

Lectures on Linear Algebra

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Chapter 4

4.2 Basis and Dimension

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CHAPTER 4 Subspaces and Their Properties

4.2 Basis and Dimension

Definition

Let V be a nonzero subspace of $\mathbb{R}^n$. A basis (plural, bases) for V is a linearly independent generating set for V .

Remark

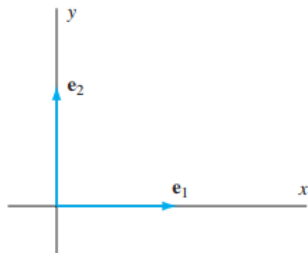
$\{e_1, e_2, \dots, e_n\}$ is a basis for $\mathbb{R}^n$. We call this basis the standard basis for $\mathbb{R}^n$ and denote it by ξ .

There are infinitely many bases for $\mathbb{R}^2$. for example

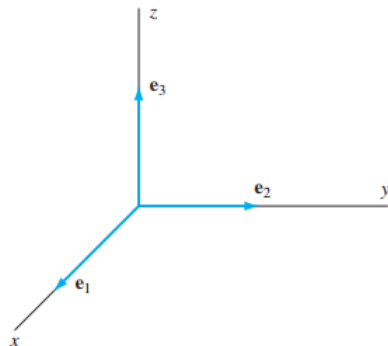
$$e_1, e_2$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

4.2 Basis and Dimension



The standard basis for $\mathcal{R}^2$



The standard basis for $\mathcal{R}^3$

4.2 Basis and Dimension

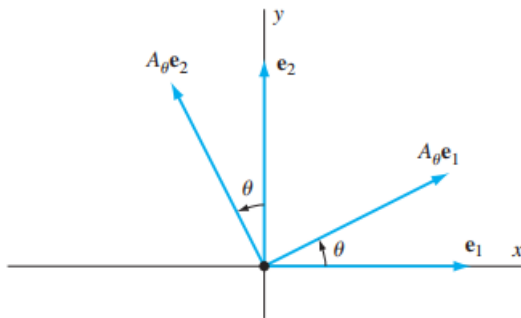


Figure 4.6 Rotated standard vectors

4.2 Basis and Dimension

Theorem

The pivot columns of a matrix form a basis for its column space.

Example

Find a basis for Col A if

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.2 Basis and Dimension

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Remark

The column space of a matrix is usually different from that of its reduced row echelon form. In fact, the column space of a matrix in reduced row echelon form always has a basis consisting of standard vectors, which is not usually the case for other matrices.

(Reduction Theorem)

Let S be a finite generating set for a nonzero subspace V of $\mathbb{R}^n$. Then S can be reduced to a basis for V by removing vectors from S .

4.2 Basis and Dimension

Example

Find a basis for $\text{Span } \mathcal{S}$ consisting of vectors in $\mathcal{S}$ if

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Solution Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & -1 & 1 & -1 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.2 Basis and Dimension

Thus the basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Remark

Every basis for $\mathcal{R}^n$ must contain exactly n vectors.

Remark

Let $\mathcal{S}$ be a finite subset of $\mathcal{R}^n$. Then the following are true:

1. If $\mathcal{S}$ is a generating set for $\mathcal{R}^n$, then $\mathcal{S}$ contains at least n vectors.
2. If $\mathcal{S}$ is linearly independent, then $\mathcal{S}$ contains at most n vectors.
3. If $\mathcal{S}$ is a basis for $\mathcal{R}^n$, then $\mathcal{S}$ contains exactly n vectors.

4.2 Basis and Dimension

(Extension Theorem)

Let S be a linearly independent subset of a nonzero subspace V of $\mathbb{R}^n$. Then S can be extended to a basis for V by inclusion of additional vectors. In particular, every nonzero subspace has a basis.

Theorem

Let V be a nonzero subspace of $\mathbb{R}^n$. Then any two bases for V contain the same number of vectors.

Remark

A basis is a generating set for a subspace containing the fewest possible vectors.

