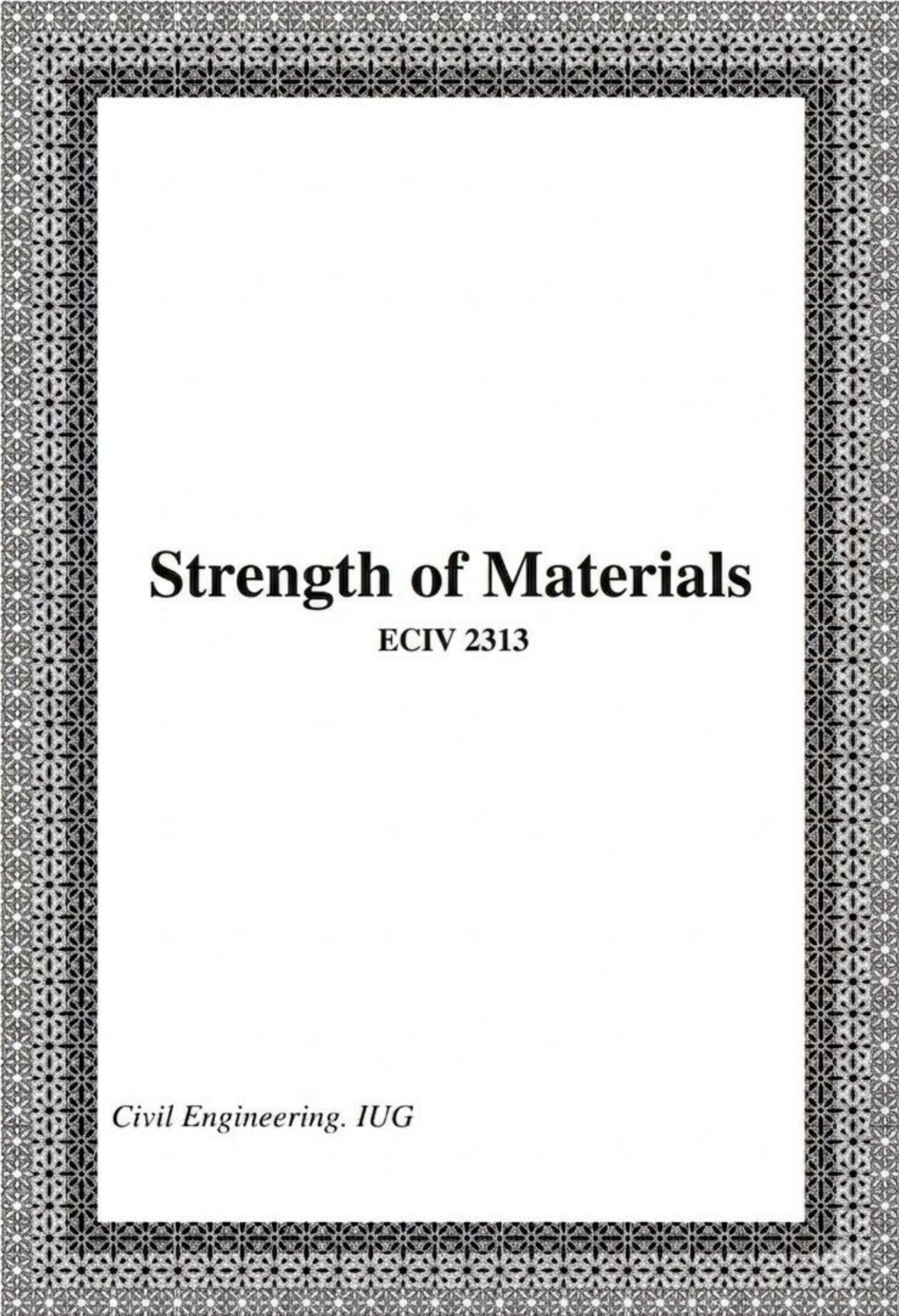




Strength of Materials

ECIV 2313

Civil Engineering. IUG

CHAPTER

1

MECHANICS OF MATERIALS

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Introduction – Concept of Stress

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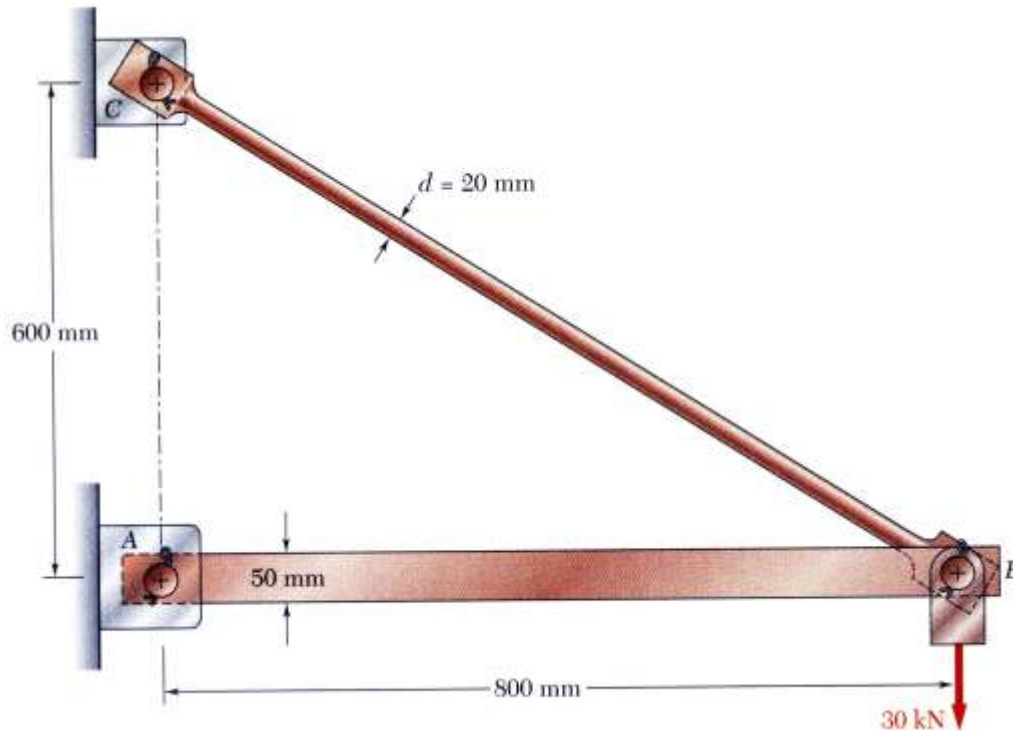


Concept of Stress

- The main objective of the study of mechanics of materials is to provide the future engineer with the means of analyzing and designing various machines and load bearing structures.
- Both the analysis and design of a given structure involve the determination of *stresses* and *deformations*. This chapter is devoted to the concept of stress.

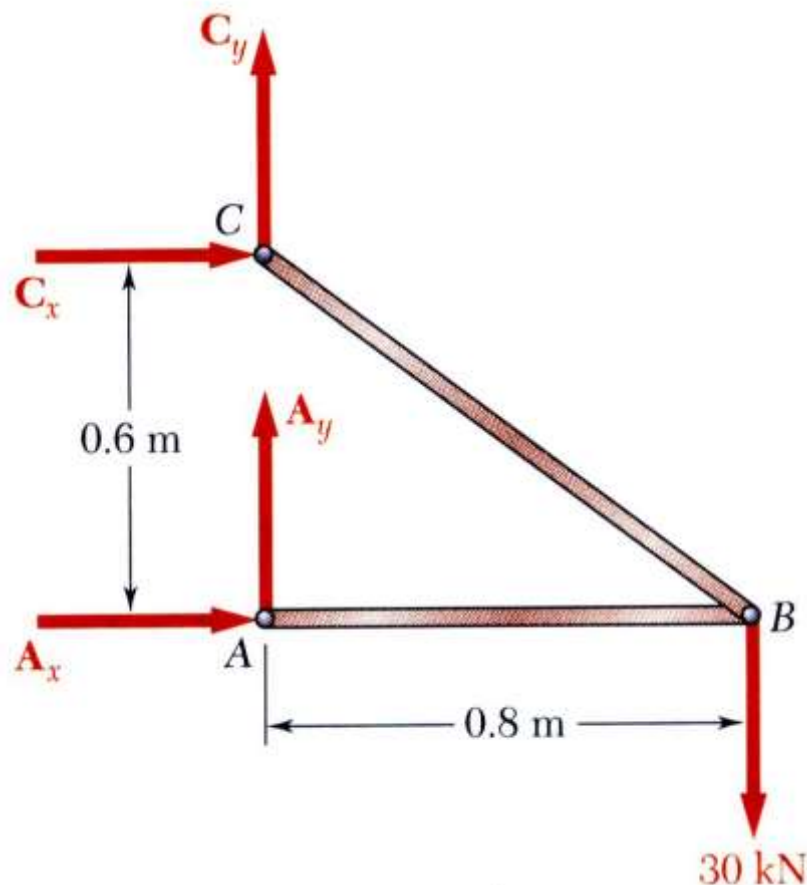


Review of Statics



- The structure is designed to support a 30 kN load
- The structure consists of a boom and rod joined by pins (zero moment connections) at the junctions and supports
- Perform a static analysis to determine the internal force in each structural member and the reaction forces at the supports

Structure Free-Body Diagram



- Structure is detached from supports and the loads and reaction forces are indicated
- Conditions for static equilibrium:
$$\sum M_C = 0 = A_x(0.6\text{ m}) - (30\text{ kN})(0.8\text{ m})$$
$$A_x = 40\text{ kN}$$
$$\sum F_x = 0 = A_x + C_x$$
$$C_x = -A_x = -40\text{ kN}$$
$$\sum F_y = 0 = A_y + C_y - 30\text{ kN} = 0$$
$$A_y + C_y = 30\text{ kN}$$
- A_y and C_y can not be determined from these equations

Component Free-Body Diagram

- In addition to the complete structure, each component must satisfy the conditions for static equilibrium
- Consider a free-body diagram for the boom:

$$\sum M_B = 0 = -A_y(0.8\text{m})$$

$$A_y = 0$$

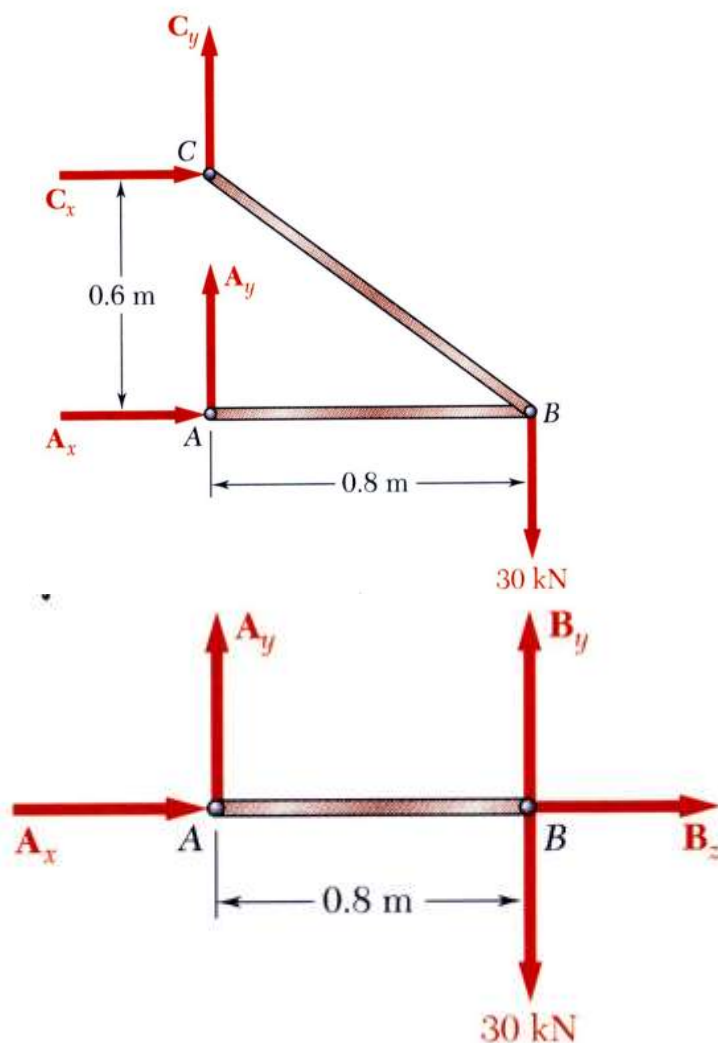
substitute into the structure equilibrium equation

$$C_y = 30\text{ kN}$$

- Results:

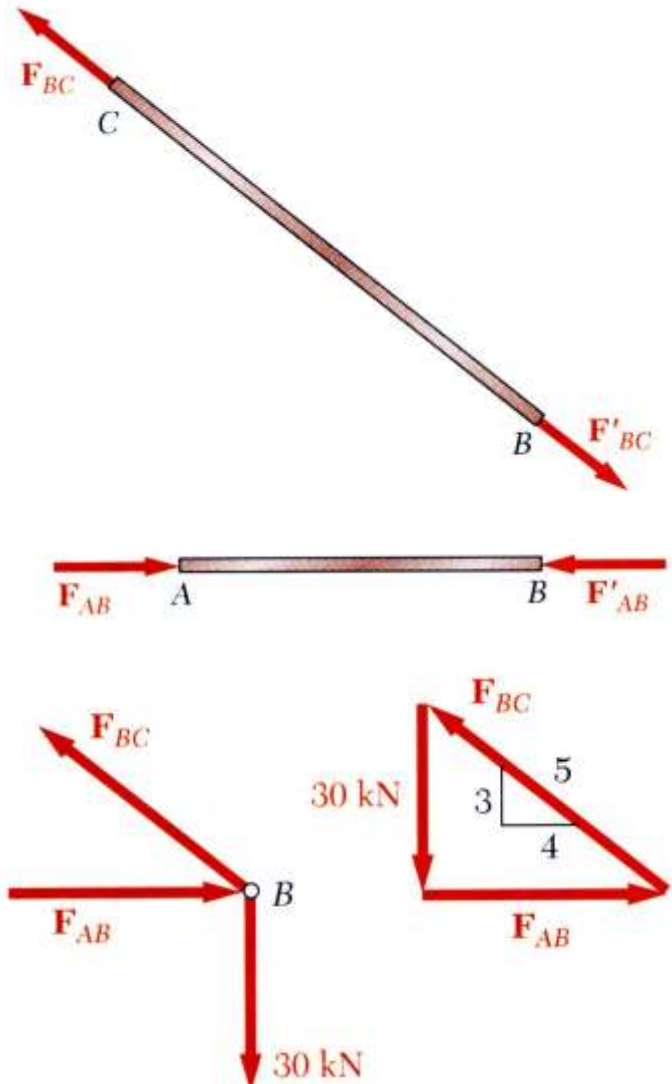
$$A = 40\text{ kN} \rightarrow \quad C_x = 40\text{ kN} \leftarrow \quad C_y = 30\text{ kN} \uparrow$$

Reaction forces are directed along boom and rod



Method of Joints

- The boom and rod are 2-force members, i.e., the members are subjected to only two forces which are applied at member ends
- For equilibrium, the forces must be parallel to to an axis between the force application points, equal in magnitude, and in opposite directions
- Joints must satisfy the conditions for static equilibrium which may be expressed in the form of a force triangle:

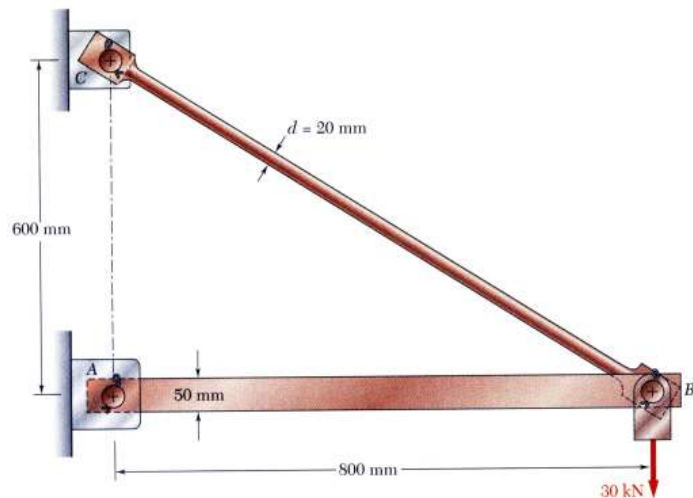


$$\sum \vec{F}_B = 0$$

$$\frac{F_{AB}}{4} = \frac{F_{BC}}{5} = \frac{30 \text{ kN}}{3}$$

$$F_{AB} = 40 \text{ kN} \quad F_{BC} = 50 \text{ kN}$$

Stress Analysis



Can the structure safely support the 30 kN load?

