\left[\begin{array}{cccccc|c} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

which is in reduced row echelon form.

1.3 SYSTEMS OF LINEAR EQUATIONS

EXERCISES

Find a general solution of the system of linear equations

$$\begin{array}{rcl} x_1 & + 2x_4 & = 7 \\ x_2 & - 3x_4 & = 8 \\ x_3 + 6x_4 & = 9. \end{array}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

Remark

Whenever an augmented matrix contains a row in which the only nonzero entry lies in the last column, the corresponding system of linear equations has no solution.

Example

$$\begin{bmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system of linear equations corresponding to this matrix is

$$\begin{aligned} x_1 - 3x_3 &= 5 \\ x_2 + 2x_3 &= 4 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 &= 0. \end{aligned}$$

1.3 SYSTEMS OF LINEAR EQUATIONS

Theorem

Every matrix can be transformed into one and only one matrix in reduced row echelon form by means of a sequence of elementary row operations.

Procedure for Solving a System of Linear Equations

1. Write the augmented matrix $[A \ \mathbf{b}]$ of the system.
2. Find the reduced row echelon form $[R \ \mathbf{c}]$ of $[A \ \mathbf{b}]$.
3. If $[R \ \mathbf{c}]$ contains a row in which the only nonzero entry lies in the last column, then $A\mathbf{x} = \mathbf{b}$ has no solution. Otherwise, the system has at least one solution. Write the system of linear equations corresponding to the matrix $[R \ \mathbf{c}]$, and solve this system for the basic variables in terms of the free variables to obtain a general solution of $A\mathbf{x} = \mathbf{b}$.

1.3 SYSTEMS OF LINEAR EQUATIONS

EXERCISE

Does the system of the following augmented matrix has solution?

$$\left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

1.3 SYSTEMS OF LINEAR EQUATIONS

EXERCISE

Does the system of the following augmented matrix has solution?

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

1.3 SYSTEMS OF LINEAR EQUATIONS

EXERCISE

Does the system of the following augmented matrix has solution?

$$\left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

EXERCISES Pages 38-40

1-76