

# Lectures on Linear Algebra

Dr. rer. nat. Faten Said Abu-Shoga

Islamic University of Gaza

Chapter 7

7.3 Basis and Dimension

April 30, 2021

# CHAPTER 7 Vector Spaces

## 7.3 Basis and Dimension

### LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

An infinite subset  $S$  of a vector space  $V$  is linearly dependent if some finite subset of  $S$  is linearly dependent. An infinite set  $S$  is linearly independent if  $S$  is not linearly dependent, that is, if every finite subset of  $S$  is linearly independent.

## 7.3 Basis and Dimension

### Example

The subset  $S = \{x^2 - 3x + 2, 3x^2 - 5x, 2x - 3\}$  of  $\mathcal{P}_2$  is linearly dependent because

$$3(x^2 - 3x + 2) + (-1)(3x^2 - 5x) + 2(2x - 3) = 0$$

where 0 is the zero polynomial. As with linearly dependent subsets of  $\mathcal{R}^n$ ,  $S$  is linearly dependent because we are able to represent the zero vector as a linear combination of the vectors in  $S$  with at least one nonzero coefficient.

## 7.3 Basis and Dimension

### Example

In Example 14 of Section 7.1, we noted that the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is a generating set for the subspace of  $2 \times 2$  matrices with trace equal to zero. We now show that this set is linearly independent. Consider any scalars  $a, b$ , and  $c$  such that

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = O$$

where  $O$  is the zero matrix of  $\mathcal{M}_{2 \times 2}$ . In Example 14 of Section 7.1, we observed that this linear combination equals

## 7.3 Basis and Dimension

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

and hence

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By equating corresponding entries, we find that  $a = 0$ ,  $b = 0$ , and  $c = 0$ . It follows that  $S$  is linearly independent.

### Exercise

Determine if the set

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

is a linearly independent subset of  $\mathcal{M}_{2 \times 2}$ .

## 7.3 Basis and Dimension

## 7.3 Basis and Dimension

### Infinite subset

The infinite subset  $\{1, x, x^2, \dots, x^n, \dots\}$  of the vector space  $\mathcal{P}$  is linearly independent. Observe that any linear combination of a nonempty finite subset of this set is not the zero polynomial unless all of the coefficients are zeros. So every nonempty finite subset is linearly independent.

### Infinite subset

The infinite subset

$$\{1 + x, 1 - x, 1 + x^2, 1 - x^2, \dots, 1 + x^n, 1 - x^n, \dots\}$$

of  $\mathcal{P}$  is linearly dependent because it contains a finite linearly dependent subset. One example of such a set is

$\{1 + x, 1 - x, 1 + x^2, 1 - x^2\}$ , because

$$1(1 + x) + 1(1 - x) + (-1)(1 + x^2) + (-1)(1 - x^2) = 0$$

## 7.3 Basis and Dimension

### Theorem

Let  $V$  be a vector space with a finite basis. Then every basis for  $V$  is finite and contains the same number of vectors.

### Properties of Finite-Dimensional Vector Spaces

Let  $V$  be an  $n$ -dimensional vector space.

- 1 Any linearly independent subset of  $V$  contains at most  $n$  vectors.
- 2 Any linearly independent subset of  $V$  containing exactly  $n$  vectors is a basis for  $V$ .
- 3 Any generating set for  $V$  contains at least  $n$  vectors.
- 4 Any generating set for  $V$  containing exactly  $n$  vectors is a basis for  $V$ .

## 7.3 Basis and Dimension

### Example

Recall  $\mathcal{M}_{m \times n}$ , the vector space of  $m \times n$  matrices in Example 1 of Section 7.1. Let  $m = n = 2$ , and define

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\text{and} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

any matrix  $A$  in  $\mathcal{M}_{2 \times 2}$  can be written as a linear combination

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

## 7.3 Basis and Dimension

$$= a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$$

of the vectors in  $S = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ . Furthermore, if  $O$  is written as a linear combination of the matrices in  $S$ , then all of the coefficients must be zeros, and so  $S$  is linearly independent. It follows that  $S$  is a basis for  $\mathcal{M}_{2 \times 2}$ .

### EXERCISES Pages 518

1-24