

Lectures on Linear Algebra

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Chapter 1

1.4 Gaussian Elimination

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Chapter 1 MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

1.4 Gaussian Elimination

Example: Solve the following system of linear equations

$$\begin{aligned}x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9\end{aligned}$$

Solution: The augmented matrix of this system is

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array} \right]$$

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$$\begin{array}{l} \mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2 \\ -2\mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3 \\ 3\mathbf{r}_1 + \mathbf{r}_4 \rightarrow \mathbf{r}_4 \end{array} \left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{array} \right]$$

$$\begin{array}{l} \mathbf{r}_2 \leftrightarrow \mathbf{r}_3 \\ \xrightarrow{} \end{array} \left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{array} \right]$$

$$\begin{array}{l} -\mathbf{r}_2 + \mathbf{r}_4 \rightarrow \mathbf{r}_4 \\ \xrightarrow{} \end{array} \left[\begin{array}{cccccc} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 8 & 8 & 16 \end{array} \right]$$

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$$\underline{-2r_3 + r_4 \rightarrow r_4} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{4}r_3 \rightarrow r_3 \\ \longrightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{-r_2 \rightarrow r_2} \begin{bmatrix} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\underline{r_2 + r_1 \rightarrow r_1} \quad \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 2x_2 - x_5 = -5$$

$$x_3 = -3$$

$$x_4 + x_5 = 2$$

The general solution of this system is

$$x_1 = -5 - 2x_2 + x_5$$

$$x_2 \text{ and } x_5 \text{ free}$$

$$x_3 = -3$$

$$x_4 = 2 - x_5$$

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EXERCISES

Find the general solution of

$$x_1 - x_2 - 3x_3 + x_4 - x_5 = -2$$

$$-2x_1 + 2x_2 + 6x_3 - 6x_5 = -6$$

$$3x_1 - 2x_2 - 8x_3 + 3x_4 - 5x_5 = -7$$

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THE RANK AND NULLITY OF A MATRIX

Definition

The rank of an $m \times n$ matrix A , denoted by $\text{rank } A$, is defined to be the number of nonzero rows in the reduced row echelon form of A . The nullity of A , denoted by $\text{nullity } A$, is defined to be $n - \text{rank } A$.

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THE RANK AND NULLITY OF A MATRIX

Example

For the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

the reduced row echelon form is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the rank of the matrix is 3. The nullity of the matrix is $6 - 3 = 3$.

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Remarks

- 1 The rank of a matrix equals the number of pivot columns in the matrix.
- 2 The nullity of a matrix equals the number of nonpivot columns in the matrix.
- 3 If an $n \times n$ matrix has rank n , then its reduced row echelon form is I_n

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Remarks

If $Ax = b$ is the matrix form of a consistent system of linear equations, then

- (a) the number of basic variables in a general solution of the system equals the rank of A ;
- (b) the number of free variables in a general solution of the system equals the nullity of A .

Thus

- ① a consistent system of linear equations has a unique solution if and only if the nullity of its coefficient matrix equals 0.
- ② a consistent system of linear equations has infinitely many solutions if and only if the nullity of its coefficient matrix is positive.

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Exercise

Consider the following system of linear equations:

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + 3x_3 = -2 + s$$

$$x_1 - x_2 + rx_3 = 3$$

- (a) For what values of r and s is this system of linear equations inconsistent?
- (b) For what values of r and s does this system of linear equations have infinitely many solutions?
- (c) For what values of r and s does this system of linear equations have a unique solution?

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Exercise

Consider the following system of linear equations:

$$x_1 + 3x_2 = 1 + s$$

$$x_1 + rx_2 = 5$$

- (a) For what values of r and s is this system of linear equations inconsistent?
- (b) For what values of r and s does this system of linear equations have infinitely many solutions?
- (c) For what values of r and s does this system of linear equations have a unique solution?

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Theorem (Test for Consistency)

The following conditions are equivalent:

- (a) The matrix equation $Ax = b$ is consistent.
- (b) The vector b is a linear combination of the columns of A .
- (c) The reduced row echelon form of the augmented matrix $[A \ b]$ has no row of the form $[0 \ 0 \ \cdots \ 0 \ d]$, where $d \neq 0$.

EXERCISES Pages 52-55

1-42 , 47-50 and 53-80.