

Lectures on Linear Algebra

Dr. rer. nat. Faten Said Abu-Shoga

Islamic University of Gaza

Chapter 5

5.2 The Characteristic Polynomial

April 27, 2021

CHAPTER 5 Eigenvalues, Eigenvectors, and Diagonalization

5.2 The Characteristic Polynomial

Definitions

- 1 The eigenvalues of a square matrix A are the values of t that satisfy

$$\det(A - tI_n) = 0$$

- 2 The equation $\det(A - tI_n) = 0$ is called the characteristic equation of A .
- 3 $\det(A - tI_n)$ is called the characteristic polynomial of A .
- 4 The eigenvalues of the matrix A are the (real) roots of the characteristic polynomial of A .

5.2 The Characteristic Polynomial

Example

Determine the eigenvalues of $A = \begin{bmatrix} -4 & -3 \\ 3 & 6 \end{bmatrix}$ and then find a basis for each eigenspace.

Solution We begin by forming the matrix

$$A - tI_2 = \begin{bmatrix} -4 - t & -3 \\ 3 & 6 - t \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned} \det(A - tI_2) &= (-4 - t)(6 - t) - (-3) \cdot 3 \\ &= (-24 - 2t + t^2) + 9 = t^2 - 2t - 15 \\ &= (t + 3)(t - 5) \end{aligned}$$

5.2 The Characteristic Polynomial

Therefore the eigenvalues of A are -3 and 5 .

Solve $(A + 3I_2)\mathbf{x} = \mathbf{0}$ and $(A - 5I_2)\mathbf{x} = \mathbf{0}$ to find bases for the eigenspaces. Since the reduced row echelon form of $A + 3I_2$ is

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

the vector form of the general solution of $(A + 3I_2)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

the basis for the eigenspace of A corresponding to the eigenvalue -3 is

$$\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

5.2 The Characteristic Polynomial

In a similar manner, the reduced row echelon form of $A - 5I_2$, which is

$$\begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

produces the basis

$$\left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$$

for the eigenspace of A corresponding to the eigenvalue 5 .

5.2 The Characteristic Polynomial

Remark

The eigenvalues of an upper triangular or lower triangular matrix are its diagonal entries.

Example

Determine the eigenvalues of the matrix

$$A = \begin{bmatrix} -3 & -1 & -7 & 1 \\ 0 & 6 & 9 & -2 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Solution Because A is an upper triangular matrix, its eigenvalues are its diagonal entries, which are $-3, 6, -5$, and 8 .

5.2 The Characteristic Polynomial

Exercise

Determine the eigenvalues of the matrix

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 8 & 7 & -2 & 0 \\ 9 & -5 & 6 & 3 \end{bmatrix}$$

5.2 The Characteristic Polynomial

THE MULTIPLICITY OF AN EIGENVALUE

Definition

If λ is an eigenvalue of an $n \times n$ matrix M , then the largest positive integer k such that $(t - \lambda)^k$ is a factor of the characteristic polynomial of M is called the multiplicity of λ .

Exercise

If the characteristic polynomial of a matrix is

$$-(t - 3)(t + 5)^2(t - 8)^4$$

determine the eigenvalues of the matrix and their multiplicities.

5.2 The Characteristic Polynomial

Remark

Let λ be an eigenvalue of a matrix A . The dimension of the eigenspace of A corresponding to λ is less than or equal to the multiplicity of λ .

Example

Determine the eigenvalues, their multiplicities, and a basis for each eigenspace of the linear operator T on $\mathcal{R}^3$ defined by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 \\ 2x_1 - x_2 - x_3 \\ -x_3 \end{bmatrix}$$

Solution The standard matrix of T is $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$.

5.2 The Characteristic Polynomial

Hence the characteristic polynomial of T is

$$\begin{aligned}\det(A - tI_3) &= \det \begin{bmatrix} -1-t & 0 & 0 \\ 2 & -1-t & -1 \\ 0 & 0 & -1-t \end{bmatrix} \\ &= (-1-t) \cdot \det \begin{bmatrix} -1-t & -1 \\ 0 & -1-t \end{bmatrix} \\ &= (-1-t)^3 = -(t+1)^3.\end{aligned}$$

Therefore the only eigenvalue of T is -1 , and its multiplicity is 3. The eigenspace of T corresponding to -1 is the solution set of $(A + I_3)\mathbf{x} = \mathbf{0}$. Since the reduced row echelon form of $A + I_3$ is

$$\begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

5.2 The Characteristic Polynomial

we see that

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

is a basis for the eigenspace of T corresponding to -1 . Note that this eigenspace is 2-dimensional and the multiplicity of -1 is 3,

5.2 The Characteristic Polynomial

THE EIGENVALUES OF SIMILAR MATRICES

Definition

Two matrices A and B are called similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

Remark

Similar matrices have the same characteristic polynomial and hence have the same eigenvalues and multiplicities. In addition, their eigenspaces corresponding to the same eigenvalue have the same dimension.

EXERCISES Pages 309 and 311

1-72