

Lectures on Linear Algebra

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Chapter 3

3.1 Cofactor Expansion

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CHAPTER 3 Determinants

3.1 Cofactor Expansion

Consider the 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We see that

$$\begin{aligned} AC &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

3.1 Cofactor Expansion

and similarly,

$$\begin{aligned}CA &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\&= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\&= (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

Result

If $ad - bc \neq 0$, the matrix $\frac{1}{ad-bc}C$ is the inverse of A , and so A is invertible.

Thus the scalar $ad - bc$ determines whether or not the preceding matrix A is invertible. We call the scalar $ad - bc$ the determinant of A and denote it by $\det A$ or $|A|$.

3.1 Cofactor Expansion

Exercise

Evaluate the determinant of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$,
and $C = \begin{bmatrix} 8 & 3 \\ -6 & 5 \end{bmatrix}$. Are these matrices invertible? If so,
compute their inverses.

3.1 Cofactor Expansion

Exercise

Determine the scalars c for which $A - cI_2$ is not invertible, where

$$A = \begin{bmatrix} 11 & 12 \\ -8 & -9 \end{bmatrix}$$

Remark

We see that $[a]$ is invertible if and only if $a \neq 0$. Hence for a 1×1 matrix $[a]$, we define the determinant of $[a]$ by $\det [a] = a$.

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$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{row } i \\ \uparrow \text{column } j \end{array}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix}, A_{21} = \begin{bmatrix} 2 & 3 \\ 9 & 8 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 9 \end{bmatrix}, \text{ and } A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

3.1 Cofactor Expansion

Remark

We define the determinant of an $n \times n$ matrix A for $n \geq 3$ by

$$\det A = a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + \dots + (-1)^{1+n} a_{1n} \cdot \det A_{1n}$$

We denote the determinant of A by $\det A$ or $|A|$.

Definition

If we let $c_{ij} = (-1)^{i+j} \cdot \det A_{ij}$, then our definition of the determinant of A can be written as

$$\det A = a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n} \quad (*)$$

The number c_{ij} is called the (i, j) -cofactor of A , and equation $(*)$ is called the cofactor expansion of A along the first row.

3.1 Cofactor Expansion

Evaluate the determinant of A by using the cofactor expansion along the first row, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix}.$$

$$\begin{aligned}\det A &= a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} \\&= 1(-1)^{1+1} \det A_{11} + 2(-1)^{1+2} \det A_{12} + 3(-1)^{1+3} \det A_{13} \\&= 1(-1)^{1+1} \det \begin{bmatrix} 5 & 6 \\ 9 & 8 \end{bmatrix} + 2(-1)^{1+2} \det \begin{bmatrix} 4 & 6 \\ 7 & 8 \end{bmatrix} \\&\quad + 3(-1)^{1+3} \det \begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix} \\&= 1(1)(5 \cdot 8 - 6 \cdot 9) + 2(-1)(4 \cdot 8 - 6 \cdot 7) + 3(1)(4 \cdot 9 - 5 \cdot 7) \\&= 1(1)(-14) + (2)(-1)(-10) + 3(1)(1) = -14 + 20 + 3 = 9.\end{aligned}$$

3.1 Cofactor Expansion

Exercise

Evaluate the determinant of A by using the cofactor expansion.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 8 \\ 4 & 5 & 6 & 1 & 3 \\ 7 & 9 & 8 & 4 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3.1 Cofactor Expansion

Theorem

The determinant of an upper triangular $n \times n$ matrix or a lower triangular $n \times n$ matrix equals the product of its diagonal entries.

Exercise

Compute the determinants of

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 8 & 7 & 0 & 0 \\ -6 & -1 & -3 & 0 \\ 4 & 3 & 9 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

3.1 Cofactor Expansion

GEOMETRIC APPLICATIONS OF THE DETERMINANT

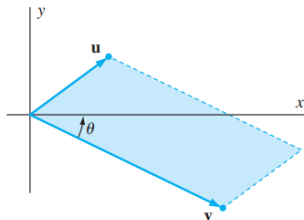


Figure 3.1 The parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$

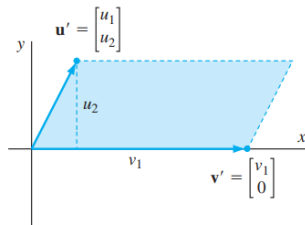


Figure 3.2 A rotation of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$

The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is $|\det[\mathbf{u} \ \mathbf{v}]|$

3.1 Cofactor Expansion

Example

The area of the parallelogram in $\mathcal{R}^2$ determined by the vectors

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

is

$$\left| \det \begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix} \right| = |(-2)(5) - (1)(3)| = |-13| = 13$$

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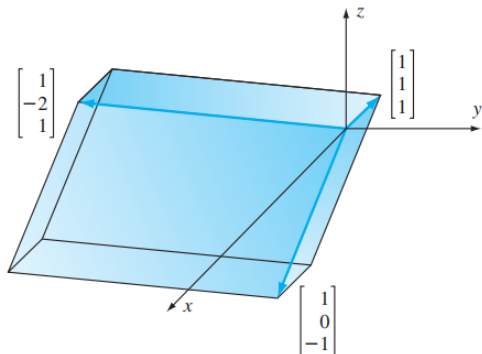


Figure 3.3 The parallelepiped determined by 3 vectors in $\mathcal{R}^3$

3.1 Cofactor Expansion

Example

The volume of the parallelepiped in $\mathcal{R}^3$ determined by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is

$$\left| \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \right| = 6$$

3.1 Cofactor Expansion

Remarks

- 1 If a $n \times n$ matrix has identical rows, then its determinant is zero.
- 2 For any $n \times n$ matrix A , $\det A^T = \det A$.
- 3 For any $n \times n$ matrices A and B , $\det AB = (\det A)(\det B)$.
- 4 The determinant of an $n \times n$ matrix with a zero row is zero.
- 5 $\det A^{-1} = \frac{1}{\det A}$

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