

Lectures on Linear Algebra

Dr. rer. nat. Faten Said Abu-Shoga

Islamic University of Gaza

Chapter 7

7.1 Vector Spaces and Their Subspaces

April 30, 2021

CHAPTER 7 Vector Spaces

7.1 Vector Spaces and Their Subspaces

Definition

A (real) vector space is a set V on which two operations, called vector addition and scalar multiplication, are defined so that for any elements u , v , and w in V and any scalars a and b , the sum $u + v$ and the scalar multiple au are unique elements of V , and such that the following axioms hold:



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Axioms of a Vector Space

- ① $u + v = v + u$. (commutative law of vector addition)
- ② $(u + v) + w = u + (v + w)$. (associative law of vector addition)
- ③ There is an element 0 in V such that $u + 0 = u$.
- ④ There is an element $-u$ in V such that $u + (-u) = 0$.
- ⑤ $1u = u$.
- ⑥ $(ab)u = a(bu)$.
- ⑦ $a(u + v) = au + av$.
- ⑧ $(a + b)u = au + bu$.

Remark

The vector 0 in axioms 3 and 4 is called **the zero vector** and it is unique. For any vector u in a vector space V , the vector $-u$ in axiom 4 is called **the additive inverse** of u and is unique.

7.1 Vector Spaces and Their Subspaces

Example

For any given positive integers m and n , let $\mathcal{M}_{m \times n}$ denote the set of all $m \times n$ matrices. Then $\mathcal{M}_{m \times n}$ is a vector space with the operations of matrix addition and multiplication of a matrix by a scalar. In this case, the $m \times n$ zero matrix plays the role of the zero vector.

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Example

For any given positive integers m and n , let $\mathcal{L}(\mathcal{R}^n, \mathcal{R}^m)$ denote the set of all linear transformations from $\mathcal{R}^n$ to $\mathcal{R}^m$. Let T and U be in $\mathcal{C}(\mathcal{R}^n, \mathcal{R}^m)$ and c be a scalar. Define $(T + U) : \mathcal{R}^n \rightarrow \mathcal{R}^m$ and $cT : \mathcal{R}^n \rightarrow \mathcal{R}^m$ by

$$(T + U)(\mathbf{x}) = T(\mathbf{x}) + U(\mathbf{x}) \text{ and } (cT)(\mathbf{x}) = cT(\mathbf{x})$$

for all $\mathbf{x}$ in $\mathcal{R}^n$. By Exercises 83 and 84 of Section 2.7, $T + U$ and cT are in $\mathcal{C}(\mathcal{R}^n, \mathcal{R}^m)$. It can be shown that $\mathcal{L}(\mathcal{R}^n, \mathcal{R}^m)$ is a vector space under these operations. The zero transformation T_0 plays the role of the zero vector, and for any transformation T , we have that $(-1)T$ is the additive inverse $-T$.

7.1 Vector Spaces and Their Subspaces

Example

Let $\mathcal{P}$ denote the set of all polynomials

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

where n is a nonnegative integer and $a_0, a_1, \dots, a_n$ are real numbers. For each i , the scalar a_i is called the coefficient of x^i . Furthermore, if $a_i = 0$, we often omit the term a_ix^i entirely. The unique polynomial $p(x)$ with only zero coefficients is called the zero polynomial. The degree of a nonzero polynomial $p(x)$ is defined to be the largest exponent of x that appears in the representation

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

with a nonzero coefficient. The zero polynomial is not assigned a degree, and a constant polynomial has degree 0 .

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If $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_mx^m$ then $p(x)$ and $q(x)$ are equal if and only if $m = n$ and $a_i = b_i$ for $i = 0, 1, \dots, n$.

For any polynomials

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

and

$$q(x) = b_0 + b_1x + \cdots + b_nx^n$$

(not necessarily of the same degree), and any scalar a , we define the sum $p(x) + q(x)$ and the scalar multiple $ap(x)$ by

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

$$ap(x) = (a \cdot a_0) + (a \cdot a_1)x + \cdots + (a \cdot a_n)x^n$$

7.1 Vector Spaces and Their Subspaces

Example

Let S be the set $\mathcal{R}^2$ with addition and scalar multiplication defined as

$$(a, b) \oplus (c, d) = (a + c, 0) \quad \text{and} \quad k \odot (a, b) = (ka, kb)$$

for all (a, b) and (c, d) in $\mathcal{R}^2$. Show that S is not a vector space. Solution We show that S has no zero vector. Suppose that (z, w) is a zero vector for S . Then

$$(1, 1) \oplus (z, w) = (1 + z, 0) \neq (1, 1).$$

So (z, w) does not satisfy axiom 3, and hence S has no zero vector.