4.2 Basis and Dimension

Remark

If we adjoin an additional vector to a basis, then the larger set cannot be contained in a basis, because any two bases for a subspace must contain the same number of vectors.

Remark

A basis is a linearly independent subset of a subspace that is as large as possible.

Definition

The number of vectors in a basis for a nonzero subspace V of $\mathbb{R}^n$ is called the dimension of V and is denoted by $\dim V$. It is convenient to define the dimension of the zero subspace of $\mathbb{R}^n$ to be 0.

4.2 Basis and Dimension

Remark

Since the standard basis for $\mathcal{R}^n$ contains n vectors, $\dim \mathcal{R}^n = n$.

Exercise

Is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ a basis for $\mathcal{R}^3$? Justify your answer.

Theorem

Let V be a subspace of $\mathcal{R}^n$ with dimension k . Then every linearly independent subset of V contains at most k vectors; or equivalently, any finite subset of V containing more than k vectors is linearly dependent.

4.2 Basis and Dimension

Exercise

Find a basis for the subspace

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{R}^4 : x_1 - 3x_2 + 5x_3 - 6x_4 = 0 \right\}$$

of $\mathcal{R}^4$, and determine the dimension of V . (Since V is defined as the set of solutions of a homogeneous system of linear equations, V is, in fact, a subspace.)

Solution The vectors in V are solutions of $x_1 - 3x_2 + 5x_3 - 6x_4 = 0$,

$$x_1 = 3x_2 - 5x_3 + 6x_4$$

the vector form of the general solution of this system is

4.2 Basis and Dimension

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 + 6x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

As noted in Section 1.7, the set

$$\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

containing the vectors in the vector form is both a generating set for V and a linearly independent set. Therefore $\mathcal{S}$ is a basis for V . Since $\mathcal{S}$ is a basis for V containing 3 vectors, $\dim V = 3$.

4.2 Basis and Dimension

CONFIRMING THAT A SET IS A BASIS FOR A SUBSPACE

Theorem

Let V be a k -dimensional subspace of $\mathbb{R}^n$. Suppose that S is a subset of V with exactly k vectors. Then S is a basis for V if either S is linearly independent or S is a generating set for V .

Steps to Show that a Set B is a Basis for a Subspace V of $\mathbb{R}^n$

1. Show that B is contained in V .
2. Show that B is linearly independent (or that B is a generating set for V).
3. Compute the dimension of V , and confirm that the number of vectors in B equals the dimension of V .

4.2 Basis and Dimension

Example

Show that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathcal{R}^4 : v_1 + v_2 + v_4 = 0 \right\}$$

Solution Clearly, the components of the three vectors in $\mathcal{B}$ all satisfy the equation $v_1 + v_2 + v_4 = 0$. Thus $\mathcal{B}$ is a subset of V , so step 1 is satisfied.

4.2 Basis and Dimension

Because the reduced row echelon form of

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

it follows that $\mathcal{B}$ is linearly independent, so step 2 is satisfied. As in Example 3, we find that

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for V . Hence the dimension of V is 3. But $\mathcal{B}$ contains three vectors, so step 3 is satisfied, and hence $\mathcal{B}$ is a basis for V .

4.2 Basis and Dimension

Example

Let W be the span of $\mathcal{S}$, where

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

Show that a basis for W is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

4.2 Basis and Dimension

Solution

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \end{bmatrix}.$$

You should check that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in B . Therefore B is a subset of W , so step 1 is satisfied. The reduced row echelon form of B is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so B is linearly independent. Thus step 2 is satisfied.

4.2 Basis and Dimension

Since the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first 3 columns of A are its pivot columns and hence are a basis for $\text{Col } A = W$. Thus $\dim W = 3$, which equals the number of vectors contained in $\mathcal{B}$, and step 3 is satisfied. Hence $\mathcal{B}$ is a basis for W .

EXERCISES Pages 250-253

1-70 and 79-84