- From a statics analysis

$$F_{AB} = 40 \text{ kN (compression)}$$

$$F_{BC} = 50 \text{ kN (tension)}$$

- At any section through member BC, the internal force is 50 kN with a force intensity or stress of

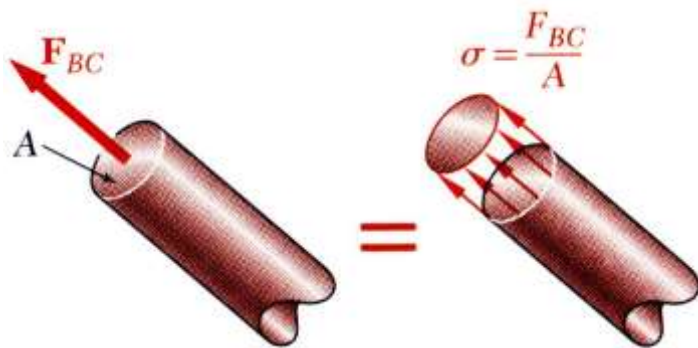
$$d_{BC} = 20 \text{ mm}$$

$$\sigma_{BC} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{314 \times 10^{-6} \text{ m}^2} = 159 \text{ MPa}$$

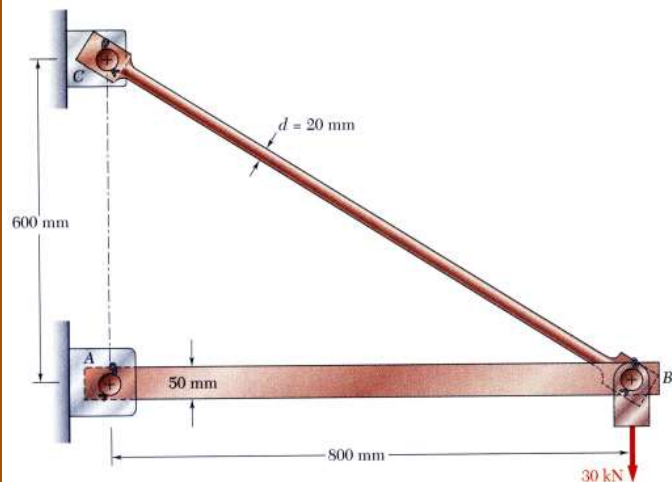
- From the material properties for steel, the allowable stress is

$$\sigma_{\text{all}} = 165 \text{ MPa}$$

- Conclusion: the strength of member BC is adequate



Design



- Design of new structures requires selection of appropriate materials and component dimensions to meet performance requirements
- For reasons based on cost, weight, availability, etc., the choice is made to construct the rod from aluminum ($\sigma_{all} = 100 \text{ MPa}$). What is an appropriate choice for the rod diameter?

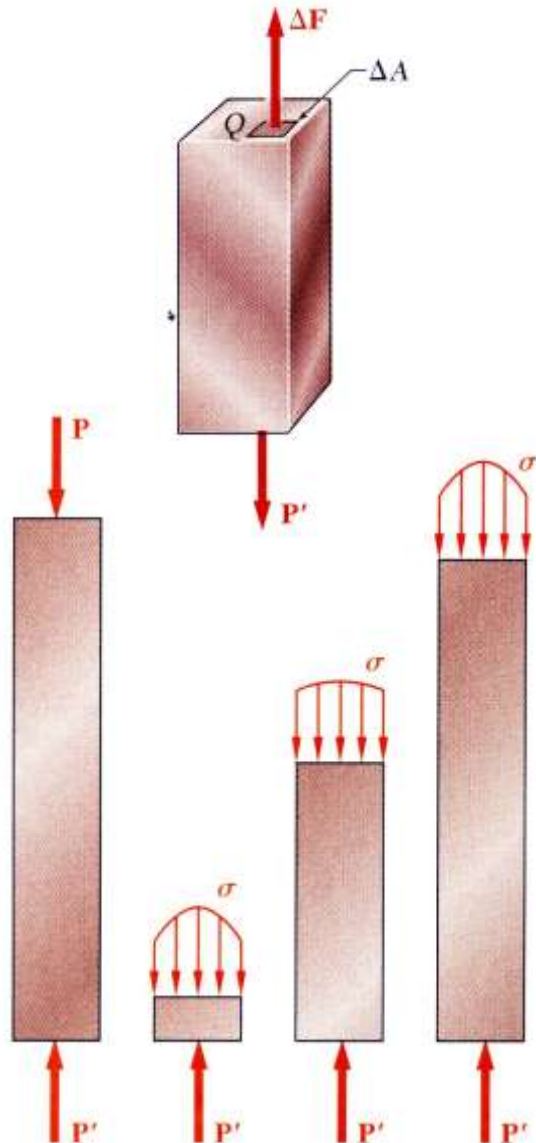
$$\sigma_{all} = \frac{P}{A} \quad A = \frac{P}{\sigma_{all}} = \frac{50 \times 10^3 \text{ N}}{100 \times 10^6 \text{ Pa}} = 500 \times 10^{-6} \text{ m}^2$$

$$A = \pi \frac{d^2}{4}$$

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4(500 \times 10^{-6} \text{ m}^2)}{\pi}} = 2.52 \times 10^{-2} \text{ m} = 25.2 \text{ mm}$$

- An aluminum rod 26 mm or more in diameter is adequate

Axial Loading: Normal Stress



- The resultant of the internal forces for an axially loaded member is *normal* to a section cut perpendicular to the member axis.
- The force intensity on that section is defined as the normal stress.

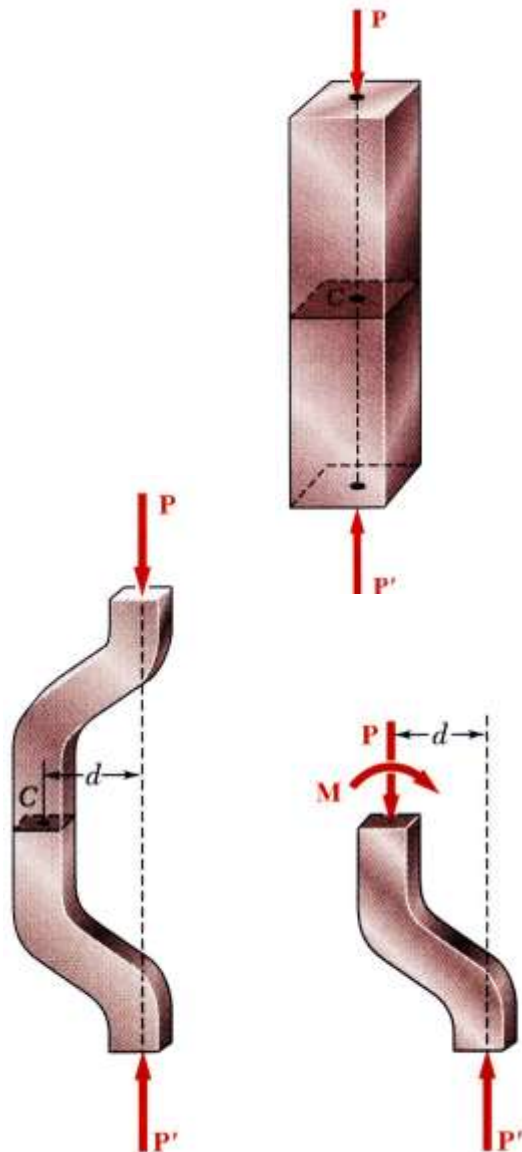
$$\sigma = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A} \quad \sigma_{ave} = \frac{P}{A}$$

- The normal stress at a particular point may not be equal to the average stress but the resultant of the stress distribution must satisfy

$$P = \sigma_{ave} A = \int_A dF = \int_A \sigma dA$$

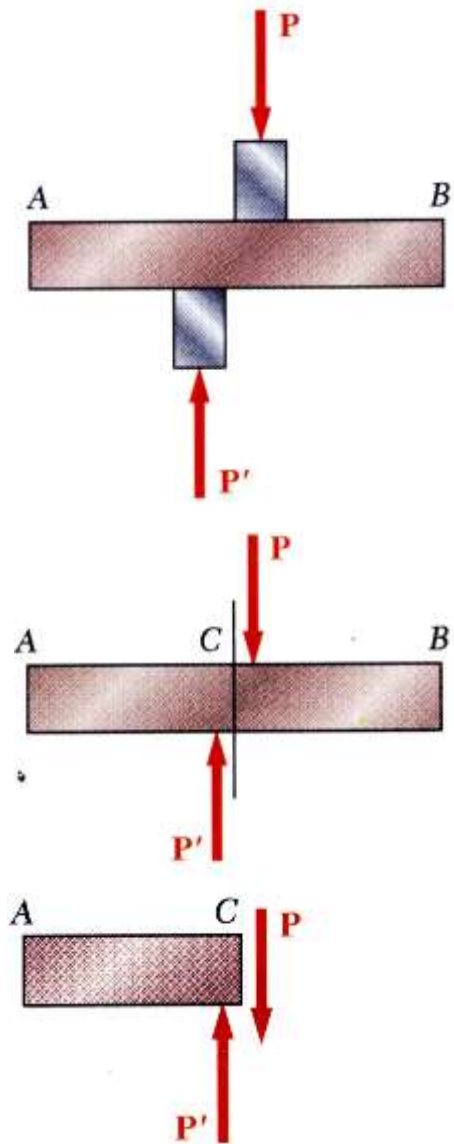
- The detailed distribution of stress is statically indeterminate, i.e., can not be found from statics alone.

Centric & Eccentric Loading



- A uniform distribution of stress in a section infers that the line of action for the resultant of the internal forces passes through the centroid of the section.
- A uniform distribution of stress is only possible if the concentrated loads on the end sections of two-force members are applied at the section centroids. This is referred to as *centric loading*.
- If a two-force member is *eccentrically loaded*, then the resultant of the stress distribution in a section must yield an axial force and a moment.
- The stress distributions in eccentrically loaded members cannot be uniform or symmetric.

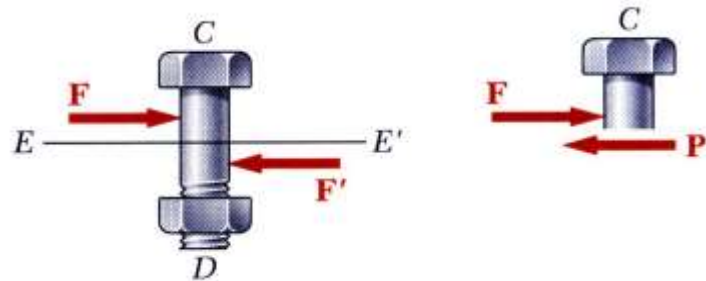
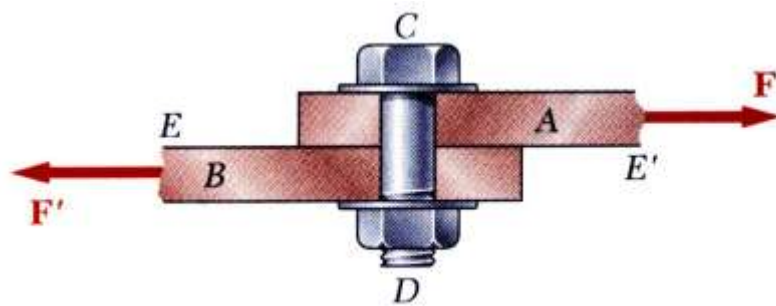
Shearing Stress



- Forces P and P' are applied transversely to the member AB .
- Corresponding internal forces act in the plane of section C and are called *shearing* forces.
- The resultant of the internal shear force distribution is defined as the *shear* of the section and is equal to the load P .
- The corresponding average shear stress is,
$$\tau_{\text{ave}} = \frac{P}{A}$$
- Shear stress distribution varies from zero at the member surfaces to maximum values that may be much larger than the average value.
- The shear stress distribution cannot be assumed to be uniform.

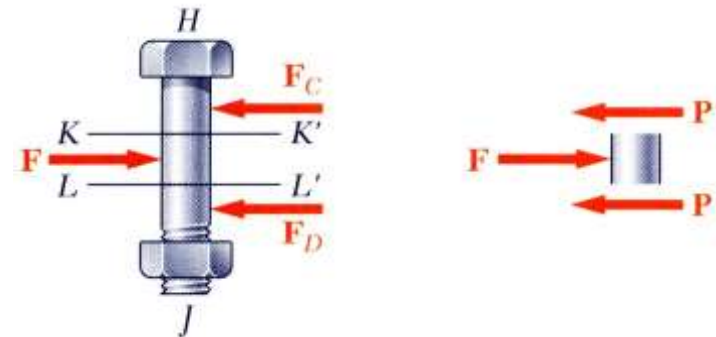
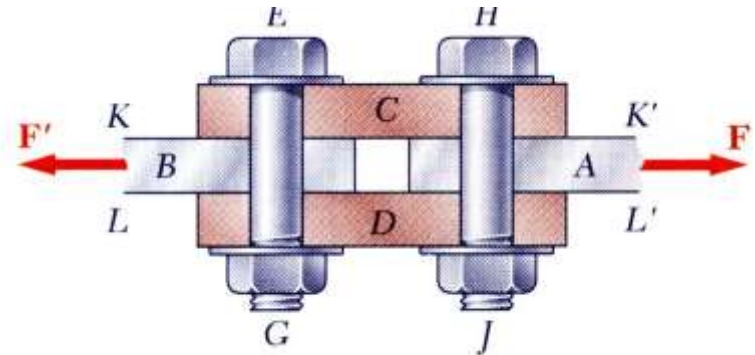
Shearing Stress Examples

Single Shear



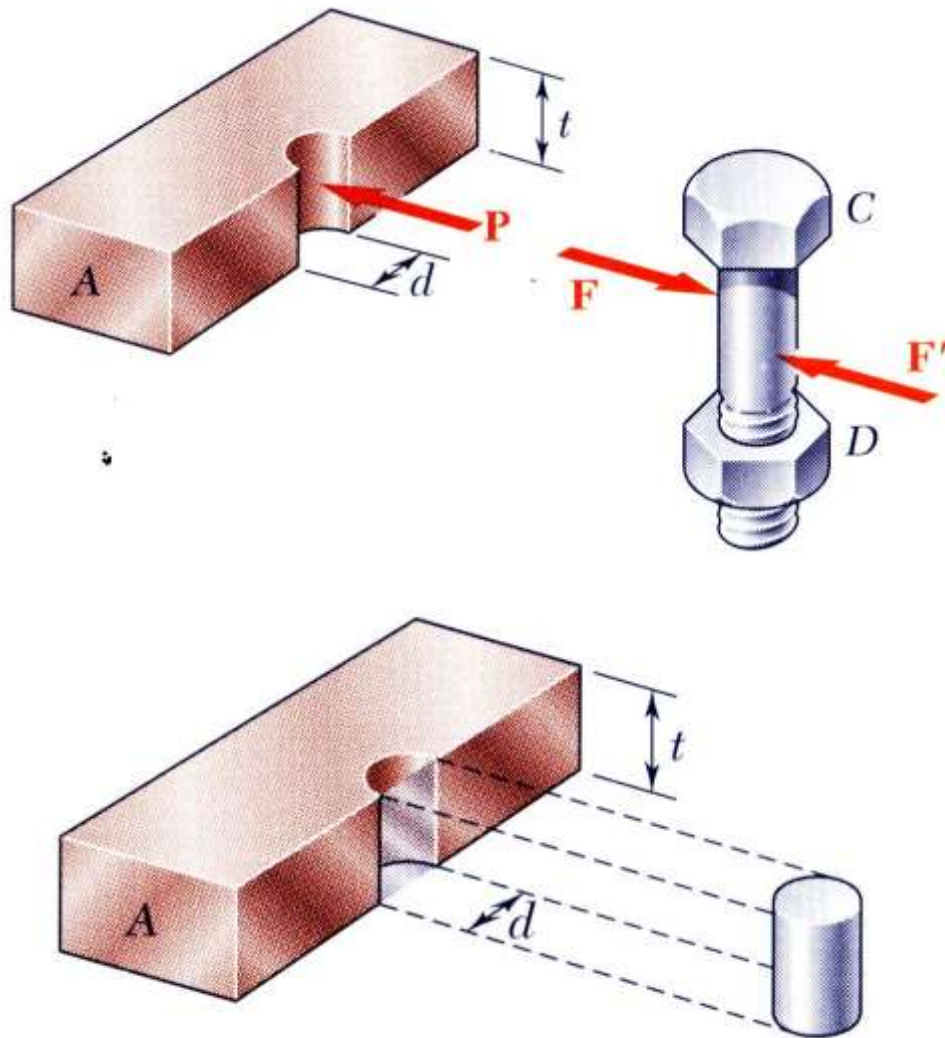
$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{A}$$

