



The Islamic University of Gaza

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Linear Algebra

MATH2341

Midterm Exam
Date: 25/03/2018

Student Name:	Question	1	2	3	4	Total
	Points	4	10	6	10	30
Student Number:	Score					

Solve the following questions

1. Choose the correct answer for each of the following questions. For each question, only one answer is correct and only one answer is allowed.

(a) (1 point) If A and B are invertible matrices of the same size, then

- (A) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
 (B) AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$
 (C) both a) and b) are valid since the order of the matrices does not matter
 (D) AB is not necessarily invertible since $\det(AB)$ could be zero.
 (E) none of the above

(b) (1 point) An elementary matrix

- (A) is the same as an identity matrix
 (B) is a matrix that can be written as the product $E_1E_2 \dots E_{k-1}$
 (C) is a 1×1 matrix
 (D) differs from an identity matrix by one elementary row or column operation
 (E) differs from an identity matrix by one elementary row operation

(c) (1 point) Let $A = \begin{bmatrix} u & u-1 \\ 1 & 1 \end{bmatrix}$ where $u \in \mathbb{R}$, then $A + A^{-1}$ is equal to

- (A) $0_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (B) $I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & -u \\ -1 & u \end{bmatrix}$
 (D) scalar multiple of $I_{2 \times 2}$ (E) none of the above

(d) (1 point) The addition of the matrices $\begin{bmatrix} 3 & 7 & 2 \\ 1 & 0 & 0 \\ 5 & 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ gives a

- (A) 3×3 matrix (B) 3×2 matrix (C) 2×3 matrix
 (D) 2×2 matrix (E) none of the above

2. TRUE/FALSE: For each statement below, please answer true or false.

(a) (1 point) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $A\mathbf{x} = \mathbf{0}$, then $7\mathbf{x}_1 + 3\mathbf{x}_2$ is also a solution to $A\mathbf{x} = \mathbf{0}$.

☒ True ☐ False

(b) (1 point) If A, B , and C are $n \times n$ matrices, and if $AB = AC$, then it must be true that $B = C$.

☐ True ☒ False

(c) (1 point) All elementary matrices are invertible.

☒ True ☐ False

(d) (1 point) A set of five vectors of $\mathbb{R}^3$ must be linearly dependent.

☒ True ☐ False

(e) (1 point) If A and B are $n \times n$ matrices, and if $AB = B$, then B is the identity matrix.

☐ True ☒ False

(f) (1 point) If the coefficient matrix A of the system $A\mathbf{x} = \mathbf{b}$ is invertible, then the system has infinitely many solutions.

☐ True ☒ False

(g) (1 point) If A is invertible, then $ABA^{-1} = B$.

☐ True ☒ False

(h) (1 point) If A is invertible matrix such that $A^2 = A$, then A must be the identity matrix.

☒ True ☐ False

(i) (1 point) If $x_1 = 0, x_2 = 0, x_3 = 1$ is a solution to a homogeneous system of linear equation, then the system has infinitely many solutions.

☒ True ☐ False

(j) (1 point) If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+1 \\ 3y \end{bmatrix}$$

then T is a linear transformation.

☐ True ☒ False

3. (6 points) Consider the homogeneous system of linear equations $Ax = 0$, whose coefficient matrix is given by the following matrix A . Find the vector form for the general solution of the system.

$$A = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & 1 & -2 & -7 \\ 3 & 0 & -3 & -6 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution:

The homogeneous system is given by:

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We apply elementary row operations to the augmented matrix of the system as follows.

$$[A|\vec{0}] = \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 2 & 1 & -2 & -7 & 0 \\ 3 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & -3 & 0 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_4} \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that the general solution is given by

$$x_1 = x_3 + 2x_4$$

$$x_2 = 3x_4$$

x_3 and x_4 are free variables

The vector form of the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

4. Let A be the coefficient matrix of the system of linear equations

$$-x_1 - 2x_2 = 1$$

$$2x_1 + 3x_2 = -1.$$

(a) (4 points) Use the augmented matrix $[A | I]$ to find A^{-1} .

$$\begin{aligned} \left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array} \right] &\xrightarrow{-R_1} \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array} \right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{array} \right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{array} \right] \\ &\xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & -1 \end{array} \right] \\ \therefore A^{-1} &= \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \end{aligned}$$

$$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

(b) (2 points) Write A^{-1} as a product of the elementary matrices obtained in part(a).

$$\begin{aligned} A^{-1} &= E_4 E_3 E_2 E_1 \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

(c) (2 points) Solve the system of part (a) by using the inverse matrix A^{-1} .

$$\begin{aligned} A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \text{multiply from left by } A^{-1} \Rightarrow I_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \Rightarrow I_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

(d) (2 points) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the solution of the system obtained in part(c). Calculate and simplify $A^{56}\mathbf{x}$.

Since $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the solution of the system $A\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we see that

$A\vec{x} = \vec{x}$. From this formula we have

$$A^2\vec{x} = A(A\vec{x}) = A\vec{x} = \vec{x}$$

$$A^3\vec{x} = A(A^2\vec{x}) = A\vec{x} = \vec{x}$$

$$A^4\vec{x} = A(A^3\vec{x}) = A\vec{x} = \vec{x}$$

$$\text{Thus we have } A^{56}\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \#$$