

The Islamic University of Gaza

Mathematics Departement
Linear Algebra (MATH2341)

Midterm Exam
Date: 06/04/2023

Student Name:
Student Number:

Question	1	2	3	4	Total
Points	10	16	16	18	60
Score	10	16	16	16	58

Solve the following questions

Q.1) Choose the correct answer for each of the following questions.

1. Let $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation induced by A . If T_A is a one to one transformation, then

- ✓ (a) rank $A = m$.
- ✗ (b) The columns of A are a generating set of $\mathbb{R}^m$.
- ✗ (c) The linear system $Ax = b$ has always unique solution.
- ☒ (d) $T(v) = 0$ if and only if $v = 0$.

2. Suppose that A , B and C are invertible matrices of the same size. Which of the $(A^{-1})^T (A^T)^{-1}$ following need not be invertible?

- ☒ (A) $A + C^T$
- ✓ (B) $B^T C^{-1}$
- ✓ (C) AB^T
- ✓ (D) $AC^{-1}B$

3. If A is symmetric matrix of size $n \times n$, then ? $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow A = A^T$

- (A) $AA^T = 2A$
- ☒ (B) $A + A^T = 2A$
- (C) $A - A^T = I_n$
- (D) $A + A^T = 0$

4. Find the values of x and y in the matrix below if the matrix is a skew symmetric matrix.

$$x+y=-3 \Rightarrow y=3$$

$$x=-6$$

$$P = \begin{bmatrix} 0 & x+y & 6 \\ 3 & 0 & -9 \\ x & 9 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & -3 & 6 \\ 3 & 0 & 9 \\ -6 & 9 & 0 \end{bmatrix}$$

- ☒ (A) $x = -6, y = 3$
- (B) $x = 3, y = 3$
- (C) $x = 6, y = -3$
- (D) $x = 0, y = 3$

5. If $X = \begin{bmatrix} f & 2 \\ 3 & a \end{bmatrix}$, $Y = \begin{bmatrix} 4 & 5 & 8 \\ 6 & 7 & b \end{bmatrix}$ and $Z = \begin{bmatrix} 6 & c & 3 \\ 8 & d & e \end{bmatrix}$,

Which of the following matrix operations CAN NOT be performed?

- ✗ ☒ (A) $2X + 3Z$
- ✓ (B) XY
- ✓ (C) XZ
- ✓ (D) $3Y + 5Z$
- (E) All of the above operations can be performed.

Q.11) Consider the linear system $AX = b$, where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 1 \\ -2 & -5 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix}$$

1. Find A^{-1}

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -7 & 1 & 0 & 1 & 0 \\ -2 & -5 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 = 2R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -7 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array} \right] \xrightarrow{R_2 = 7R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 14 & 1 & 7 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{array} \right] \\ & \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 14 & 1 & 7 \end{array} \right] \xrightarrow{R_1 = 2R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 29 & 2 & 14 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 14 & 1 & 7 \end{array} \right] \\ & \xrightarrow{R_1 = -3R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23 & 2 & 11 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 14 & 1 & 7 \end{array} \right] \end{aligned}$$

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So, $A^{-1} = \begin{bmatrix} 23 & 2 & 11 \\ 2 & 0 & 1 \\ 14 & 1 & 7 \end{bmatrix}$

2. Use A^{-1} to solve the system $Ax = b$

$Ax = b$ multiplying by A^{-1} from the left

$$x = A^{-1}b = \begin{bmatrix} 23 & 2 & 11 \\ 2 & 0 & 1 \\ 14 & 1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix} = \begin{bmatrix} -20 \\ -2 \\ -14 \end{bmatrix} \quad (3)$$

3. What is the solution of the system $Ax = 0$

$$\left[\begin{array}{ccc|c} R & 0 \\ of A \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \Rightarrow \text{the solution is } x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(3)

since A is invertible, then its nullspace is $\vec{0}$ only.

Q.III) Suppose that $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear transformation such that
 $T(e_1) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, and $T(e_3) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

1. Determine $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ for any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$.

$$A = [T(e_1) \ T(e_2) \ T(e_3)] = \begin{bmatrix} -2 & 0 & 2 \\ 1 & -3 & 4 \end{bmatrix}$$

$$T = AX = \begin{bmatrix} -2 & 0 & 2 \\ 1 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T = \begin{bmatrix} -2x_1 + 2x_3 \\ x_1 - 3x_2 + 4x_3 \end{bmatrix}$$

2. Determine a generating set for the range of T .

$$\text{Range} = \text{Span}(\text{columns of } A)$$

$$= \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

3. Determine a generating set for the null space of T .

$$[A:0] = \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 1 & -3 & 4 & 0 \end{array} \right] \Rightarrow \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & -3 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{-R_1 + R_2 = R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -3 & 5 & 0 \end{array} \right] \xrightarrow{-\frac{R_2}{3} = R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{5}{3} & 0 \end{array} \right] \Rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = \frac{5}{3}x_3 \\ x_3 \text{ free} \end{matrix} \Rightarrow x_3 \begin{bmatrix} 1 \\ 5/3 \\ 1 \end{bmatrix} = \text{solution}$$

4. Determine whether T is onto or one-to-one? Why?

T is onto, since $\text{Rank}(A) = 2 = \# \text{ of rows of } A$
 T is a generating set of $\mathbb{R}^2$.

Answer: $\text{span}\left(\begin{bmatrix} 1 \\ 5/3 \end{bmatrix}\right)$

T is not one-to-one, since $\text{Rank}(A) = 2 \neq 3 = \# \text{ of columns of } A$.
columns of A are linearly dependent.

Q.IV)

1. Let $A = \begin{bmatrix} 1 & -2 & 3 & 9 & 1 \\ 2 & -4 & 1 & 3 & 0 \\ -3 & 6 & 0 & 0 & 1 \\ 1 & -2 & 5 & 15 & 0 \\ 4 & -8 & 2 & 6 & 1 \end{bmatrix}$ be row reduces to $R = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Calculate $\text{trac}(3A - 2A^T)$ without finding A^T . Explain your answer.

$\text{trace}(A) = \text{trace}(A^T)$ since the ^{main} diagonal entries didn't change.

$$\text{trace}(3A - 2A^T) = 3\text{trace}(A) - 2\text{trace}(A^T)$$

$$= \text{trace}(A) = 1 - 4 + 0 + 15 + 1 = 13$$

(b) List the linearly independent columns of A . Explain your answer.

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \\ 4 \end{bmatrix}, a_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 5 \\ 2 \end{bmatrix}, a_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

since they're pivot columns.

(all other columns could be constructed from these columns).
generated

2. Determine whether the following statements is True or False:

(T) If A and B are diagonal matrices, then $AB = BA$.

(T) If $A^2 = I_n$, then $A = I_n$ or $A = -I_n$.

(F) A homogenous system with more variables than equations has a finite number of solutions. infinite (because of free variables).

(T) The inverse of elementary matrix is elementary.

(F) The space $\mathbb{R}^5$ can be spanned by at least 5 vectors Linearly dependent. independent

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & -k \\ 0 & k \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix} \checkmark$$