Double Shear



$$\tau_{\text{ave}} = \frac{P}{A} = \frac{F}{2A}$$

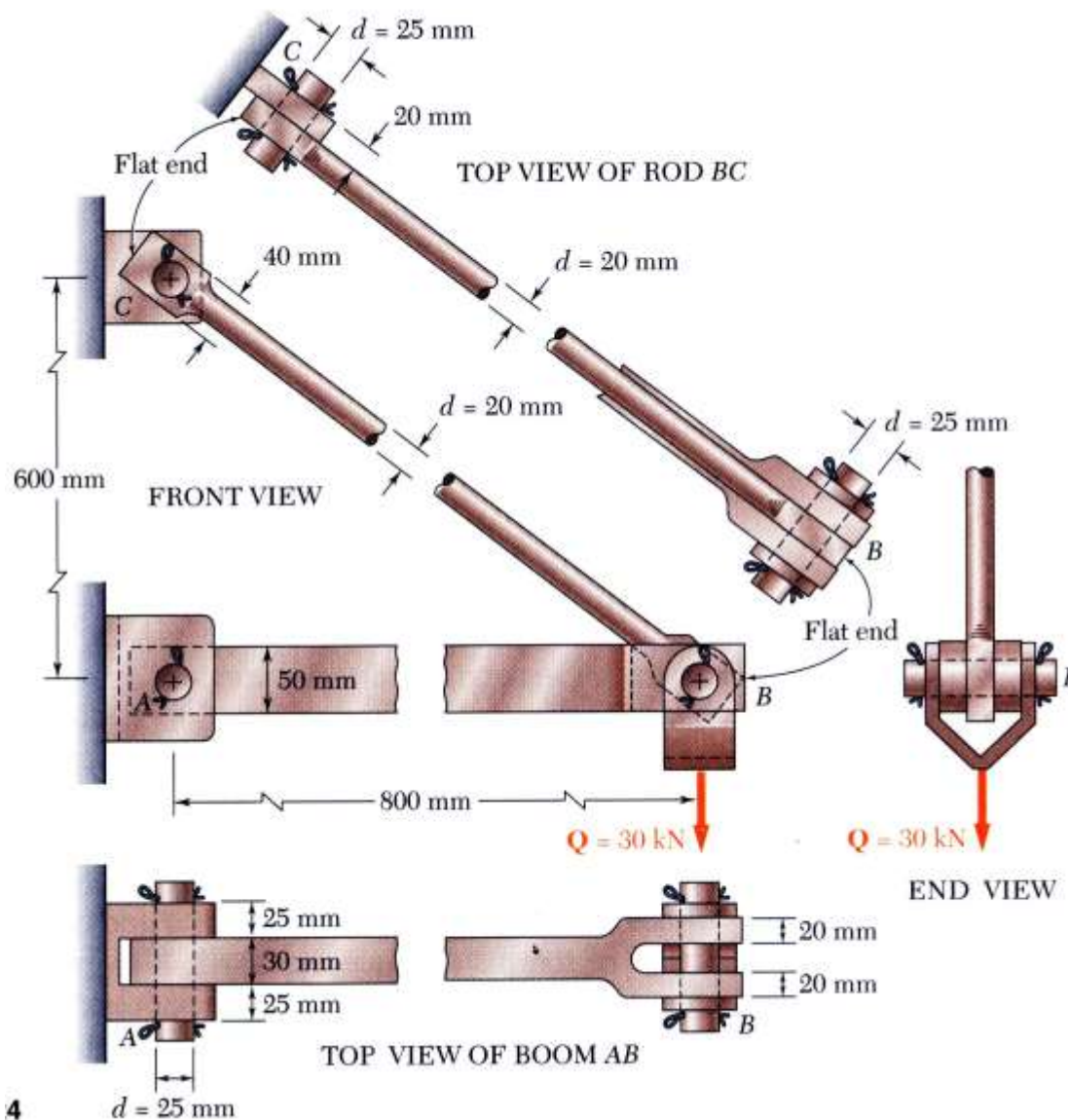
Bearing Stress in Connections



- Bolts, rivets, and pins create stresses on the points of contact or *bearing surfaces* of the members they connect.
- The resultant of the force distribution on the surface is equal and opposite to the force exerted on the pin.
- Corresponding average force intensity is called the bearing stress,

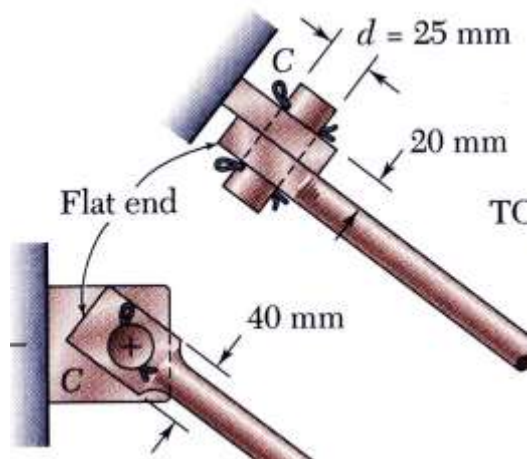
$$\sigma_b = \frac{P}{A} = \frac{P}{td}$$

Stress Analysis & Design Example



- Would like to determine the stresses in the members and connections of the structure shown.
- From a statics analysis:
 $F_{AB} = 40$ kN (compression)
 $F_{BC} = 50$ kN (tension)
- Must consider maximum normal stresses in AB and BC , and the shearing stress and bearing stress at each pinned connection

Rod & Boom Normal Stresses



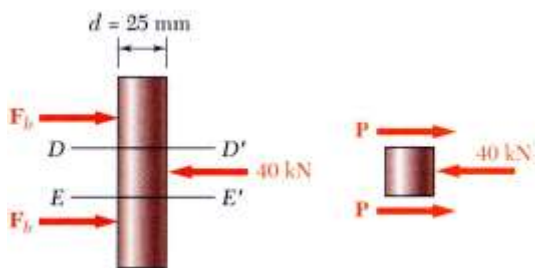
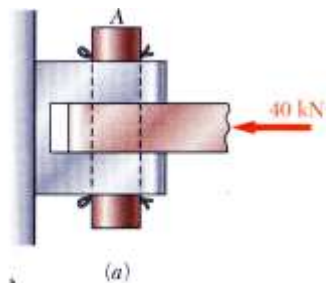
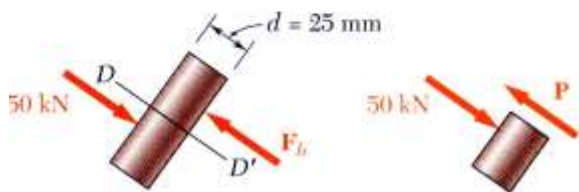
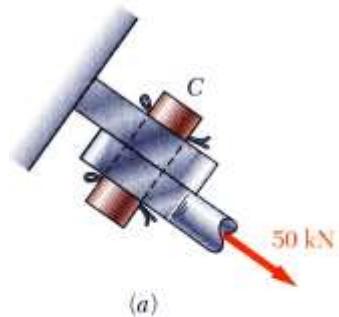
- The rod is in tension with an axial force of 50 kN.
- At the rod center, the average normal stress in the circular cross-section ($A = 314 \times 10^{-6} \text{ m}^2$) is $\sigma_{BC} = +159 \text{ MPa}$.
- At the flattened rod ends, the smallest cross-sectional area occurs at the pin centerline,

$$A = (20 \text{ mm})(40 \text{ mm} - 25 \text{ mm}) = 300 \times 10^{-6} \text{ m}^2$$

$$\sigma_{BC,end} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{300 \times 10^{-6} \text{ m}^2} = 167 \text{ MPa}$$

- The boom is in compression with an axial force of 40 kN and average normal stress of -26.7 MPa .
- The minimum area sections at the boom ends are unstressed since the boom is in compression.

Pin Shearing Stresses



- The cross-sectional area for pins at A, B, and C,

$$A = \pi r^2 = \pi \left(\frac{25 \text{ mm}}{2} \right)^2 = 491 \times 10^{-6} \text{ m}^2$$

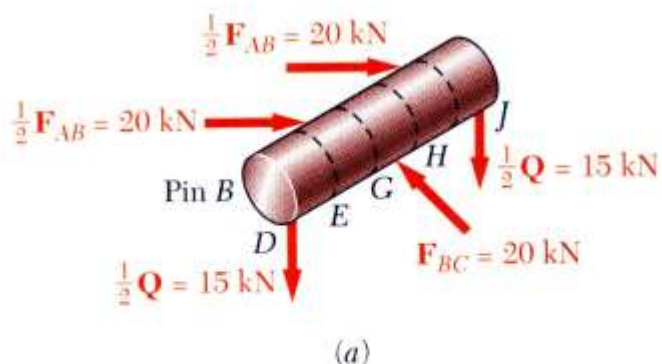
- The force on the pin at C is equal to the force exerted by the rod BC,

$$\tau_{C,ave} = \frac{P}{A} = \frac{50 \times 10^3 \text{ N}}{491 \times 10^{-6} \text{ m}^2} = 102 \text{ MPa}$$

- The pin at A is in double shear with a total force equal to the force exerted by the boom AB,

$$\tau_{A,ave} = \frac{P}{A} = \frac{20 \text{ kN}}{491 \times 10^{-6} \text{ m}^2} = 40.7 \text{ MPa}$$

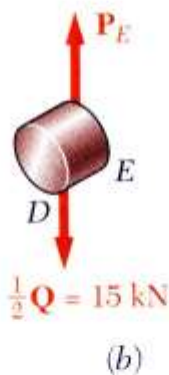
Pin Shearing Stresses



- Divide the pin at B into sections to determine the section with the largest shear force,

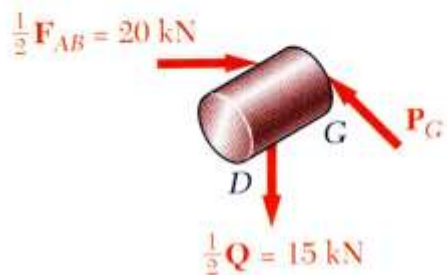
$$P_E = 15 \text{ kN}$$

$$P_G = 25 \text{ kN (largest)}$$

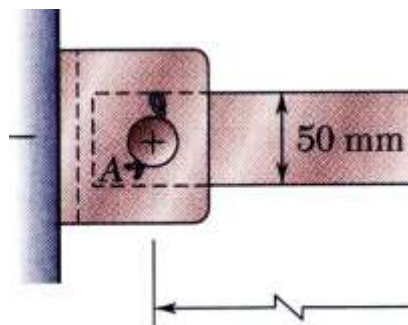


- Evaluate the corresponding average shearing stress,

$$\tau_{B,ave} = \frac{P_G}{A} = \frac{25 \text{ kN}}{491 \times 10^{-6} \text{ m}^2} = 50.9 \text{ MPa}$$

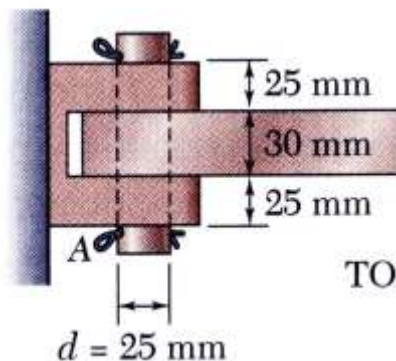


Pin Bearing Stresses



- To determine the bearing stress at A in the boom AB, we have $t = 30 \text{ mm}$ and $d = 25 \text{ mm}$,

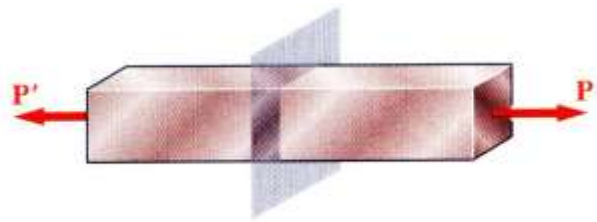
$$\sigma_b = \frac{P}{td} = \frac{40 \text{ kN}}{(30 \text{ mm})(25 \text{ mm})} = 53.3 \text{ MPa}$$



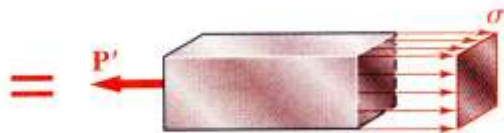
- To determine the bearing stress at A in the bracket, we have $t = 2(25 \text{ mm}) = 50 \text{ mm}$ and $d = 25 \text{ mm}$,

$$\sigma_b = \frac{P}{td} = \frac{40 \text{ kN}}{(50 \text{ mm})(25 \text{ mm})} = 32.0 \text{ MPa}$$

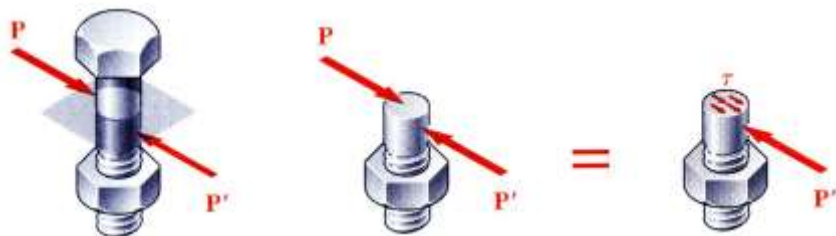
Stress in Two Force Members



(a)

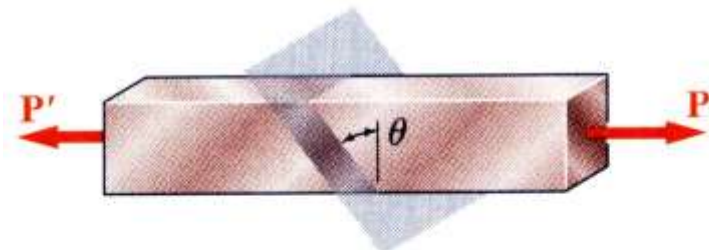


(b)



- Axial forces on a two force member result in only normal stresses on a plane cut perpendicular to the member axis.
- Transverse forces on bolts and pins result in only shear stresses on the plane perpendicular to bolt or pin axis.
- Will show that either axial or transverse forces may produce both normal and shear stresses with respect to a plane other than one cut perpendicular to the member axis.

Stress on an Oblique Plane



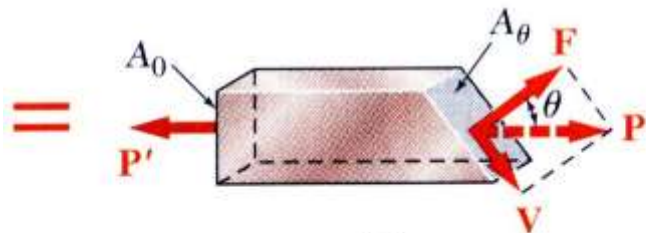
(a)

- Pass a section through the member forming an angle θ with the normal plane.



(b)

- From equilibrium conditions, the distributed forces (stresses) on the plane must be equivalent to the force P .



(c)

- Resolve P into components normal and tangential to the oblique section,

$$F = P \cos \theta \quad V = P \sin \theta$$

- The average normal and shear stresses on the oblique plane are

$$\sigma = \frac{F}{A_\theta} = \frac{P \cos \theta}{A_0 / \cos \theta} = \frac{P}{A_0} \cos^2 \theta$$

$$\tau = \frac{V}{A_\theta} = \frac{P \sin \theta}{A_0 / \cos \theta} = \frac{P}{A_0} \sin \theta \cos \theta$$

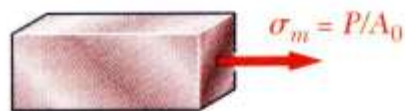


(d)

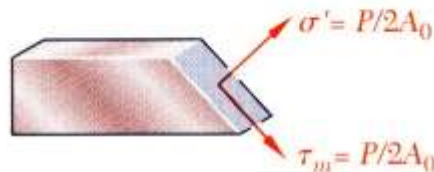
Maximum Stresses



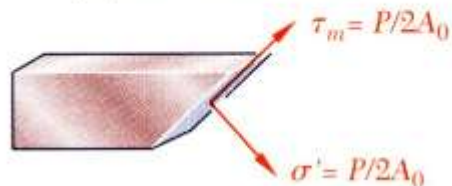
(a) Axial loading



(b) Stresses for $\theta = 0$



(c) Stresses for $\theta = 45^\circ$



(d) Stresses for $\theta = -45^\circ$

- Normal and shearing stresses on an oblique plane

$$\sigma = \frac{P}{A_0} \cos^2 \theta \quad \tau = \frac{P}{A_0} \sin \theta \cos \theta$$

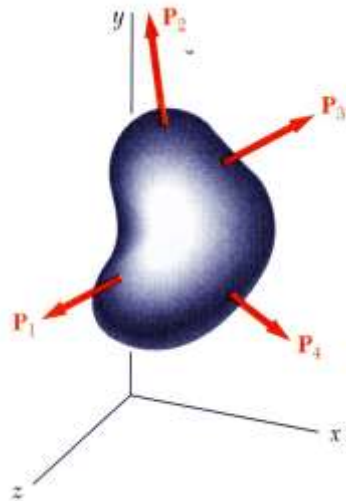
- The maximum normal stress occurs when the reference plane is perpendicular to the member axis,

$$\sigma_m = \frac{P}{A_0} \quad \tau' = 0$$

- The maximum shear stress occurs for a plane at $\pm 45^\circ$ with respect to the axis,

$$\tau_m = \frac{P}{A_0} \sin 45 \cos 45 = \frac{P}{2A_0} = \sigma'$$

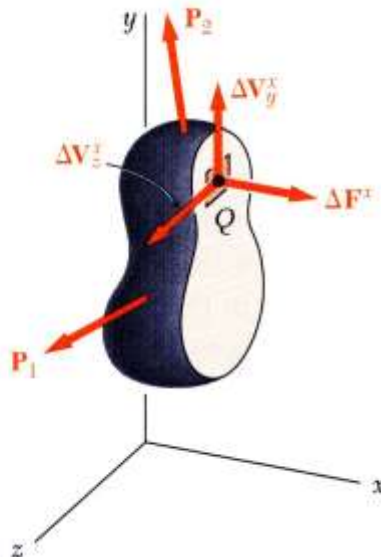
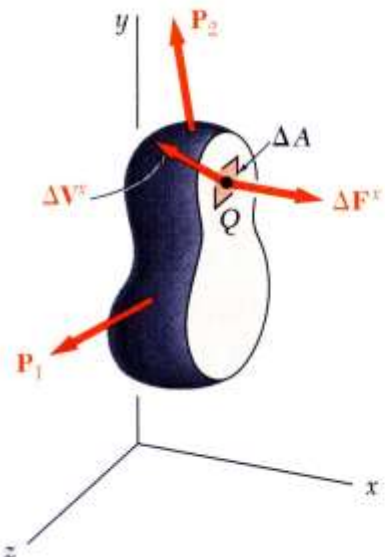
Stress Under General Loadings