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PROPERTIES OF VECTOR SPACES

Theorem :PROPERTIES OF VECTOR SPACES

Let V be a vector space. For any $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and any scalar a , the following statements are true:

- 1 If $\mathbf{u} + \mathbf{v} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{w}$ (right cancellation law)
- 2 If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. (left cancellation law)
- 3 The zero vector $\mathbf{0}$ is unique; that is, it is the only vector in V that satisfies axiom 3 .
- 4 Each vector in V has exactly one additive inverse.
- 5 $0\mathbf{v} = \mathbf{0}$
- 6 $a\mathbf{0} = \mathbf{0}$.
- 7 $(-1)\mathbf{v} = -\mathbf{v}$
- 8 $(-a)\mathbf{v} = a(-\mathbf{v}) = -(a\mathbf{v})$

7.1 Vector Spaces and Their Subspaces

Example 1

Let

$$\mathcal{V} = \{(x, y, z) : x, y, z \in \mathbb{R}\},$$

define addition and scalar multiplication on $\mathcal{V}$ by:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \text{ and} \\ \alpha(x, y, z) = (\alpha x, y, z).$$

Then $\mathcal{V}$ is not a vector space. One can easily show that $(\alpha + \beta)(x, y, z) \neq \alpha(x, y, z) + \beta(x, y, z)$.

Example 3

Let

$$\mathcal{V} = \{(1, x, y) : x, y \in \mathbb{R}\},$$

under the usual addition of vectors and the usual multiplication of a scalar. Then $\mathcal{V}$ is not a vector space. One can easily show that for $u = (1, x_1, y_1)$ and $v = (1, x_2, y_2)$, $u + v \notin \mathcal{V}$.

7.1 Vector Spaces and Their Subspaces

Example 4

Let

$$\mathcal{V} = \{A : A \text{ is a } 2 \times 2 \text{ matrix with } \det(A) \neq 0\},$$

under the usual addition and the usual scalar multiplication of matrices. Then $\mathcal{V}$ is not a vector space. Clearly, the zero matrix $\mathbf{0}_{2 \times 2} \notin \mathcal{V}$.

Example 5

Let $\mathcal{V} = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$, define addition and scalar multiplication on $\mathcal{V}$ by:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 - y_2), \text{ and} \\ \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2).$$

Then $\mathcal{V}$ is not a vector space. Clearly, $u + v \neq v + u$.

7.1 Vector Spaces and Their Subspaces

SUBSPACES

Definition

A subset W of a vector space V is called a subspace of V if W satisfies the following three properties:

- 1 The zero vector of V is in W .
- 2 Whenever u and v belong to W , then $u + v$ belongs to W . (In this case, we say that W is closed under (vector) addition.)
- 3 Whenever u belongs to W and c is a scalar, then cu belongs to W . (In this case, we say that W is closed under scalar multiplication.)

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Example

Let S be a nonempty set, and let W be the subset of $\mathcal{F}(S)$ consisting of all functions f such that $f(s_0) = 0$ for some particular element s_0 in S . Clearly, the zero function lies in W . For any functions f and g in S , and any scalar a .

$$(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0$$

and

$$(af)(s_0) = af(s_0) = a \cdot 0 = 0$$

Hence $f + g$ and af are in W . We conclude that W is closed under the operations of $\mathcal{F}(S)$. Therefore W is a subspace of V .

7.1 Vector Spaces and Their Subspaces

Example

Let W be the set of all $n \times n$ matrices with trace equal to zero. Show that W is a subspace of $M_{n \times n}$

Solution Since the $n \times n$ zero matrix has trace equal to zero, it belongs to W . Suppose that A and B are matrices in W . Then

$$\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B) = 0 + 0 = 0$$

and, for any scalar c .

$$\text{trace}(cA) = c \cdot \text{trace}(A) = c \cdot 0 = 0$$

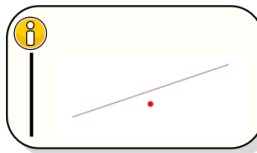
Therefore $A + B$ and cA are in W . We conclude that W is closed under the operations of $M_{n \times n}$. Therefore W is a subspace of $M_{n \times n}$

7.1 Vector Spaces and Their Subspaces

Non-Example

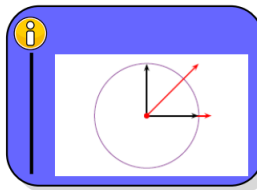
A line L (or any other set) that doesn't contain the origin is not a subspace.

Fails: 1.



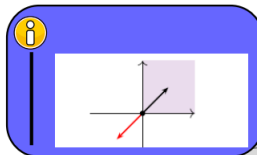
Non-Example

A circle C is not a subspace. Fails: 1,2,3.
Think: a circle isn't a "linear space."



Non-Example

The first quadrant in $\mathbb{R}^2$ is not a subspace. Fails: 3 only.



