

Lectures on Linear Algebra

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Chapter 3

3.2 Properties of Determinants

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CHAPTER 3 Determinants

3.2 Properties of Determinants

The following sequence of elementary row operations transforms

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix}$$

into an upper triangular matrix U :

$$\begin{aligned} A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix} &\xrightarrow{-4\mathbf{r}_1 + \mathbf{r}_2 \rightarrow \mathbf{r}_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 9 & 8 \end{bmatrix} \\ &\xrightarrow{-7\mathbf{r}_1 + \mathbf{r}_3 \rightarrow \mathbf{r}_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -13 \end{bmatrix} \xrightarrow{-\frac{5}{3}\mathbf{r}_2 + \mathbf{r}_3 \rightarrow \mathbf{r}_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -3 \end{bmatrix} = U \end{aligned}$$

$$\det A = \det U = 1(-3)(-3) = 9.$$

3.2 Properties of Determinants

Theorem : Let A be an $n \times n$ matrix

- (a) If B is a matrix obtained by interchanging two rows of A , then $\det B = -\det A$.
- (b) If B is a matrix obtained by multiplying each entry of some row of A by a scalar k , then $\det B = k \cdot \det A$.
- (c) If B is a matrix obtained by adding a multiple of some row of A to a different row, then $\det B = \det A$.
- (d) For any $n \times n$ elementary matrix E , we have $\det EA = (\det E)(\det A)$

If an $n \times n$ matrix A is transformed into an upper triangular matrix U by elementary row operations other than scaling operations, then

$$\det A = (-1)^r u_{11} u_{22} \cdots u_{nn}$$

where r is the number of row interchanges performed.

3.2 Properties of Determinants

Example

Use elementary row operations to compute the determinant of

$$A = \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 4 & -4 & 4 & 15 \end{bmatrix}$$

Solution We apply steps 1 – 4 of the Gaussian elimination algorithm to transform A into an upper triangular matrix U .

$$A = \begin{bmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 4 & -4 & 4 & 15 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 4 & -4 & 4 & 15 \end{bmatrix}$$

3.2 Properties of Determinants

$$\begin{aligned} &\xrightarrow{2r_1+r_4} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 0 & 4 & -2 \\ 0 & 1 & 3 & -3 \\ 0 & -4 & 12 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & -4 & 12 & 1 \end{bmatrix} \\ &\xrightarrow{4r_2+r_4} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 24 & -11 \end{bmatrix} \xrightarrow{-6r_3+r_4} \begin{bmatrix} -2 & 0 & 4 & -7 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U \end{aligned}$$

Since U is an upper triangular matrix, we have $\det U = (-2) \cdot 1 \cdot 4 \cdot 1 = -8$. During the transformation of A into U , two row interchanges were performed. Thus

$$\det A = (-1)^2 \cdot \det U = -8$$

3.2 Properties of Determinants

FOUR PROPERTIES OF DETERMINANTS

Theorem

Let A and B be square matrices of the same size. The following statements are true:

- (a) A is invertible if and only if $\det A \neq 0$.
- (b) $\det AB = (\det A)(\det B)$.
- (c) $\det A^T = \det A$
- (d) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

3.2 Properties of Determinants

Example

For what scalar c is the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix}$$

not invertible?

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & c \\ 2 & 1 & 7 \end{bmatrix} &\xrightarrow{r_1+r_2 \rightarrow r_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 2 & 1 & 7 \end{bmatrix} \xrightarrow{-2r_1+r_3 \rightarrow r_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 0 & 3 & 3 \end{bmatrix} \\ &\xrightarrow{3r_2+r_3 \rightarrow r_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & c+2 \\ 0 & 0 & 3c+9 \end{bmatrix} \end{aligned}$$

Hence $\det A = 1(-1)(3c+9) = -3c-9$. Thus A is not invertible if and only if $0 = -3c-9$, that is, if and only if $c = -3$.

3.2 Properties of Determinants

CAUTION

Let A and B be arbitrary $n \times n$ matrices. Although $\det AB = (\det A)(\det B)$ by Theorem 3.4(b), the corresponding property for matrix addition is not true. Consider, for instance, the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly, $\det A = \det B = 0$, whereas $\det(A + B) = \det I_2 = 1$.
Therefore

$$\det(A + B) \neq \det A + \det B$$

So the determinant of a sum of matrices need not be equal to the sum of their determinants.

3.2 Properties of Determinants

CRAMER'S RULE

Theroem (Cramer's Rule)

Let A be an invertible $n \times n$ matrix, $\mathbf{b}$ be in $\mathcal{R}^n$, and M_i be the matrix obtained from A by replacing column i of A by $\mathbf{b}$. Then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{u}$ in which the components are given by

$$u_i = \frac{\det M_i}{\det A} \text{ for } i = 1, 2, \dots, n$$

3.2 Properties of Determinants

Example

Use Cramer's rule to solve the system of equations

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 + x_3 = 3$$

$$x_1 + x_2 - x_3 = 1$$

Solution The coefficient matrix of this system is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

3.2 Properties of Determinants

Example

Since $\det A = 6$, A is invertible by Theorem 3.4(a), and hence Cramer's rule can be used. In the notation of Theorem 3.5, we have

$$M_1 = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \text{ and } M_3 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore the unique solution of the given system is the vector with components

$$u_1 = \frac{\det M_1}{\det A} = \frac{15}{6} = \frac{5}{2}, u_2 = \frac{\det M_2}{\det A} = \frac{-6}{6} = -1, \text{ and } u_3 = \frac{\det M_3}{\det A} = \frac{3}{6} = \frac{1}{2}$$

It is readily checked that these values satisfy each of the equations in the given system.

3.2 Properties of Determinants

Exercise

Solve the following system of linear equations using Cramer's rule:

$$3x_1 + 8x_2 = 4$$

$$2x_1 + 6x_2 = 2$$

EXERCISES Pages 220 and 221

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