

Lectures on Linear Algebra

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Chapter 1

1.7 Linear Dependence and Linear Independence

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Chapter 1 MATRICES, VECTORS, AND SYSTEMS OF LINEAR EQUATIONS

1.7 Linear Dependence and Linear Independence

Definition

A set of k vectors $\{u_1, u_2, \dots, u_k\}$ in $\mathbb{R}^n$ is called **linearly dependent** if there exist scalars $c_1, c_2, \dots, c_k$, not all 0, such that

$$c_1 u_1 + c_2 u_2 + \dots + c_k u_k = 0$$

In this case, we also say that the vectors $u_1, u_2, \dots, u_k$ are **linearly dependent**.

A set of k vectors $\{u_1, u_2, \dots, u_k\}$ is called **linearly independent** if the only scalars $c_1, c_2, \dots, c_k$ such that

$$c_1 u_1 + c_2 u_2 + \dots + c_k u_k = 0 \text{ are } c_1 = c_2 = \dots = c_k = 0.$$

In this case, we also say that the vectors $u_1, u_2, \dots, u_k$ are **linearly independent**.

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Remark

Note that a set is linearly independent if and only if it is not linearly dependent.

Example

Show that the sets $\mathcal{S}_1 = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ are linearly dependent.

Solution: The equation

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is true with $c_1 = 2$, $c_2 = -1$, and $c_3 = 1$. Since not all the coefficients in the preceding linear combination are 0, $\mathcal{S}_1$ is linearly dependent.

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Example

Show that the sets $S_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ are linearly dependent.

Solution: Because

$$1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and at least one of the coefficients in this linear combination is nonzero, S_2 is also linearly dependent.

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Remark

Any finite subset $S = \{0, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ of $\mathcal{R}^n$ that contains the zero vector is linearly dependent because

$$1 \cdot 0 + 0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_k = 0$$

is a linear combination of the vectors in S in which at least one coefficient is nonzero.

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CAUTION

While the equation

$$0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is true, it tells us nothing about the linear independence or dependence of the set $\mathcal{S}_1$ in Example 1 . A similar statement is true for any set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$:

$$0\mathbf{u}_1 + 0\mathbf{u}_2 + \dots + 0\mathbf{u}_k = \mathbf{0}$$

For a set of vectors to be linearly dependent, the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_k\mathbf{u}_k = \mathbf{0}$$

must be satisfied with at least one nonzero coefficient

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The equation $c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_i \mathbf{u}_i = 0$ can be written as a matrix vector product

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \end{bmatrix} = 0.$$

Remark

The set $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_i\}$ is linearly dependent if and only if there exists a nonzero solution of $A\mathbf{x} = 0$, where

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_i \end{bmatrix}.$$

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Example

Determine whether the set

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

is linearly dependent or linearly independent?

Solution We must determine whether $Ax = 0$ has a nonzero solution. where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

is the matrix whose columns are the vectors in S .

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The augmented matrix of $A\mathbf{x} = 0$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix}$$

and its reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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Hence the general solution of this system is

$$x_1 = -2x_3$$

$$x_2 = x_3$$

$$x_3 \text{ free}$$

$$x_4 = 0$$

Because the solution of $Ax = 0$ contains a free variable, this system of linear equations has infinitely many solutions. and we can obtain a nonzero solution by choosing any nonzero value of the free variable.

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Example

Determine whether the set

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly dependent or linearly independent.

Solution: We must check whether $A\mathbf{x} = 0$ has a nonzero solution, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Note that the system $A\mathbf{x} = 0$ has nonzero solutions if and only if its general solution contains a free variable.

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Since the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the rank of A is 3, and the nullity of A is $3 - 3 = 0$. Thus the general solution of $A\mathbf{x} = 0$ has no free variables. So $A\mathbf{x} = 0$ has no nonzero solutions, and hence $\mathcal{S}$ is linearly independent.

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Theorem

The following statements about an $m \times n$ matrix A are equivalent:

- (a) The columns of A are linearly independent.*
- (b) The equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathbb{R}^m$.*
- (c) The nullity of A is zero.*
- (d) The rank of A is n , the number of columns of A .*
- (e) The columns of the reduced row echelon form of A are distinct standard vectors in $\mathbb{R}^m$.*
- (f) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.*
- (g) There is a pivot position in each column of A .*

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Exercise

Is some vector in the set

$$\mathcal{S} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

a linear combination of the others?

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Definition

The equation $A\mathbf{x} = \mathbf{b}$ is called homogeneous if $\mathbf{b} = \mathbf{0}$.

Remark

- In checking if a subset is linearly independent, we are led to a homogeneous equation.
- A homogeneous equation must be consistent because $\mathbf{0}$ is a solution of $A\mathbf{x} = \mathbf{0}$.
- A homogeneous system of linear equations with more variables than equations has infinitely many solutions.
- A homogeneous system of linear equations of unique solution, the solution is the zero only.
- The number of solutions of $A\mathbf{x} = \mathbf{0}$ determines the linear dependence or independence of the columns of A .

