

Lectures on Linear Algebra

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Chapter 2

2.4 The Inverse of a Matrix

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CHAPTER 2 Matrices and Linear Transformations

2.4 The Inverse of a Matrix

Theorem

Let A be an $n \times n$ matrix. Then A is invertible if and only if the reduced row echelon form of A is I_n .

Algorithm for Matrix Inversion

Let A be an $n \times n$ matrix. Use elementary row operations to transform $[A \ I_n]$ into the form $[RB]$, where R is a matrix in reduced row echelon form. Then either

- (a) $R = I_n$, in which case A is invertible and $B = A^{-1}$; or
- (b) $R \neq I_n$, in which case A is not invertible.

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Example

$$[A \ I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{-2r_1 + r_2 \rightarrow r_2 \\ -3r_1 + r_3 \rightarrow r_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{2r_2 + r_3 \rightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array} \right]$$

$$\xrightarrow{-r_3 \rightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array} \right]$$

$$\xrightarrow{-3r_3 + r_1 \rightarrow r_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array} \right]$$

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Example

$$\xrightarrow{-2r_2+r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array} \right] = \left[\begin{array}{ccc} I_3 & B \end{array} \right]$$

$$A^{-1} = B = \begin{bmatrix} -16 & 4 & 3 \\ -2 & 1 & 0 \\ 7 & -2 & -1 \end{bmatrix}$$

Remark

If $[R \ B]$ is the reduced row echelon form of $[A \ I_n]$, then either:

- (a) $R = I_n$, in which case A is invertible and $B = A^{-1}$; or
- (b) $R \neq I_n$, in which case A is not invertible.

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Example

Does the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ have an inverse?

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Exercise

For each of the matrices

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \\ -2 & -5 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 1 \\ -1 & 4 & 4 \end{bmatrix}$$

determine whether the matrix is invertible. If so, find its inverse.

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(Invertible Matrix Theorem)

Let A be an $n \times n$ matrix. The following statements are equivalent:

- (a) A is invertible.
- (b) The reduced row echelon form of A is I_n .
- (c) The rank of A equals n .
- (d) The span of the columns of A is $\mathbb{R}^n$.
- (e) The equation $Ax = b$ is consistent for every b in $\mathbb{R}^n$.
- (f) The nullity of A equals zero.
- (g) The columns of A are linearly independent.
- (h) The only solution of $Ax = 0$ is 0 .
- (i) There exists an $n \times n$ matrix B such that $BA = I_n$.
- (j) There exists an $n \times n$ matrix C such that $AC = I_n$.
- (k) A is a product of elementary matrices.

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