

Lectures on Linear Algebra

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Chapter 5

5.3 Diagonalization of Matrices

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CHAPTER 5 Eigenvalues, Eigenvectors, and Diagonalization

5.3 Diagonalization of Matrices

Definition

An $n \times n$ matrix A is called diagonalizable if $A = PDP^{-1}$ for some diagonal $n \times n$ matrix D and some invertible $n \times n$ matrix P .

Theorem

An $n \times n$ matrix A is diagonalizable if and only if there is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A .

Furthermore, $A = PDP^{-1}$, where D is a diagonal matrix and P is an invertible matrix, if and only if the columns of P are a basis for $\mathbb{R}^n$ consisting of eigenvectors of A and the diagonal entries of D are the eigenvalues corresponding to the respective columns of P .

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Theorem

A set of eigenvectors of a square matrix that correspond to distinct eigenvalues is linearly independent.

Remark

Every $n \times n$ matrix having n distinct eigenvalues is diagonalizable.

Algorithm for Matrix Diagonalization

Let A be a diagonalizable $n \times n$ matrix. Combining bases for each eigenspace of A forms a basis $\mathcal{B}$ for $\mathcal{R}^n$ consisting of eigenvectors of A . Therefore, if P is the matrix whose columns are the vectors in $\mathcal{B}$ and D is a diagonal matrix whose diagonal entries are eigenvalues of A corresponding to the respective columns of P , then $A = PDP^{-1}$.

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Example

Show that the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$

Solution Before we computed the characteristic polynomial $-(t+1)^2(t-3)$ of A , and we also found that

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is basis for the eigenspace of A corresponding to eigenvalue 3.

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Similarly.

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is a basis for the eigenspace corresponding to -1 . The combined set

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

is therefore linearly independent. It follows that A is diagonalizable and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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Exercise

The characteristic polynomial of

$$A = \begin{bmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

is $-(t+1)(t-2)^2$. Show that A is diagonalizable by finding an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

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WHEN IS A MATRIX DIAGONALIZABLE?

Test for a Diagonalizable Matrix Whose Characteristic Polynomial Is Known

An $n \times n$ matrix A is diagonalizable if and only if both of the following conditions are true:

- 1 The total number of eigenvalues of A , when each eigenvalue is counted as often as its multiplicity, is equal to n .
- 2 For each eigenvalue λ of A , the dimension of the corresponding eigenspace, which is $n - \text{rank}(A - \lambda I_n)$, is equal to the multiplicity of λ .

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Exercise

Determine whether each of the following matrices is diagonalizable (using real eigenvalues):

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -7 & -3 & -6 \\ 0 & -4 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$
$$C = \begin{bmatrix} -6 & -3 & 1 \\ 5 & 2 & -1 \\ 2 & 3 & -5 \end{bmatrix} \quad M = \begin{bmatrix} -3 & 2 & 1 \\ 3 & -4 & -3 \\ -8 & 8 & 6 \end{bmatrix}$$

The respective characteristic polynomials of these matrices are

$$-(t+1)(t^2+4), \quad -(t+1)(t+4)^2, \quad -(t+1)(t+4)^2, \quad \text{and} \quad -(t+1)(t^2-4).$$

EXERCISES Pages 321 and 322

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