

Lectures on Linear Algebra

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Chapter 2

2.1 Matrix Multiplication

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CHAPTER 2 Matrices and Linear Transformations

2.1 Matrix Multiplication

Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the (matrix) product AB to be the $m \times p$ matrix whose j -th column is Ab_j . That is,

$$C = [Ab_1 \ Ab_2 \dots \ Ab_p]$$

Remark

For any $m \times n$ matrix A , any $n \times p$ matrix B , and any $p \times 1$ vector v ,

$$(AB)v = A(Bv)$$

2.1 Matrix Multiplication

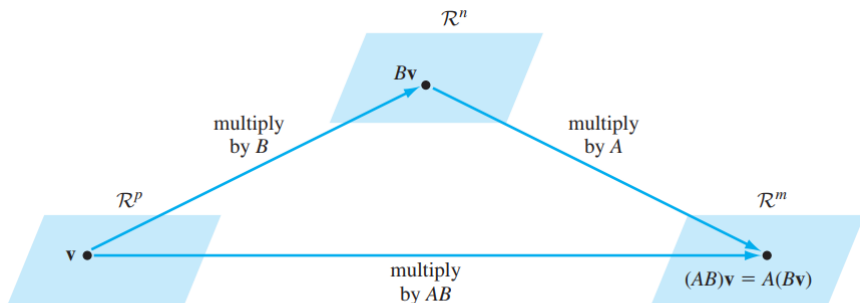


Figure 2.1 The associative law of multiplication

2.1 Matrix Multiplication

Remark

When the sizes of A and B are written side by side in the same order as the product, that is,

$$(m \times n)(n \times p)$$

the inner dimensions must be equal, and the outer dimensions give the size of the product AB . Symbolically,

$$(m \times n)(n \times p) = (m \times p)$$

2.1 Matrix Multiplication

Exercise

Suppose that A is a 2×4 matrix and B is a 2×3 matrix.

- (a) Is the product BA^T defined? If so, what is its size?
- (b) Is the product A^TB defined? If so, what is its size?

2.1 Matrix Multiplication

Exercise

For $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & -3 & 4 \\ 3 & 1 & -2 \end{bmatrix}$, compute AB .

2.1 Matrix Multiplication

Exercise

For $A = \begin{bmatrix} 0 & -1 & 4 \\ 2 & 5 & 1 \\ -1 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 3 \\ 1 & -3 & 2 \\ 3 & -1 & 0 \end{bmatrix}$, compute AB .

2.1 Matrix Multiplication

Matrix Multiplication Is Not Commutative

For arbitrary matrices A and B, AB need not equal BA

Example

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.1 Matrix Multiplication

Theorem

Let A and B be $k \times m$ matrices, C be an $m \times n$ matrix, and P and Q be $n \times p$ matrices. Then the following statements are true:

- 1 $s(AC) = (sA)C = A(sC)$ for any scalar s .
- 2 $A(CP) = (AC)P$
- 3 $(A + B)C = AC + BC$
- 4 $C(P + Q) = CP + CQ$
- 5 $I_k A = A = A I_m$
- 6 The product of any matrix and a zero matrix is a zero matrix
- 7 $(AC)^T = C^T A^T$

2.1 Matrix Multiplication

Remark

As with real numbers, we use the exponential notation A^k to denote the product of A with itself k times. By convention, $A^1 = A$ and $A^0 = I_n$.

Definition

If A and B are matrices with the same number of rows, then we denote by $[A \ B]$ the matrix whose columns are the columns of A followed by the columns of B , in order. We call $[A \ B]$ an augmented matrix.

2.1 Matrix Multiplication

Example

Let

$$A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix}, \quad \text{and } P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

Use the equation $P[A \ B] = [PA \ PB]$ to compute the product $P[A \ B]$.

Solution Observe that $PA = PI_2 = P$ and

$$PB = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3 \\ 5 & -3 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

2.1 Matrix Multiplication

Therefore

$$P \left[\begin{array}{cc} A & B \end{array} \right] = \left[\begin{array}{cc} PA & PB \end{array} \right] = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 6 & 3 \\ 2 & -1 & 5 & -3 & 1 \\ 0 & 1 & -1 & 3 & 1 \end{array} \right]$$

Exercise

Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -2 \\ 0 & 3 \\ -4 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 5 \\ -6 & 0 & 2 \end{bmatrix}$$

Find the fourth column of $A[B \ C]$ without computing the entire matrix.

2.1 Matrix Multiplication

SPECIAL MATRICES

Definition

The (i, j) -entry of a matrix A is called a diagonal entry if $i = j$. The diagonal entries form the diagonal of A . A square matrix A is called a diagonal matrix if all its nondiagonal entries are zeros.

Example

The identity matrices and square zero matrices are diagonal matrices.

2.1 Matrix Multiplication

Definition

If A and B are $n \times n$ diagonal matrices, then AB is also an $n \times n$ diagonal matrix. Moreover, the diagonal entries of AB are the products of the corresponding diagonal entries of A and B .

Example

Suppose that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

2.1 Matrix Multiplication

Properties

- 1 A square zero matrix is a diagonal matrix.
- 2 If B is a diagonal matrix, then cB is a diagonal matrix for any scalar c .
- 3 If B is a diagonal matrix, then B^T is a diagonal matrix.
- 4 if B and C are diagonal matrices of the same size, then $B + C$ is a diagonal matrix.

2.1 Matrix Multiplication

Definition

A (square) matrix A is called symmetric if $A^T = A$.

Example

- 1 Diagonal matrices are symmetric.
- 2 let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix}$$

2.1 Matrix Multiplication

Properties

- 1 The (i, j) -entry of a symmetric matrix equals the (j, i) -entry.
- 2 A square zero matrix is symmetric.
- 3 If B is a symmetric matrix, then so is cB for any scalar c .
- 4 If B is a square matrix, then $B + B^T$ is symmetric.
- 5 If B and C are $n \times n$ symmetric matrices, then so is $B + C$.

2.1 Matrix Multiplication

Definition

A (square) matrix A is called skew-symmetric if $A^T = -A$.

Example

let

$$A = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}$$

2.1 Matrix Multiplication

Definition

The trace of an $n \times n$ matrix A , written $\text{trace}(A)$, is defined to be the sum

$$\text{trace}(A) = a_{11} + a_{22} + \dots + a_{nn}$$

Remark

For any $n \times n$ matrices A and B and scalar c , the following statements are true:

- (a) $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$.
- (b) $\text{trace}(cA) = c \cdot \text{trace}(A)$.
- (c) $\text{trace}(A^T) = \text{trace}(A)$.

2.1 Matrix Multiplication

Example

$$\text{let } A = \begin{bmatrix} -5 & 2 & 4 & 9 \\ -2 & 0 & -1 & 5 \\ -4 & 1 & 7 & -2 \\ 8 & 3 & 7 & -6 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -2 & 3 & 13 \\ -2 & 7 & -1 & -2 \\ 3 & 5 & -10 & -7 \\ -8 & 4 & 7 & 8 \end{bmatrix}$$

Find $\text{trace}(A)$ and $\text{trace}(2A + B^T)$.

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