

Lectures on Linear Algebra

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Chapter 5

5.1 Eigenvalues and Eigenvectors

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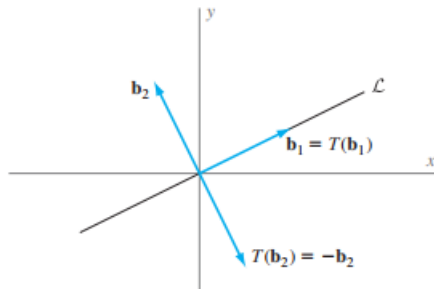
CHAPTER 5 Eigenvalues, Eigenvectors, and Diagonalization

5.1 Eigenvalues and Eigenvectors

Consider the vectors

$$\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

apply a reflection on the line $\mathcal{L}$



5.1 Eigenvalues and Eigenvectors

Nonzero vectors that are mapped to a multiple of themselves play an important role in understanding the behavior of linear operators and square matrices.

Definitions

Let T be a linear operator on $\mathcal{R}^n$. A nonzero vector $\mathbf{v}$ in $\mathcal{R}^n$ is called an eigenvector of T if $T(\mathbf{v})$ is a multiple of $\mathbf{v}$; that is, $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ . The scalar λ is called the eigenvalue of T that corresponds to $\mathbf{v}$.

5.1 Eigenvalues and Eigenvectors

For the reflection T of $\mathcal{R}^2$ about line $\mathcal{L}$ and the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, where $\mathbf{b}_1$ is in the direction of $\mathcal{L}$ and $\mathbf{b}_2$ is in a direction perpendicular to $\mathcal{L}$, we have

$$T(\mathbf{b}_1) = \mathbf{b}_1 = 1\mathbf{b}_1 \quad \text{and} \quad T(\mathbf{b}_2) = -\mathbf{b}_2 = (-1)\mathbf{b}_2$$

Therefore $\mathbf{b}_1$ is an eigenvector of T with corresponding eigenvalue 1, and $\mathbf{b}_2$ is an eigenvector of T with corresponding eigenvalue -1 . No nonzero vectors other than multiples of $\mathbf{b}_1$ and $\mathbf{b}_2$ have this property.

5.1 Eigenvalues and Eigenvectors

Definitions

Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v}$ in $\mathcal{R}^n$ is called an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ . The scalar λ is called the eigenvalue of A that corresponds to $\mathbf{v}$.

Remark

The eigenvectors and corresponding eigenvalues of a linear operator are the same as those of its standard matrix.

5.1 Eigenvalues and Eigenvectors

Example

For

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix}$$

show that $\mathbf{v}$ is an eigenvector of A .

Solution

$$A\mathbf{v} = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 4\mathbf{v}$$

we see that $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue 4.

5.1 Eigenvalues and Eigenvectors

Exercise

Show that

$$\mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

are eigenvectors of

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix}$$

To what eigenvalues do $\mathbf{u}$ and $\mathbf{v}$ correspond?

5.1 Eigenvalues and Eigenvector

Remark

Note that $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ if and only if $\mathbf{v}$ is a nonzero vector such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

$$A\mathbf{v} - \lambda I_n \mathbf{v} = \mathbf{0}$$

$$(A - \lambda I_n) \mathbf{v} = \mathbf{0}.$$

Thus $\mathbf{v}$ is a nonzero solution of the system of linear equations $(A - \lambda I_n) \mathbf{x} = \mathbf{0}$.

5.1 Eigenvalues and Eigenvectors

Remark

Let A be an $n \times n$ matrix with eigenvalue λ . The eigenvectors of A corresponding to λ are the nonzero solutions of $(A - \lambda I_n)x = 0$

Definitions

- 1 The set of solutions of $(A - \lambda I_n)x = 0$ is called the eigenspace of A corresponding to the eigenvalue λ . This is just the null space of $(A - \lambda I_n)$, and hence it is a subspace of $\mathbb{R}^n$.
- 2 If λ is an eigenvalue of a linear operator T on $\mathbb{R}^n$, the set of vectors v in $\mathbb{R}^n$ such that $T(v) = \lambda v$ is called the eigenspace of T corresponding to λ .

Remark

The eigenspace of A corresponding to λ consists of the zero vector and all the eigenvectors corresponding to λ .

5.1 Eigenvalues and Eigenvectors

Exercise

Show that 3 and -2 are eigenvalues of the linear operator T on $\mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_2 \\ -3x_1 + x_2 \end{bmatrix}$$

and find bases for the corresponding eigenspaces.

5.1 Eigenvalues and Eigenvectors

Exercise

Show that 1 is an eigenvalue of the linear operator on $\mathbb{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 - x_2 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

and find a basis for the corresponding eigenspace.

5.1 Eigenvalues and Eigenvectors

Exercise

Show that 3 is an eigenvalue of

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

and find a basis for the corresponding eigenspace.

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