

Lectures on Linear Algebra

Dr. rer. nat. Faten Said Abu-Shoga

Islamic University of Gaza

Chapter 2

2.8 Composition and Invertibility of Linear Transformations

April 1, 2021

CHAPTER 2 Matrices and Linear Transformations

2.8 Composition and Invertibility of Linear Transformations

ONTO AND ONE-TO-ONE FUNCTIONS

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - 4x_2 \\ 2x_1 + x_3 \end{bmatrix}$$

The standard matrix of T is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 3 & -4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Now $\mathbf{w}$ is in the range of T if and only if $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in $\mathbb{R}^3$. Writing $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$, we see that

$$\mathbf{w} = T(\mathbf{v}) = T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = v_1T(\mathbf{e}_1) + v_2T(\mathbf{e}_2) + v_3T(\mathbf{e}_3)$$

2.8 Composition and Invertibility of Linear Transformations

We conclude that the range of T equals the span of the columns of A

$$\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Remark

The range of a linear transformation equals the span of the columns of its standard matrix.

Definition

Definition A function $f : \mathcal{R}^n \rightarrow \mathcal{R}^m$ is said to be onto if its range is all of $\mathcal{R}^m$ that is, if every vector in $\mathcal{R}^m$ is an image.

2.8 Composition and Invertibility of Linear Transformations

Example

Determine if the linear transformation $T : \mathcal{R}^3 \rightarrow \mathcal{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \\ 2x_1 + 5x_2 + 10x_3 \end{bmatrix}$$

is onto.

Solution The standard matrix A of T and its reduced row echelon form R are

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 2 & 5 & 10 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\text{rank } A = 2 \neq 3$, and therefore T is not onto.

2.8 Composition and Invertibility of Linear Transformations

Determine if the linear transformation $T: \mathcal{R}^3 \rightarrow \mathcal{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 + 4x_3 \\ x_1 + 3x_2 + 6x_3 \\ 2x_1 + 5x_2 + 10x_3 \end{bmatrix}$$

is onto.

Solution For T to be onto, the rank of its standard matrix must equal 3. But the standard matrix A of T and its reduced row echelon form R are

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 6 \\ 2 & 5 & 10 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\text{rank } A = 2 \neq 3$, and therefore T is not onto.

2.8 Composition and Invertibility of Linear Transformations

Theorem

Let $T : \mathcal{R}^n \rightarrow \mathcal{R}^m$ be a linear transformation with standard matrix A . The following conditions are equivalent:

- (a) T is onto; that is, the range of T is $\mathcal{R}^m$.
- (b) The columns of A form a generating set for $\mathcal{R}^m$.
- (c) $\text{rank } A = m$.

There is a close relationship between the range of a matrix transformation and the consistency of a system of linear equations.

2.8 Composition and Invertibility of Linear Transformations

Consider the system

$$\begin{aligned}x_1 &= 1 \\2x_1 + x_2 &= 3 \\x_1 - x_2 &= 1\end{aligned}$$

The system is equivalent to the matrix equation $A\mathbf{x} = \mathbf{b}$, where

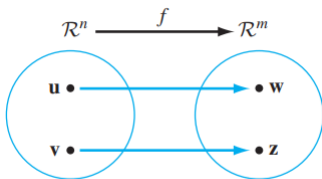
$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Because we can write the matrix equation as $T_A(\mathbf{x}) = \mathbf{b}$, the system has a solution if and only if $\mathbf{b}$ is in the range of T_A .

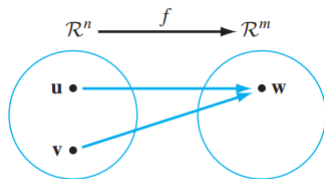
2.8 Composition and Invertibility of Linear Transformations

Definition

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be *one-to-one* if every pair of distinct vectors in $\mathbb{R}^n$ has distinct images. That is, if u and v are distinct vectors in $\mathbb{R}^n$, then $f(u)$ and $f(v)$ are distinct vectors in $\mathbb{R}^m$.



(a) Vectors with distinct images

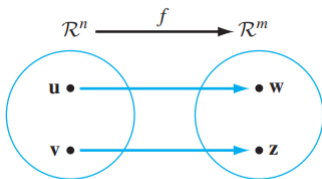


(b) Vectors with identical images

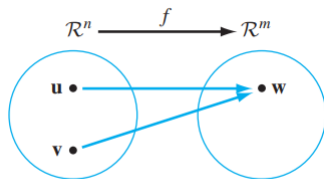
2.8 Composition and Invertibility of Linear Transformations

Definition

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be *one-to-one* if every pair of distinct vectors in $\mathbb{R}^n$ has distinct images. That is, if u and v are distinct vectors in $\mathbb{R}^n$, then $f(u)$ and $f(v)$ are distinct vectors in $\mathbb{R}^m$.



