

Lectures on Linear Algebra

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Chapter 2

2.1 Matrix Multiplication

March 16, 2021

CHAPTER 2 Matrices and Linear Transformations

2.3 Invertibility and Elementary Matrices

Definition

An $n \times n$ matrix A is called invertible if there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

In this case, B is called an inverse of A .

Theorem

If A is an invertible matrix, then its inverse is unique.

Proof.

For if both B and C are inverses of A , then

$$AB = BA = I_n \quad \text{and} \quad AC = CA = I_n$$

Hence $B = BI_n = B(AC) = (BA)C = I_n C = C$



2.3 Invertibility and Elementary Matrices

Remark

When A is invertible, we denote the unique inverse of A by A^{-1} , so that $AA^{-1} = A^{-1}A = I_n$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$BA = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

So A is invertible, and B is the inverse of A , that is, $A^{-1} = B$.

2.3 Invertibility and Elementary Matrices

Remark

The $n \times n$ zero matrix O has no inverse because $OB = O \neq I_n$ for any $n \times n$ matrix B .

But there are also other square matrices that are not invertible;

for example, $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

2.3 Invertibility and Elementary Matrices

Remark

If A is invertible, then we can solve the matrix equation $Ax = b$ as follows:

$$\begin{aligned}Ax &= b \\A^{-1}(Ax) &= A^{-1}b \\(A^{-1}A)x &= A^{-1}b \\I_n x &= A^{-1}b \\x &= A^{-1}b\end{aligned}$$

Theorem

If A is an invertible $n \times n$ matrix, then for every $\mathbf{b}$ in $\mathcal{R}^n$, $A\mathbf{x} = \mathbf{b}$ has the unique solution $A^{-1}\mathbf{b}$.

2.3 Invertibility and Elementary Matrices

Exercise

Use a matrix inverse to solve the system of linear equations

$$x_1 + 2x_2 = 4$$

$$3x_1 + 5x_2 = 7$$

2.3 Invertibility and Elementary Matrices

The rotation matrix

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- ❶ For $\theta = 0^\circ$, $A_\theta = I_2$.
- ❷ For any angle α , $A_\alpha A_{-\alpha} = A_{\alpha+(-\alpha)} = A_{0^\circ} = I_2$
- ❸ $A_{-\alpha} A_\alpha = I_2$.
- ❹ A_α satisfies the definition of an invertible matrix with inverse $A_{-\alpha}$. $(A_\alpha)^{-1} = A_{-\alpha}$.

Remark

Another way of viewing $A_{-\alpha} A_\alpha$ is that it represents a rotation by α , followed by a rotation by $-\alpha$, which results in a net rotation of 0° . This is the same as multiplying by the identity matrix.

2.3 Invertibility and Elementary Matrices

Theorem

Let A and B be $n \times n$ matrices.

- (a) If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- (b) If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- (c) If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Remark

Let $A_1, A_2, \dots, A_k$ be $n \times n$ invertible matrices. Then the product $A_1 A_2 \dots A_k$ is invertible, and

$$(A_1 A_2 \dots A_k)^{-1} = (A_k)^{-1} (A_{k-1})^{-1} \dots (A_1)^{-1}$$

2.3 Invertibility and Elementary Matrices

ELEMENTARY MATRICES

Remark

Every elementary row operation can be performed by matrix multiplication.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We can multiply row 2 of the matrix A by the scalar k by means of the matrix product.

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

2.3 Invertibility and Elementary Matrices

We can interchange rows 1 and 2 by means of the matrix product

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

We can add k times row 1 of A to row 2 by means of the matrix product

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$$

2.3 Invertibility and Elementary Matrices

Definition

An $n \times n$ matrix E is called an elementary matrix if we can obtain E from I_n by a single elementary row operation.

Example

The matrices are examples of elementary matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

2.3 Invertibility and Elementary Matrices

Let A be an $m \times n$ matrix, and let E be an $m \times m$ elementary matrix resulting from an elementary row operation on I_m . Then the product EA can be obtained from A by the identical elementary row operation on A .

Example

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

then

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 7 & 10 \end{bmatrix}$$

2.3 Invertibility and Elementary Matrices

Theorem

Every elementary matrix is invertible. Furthermore, the inverse of an elementary matrix is also an elementary matrix.

Facts

If A is an $m \times n$ matrix with reduced row echelon form R , there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$E_k E_{k-1} \dots E_1 A = R$$

If $P = E_k E_{k-1} \dots E_1$ is a product of elementary matrices, so P is an invertible matrix and write

$$PA = R$$

2.3 Invertibility and Elementary Matrices

Theorem

Let A be an $m \times n$ matrix with reduced row echelon form R . Then there exists an invertible $m \times m$ matrix P such that $PA = R$.

corollary

The matrix equation $Ax = b$ has the same solutions as $Rx = c$, where $[Rc]$ is the reduced row echelon form of the augmented matrix $[Ab]$.

Remark

As a special case of the preceding corollary, we note that if $b = 0$, then $c = 0$. Therefore $Ax = 0$ and $Rx = 0$ are equivalent.

2.3 Invertibility and Elementary Matrices

Column Correspondence Property

Let A be a matrix and R its reduced row echelon form. If column j of R is a linear combination of other columns of R , then column j of A is a linear combination of the corresponding columns of A using the same coefficients, and vice versa.

Theorem

The following statements are true for any matrix A :

- (a) The pivot columns of A are linearly independent.
- (b) Each nonpivot column of A is a linear combination of the previous pivot columns of A , where the coefficients of the linear combination are the entries of the corresponding column of the reduced row echelon form of A .

2.3 Invertibility and Elementary Matrices

Example

The reduced row echelon form of a matrix A is

$$R = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine A , given that the first and third columns of A are

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Solution: Since the first and third columns of R are the pivot columns, these are also the pivot columns of A .

2.3 Invertibility and Elementary Matrices

Now observe that the second column of R is $2\mathbf{r}_1$, and hence, by the column correspondence property, the second column of A is

$$\mathbf{a}_2 = 2\mathbf{a}_1 = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

Furthermore, the fourth column of R is $\mathbf{r}_4 = (-1)\mathbf{r}_1 + \mathbf{r}_3$, and so, again by the column correspondence property,

$$\mathbf{a}_4 = (-1)\mathbf{a}_1 + \mathbf{a}_3 = (-1) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

2.3 Invertibility and Elementary Matrices

Thus

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

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