- A member subjected to a general combination of loads is cut into two segments by a plane passing through Q
- The distribution of internal stress components may be defined as,

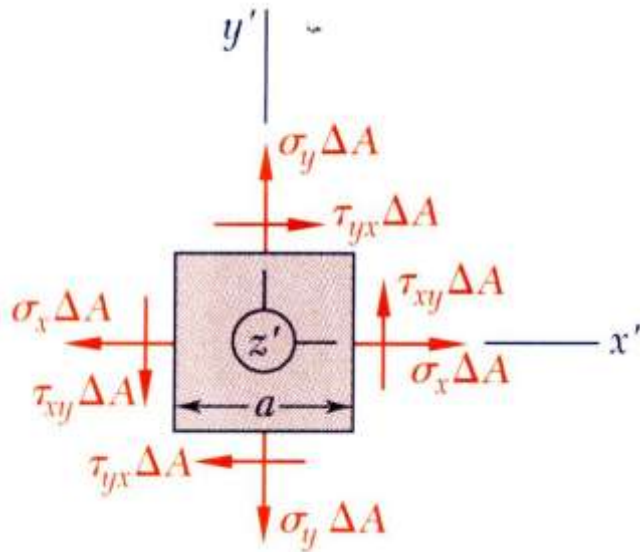
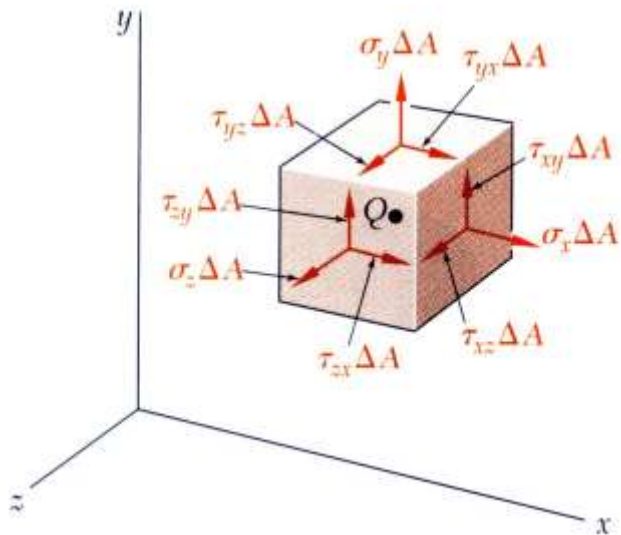
$$\sigma_x = \lim_{\Delta A \rightarrow 0} \frac{\Delta F^x}{\Delta A}$$

$$\tau_{xy} = \lim_{\Delta A \rightarrow 0} \frac{\Delta V_y^x}{\Delta A} \quad \tau_{xz} = \lim_{\Delta A \rightarrow 0} \frac{\Delta V_z^x}{\Delta A}$$



- For equilibrium, an equal and opposite internal force and stress distribution must be exerted on the other segment of the member.

State of Stress



- Stress components are defined for the planes cut parallel to the x , y and z axes. For equilibrium, equal and opposite stresses are exerted on the hidden planes.

- The combination of forces generated by the stresses must satisfy the conditions for equilibrium:

$$\sum F_x = \sum F_y = \sum F_z = 0$$

$$\sum M_x = \sum M_y = \sum M_z = 0$$

- Consider the moments about the z axis:

$$\sum M_z = 0 = (\tau_{xy}\Delta A)a - (\tau_{yx}\Delta A)a$$

$$\tau_{xy} = \tau_{yx}$$

$$\text{similarly, } \tau_{yz} = \tau_{zy} \quad \text{and} \quad \tau_{xz} = \tau_{zx}$$

- It follows that only 6 components of stress are required to define the complete state of stress

Factor of Safety

Structural members or machines must be designed such that the working stresses are less than the ultimate strength of the material.

FS = Factor of safety

$$FS = \frac{\sigma_u}{\sigma_{all}} = \frac{\text{ultimate stress}}{\text{allowable stress}}$$

Factor of safety considerations:

- uncertainty in material properties
- uncertainty of loadings
- uncertainty of analyses
- number of loading cycles
- types of failure
- maintenance requirements and deterioration effects
- importance of member to structures integrity
- risk to life and property
- influence on machine function



CHAPTER

2

MECHANICS OF MATERIALS

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Stress and Strain – Axial Loading

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Stress & Strain: Axial Loading

- Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.
- Considering structures as deformable allows determination of member forces and reactions which are statically indeterminate.
- Determination of the stress distribution within a member also requires consideration of deformations in the member.
- Chapter 2 is concerned with deformation of a structural member under axial loading. Later chapters will deal with torsional and pure bending loads.



Normal Strain

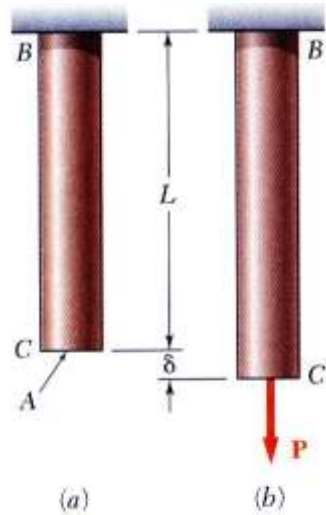


Fig. 2.1

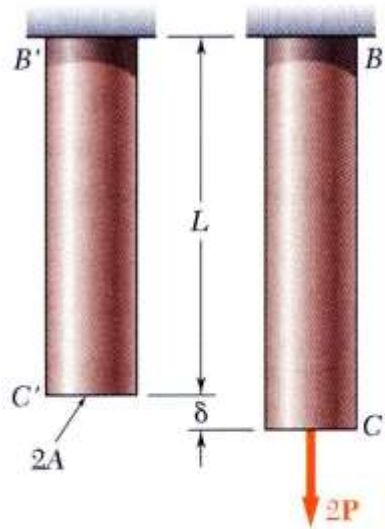


Fig. 2.3

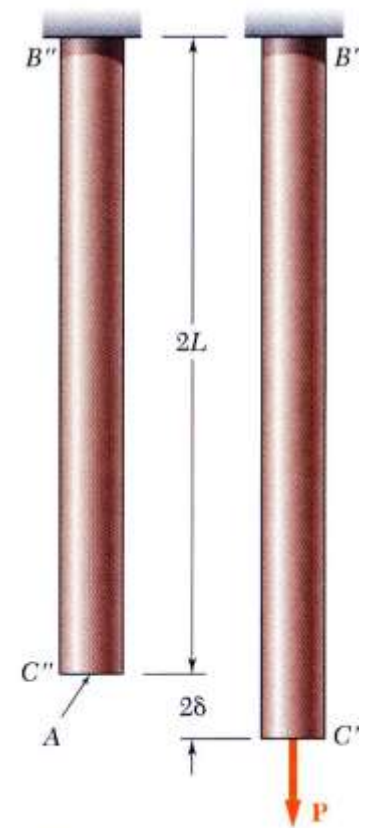


Fig. 2.4

$$\sigma = \frac{P}{A} = \text{stress}$$

$$\epsilon = \frac{\delta}{L} = \text{normal strain}$$

$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$

$$\epsilon = \frac{\delta}{L}$$

$$\sigma = \frac{P}{A}$$

$$\epsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Stress-Strain Test

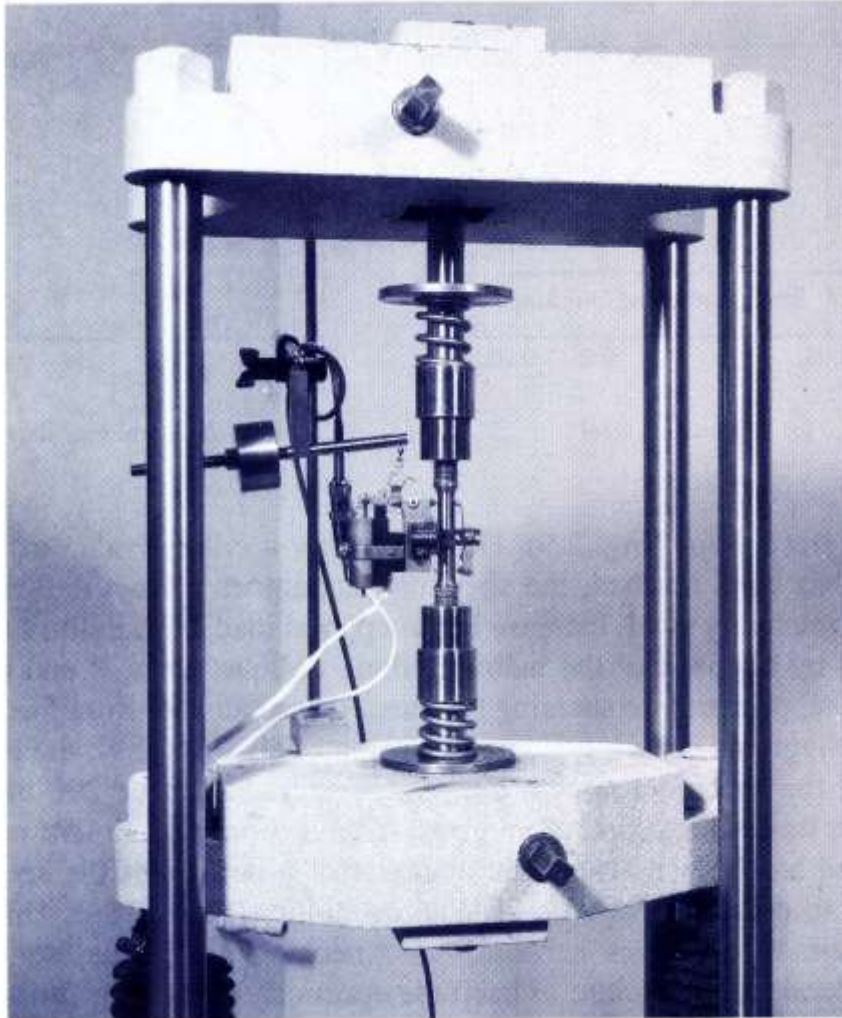


Fig. 2.7 This machine is used to test tensile test specimens, such as those shown in this chapter.

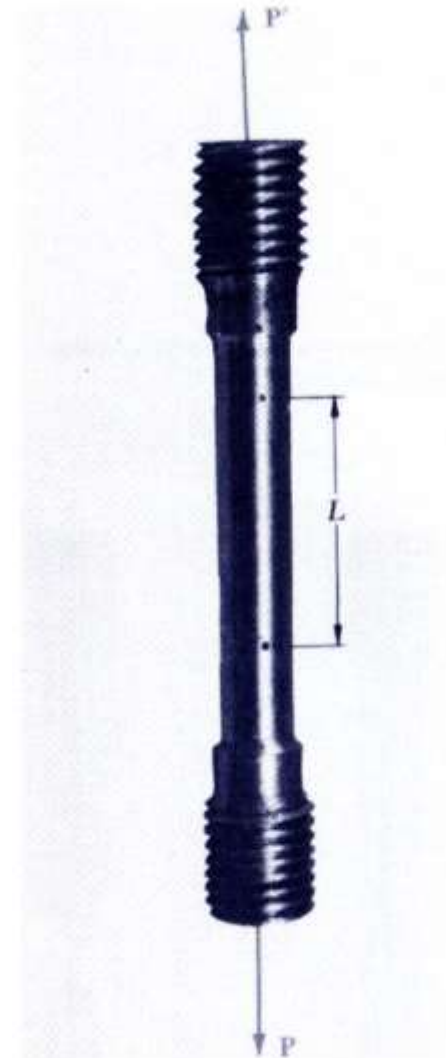
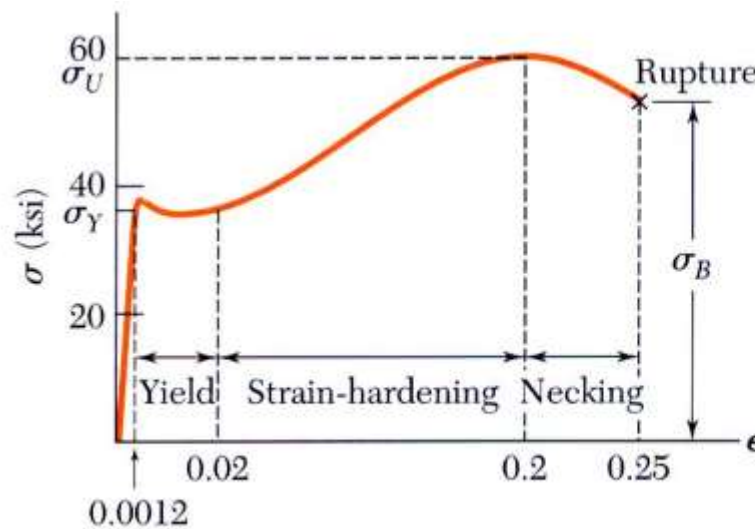
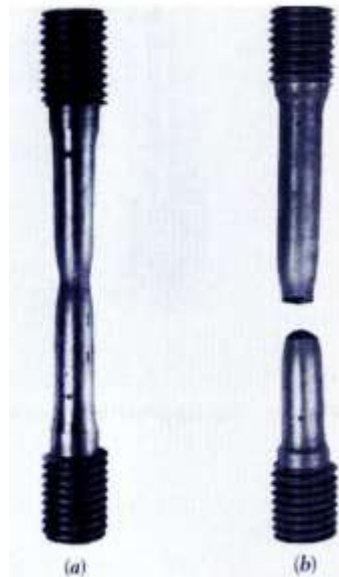
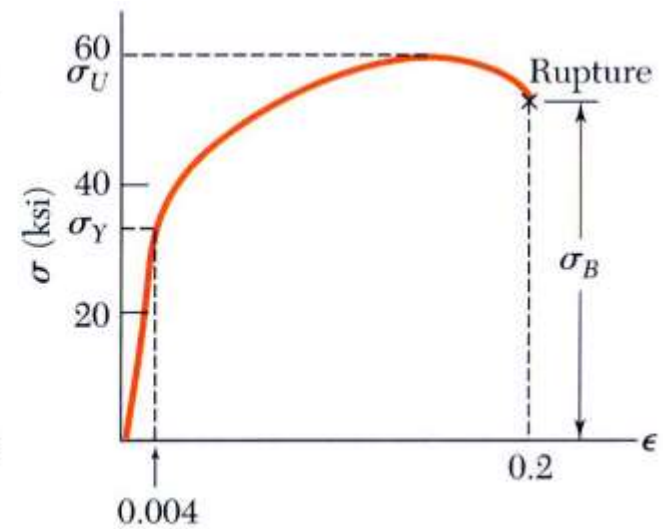


Fig. 2.8 Test specimen with tensile load.