(a) Vectors with distinct images



(b) Vectors with identical images

2.8 Composition and Invertibility of Linear Transformations

Remark

Suppose that $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a one-to-one linear transformation. If w is a nonzero vector in $\mathbb{R}^n$, then $T(w) \neq T(0) = 0$, and hence 0 is the only vector in $\mathbb{R}^n$ whose image under T is the zero vector of $\mathbb{R}^m$.

Definition

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear. The null space of T is the set of all v in $\mathbb{R}^n$ such that $T(v) = 0$.

Remark

A linear transformation is one-to-one if and only if its null space contains only 0.

2.8 Composition and Invertibility of Linear Transformations

Example

Suppose that $T : \mathcal{R}^3 \rightarrow \mathcal{R}^2$ is defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \end{bmatrix}$$

Find a generating set for the null space of T .

Solution The standard matrix of T is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -3 \end{bmatrix}$$

Because the null space of T is the set of solutions of $A\mathbf{x} = \mathbf{0}$,

2.8 Composition and Invertibility of Linear Transformations

the reduced row echelon form of A is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This matrix corresponds to the system

$$x_1 - x_2 = 0$$

$$x_3 = 0$$

So every solution has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus a generating set for the null space of T is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

2.8 Composition and Invertibility of Linear Transformations

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with standard matrix A . Then the following statements are equivalent:

- (a) T is one-to-one.
- (b) The null space of T consists only of the zero vector.
- (c) The columns of A are linearly independent.
- (d) $\text{rank } A = n$.

2.8 Composition and Invertibility of Linear Transformations

Remark

We may write the system in the equivalent form $T_A(x) = b$. The following list compares the nature of solutions of $Ax = b$ and properties of T_A :

- (a) $Ax = b$ has a solution if and only if b is in the range of T_A .
- (b) $Ax = b$ has a solution for every b if and only if T_A is onto.
- (c) $Ax = b$ has at most one solution for every b if and only if T_A is one-to-one.

2.8 Composition and Invertibility of Linear Transformations

Exersice

Let $T : \mathcal{R}^2 \rightarrow \mathcal{R}^3$ be the linear transformation defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - x_2 \\ -x_1 + 2x_2 \\ 2x_1 \end{bmatrix}$$

- (a) Determine a generating set for the range of T .
- (b) Determine a generating set for the null space of T .
- (c) Is T onto?
- (d) Is T one-to-one?

2.8 Composition and Invertibility of Linear Transformations

COMPOSITION OF LINEAR TRANSFORMATIONS

Suppose that $f: \mathcal{R}^2 \rightarrow \mathcal{R}^3$ and $g: \mathcal{R}^3 \rightarrow \mathcal{R}^2$ are the functions defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_1 + x_2 \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ 3x_2 \end{bmatrix}.$$

Then $g \circ f: \mathcal{R}^2 \rightarrow \mathcal{R}^2$ is defined by

$$(g \circ f)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = g\left(f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\right) = g\left(\begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_1 + x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 - (x_1 + x_2) \\ 3x_1 x_2 \end{bmatrix}.$$

2.8 Composition and Invertibility of Linear Transformations

Theorem

Let $T: \mathcal{R}^n \rightarrow \mathcal{R}^n$ be a linear transformation with standard matrix A . Then T is invertible if and only if A is invertible, in which case $T^{-1} = T_{A^{-1}}$. Thus T^{-1} is linear, and its standard matrix is A^{-1} .

Suppose that $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ so that $T_A \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \begin{bmatrix} v_1 + 2v_2 \\ 3v_1 + 5v_2 \end{bmatrix}$. We saw in Example 1 of Section 2.3 that $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$. By Theorem 2.13, we have

$$T_A^{-1} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = T_{A^{-1}} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -5v_1 + 2v_2 \\ 3v_1 - v_2 \end{bmatrix}.$$

2.8 Composition and Invertibility of Linear Transformations

Property of T	The number of solutions of $Ax = b$	Property of the columns of A	Property of the rank of A
T is onto.	$Ax = b$ has at least one solution for every b in $\mathcal{R}^m$.	The columns of A are a generating set for $\mathcal{R}^m$.	$\text{rank } A = m$
T is one-to-one.	$Ax = b$ has at most one solution for every b in $\mathcal{R}^m$.	The columns of A are linearly independent.	$\text{rank } A = n$
T is invertible.	$Ax = b$ has a unique solution for every b in $\mathcal{R}^m$.	The columns of A are a linearly independent generating set for $\mathcal{R}^m$.	$\text{rank } A = m = n$

2.8 Composition and Invertibility of Linear Transformations

EXERCISES Page 188-190

1-60