

Lectures on Linear Algebra

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Chapter 4

4.5 Matrix Representations of Linear Operators

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CHAPTER 4 Subspaces and Their Properties

4.5 Matrix Representations of Linear Operators

Definition

A linear transformation where the domain and codomain equal $\mathbb{R}^n$ is called a linear operator on $\mathbb{R}^n$.

Definition

Let T be a linear operator on $\mathbb{R}^n$ and $\beta = b_1, b_2, \dots, b_n$ be a basis for $\mathbb{R}^n$. The matrix

$$[[T(b_1)]_\beta [T(b_2)]_\beta \dots [T(b_n)]_\beta]$$

is called the matrix representation of T with respect to β , or the β -matrix of T . It is denoted by $[T]_\beta$.

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Example

Let T be the linear operator on $\mathcal{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix}.$$

Calculate the matrix representation of T with respect to the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

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Solution Applying T to each of the vectors in $\mathcal{B}$, we obtain

$$T(\mathbf{b}_1) = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \quad T(\mathbf{b}_2) = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{b}_3) = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}.$$

We must now compute the *coordinate vectors* of these images with respect to $\mathcal{B}$. Let $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$. Then

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = B^{-1}T(\mathbf{b}_1) = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad [T(\mathbf{b}_2)]_{\mathcal{B}} = B^{-1}T(\mathbf{b}_2) = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix},$$

and

$$[T(\mathbf{b}_3)]_{\mathcal{B}} = B^{-1}T(\mathbf{b}_3) = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix},$$

from which it follows that the $\mathcal{B}$ -matrix of T is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & -9 & 8 \\ -1 & 3 & -3 \\ 1 & 6 & 1 \end{bmatrix}.$$

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Theorem

Let T be a linear operator on $\mathbb{R}^n$, B a basis for $\mathbb{R}^n$, B the matrix whose columns are the vectors in B , and A the standard matrix of T . Then

$$[T]_{\beta} = B^{-1}AB$$

, or equivalently,

$$A = B[T]_{\beta}B^{-1}$$

Example

Calculate $[T]_{\beta}$ if T and β are the linear operator and the basis given by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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Solution The standard matrix of T is

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ T(\mathbf{e}_3)] = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}.$$

Taking B to be the matrix whose columns are the vectors in $\mathcal{B}$, we have

$$[T]_{\mathcal{B}} = B^{-1}AB = \begin{bmatrix} 3 & -9 & 8 \\ -1 & 3 & -3 \\ 1 & 6 & 1 \end{bmatrix}.$$

Note that our answer agrees with that in Example 1.

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Exercise

Find the $\mathcal{B}$ -matrix representation of the linear operator T on $\mathcal{R}^3$, where

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + 2x_3 \\ x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

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Example

Let T be a linear operator on $\mathcal{R}^3$ such that

$$T \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad T \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Find the standard matrix of T .

Solution Let A denote the standard matrix of T ,

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

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and

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Observe that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly independent and thus is a basis for $\mathcal{R}^3$. Hence we are given the images of the vectors in a basis for $\mathcal{R}^3$. Let $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$ and $C = [\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{c}_3]$. Then $A\mathbf{b}_1 = T(\mathbf{b}_1) = \mathbf{c}_1$, $A\mathbf{b}_2 = T(\mathbf{b}_2) = \mathbf{c}_2$, and $A\mathbf{b}_3 = T(\mathbf{b}_3) = \mathbf{c}_3$. Thus

$$AB = A[\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3] = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ A\mathbf{b}_3] = [\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{c}_3] = C.$$

Because the columns of B are linearly independent, B is invertible by the Invertible Matrix Theorem. Thus

$$A = A(BB^{-1}) = (AB)B^{-1} = CB^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ .5 & 1.5 & -1.5 \\ -.5 & -.5 & 1.5 \end{bmatrix}.$$



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EXERCISES

Let $\beta = \{b_1, b_2\}$ be a basis for $\mathbb{R}^2$ and T be a linear operator on $\mathbb{R}^2$ such that $T(b_1) = b_1 + 4b_2$ and $T(b_2) = -3b_1$. Determine $[T]_\beta$.

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EXERCISES

determine (a) $[T]_{\beta}$, (b) the standard matrix of T , and (c) an explicit formula for $T(x)$ from the given information.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, T \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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