Stress-Strain Diagram: Ductile Materials



(a) Low-carbon steel



(b) Aluminum alloy

Stress-Strain Diagram: Brittle Materials

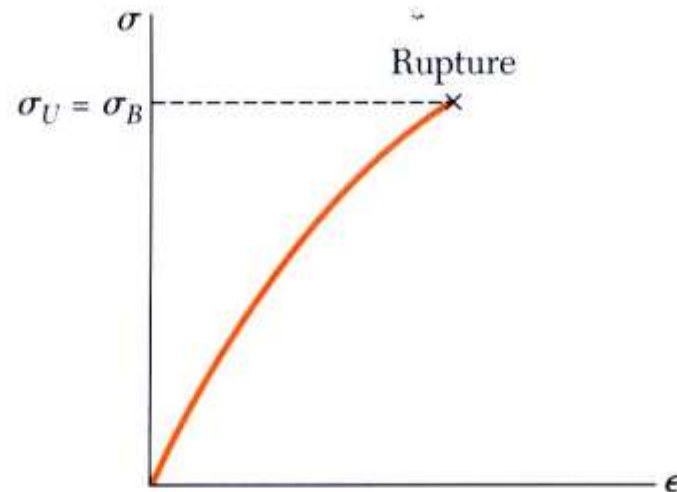
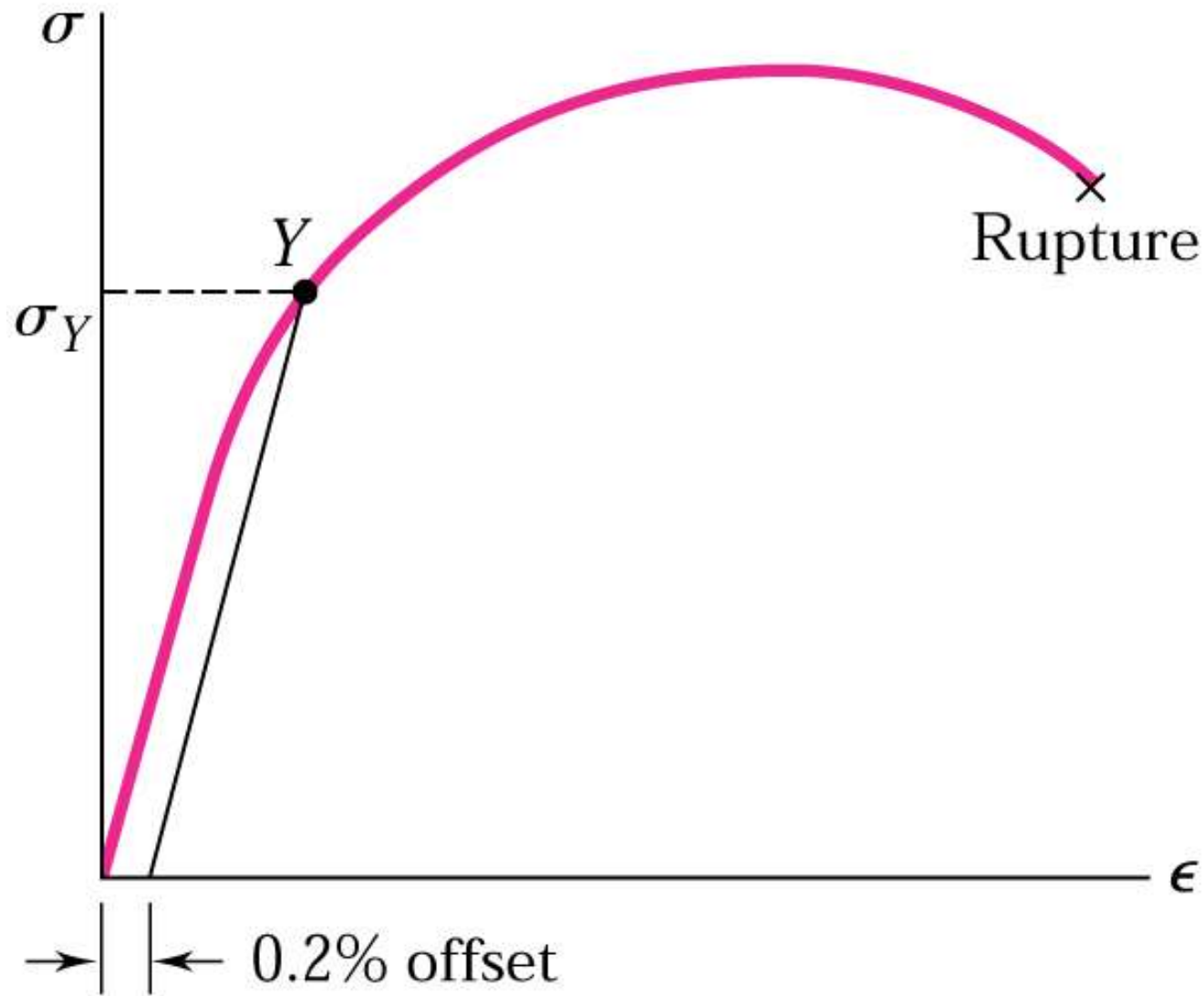
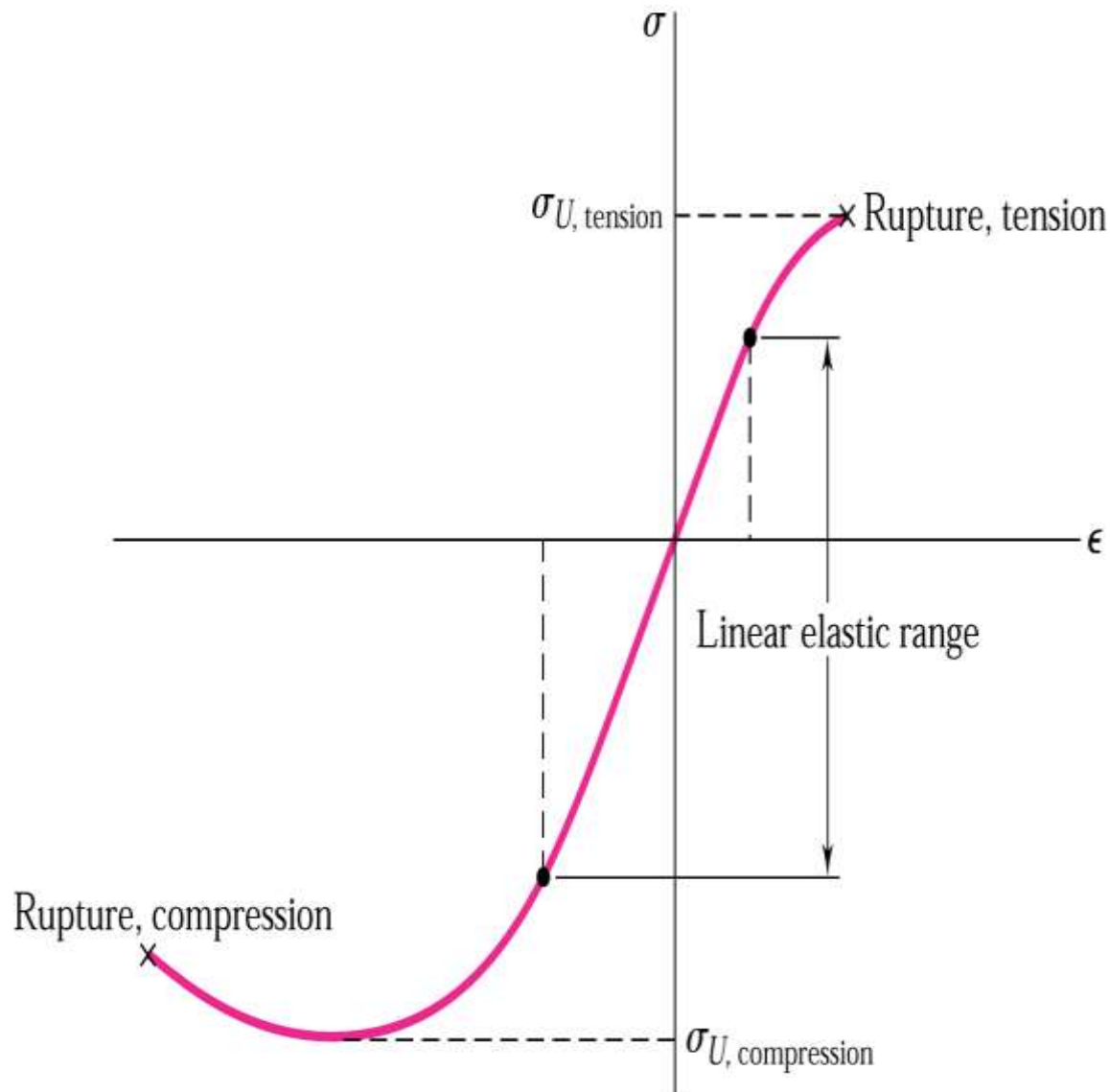


Fig. 2.11 Stress-strain diagram for a typical brittle material.

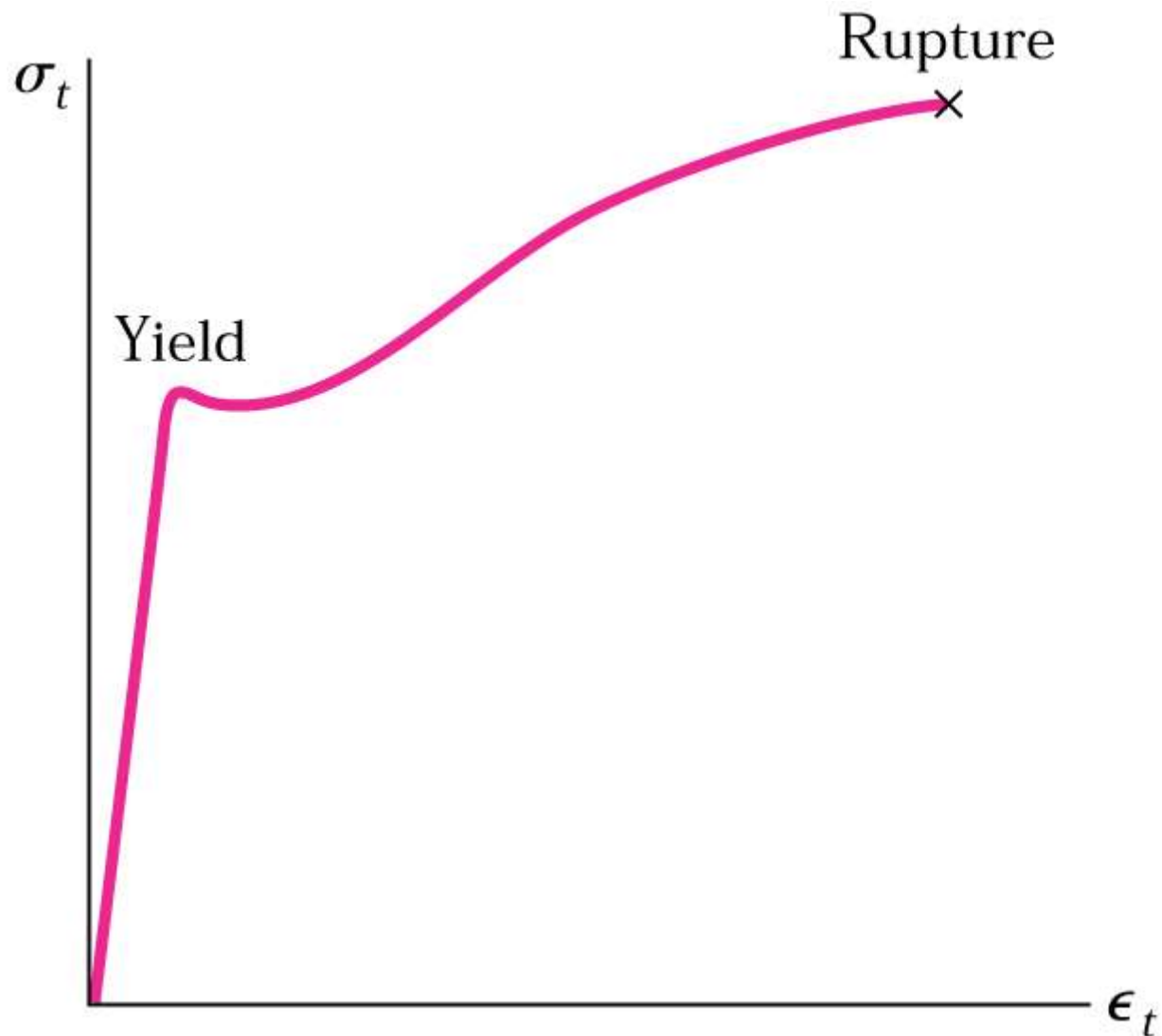
Stress-Strain Diagram: Brittle Materials



Stress-Strain Diagram: Brittle Materials



Stress-Strain Diagram: Brittle Materials



Hooke's Law: Modulus of Elasticity

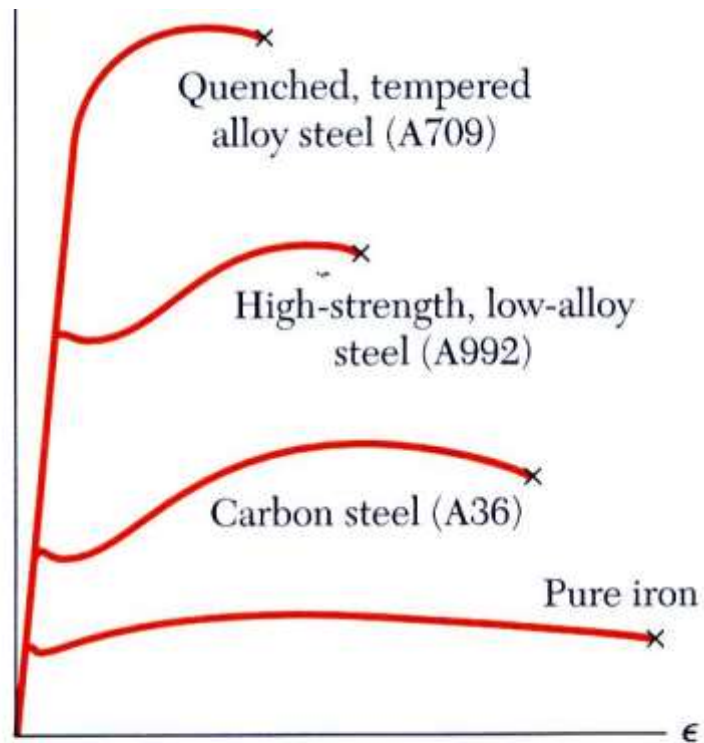


Fig. 2.16 Stress-strain diagrams for iron and different grades of steel.

- Below the yield stress

$$\sigma = E\epsilon$$

E = Young's Modulus or
Modulus of Elasticity

- Strength is affected by alloying, heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.

Elastic vs. Plastic Behavior

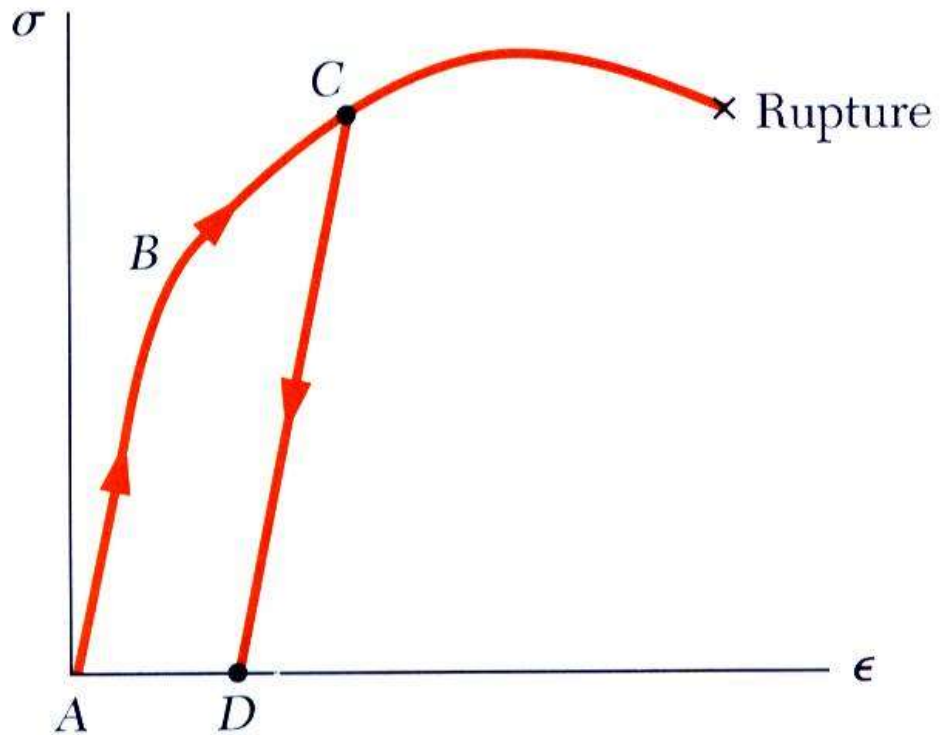


Fig. 2.18

- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

Fatigue

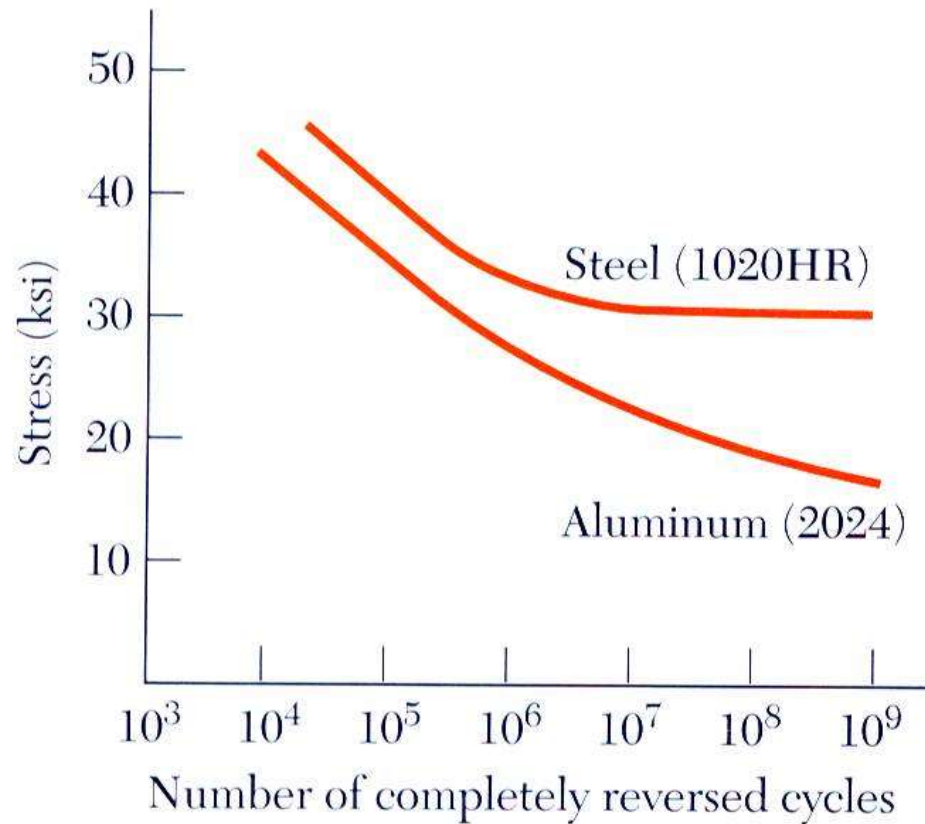


Fig. 2.21

- Fatigue properties are shown on S-N diagrams.
- A member may fail due to *fatigue* at stress levels significantly below the ultimate strength if subjected to many loading cycles.
- When the stress is reduced below the *endurance limit*, fatigue failures do not occur for any number of cycles.

