

Lectures on Linear Algebra

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Chapter 4

4.3 The Dimension of Subspaces Associated with a Matrix

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CHAPTER 4 Subspaces and Their Properties

4.3 The Dimension of Subspaces Associated with a Matrix

For the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

the set of pivot columns of A ,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\text{Col } A$. Hence the dimension of $\text{Col } A$ is 3 .

4.3 The Dimension of Subspaces Associated with a Matrix

Remark

- 1 The dimension of the column space of a matrix equals the rank of the matrix.
- 2 The dimension of the null space of a matrix equals the nullity of the matrix.

4.3 The Dimension of Subspaces Associated with a Matrix

Example

Show that

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -2 \\ -2 \\ -1 \end{bmatrix} \right\}$$

is a basis for the null space of

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix}$$

4.3 The Dimension of Subspaces Associated with a Matrix

Solution

- 1 Each vector in $\mathcal{B}$ is a solution of $A\mathbf{x} = 0$, the set $\mathcal{B}$ is contained in $\text{Null } A$
- 2 neither vector in $\mathcal{B}$ is a multiple of the other, so $\mathcal{B}$ is linearly independent.
- 3 Since the reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

the rank of A is 3 and its nullity is $5 - 3 = 2$. Hence $\text{Null } A$ has dimension 2, so $\mathcal{B}$ is a basis for $\text{Null } A$.

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Remark

It is important to note that, unlike the column space of a matrix, the row space is unaffected by elementary row operations. Consider the matrix A and its reduced row echelon form R , given by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$\text{Notice that} \quad \text{Row } A = \text{Row } R = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

but

$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \neq \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \text{Col } R$$

4.3 The Dimension of Subspaces Associated with a Matrix

Theorem

The nonzero rows of the reduced row echelon form of a matrix constitute a basis for the row space of the matrix.

Example

Recall that for the matrix $A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix}$

reduced row echelon form $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

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Hence a basis for Row A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

Thus the dimension of Row A is 3 , which is the rank of A .

Remarks

- 1 The dimension of the row space of a matrix equals its rank.
- 2 $\dim(\text{Row } A) = \dim(\text{Col } A) = \dim(\text{Row } A^T)$
- 3 The rank of any matrix equals the rank of its transpose

4.3 The Dimension of Subspaces Associated with a Matrix

SUBSPACES CONTAINED WITHIN SUBSPACES

Theorem

If V and W are subspaces of $\mathbb{R}^n$ with V contained in W , then $\dim V \leq \dim W$. Moreover, if V and W also have the same dimension, then $V = W$.

The Dimensions of the Subspaces Associated with an $m \times n$ Matrix A

Subspace	Containing Space	Dimension
Col A	$\mathcal{R}^m$	$\text{rank } A$
Null A	$\mathcal{R}^n$	$\text{nullity } A = n - \text{rank } A$
Row A	$\mathcal{R}^n$	$\text{rank } A$

4.3 The Dimension of Subspaces Associated with a Matrix

You can use the following procedures to obtain bases for these subspaces:

Bases for the Subspaces Associated with a Matrix A

Col A : The pivot columns of A form a basis for Col A .

Null A : The vectors in the vector form of the solution of $A\mathbf{x} = \mathbf{0}$ constitute a basis for Null A . (See page 80.)

Row A : The nonzero rows of the reduced row echelon form of A constitute a basis for Row A . (See page 257.)

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how that $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for the null space of
the matrix

$$\begin{bmatrix} 1 & 2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 3 & 2 \\ 2 & 4 & 7 & 0 & 1 \\ 3 & 6 & 11 & 1 & 2 \end{bmatrix}$$

Find the dimensions of the column space, null space, and row space of the matrix

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