

Lectures on Linear Algebra

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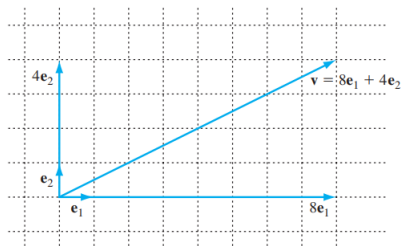
Chapter 4

4.4 Coordinate Systems

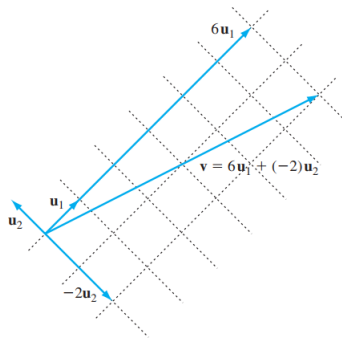
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CHAPTER 4 Subspaces and Their Properties

4.4 Coordinate Systems



(a)



(b)

Figure 4.9 Two coordinate systems for $\mathcal{R}^2$

4.4 Coordinate Systems

Theorem

Let $\beta = \{b_1, b_2, \dots, b_k\}$ be a basis for a subspace V of $\mathbb{R}^n$. Any vector v in V can be uniquely represented as a linear combination of the vectors in β ; that is, there are unique scalars $a_1, a_2, \dots, a_k$ such that

$$v = a_1 b_1 + a_2 b_2 + \dots + a_k b_k$$

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COORDINATE VECTORS

Definition

Let $\beta = \{b_1, b_2, \dots, b_n\}$ be a basis for $\mathbb{R}^n$. For each v in $\mathbb{R}^n$, there are unique scalars $c_1, c_2, \dots, c_n$ such that $v = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$. The vector

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

in $\mathbb{R}^n$ is called the coordinate vector of v relative to β , or the β -coordinate vector of v . We denote the β -coordinate vector of v by $[v]_\beta$.

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Example

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}.$$

Since $\mathcal{B}$ is a linearly independent set of 3 vectors in $\mathcal{R}^3$, $\mathcal{B}$ is a basis for $\mathcal{R}^3$. Calculate $\mathbf{u}$ if

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}.$$

Solution Since the components of $[\mathbf{u}]_{\mathcal{B}}$ are the coefficients that express $\mathbf{u}$ as a linear combination of the vectors in $\mathcal{B}$, we have

$$\mathbf{u} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 5 \end{bmatrix}.$$

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Example

$$\text{For } \mathbf{v} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\},$$

determine $[\mathbf{v}]_{\mathcal{B}}$.

Solution To find the $\mathcal{B}$ -coordinate vector of $\mathbf{v}$, we must write $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{B}$.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}.$$

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From this equation, we obtain a system of linear equations with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -4 \\ 1 & 1 & 2 & 4 \end{bmatrix}.$$

Since the reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

we see that the desired scalars are $c_1 = -6$, $c_2 = 4$, $c_3 = 3$. Thus

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}$$

is the $\mathcal{B}$ -coordinate vector of $\mathbf{v}$.

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Theorem

Let β be a basis for $\mathbb{R}^n$ and B be the matrix whose columns are the vectors in β . Then B is invertible, and for every vector v in $\mathbb{R}^n$, $B[v]_\beta = v$, or equivalently,

$$[v]_\beta = B^{-1}v$$

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Example

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix},$$

as in Example 2, and let B be the matrix whose columns are the vectors in $\mathcal{B}$. Then

$$[\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}.$$

$$\mathbf{u} = B[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ 5 \end{bmatrix},$$

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Exercise

Verify that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{R}^3$. Then find $\mathbf{u}$ if $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ -1 \end{bmatrix}$.

Find $[\mathbf{v}]_{\mathcal{B}}$ if $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ and $\mathcal{B}$ is defined as in Practice Problem 1.

EXERCISES Pages 271 and 272

1-43 and 51-54