Deformations Under Axial Loading

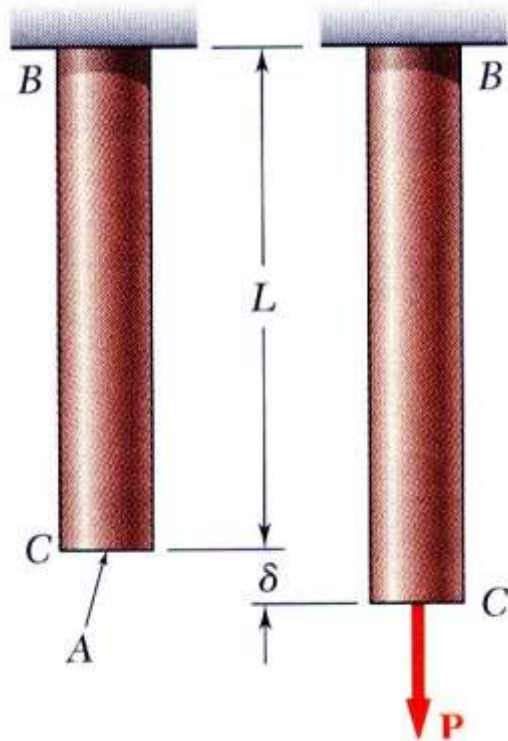


Fig. 2.22

- From Hooke's Law:

$$\sigma = E\varepsilon \quad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

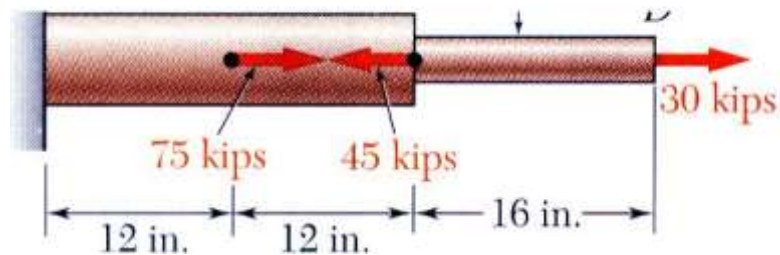
- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Example 2.01



$$E = 29 \times 10^6 \text{ psi}$$

$$D = 1.07 \text{ in.} \quad d = 0.618 \text{ in.}$$

Determine the deformation of the steel rod shown under the given loads.

SOLUTION:

- Divide the rod into components at the load application points.
- Apply a free-body analysis on each component to determine the internal force
- Evaluate the total of the component deflections.

SOLUTION:

- Divide the rod into three components:

- Apply free-body analysis to each component to determine internal forces,

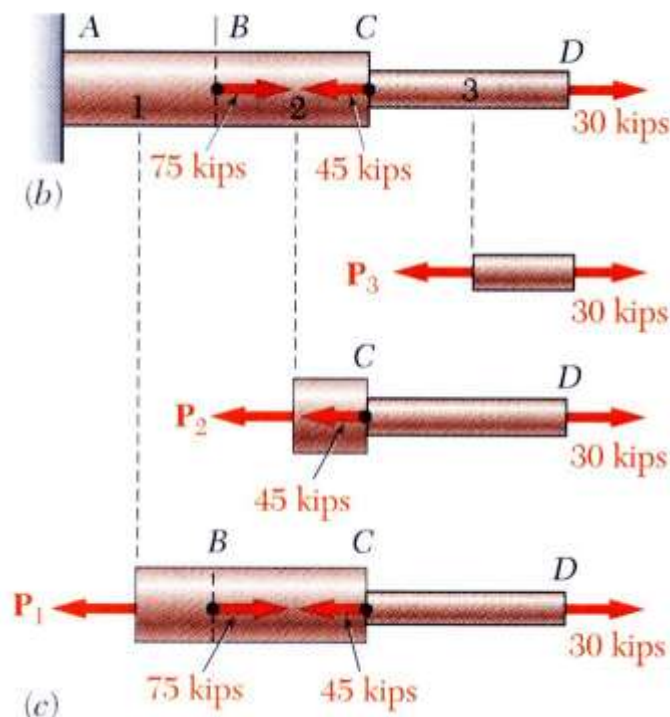
$$P_1 = 60 \times 10^3 \text{ lb}$$

$$P_2 = -15 \times 10^3 \text{ lb}$$

$$P_3 = 30 \times 10^3 \text{ lb}$$

- Evaluate total deflection,

$$\begin{aligned} \delta &= \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E} \left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3} \right) \\ &= \frac{1}{29 \times 10^6} \left[\frac{(60 \times 10^3) 12}{0.9} + \frac{(-15 \times 10^3) 12}{0.9} + \frac{(30 \times 10^3) 16}{0.3} \right] \\ &= 75.9 \times 10^{-3} \text{ in.} \end{aligned}$$

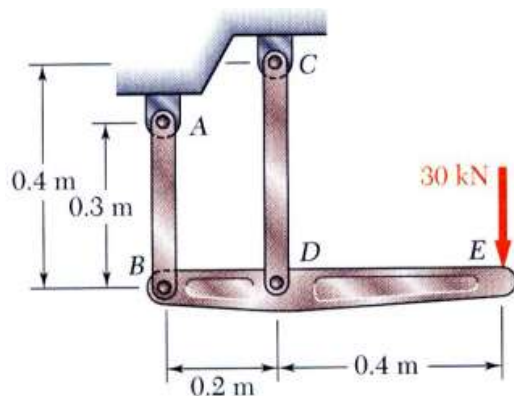


$$L_1 = L_2 = 12 \text{ in.} \quad L_3 = 16 \text{ in.}$$

$$A_1 = A_2 = 0.9 \text{ in}^2 \quad A_3 = 0.3 \text{ in}^2$$

$$\delta = 75.9 \times 10^{-3} \text{ in.}$$

Sample Problem 2.1



The rigid bar BDE is supported by two links AB and CD .

Link AB is made of aluminum ($E = 70\text{ GPa}$) and has a cross-sectional area of 500 mm^2 . Link CD is made of steel ($E = 200\text{ GPa}$) and has a cross-sectional area of (600 mm^2).

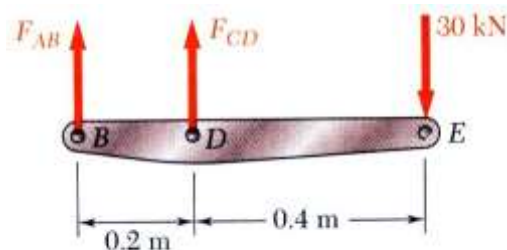
For the 30-kN force shown, determine the deflection a) of B , b) of D , and c) of E .

SOLUTION:

- Apply a free-body analysis to the bar BDE to find the forces exerted by links AB and CD .
- Evaluate the deformation of links AB and CD or the displacements of B and D .
- Work out the geometry to find the deflection at E given the deflections at B and D .

Sample Problem 2.1

SOLUTION:

Free body: Bar *BDE*

$$\sum M_B = 0$$

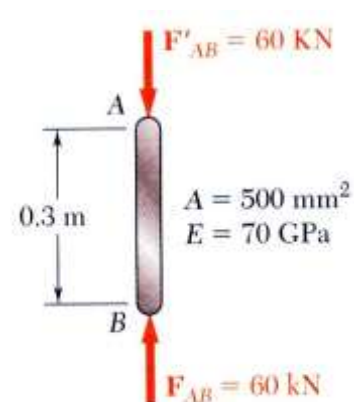
$$0 = -(30 \text{ kN} \times 0.6 \text{ m}) + F_{CD} \times 0.2 \text{ m}$$

$$F_{CD} = +90 \text{ kN} \text{ tension}$$

$$\sum M_D = 0$$

$$0 = -(30 \text{ kN} \times 0.4 \text{ m}) - F_{AB} \times 0.2 \text{ m}$$

$$F_{AB} = -60 \text{ kN} \text{ compression}$$

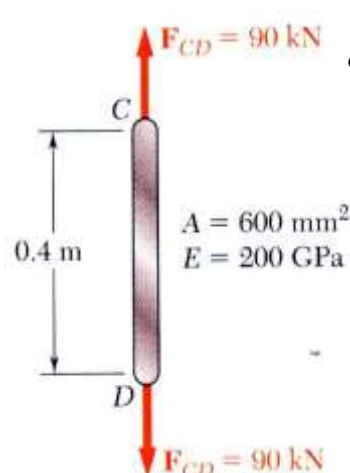
Displacement of *B*:

$$\delta_B = \frac{PL}{AE}$$

$$= \frac{(-60 \times 10^3 \text{ N})(0.3 \text{ m})}{(500 \times 10^{-6} \text{ m}^2)(70 \times 10^9 \text{ Pa})}$$

$$= -514 \times 10^{-6} \text{ m}$$

$$\delta_B = 0.514 \text{ mm} \uparrow$$

Displacement of *D*:

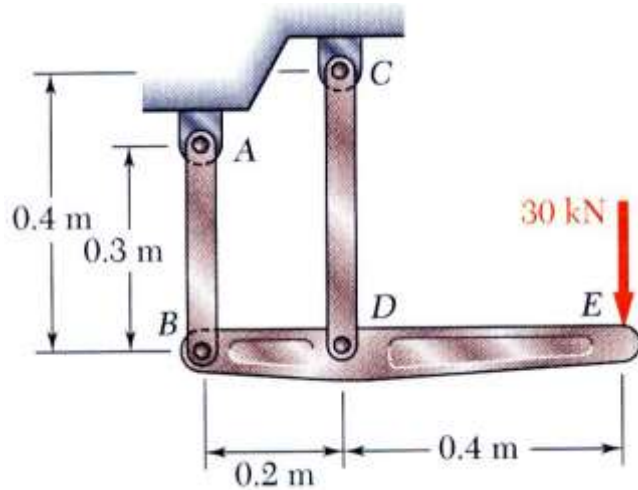
$$\delta_D = \frac{PL}{AE}$$

$$= \frac{(90 \times 10^3 \text{ N})(0.4 \text{ m})}{(600 \times 10^{-6} \text{ m}^2)(200 \times 10^9 \text{ Pa})}$$

$$= 300 \times 10^{-6} \text{ m}$$

$$\delta_D = 0.300 \text{ mm} \downarrow$$

Sample Problem 2.1



Displacement of D:

$$\frac{BB'}{DD'} = \frac{BH}{HD}$$

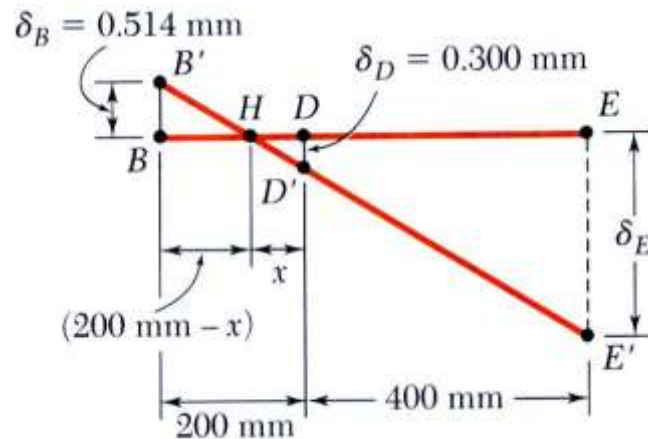
$$\frac{0.514 \text{ mm}}{0.300 \text{ mm}} = \frac{(200 \text{ mm}) - x}{x}$$

$$x = 73.7 \text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD}$$

$$\frac{\delta_E}{0.300 \text{ mm}} = \frac{(400 + 73.7) \text{ mm}}{73.7 \text{ mm}}$$

$$\delta_E = 1.928 \text{ mm}$$



$$\delta_E = 1.928 \text{ mm} \downarrow$$

Static Indeterminacy

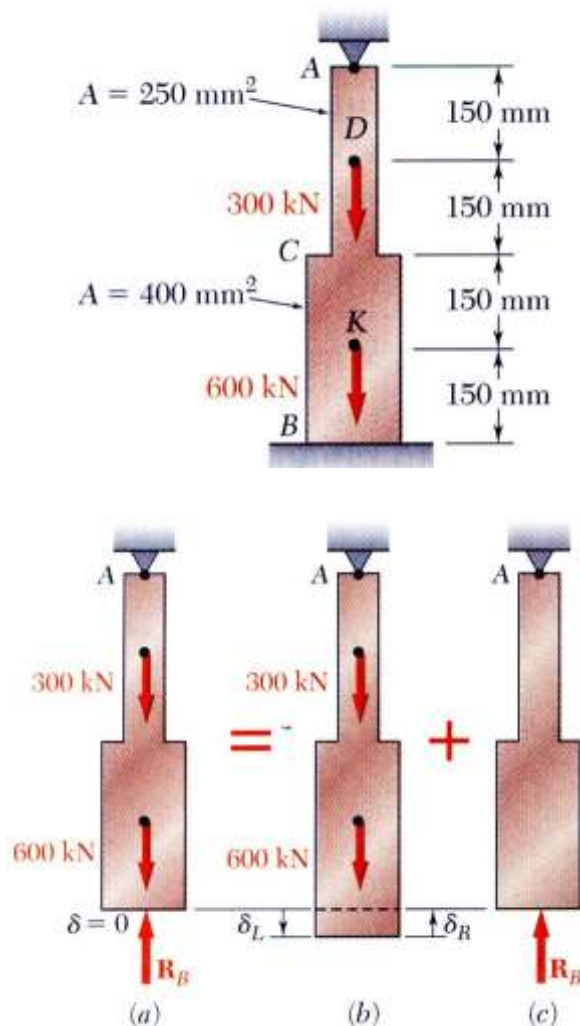
- Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.

- A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.

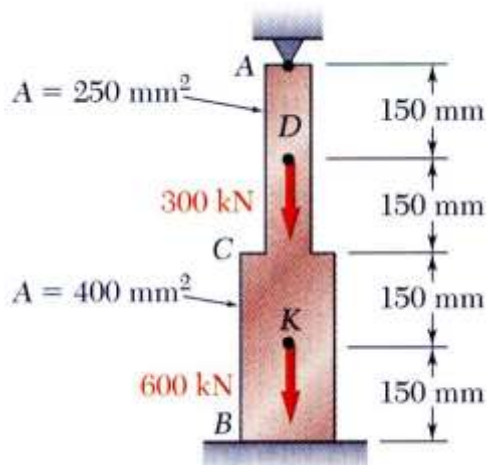
- Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.

- Deformations due to actual loads and redundant reactions are determined separately and then added or *superposed*.

$$\delta = \delta_L + \delta_R = 0$$



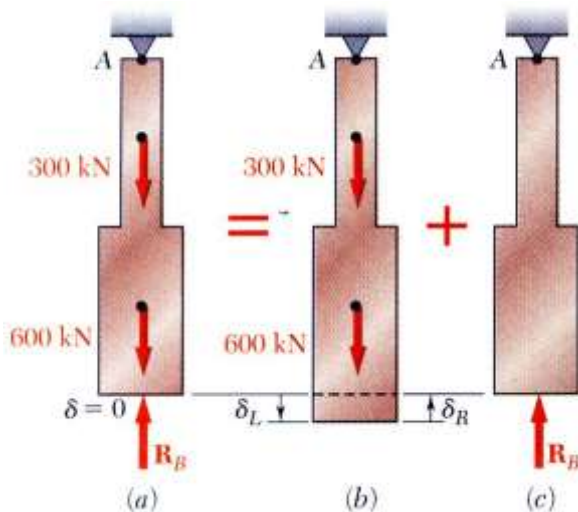
Example 2.04



Determine the reactions at A and B for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.

