

Lectures on Linear Algebra

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Chapter 4

4.1 Subspaces

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CHAPTER 4 Subspaces and Their Properties

4.1 Subspaces

Illustration

$$u, v \in \mathbb{R}^n \Rightarrow u + v \in \mathbb{R}^n$$

$$c \in \mathbb{R} \text{ and } u \in \mathbb{R}^n \Rightarrow cu \in \mathbb{R}^n$$

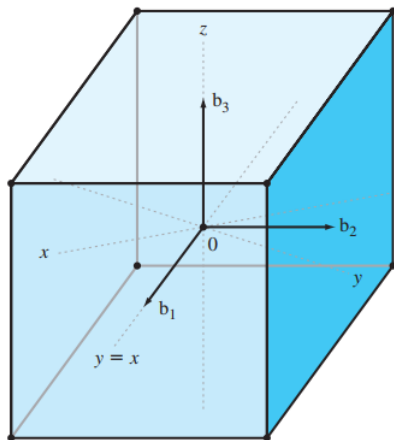
$\mathbb{R}^n$ is closed under the operations of vector addition and scalar multiplication.

$\mathbb{R}^n$ is a vector space for all n

Problem???

Which subsets of $\mathbb{R}^n$ possess this type of closure?!

4.1 Subspaces



4.1 Subspaces

Definition

A set W of vectors in $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it has the following three properties:

1. The zero vector belongs to W .
2. Whenever u and v belong to W , then $u + v$ belongs to W .
(W is closed under (vector) addition.)
3. Whenever u belongs to W and c is a scalar, then cu belongs to W . (W is closed under scalar multiplication.)

4.1 Subspaces

Example

The set $\mathcal{R}^n$ is a subspace of itself because

- 1 the zero vector belongs to $\mathcal{R}^n$,
- 2 the sum of any two vectors in $\mathcal{R}^n$ is also in $\mathcal{R}^n$,
- 3 every scalar multiple of a vector in $\mathcal{R}^n$ belongs to $\mathcal{R}^n$.

4.1 Subspaces

Example

The set W consisting of only the zero vector in $\mathcal{R}^n$ is a subspace of $\mathcal{R}^n$ called the zero subspace

- 1 0 is in W .
- 2 if $\mathbf{u}$ and $\mathbf{v}$ are vectors in W , then $\mathbf{u} = 0$ and $\mathbf{v} = 0$, so $\mathbf{u} + \mathbf{v} = 0 + 0 = 0$. Hence $\mathbf{u} + \mathbf{v}$ is in W ,
- 3 if $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u} = c0 = 0$, so $c\mathbf{u}$ is in W .

Remark

A subspace of $\mathcal{R}^n$ other than $\{0\}$ is called a nonzero subspace.

4.1 Subspaces

Example

Show that the set

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathcal{R}^3 : 6w_1 - 5w_2 + 4w_3 = 0 \right\}$$

is a subspace of $\mathcal{R}^3$

Solution:

1. Since $6(0) - 5(0) + 4(0) = 0$, the components of 0 satisfy the equation that defines W . Hence $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W .

4.1 Subspaces

2. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be in W . Then

$6u_1 - 5u_2 + 4u_3 = 0$, and also $6v_1 - 5v_2 + 4v_3 = 0$. Now

$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$. Since

$$\begin{aligned} & 6(u_1 + v_1) - 5(u_2 + v_2) + 4(u_3 + v_3) \\ &= (6u_1 - 5u_2 + 4u_3) + (6v_1 - 5v_2 + 4v_3) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

we see that the components of $\mathbf{u} + \mathbf{v}$ satisfy the equation defining W . Therefore $\mathbf{u} + \mathbf{v}$ is in W so W is closed under vector addition.

4.1 Subspaces

3. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be in W . For any scalar c , we have

$$c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}. \text{ Because}$$

$$6(cu_1) - 5(cu_2) + 4(cu_3) = c(6u_1 - 5u_2 + 4u_3) = c(0) = 0$$

the components of $c\mathbf{u}$ satisfy the equation defining W .
Therefore $c\mathbf{u}$ is in W .

Geometrically, W is a plane through the origin of $\mathbb{R}^3$.

4.1 Subspaces

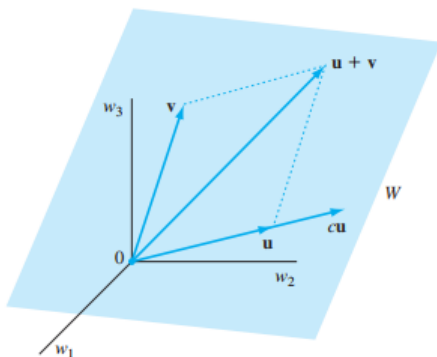


Figure 4.1 The subspace W is a plane through the origin.