7.1 Vector Spaces and Their Subspaces

LINEAR COMBINATIONS AND GENERATING SETS

Definition

A vector $\mathbf{v}$ is a linear combination of the vectors in a (possibly infinite) subset S of a vector space V if there exist vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in S and scalars $c_1, c_2, \dots, c_m$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

The scalars are called the coefficients of the linear combination.

7.1 Vector Spaces and Their Subspaces

Example

We consider examples of linear combinations of vectors of both finite and infinite sets. In the vector space of 2×2 matrices,

$$\begin{bmatrix} -1 & 8 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} + (-1) \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Hence $\begin{bmatrix} -1 & 8 \\ 2 & -2 \end{bmatrix}$ is a linear combination of the matrices

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

with coefficients 2, -1 , and 1 .

7.1 Vector Spaces and Their Subspaces

Example

Let $S = \{1, x, x^2, x^3\}$. which is a subset of the vector space $\mathcal{P}$ of all polynomials. Then the polynomial $f(x) = 2 + 3x - x^2$ is a linear combination of the vectors in S because there are scalars, namely, 2, 3, and -1 , such that

$$f(x) = (2)1 + (3)x + (-1)x^2.$$

In fact, the zero polynomial and any polynomial of degree less than or equal to 3 is a linear combination of the vectors in S . That is. the set of all linear combinations of the vectors in S is equal to $\mathcal{P}_3$, the subspace of $\mathcal{P}$.

7.1 Vector Spaces and Their Subspaces

Example

Let S be the set of real-valued functions given by

$$S = \{1, \sin t, \cos^2 t, \sin^2 t\}$$

which is a subset of $\mathcal{F}(R)$. Observe that the function $\cos 2t$ is a linear combination of the vectors in S because

$$\begin{aligned}\cos 2t &= \cos^2 t - \sin^2 t \\ &= (1) \cos^2 t + (-1) \sin^2 t.\end{aligned}$$

7.1 Vector Spaces and Their Subspaces

Example

Let

$$S = \{1, x, x^2 \dots x^n \dots\}$$

which is an infinite subset of $\mathcal{P}$. Then the polynomial $p(x) = 3 - 4x^2 + 5x^4$ is a linear combination of the vectors in S because it is a linear combination of a finite number of vectors in S , namely, 1 , x^2 , and x^4 . In fact, any polynomial

$$p(x) = a_0 + a_1x + \dots + a_nx^n$$

is a linear combination of the vectors in S because it is a linear combination of $1, x, x^2, \dots, x^n$

7.1 Vector Spaces and Their Subspaces

Example

Determine if the polynomial x is a linear combination of the polynomials $1 - x^2$ and $1 + x + x^2$ Solution Suppose that

$$\begin{aligned}x &= a(1 - x^2) + b(1 + x + x^2) \\&= (a + b) + bx + (-a + b)x^2\end{aligned}$$

for scalars a, b , and c . Then

$$a + b = 0$$

$$b = 1$$

$$-a + b = 0.$$

Since this system is inconsistent, we conclude that x is not a linear combination of the polynomials $1 - x^2$ and $1 + x + x^2$.

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Definition

The span of a nonempty subset S of a vector space V is the set of all linear combinations of vectors in S . This set is denoted by $\text{Span } S$.

By the remarks made in Example 10.

$$\text{Span } \{1, x, x^2, x^3\} = \mathcal{P}_3$$

and by the remarks in Example 12.

$$\text{Span } \{1, x, \dots, x^n, \dots\} = \mathcal{P}$$

7.1 Vector Spaces and Their Subspaces

Example

Describe the span of the subset

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Solution For any matrix A in $\text{Span } S$, there exist scalars a, b , and c , such that

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

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Therefore $\text{trace}(A) = a + (-a) = 0$. Conversely, suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix in $M_{2 \times 2}$ such that $\text{trace}(A) = 0$. Then $a + d = 0$, and hence $d = -a$. It follows that $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, which is in S by the preceding calculation. Therefore S is the subset of all 2×2 matrices with trace equal to zero. Since this set was proved to be a subspace of $M_{2 \times 2}$ in Example 6, S is a subspace of $M_{2 \times 2}$.

Theorem

The span of a nonempty subset of a vector space V is a subspace of V .

7.1 Vector Spaces and Their Subspaces

Remark

This result gives us a convenient way to define vector spaces. For example, we let $\mathcal{T}[0, 2\pi]$ denote the subspace of $\mathcal{F}([0, 2\pi])$ defined by

$$\mathcal{T}[0, 2\pi] = \text{Span}\{1, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt, \dots\}.$$

This vector space is called the space of trigonometric polynomials and is studied in Section 7.5.

7.1 Vector Spaces and Their Subspaces

Exercise

Determine whether the matrix $\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$ is in the span of the set

$$S = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \right\}.$$

EXERCISES Pages 499-500

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