

Lectures on Linear Algebra

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Chapter 1

1.6 The Span of a Set of Vectors

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Chapter 1 MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

1.6 The Span of a Set of Vectors

Definition

For a nonempty set $S = \{u_1, u_2, \dots, u_k\}$ of vectors in $\mathbb{R}^n$, we define the span of S to be the set of all linear combinations of $u_1, u_2, \dots, u_k$ in $\mathbb{R}^n$. This set is denoted by $\text{Span } S$ or

$$\text{Span}\{u_1, u_2, \dots, u_k\}$$

Examples

- 1 A linear combination of a single vector is just a multiple of that vector.
- 2 The span of $\{0\}$ is $\{0\}$.
- 3 If S contains even one nonzero vector, then $\text{Span } S$ contains infinitely many vectors.

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Exercises

Describe the spans of the following subsets of $\mathbb{R}^2$

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \quad \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\},$$

$$\mathcal{S}_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

and

$$\mathcal{S}_4 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

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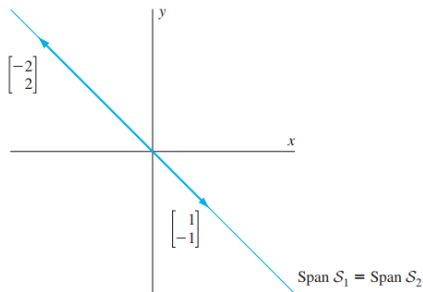


Figure 1.24 The span of S_2

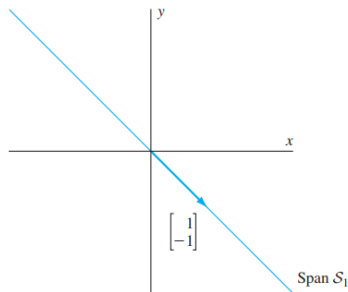


Figure 1.23 The span of S_1

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Exercises : For the standard vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbb{R}^3$, we see that the span of $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the set of vectors of the form

$$a\mathbf{e}_1 + b\mathbf{e}_2 = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

- 1 If u and v are nonparallel vectors in $\mathbb{R}^3$, then the span of $\{u, v\}$ is a plane through the origin.
- 2 $\text{Span}\{\mathbf{e}_3\}$ is the set of vectors that lie along the z -axis in $\mathbb{R}^3$.

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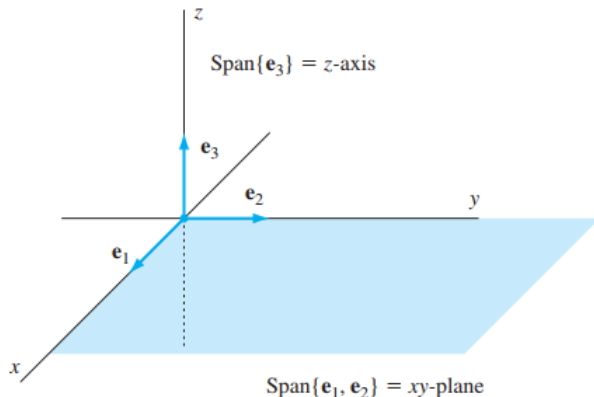


Figure 1.25 The span of $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathcal{R}^3$

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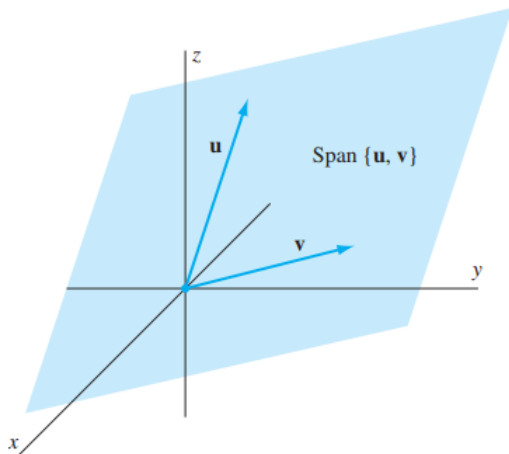


Figure 1.26 The span of $\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}$ and $\mathbf{v}$ are nonparallel vectors in $\mathcal{R}^3$

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Remark

Let $S = \{u_1, u_2, \dots, u_k\}$ be a set of vectors from $\mathbb{R}^n$, and let A be the matrix whose columns are $u_1, u_2, \dots, u_k$. Then a vector v from $\mathbb{R}^n$ is in the span of S if and only if the equation $Ax = v$ is consistent.