4.1 Subspaces

Example

Let w be a nonzero vector in $\mathbb{R}^3$. Show that the set W of all multiples of w is a subspace of $\mathbb{R}^3$.

Solution:

1. $0 = 0w$ is in W .
2. let u and v be vectors in W . Then $u = aw$ and $v = bw$ for some scalars a and b . Since

$$u + v = aw + bw = (a + b)w$$

Hence $u + v$ is in W .

3. for any scalar c , $cu = c(aw) = (ca)w$ is a multiple of w .
Thus cu is in W ,

Therefore W is a subspace of $\mathbb{R}^3$. Note that W can be depicted as a line in $\mathbb{R}^3$ through the origin.

4.1 Subspaces

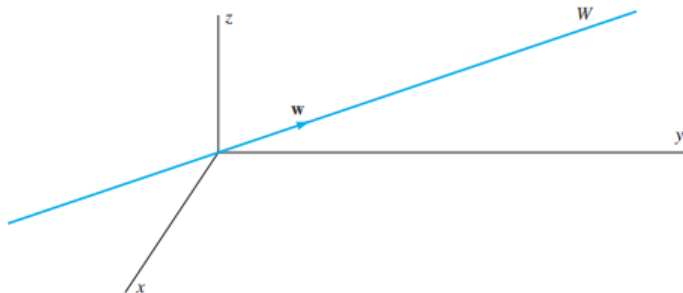


Figure 4.2 W is the set of all multiples of w .

Remark

The set of vectors in $\mathbb{R}^3$ that lie along a line through the origin is a subspace of $\mathbb{R}^3$. However, the set of vectors on a line in $\mathbb{R}^3$ that does not pass through the origin is not a subspace, for such a set does not contain the zero vector of $\mathbb{R}^3$.

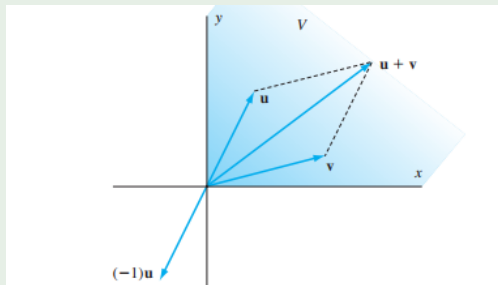


4.1 Subspaces

Example

Let V be the subset of $\mathcal{R}^2$ defined by

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{R}^2 : v_1 \geq 0 \text{ and } v_2 \geq 0 \right\}$$



4.1 Subspaces

The vectors in V are those that lie in the first quadrant of $\mathcal{R}^2$, and the nonnegative parts of the x - and y -axes.

Clearly, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in V . Suppose that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and

$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are in V . Then $u_1 \geq 0$, $u_2 \geq 0$, $v_1 \geq 0$, and $v_2 \geq 0$.

Hence $u_1 + v_1 \geq 0$ and $u_2 + v_2 \geq 0$, so that

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

is in V . Thus V is closed under vector addition.

However, V is not closed under scalar multiplication, because

$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ belongs to V , but $(-1)\mathbf{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ does not. Thus V

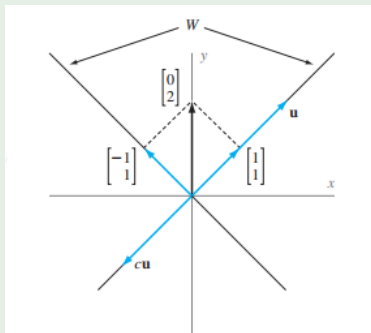
is not a subspace of $\mathcal{R}^2$

4.1 Subspaces

Example

Let W be the subset of $\mathcal{R}^2$ defined by

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1^2 = w_2^2 \right\}.$$



4.1 Subspaces

Consider a vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ in W . Since $u_1^2 = u_2^2$, it follows that $u_2 = \pm u_1$.

Hence the vector $\mathbf{u}$ lies along one of the lines $y = x$ or $y = -x$.

Clearly, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ belongs to W . Moreover, if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is in

W , then $u_1^2 = u_2^2$. So, for any scalar c , $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$ is in W

because $(cu_1)^2 = c^2 u_1^2 = c^2 u_2^2 = (cu_2)^2$. Thus W is closed under scalar multiplication. However, W is not closed under

vector addition. For example, the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

belong to W , but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ does not.

