

Lectures on Linear Algebra

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Chapter 2

2.7 Linear Transformations and Matrices

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CHAPTER 2 Matrices and Linear Transformations

2.7 Linear Transformations and Matrices

Definitions

Let S_1 and S_2 be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

- 1 A function f from S_1 to S_2 , written $f : S_1 \rightarrow S_2$, is a rule that assigns to each vector v in S_1 a unique vector $f(v)$ in S_2 .
- 2 The vector $f(v)$ is called the image of v (under f).
- 3 The set S_1 is called the domain of a function f , and the set S_2 is called the codomain of f .
- 4 The range of f is defined to be the set of images $f(v)$ for all v in S_1 .

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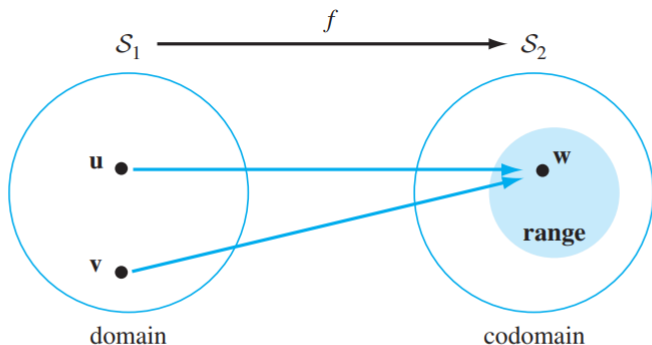


Figure 2.9 The domain, codomain, and range of a function

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Example

Define $f : \mathcal{R}^3 \rightarrow \mathcal{R}^2$ by the rule

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1^2 \end{bmatrix}$$

Then f is a function whose domain is $\mathcal{R}^3$ and codomain is $\mathcal{R}^2$. Notice that

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

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Example

So $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is the image of both $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$. However, not every vector in $\mathcal{R}^2$ is an image of a vector in $\mathcal{R}^3$ because every image must have a nonnegative second component.

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Example

Let A be the 3×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Define the function $T_A : \mathcal{R}^2 \rightarrow \mathcal{R}^3$ by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

$$T_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

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Definition

Let A be an $m \times n$ matrix. The function $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T_A(x) = Ax$ for all $x \in \mathbb{R}^n$ is called the matrix transformation induced by A .

Exercise

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & 4 \end{bmatrix}$

- (a) What is the domain of T_A ?
- (b) What is the codomain of T_A ?
- (c) Compute $T_A\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)$.

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Example

Let A be the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $T_A : \mathcal{R}^3 \rightarrow \mathcal{R}^3$ is defined by

$$T_A \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

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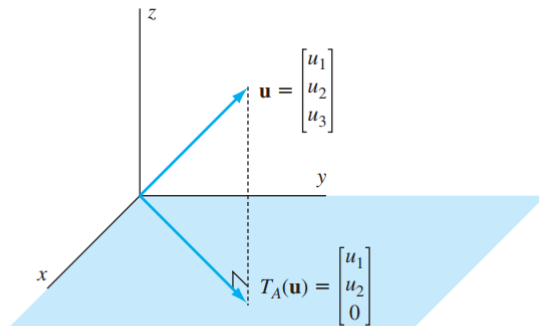


Figure 2.11 The orthogonal projection of $\mathbf{u}$ on the xy -plane

We can see from Figure 2.11 that $T_A(\mathbf{u})$ is the orthogonal projection of $\mathbf{u}$ on the xy -plane. The range of T_A is the xy -plane in $\mathcal{R}^3$.

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Example

Let k be a scalar and $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. The function $T_A : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ is defined by $T_A \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix}$ and is called a shear transformation. Notice the effect in Figure 2.12(a) and 2.12(b).

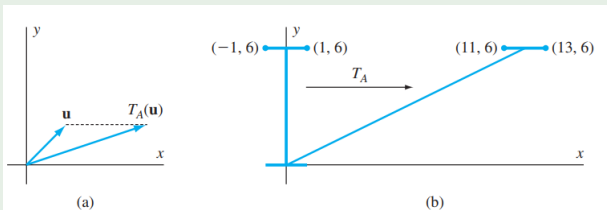


Figure 2.12 A shear transformation

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Theorem

For any $m \times n$ matrix A and any vectors u and v in $\mathbb{R}^n$, the following statements are true:

- (a) $T_A(u + v) = T_A(u) + T_A(v)$.
- (b) $T_A(cu) = cT_A(u)$ for every scalar c .

Definition

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a linear transformation (or simply linear) if, for all vectors u and v in $\mathbb{R}^n$ and all scalars c , both of the following conditions hold:

- (i) $T(u + v) = T(u) + T(v)$. (In this case, we say that T preserves vector addition.)
- (ii) $T(cu) = cT(u)$. (In this case, we say that T preserves scalar multiplication.)

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Remark

- 1 Every matrix transformation is linear.
- 2 The identity transformation $I : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, which is defined by $I(x) = x$ for all x in $\mathbb{R}^n$. It is easy to show that I is linear and its range is all of $\mathbb{R}^n$.
- 3 The zero transformation $T_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which is defined by $T_0(x) = 0$ for all x in $\mathbb{R}^n$. T_0 is linear. The range of T_0 consists precisely of the zero vector.

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Theorem

For any linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, the following statements are true:

- (a) $T(0) = 0$.
- (b) $T(-u) = -T(u)$ for all vectors u in $\mathbb{R}^n$.
- (c) $T(u - v) = T(u) - T(v)$ for all vectors u and v in $\mathbb{R}^n$.
- (d) $T(au + bv) = aT(u) + bT(v)$ for all vectors u and v in $\mathbb{R}^n$ and all scalars a and b .

Remark

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ are vectors in $\mathbb{R}^n$ and $a_1, a_2, \dots, a_k$ are scalars, then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + \cdots + a_kT(\mathbf{u}_k)$$

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Exercise

Suppose that $T : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

- (a) Find $T\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right)$.
- (b) Find $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and use the results to determine $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

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Exercise

Define $T : \mathcal{R}^2 \rightarrow \mathcal{R}^2$ by $T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \end{bmatrix}$. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathcal{R}^2$. Then $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

Prove that T is linear.

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Theorem

Let $T : \mathcal{R}^n \rightarrow \mathcal{R}^m$ be linear. Then there is a unique $m \times n$ matrix

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n)]$$

whose columns are the images under T of the standard vectors for $\mathcal{R}^n$, such that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v}$ in $\mathcal{R}^n$

Example

Let $T : \mathcal{R}^3 \rightarrow \mathcal{R}^2$ be defined by $T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix}$.

It is straightforward to show that T is linear. Find the standard matrix of T , .

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