(that is, v is a linear combination of $u_1, u_2, \dots, u_k$)

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Example

Is

$$\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix} \quad \text{or} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

a vector in the span of

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} \right\} ?$$

If so express it as a linear combination of the vectors in S .

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Solution: The vector $\mathbf{v}$ belongs to the span of $\mathcal{S}$ if and only if $\mathbf{Ax} = \mathbf{v}$ is consistent.

$$[A \ \mathbf{v}] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 1 & 8 & 0 \\ 1 & -2 & -1 & 5 \\ 1 & 1 & 5 & -1 \end{bmatrix}$$

the reduced row echelon form of $[A \ \mathbf{v}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{Ax} = \mathbf{v}$ is consistent. Hence $\mathbf{v}$ belongs to the span of $\mathcal{S}$.

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To express $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{S}$, we need to find the actual solution of $A\mathbf{x} = \mathbf{v}$. The general solution of this equation is

$$\begin{aligned}x_1 &= 1 - 3x_3 \\x_2 &= -2 - 2x_3 \\x_3 &\quad \text{free}\end{aligned}$$

For example, by taking $x_3 = 0$, we find that

$$1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 8 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \end{bmatrix} = \mathbf{v}$$

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In the same manner, $\mathbf{w}$ belongs to the span of S if and only if $A\mathbf{x} = \mathbf{w}$ is consistent. Because the reduced row echelon form of $[A \ \mathbf{w}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 1.5 shows that $A\mathbf{x} = \mathbf{w}$ is not consistent. Thus $\mathbf{w}$ does not belong to the span of S

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Exercise

Are $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in the span of

$$\mathcal{S} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}?$$

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Exercise

Let

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$$

Show that $\text{Span } \mathcal{S} = \mathcal{R}^3$.

Solution

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

R is the reduced row echelon form of A

$$R = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

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R has no zero row, there can be no row in $[R\mathbf{c}]$ in which the only nonzero entry lies in the last column.

Thus $A\mathbf{x} = \mathbf{v}$ is consistent and so $\mathbf{v}$ belongs to $\text{Span } \mathcal{S}$.

Since $\mathbf{v}$ is an arbitrary vector in $\mathcal{R}^3$, it follows that $\text{Span } \mathcal{S} = \mathcal{R}^3$

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Theorem

The following statements about an $m \times n$ matrix A are equivalent:

- (a) *The span of the columns of A is $\mathcal{R}^m$.*
- (b) *The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution (that is, $A\mathbf{x} = \mathbf{b}$ is consistent) for each $\mathbf{b}$ in $\mathcal{R}^m$*
- (c) *The rank of A is m , the number of rows of A .*
- (d) *The reduced row echelon form of A has no zero rows.*
- (e) *There is a pivot position in each row of A .*

Theorem

Let $S = \{u_1, u_2, \dots, u_k\}$ be a set of vectors from $\mathbb{R}^n$, and let v be a vector in $\mathbb{R}^n$. Then

$\text{Span}\{u_1, u_2, \dots, u_k, v\} = \text{Span}\{u_1, u_2, \dots, u_k\}$

if and only if v belongs to the span of S .

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Exersice

Is $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ a generating set for $\mathcal{R}^3$.

Solution:

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 1 & 2 & 9 & 1 \end{bmatrix}$$
$$\Rightarrow R = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

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Exercise

For the set S above, find a subset with the fewest vectors such that $\text{Span } S = \mathbb{R}^3$.

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