Thus W is not a subspace of $\mathbb{R}^2$.

4.1 Subspaces

Theorem

The span of a finite nonempty subset of $\mathcal{R}^n$ is a subspace of $\mathcal{R}^n$

4.1 Subspaces

Example

The set of vectors of the form

$$W = \left\{ \begin{bmatrix} 2a - 3b \\ b \\ -a + 4b \end{bmatrix} \in \mathcal{R}^3 : a \text{ and } b \text{ are scalars} \right\}$$

is a subspace of $\mathcal{R}^3$. Simply observe that

$$\begin{bmatrix} 2a - 3b \\ b \\ -a + 4b \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$$

so $W = \text{Span } \mathcal{S}$, where $\mathcal{S} = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \right\}$ Therefore W

is a subspace of $\mathcal{R}^3$.

4.1 Subspaces

SUBSPACES ASSOCIATED WITH A MATRIX

Definition

The null space of a matrix A is the solution set of $Ax = 0$. It is denoted by $Null\ A$.

Theorem

If A is an $m \times n$ matrix, then $Null\ A$ is a subspace of $\mathbb{R}^n$.

Definition

The column space of a matrix A is the *span* of its columns. It is denoted by $Col\ A$.

4.1 Subspaces

Example**

Find a generating set for the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Is $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in $\text{Col } A$? Is $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ in $\text{Col } A$?

Solution The column space of A is the span of the columns of A . Hence one generating set for $\text{Col } A$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix} \right\}$$

4.1 Subspaces

To see whether the vector $\mathbf{u}$ lies in the column space of A , we must determine whether $A\mathbf{x} = \mathbf{u}$ is consistent. Since the reduced row echelon form of $\begin{bmatrix} A & \mathbf{u} \end{bmatrix}$ is

$$\begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we see that the system is inconsistent, and hence $\mathbf{u}$ is not in $\text{Col } A$. On the other hand, the reduced row echelon form of $[A \mathbf{v}]$ is

$$\begin{bmatrix} 1 & 2 & 0 & -4 & 0.5 \\ 0 & 0 & 1 & 3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the system $A\mathbf{x} = \mathbf{v}$ is consistent, so $\mathbf{v}$ is in $\text{Col } A$.

4.1 Subspaces

Example

Find a generating set for the null space of the matrix A in

Example**. Is $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 3 \\ -1 \end{bmatrix}$ in Null A ? Is $\mathbf{v} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix}$ in Null A ?

Solution We must solve $A\mathbf{x} = 0$. Because the reduced row

echelon form of A is $\begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

the vector form of the general solution of $A\mathbf{x} = 0$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 4x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

4.1 Subspaces

It follows that

$$\text{Null } A = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

To see if the vector $\mathbf{u}$ lies in the null space of A , we must determine whether $A\mathbf{u} = \mathbf{0}$. An easy calculation confirms this; so $\mathbf{u}$ belongs to $\text{Null } A$. On the other hand,

$$A\mathbf{v} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$$

Since $A\mathbf{v} \neq \mathbf{0}$, we see that $\mathbf{v}$ is not in $\text{Null } A$.

4.1 Subspaces

SUBSPACES ASSOCIATED WITH A LINEAR TRANSFORMATION

Remark

The range of a linear transformation is the same as the column space of its standard matrix.

Example★

Determine a generating set for the range of the linear transformation $T : \mathcal{R}^4 \rightarrow \mathcal{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$$

4.1 Subspaces

Solution The standard matrix of T is

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Since the range of T is the same as the column space of A , a generating set for the range of T is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ -6 \end{bmatrix} \right\}$$

the set of columns of A .

4.1 Subspaces

Remark

The null space of a linear transformation is the same as the null space of its standard matrix.

This result implies that the null space of a linear transformation $T : \mathcal{R}^n \rightarrow \mathcal{R}^m$ is a subspace of $\mathcal{R}^n$

Example

Determine a generating set for the null space of the linear transformation in Example*.

Solution The standard matrix of T is given in Example*. Its reduced row echelon form is the matrix

$$\begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in Example**.

4.1 Subspaces

From the latter example, we see that

$$\text{Nul } A = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

4.1 Subspaces

Exercise

For the linear transformation $T : \mathcal{R}^4 \rightarrow \mathcal{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_3 + 2x_4 \\ -x_2 + x_3 + x_4 \\ 2x_1 + 3x_2 - x_3 + x_4 \end{bmatrix}$$

find generating sets for the null space and range.

EXERCISES Pages 237-240

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