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Example

let us consider this equation for the matrix

$$A = \begin{bmatrix} 1 & -4 & 2 & -1 & 2 \\ 2 & -8 & 3 & 2 & -1 \end{bmatrix}$$

Since the reduced row echelon form of $\begin{bmatrix} A & 0 \end{bmatrix}$ is

$$\begin{bmatrix} 1 & -4 & 0 & 7 & -8 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

the general solution of $A\mathbf{x} = 0$ is

$$\begin{aligned} x_1 &= 4x_2 - 7x_4 + 8x_5 \\ x_2, \quad x_4, \quad x_5 &\text{ free} \\ x_3 &= 4x_4 - 5x_5 \end{aligned}$$

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Expressing the solutions of $A\mathbf{x} = 0$ in vector form yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4x_2 - 7x_4 + 8x_5 \\ x_2 + 4x_4 - 5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 8 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Thus the solution of $A\mathbf{x} = 0$ is the span of

$$\mathcal{S} = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

1.7 Linear Dependence and Linear Independence

Remark

When a vector form of the general solution of $A\mathbf{x} = 0$ is obtained by the method described in Section 1.3, the vectors that appear in the vector form are linearly independent.

Exercise

Determine a vector form for the general solution of

$$x_1 - 3x_2 - x_3 + x_4 - x_5 = 0$$

$$2x_1 - 6x_2 + x_3 - 3x_4 - 9x_5 = 0$$

$$-2x_1 + 6x_2 + 3x_3 + 2x_4 + 11x_5 = 0$$

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LINEARLY DEPENDENT AND LINEARLY INDEPENDENT SETS

Theorem

Vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ in R^n are linearly dependent if and only if $\mathbf{u}_1 = \mathbf{0}$ or there exists an $i \geq 2$ such that $\mathbf{u}_i$ is a linear combination of the preceding vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{i-1}$

1.7 Linear Dependence and Linear Independence

Properties of Linearly Dependent and Independent Sets

- 1 A set consisting of a single nonzero vector is linearly independent, but $\{0\}$ is linearly dependent.
- 2 A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other or one of the vectors is zero.
- 3 Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_z\}$ be a linearly independent subset of $\mathbb{R}^n$, and $\mathbf{v}$ be in $\mathbb{R}^n$. Then V does not belong to the span of S if and only if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, V\}$ is linearly independent.
- 4 Every subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent.
- 5 If S is a subset of $\mathbb{R}^n$ and no vector can be removed from S without changing its span, then S is linearly independent.

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CAUTION

The result mentioned in item 2 is valid only for sets containing two vectors. For example, in $\mathcal{R}^3$, the set $\{e_1, e_2, e_1 + e_2\}$ is linearly dependent but no vector in the set is a multiple of another.

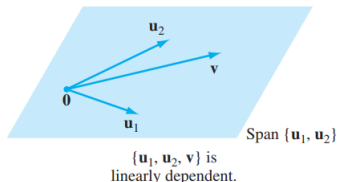
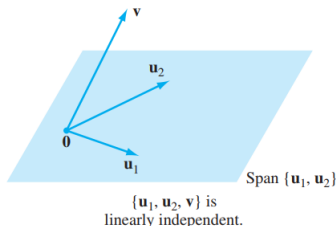


Figure 1.27 Linearly independent and linearly dependent sets of 3 vectors

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Exercise

Determine by inspection whether the following sets are linearly dependent or linearly independent:

$$S_1 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} -4 \\ 12 \\ 6 \end{bmatrix}, \begin{bmatrix} -10 \\ 30 \\ 15 \end{bmatrix} \right\}$$

$$S_3 = \left\{ \begin{bmatrix} -3 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}, \quad \text{and}$$

$$S_4 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \right\}$$

1.7 Linear Dependence and Linear Independence

Exercise

Determine by inspection whether the following sets are linearly dependent or linearly independent:

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{S}_2 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\},$$

$$\mathcal{S}_3 = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \right\}, \quad \text{and} \quad \mathcal{S}_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right\}$$

1.7 Linear Dependence and Linear Independence

The rank of A	The number of solutions of $Ax = b$	The columns of A	The reduced row echelon form R of A
$\text{rank } A = m$	$Ax = b$ has at least one solution for every b in $\mathcal{R}^m$.	The columns of A are a generating set for $\mathcal{R}^m$.	Every row of R contains a pivot position.
$\text{rank } A = n$	$Ax = b$ has at most one solution for every b in $\mathcal{R}^m$.	The columns of A are linearly independent.	Every column of R contains a pivot position.

EXERCISES Pages 84-86

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