SOLUTION:

- Consider the reaction at B as redundant, release the bar from that support, and solve for the displacement at B due to the applied loads.
- Solve for the displacement at B due to the redundant reaction at B .
- Require that the displacements due to the loads and due to the redundant reaction be compatible, i.e., require that their sum be zero.
- Solve for the reaction at A due to applied loads and the reaction found at B .



Example 2.04

SOLUTION:

- Solve for the displacement at B due to the applied loads with the redundant constraint released,

$$P_1 = 0 \quad P_2 = P_3 = 600 \times 10^3 \text{ N} \quad P_4 = 900 \times 10^3 \text{ N}$$

$$A_1 = A_2 = 400 \times 10^{-6} \text{ m}^2 \quad A_3 = A_4 = 250 \times 10^{-6} \text{ m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150 \text{ m}$$

$$\delta_L = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1.125 \times 10^9}{E}$$

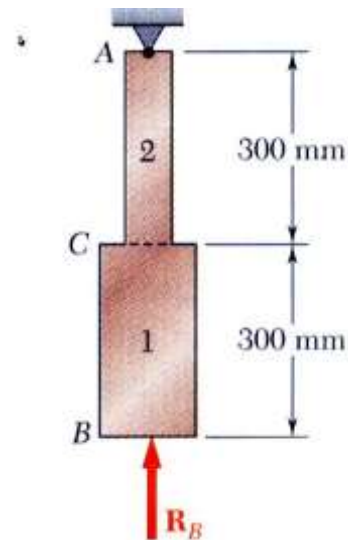
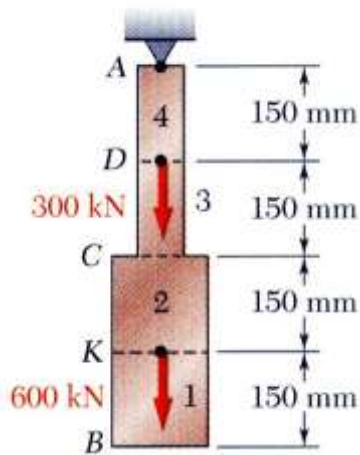
- Solve for the displacement at B due to the redundant constraint,

$$P_1 = P_2 = -R_B$$

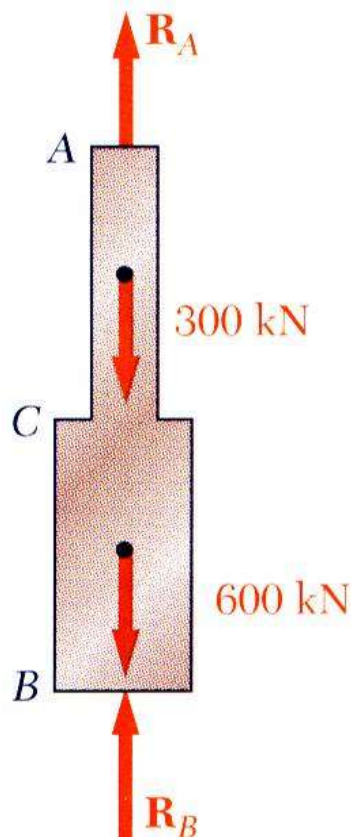
$$A_1 = 400 \times 10^{-6} \text{ m}^2 \quad A_2 = 250 \times 10^{-6} \text{ m}^2$$

$$L_1 = L_2 = 0.300 \text{ m}$$

$$\delta_R = \sum_i \frac{P_i L_i}{A_i E_i} = - \frac{(1.95 \times 10^3) R_B}{E}$$



Example 2.04



- Require that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

- Find the reaction at A due to the loads and the reaction at B

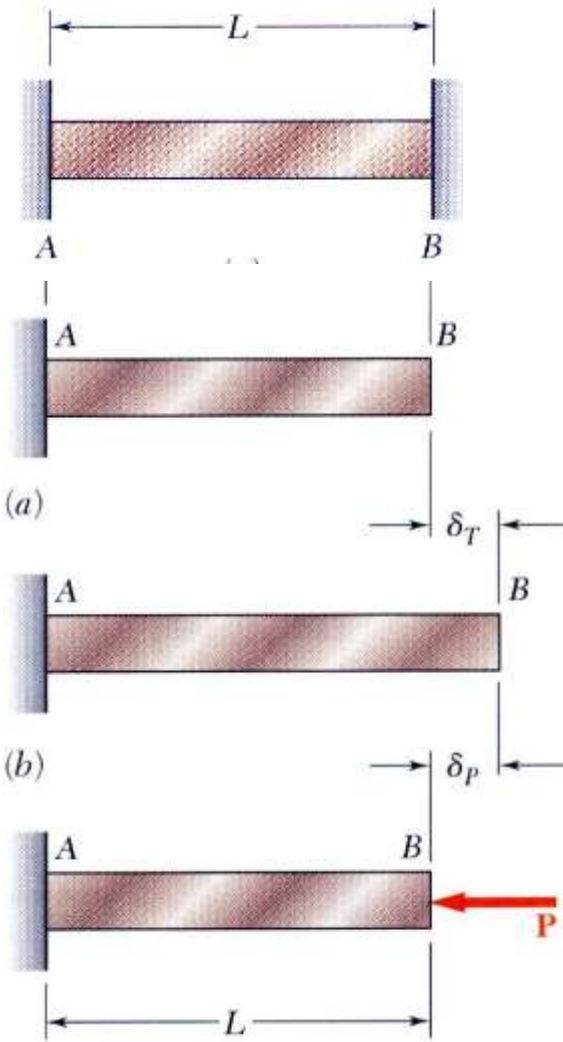
$$\Sigma F_y = 0 = R_A - 300 \text{ kN} - 600 \text{ kN} + 577 \text{ kN}$$

$$R_A = 323 \text{ kN}$$

$$R_A = 323 \text{ kN}$$

$$R_B = 577 \text{ kN}$$

Thermal Stresses



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.
- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L$$

$$\delta_P = \frac{PL}{AE}$$

α = thermal expansion coef.

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

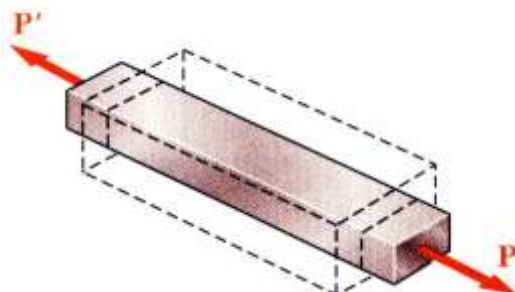
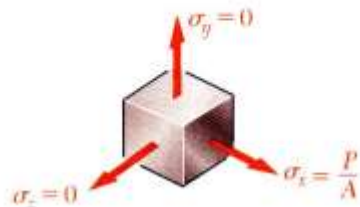
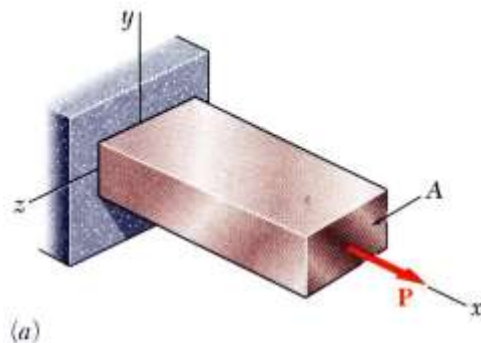
$$\delta = \delta_T + \delta_P = 0$$

$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Poisson's Ratio



- For a slender bar subjected to axial loading:

$$\epsilon_x = \frac{\sigma_x}{E} \quad \sigma_y = \sigma_z = 0$$

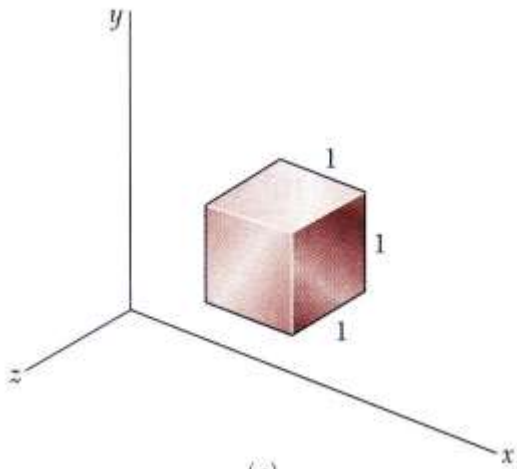
- The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is isotropic (no directional dependence),

$$\epsilon_y = \epsilon_z \neq 0$$

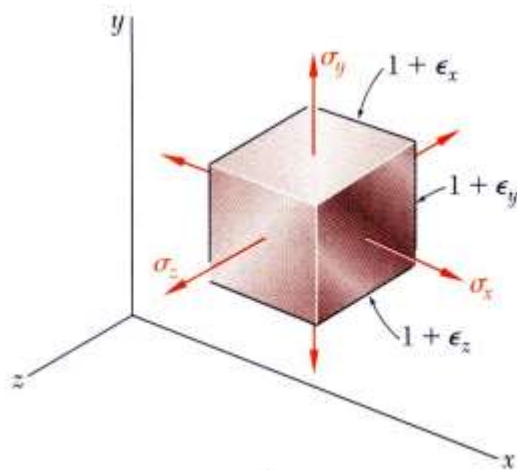
- Poisson's ratio is defined as

$$\nu = \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x}$$

Generalized Hooke's Law



(a)



(b)

- For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:

- 1) strain is linearly related to stress
- 2) deformations are small

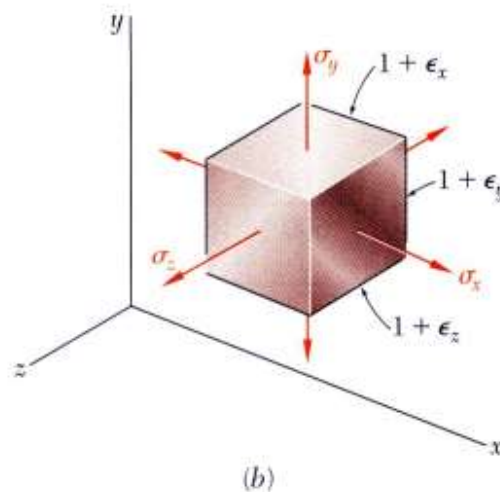
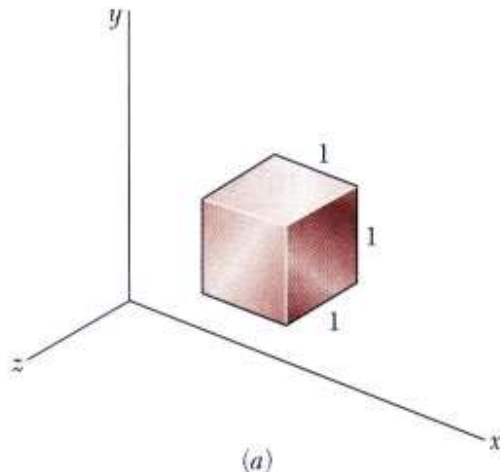
- With these restrictions:

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Dilatation: Bulk Modulus



- Relative to the unstressed state, the change in volume is

$$e = 1 - [(1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)] = 1 - [1 + \epsilon_x + \epsilon_y + \epsilon_z]$$

$$= \epsilon_x + \epsilon_y + \epsilon_z$$

$$= \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y + \sigma_z)$$

$$= \text{dilatation (change in volume per unit volume)}$$

- For element subjected to uniform hydrostatic pressure,

$$e = -p \frac{3(1 - 2\nu)}{E} = -\frac{p}{k}$$

$$k = \frac{E}{3(1 - 2\nu)} = \text{bulk modulus}$$

- Subjected to uniform pressure, dilatation must be negative, therefore

$$0 < \nu < \frac{1}{2}$$

Shearing Strain

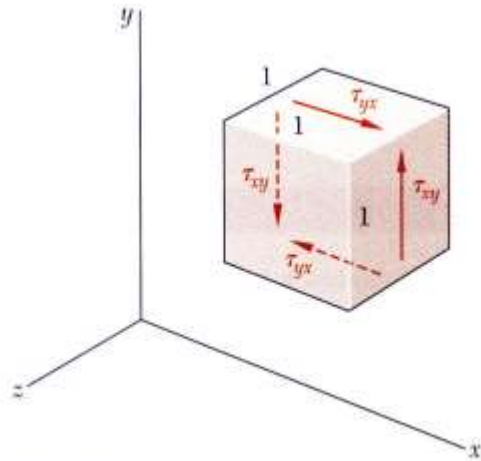


Fig. 2.46

- A cubic element subjected to a shear stress will deform into a rhomboid. The corresponding *shear strain* is quantified in terms of the change in angle between the sides,

$$\tau_{xy} = f(\gamma_{xy})$$

- A plot of shear stress vs. shear strain is similar the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For small strains,

$$\tau_{xy} = G \gamma_{xy} \quad \tau_{yz} = G \gamma_{yz} \quad \tau_{zx} = G \gamma_{zx}$$

where G is the modulus of rigidity or shear modulus.

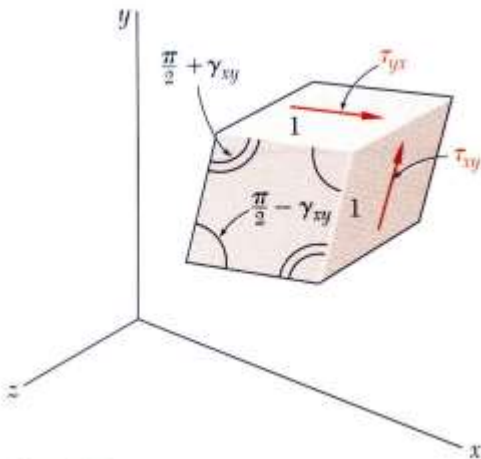
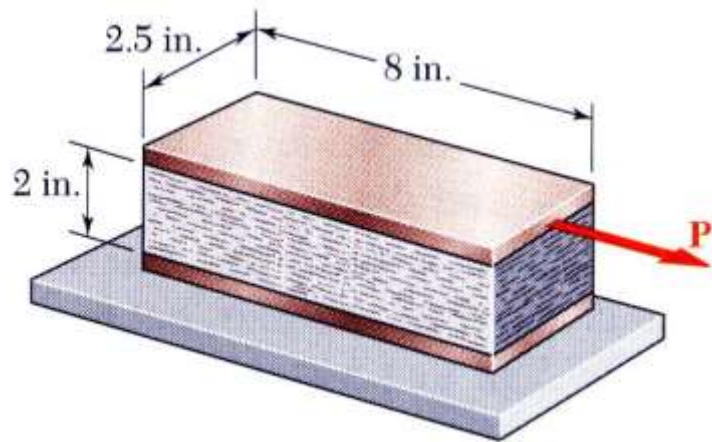


Fig. 2.47

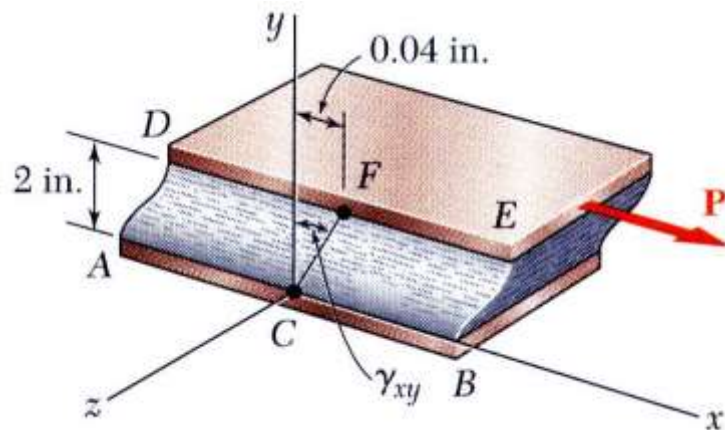
Example 2.10



A rectangular block of material with modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force P . Knowing that the upper plate moves through 0.04 in. under the action of the force, determine a) the average shearing strain in the material, and b) the force P exerted on the plate.

SOLUTION:

- Determine the average angular deformation or shearing strain of the block.
- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.
- Use the definition of shearing stress to find the force P .



- Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04 \text{ in.}}{2 \text{ in.}} \quad \gamma_{xy} = 0.020 \text{ rad}$$

- Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

$$\tau_{xy} = G\gamma_{xy} = (90 \times 10^3 \text{ psi})(0.020 \text{ rad}) = 1800 \text{ psi}$$

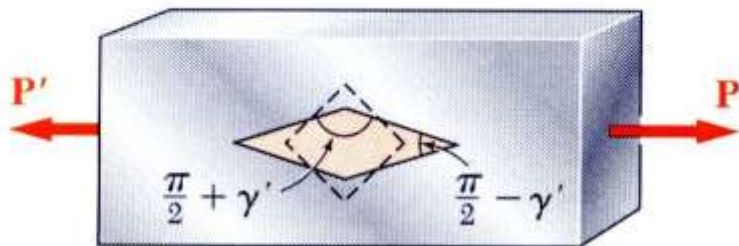
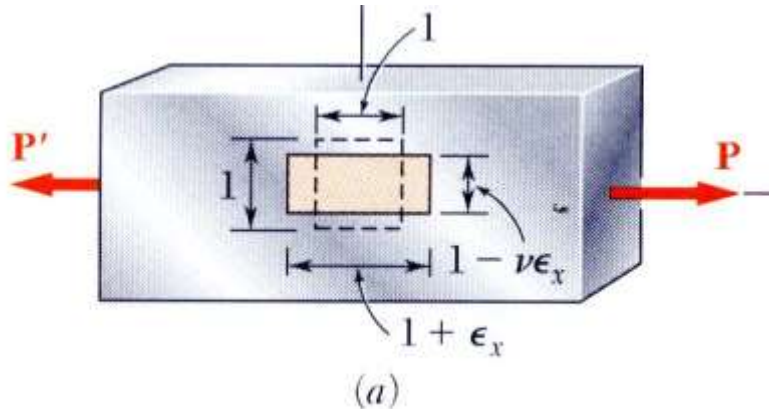
- Use the definition of shearing stress to find the force P .

$$P = \tau_{xy}A = (1800 \text{ psi})(8 \text{ in.})(2.5 \text{ in.}) = 36 \times 10^3 \text{ lb}$$

$$P = 36.0 \text{ kips}$$



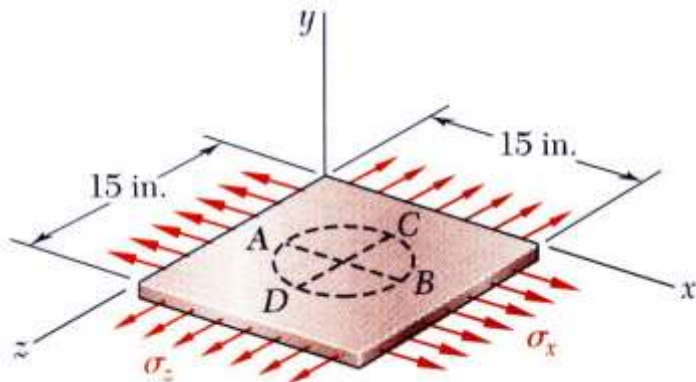
Relation Among E , ν , and G



- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1 + \nu)$$

Sample Problem 2.5



A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = 3/4$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi.

For $E = 10 \times 10^6$ psi and $\nu = 1/3$, determine the change in:

- the length of diameter AB ,
- the length of diameter CD ,
- the thickness of the plate, and
- the volume of the plate.



SOLUTION:

- Apply the generalized Hooke's Law to find the three components of normal strain.
- Evaluate the deformation components.

$$\begin{aligned}\varepsilon_x &= +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ &= \frac{1}{10 \times 10^6 \text{ psi}} \left[(12 \text{ ksi}) - 0 - \frac{1}{3}(20 \text{ ksi}) \right] \\ &= +0.533 \times 10^{-3} \text{ in./in.}\end{aligned}$$

$$\begin{aligned}\varepsilon_y &= -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \\ &= -1.067 \times 10^{-3} \text{ in./in.}\end{aligned}$$

$$\begin{aligned}\varepsilon_z &= -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} \\ &= +1.600 \times 10^{-3} \text{ in./in.}\end{aligned}$$

$$\delta_{B/A} = \varepsilon_x d = (+0.533 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{B/A} = +4.8 \times 10^{-3} \text{ in.}$$

$$\delta_{C/D} = \varepsilon_z d = (+1.600 \times 10^{-3} \text{ in./in.})(9 \text{ in.})$$

$$\delta_{C/D} = +14.4 \times 10^{-3} \text{ in.}$$

$$\delta_t = \varepsilon_y t = (-1.067 \times 10^{-3} \text{ in./in.})(0.75 \text{ in.})$$

$$\delta_t = -0.800 \times 10^{-3} \text{ in.}$$

- Find the change in volume

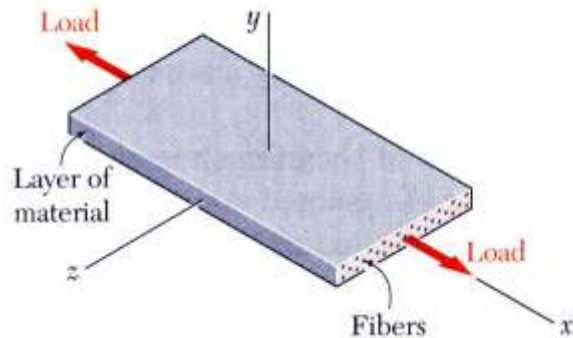
$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067 \times 10^{-3} \text{ in}^3/\text{in}^3$$

$$\Delta V = eV = 1.067 \times 10^{-3} (15 \times 15 \times 0.75) \text{ in}^3$$

$$\Delta V = +0.187 \text{ in}^3$$

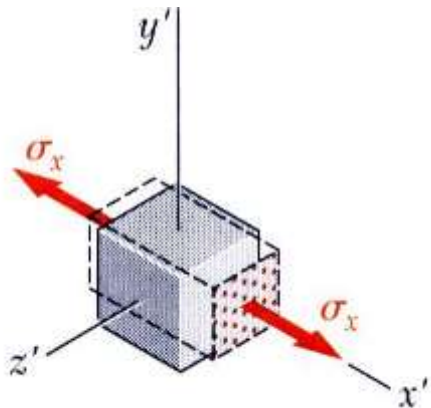


Composite Materials



- *Fiber-reinforced composite materials* are formed from *lamina* of fibers of graphite, glass, or polymers embedded in a resin matrix.
- Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

$$E_x = \frac{\sigma_x}{\epsilon_x} \quad E_y = \frac{\sigma_y}{\epsilon_y} \quad E_z = \frac{\sigma_z}{\epsilon_z}$$

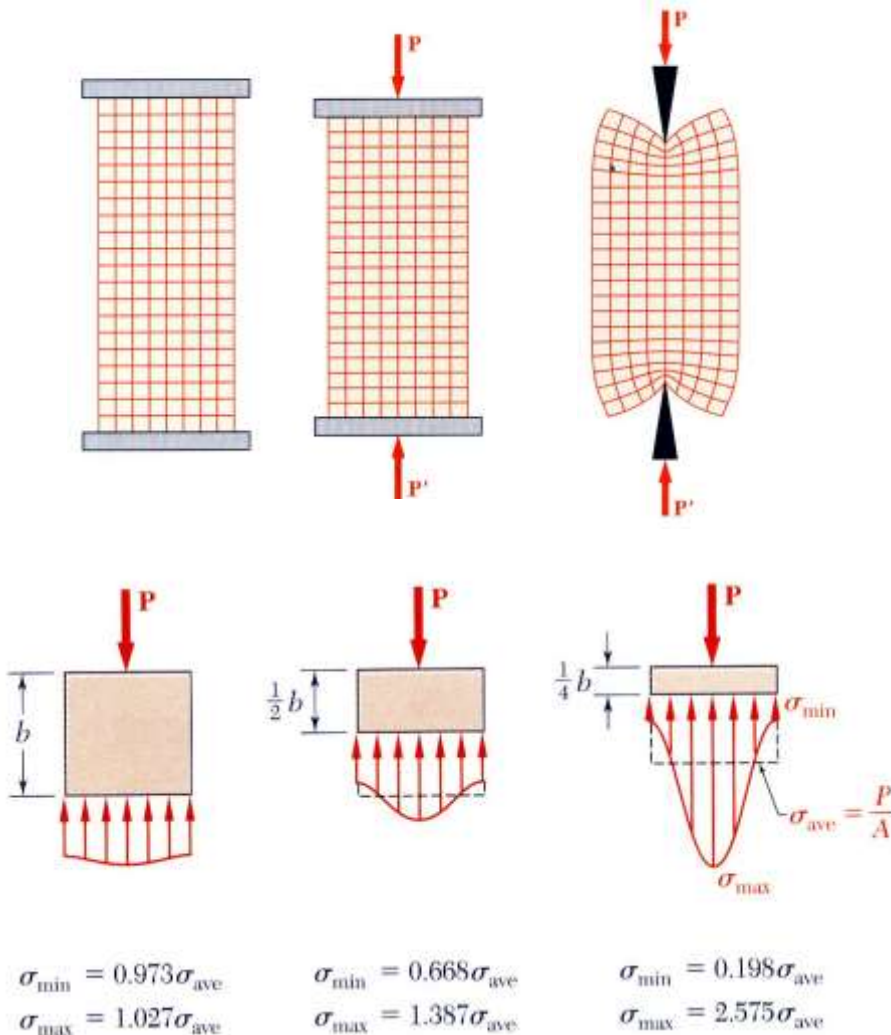


- Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy} = -\frac{\epsilon_y}{\epsilon_x} \quad \nu_{xz} = -\frac{\epsilon_z}{\epsilon_x}$$

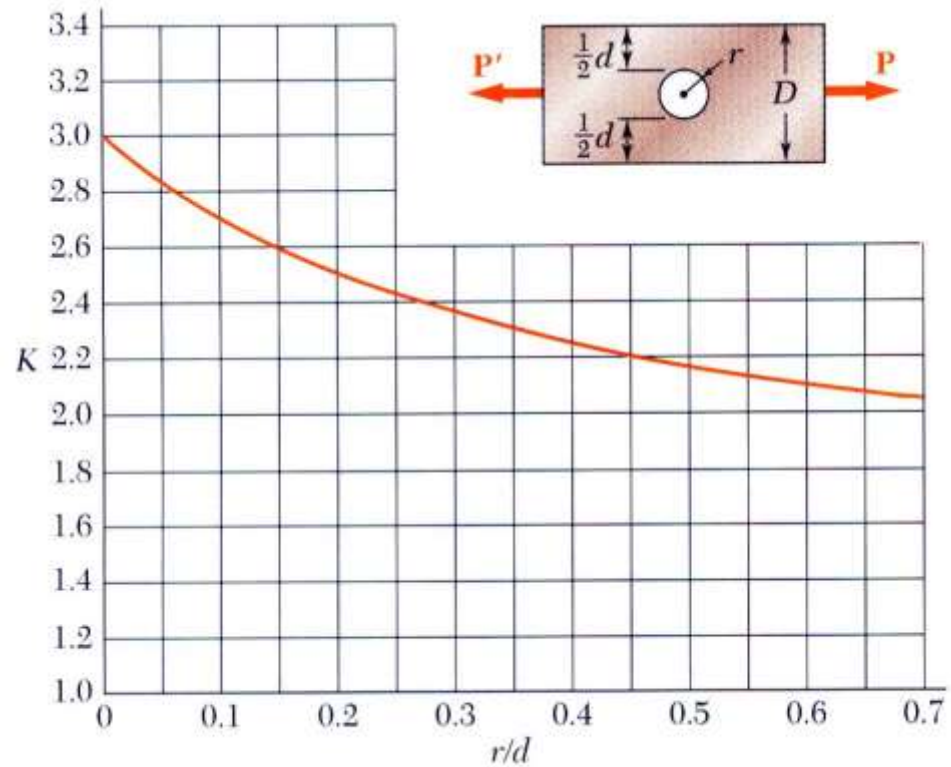
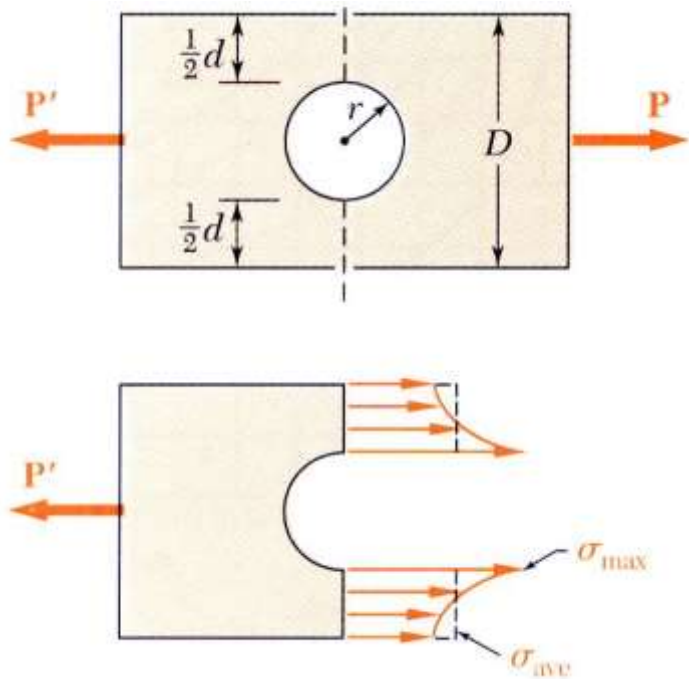
- Materials with directionally dependent mechanical properties are *anisotropic*.

Saint-Venant's Principle



- Loads transmitted through rigid plates result in uniform distribution of stress and strain.
- Concentrated loads result in large stresses in the vicinity of the load application point.
- Stress and strain distributions become uniform at a relatively short distance from the load application points.
- **Saint-Venant's Principle:** Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.

Stress Concentration: Hole

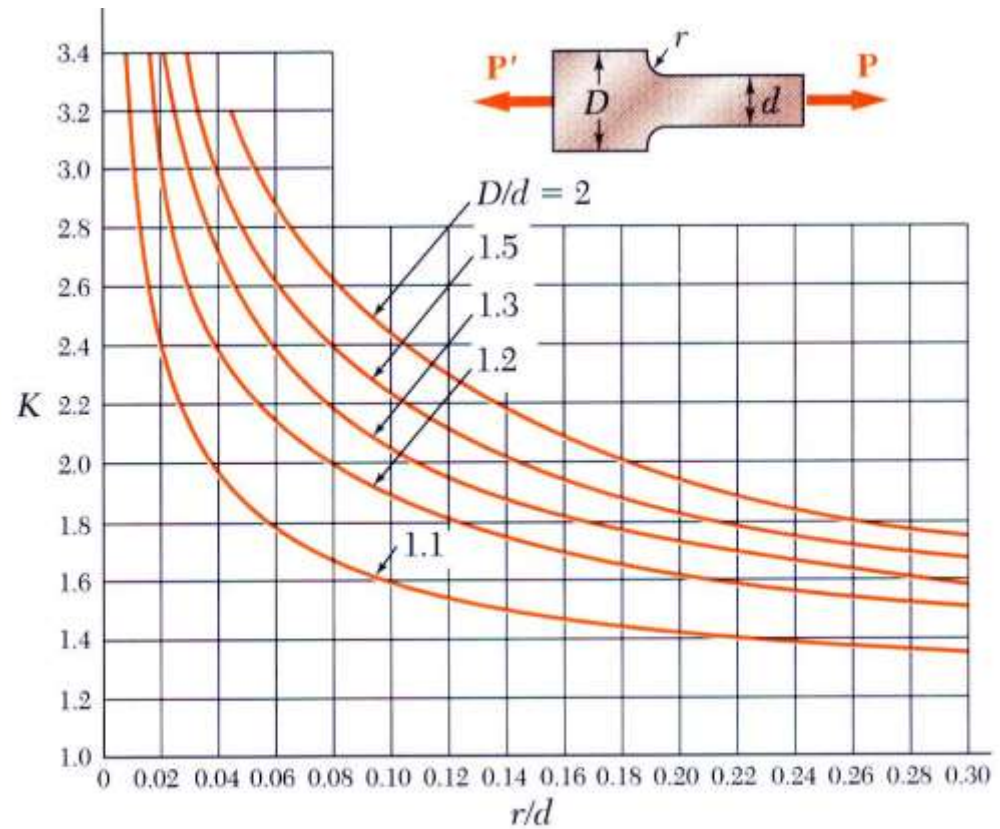
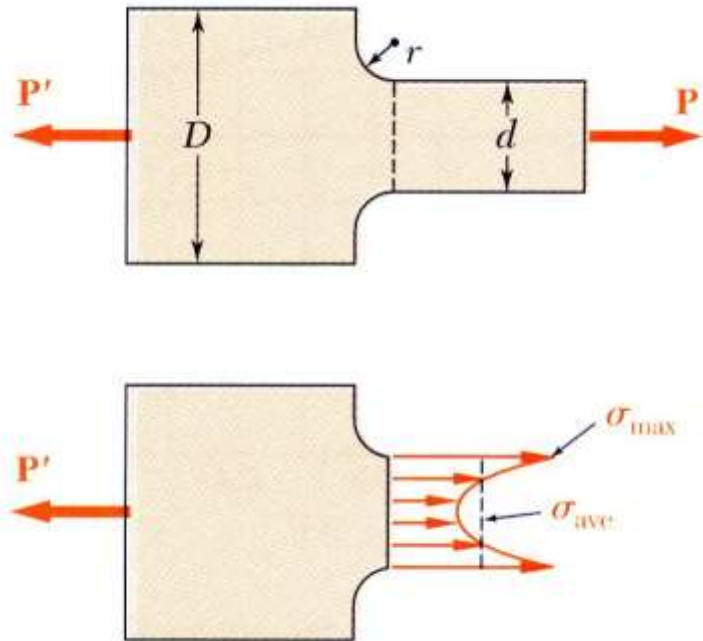


(a) Flat bars with holes

Discontinuities of cross section may result in high localized or *concentrated* stresses.

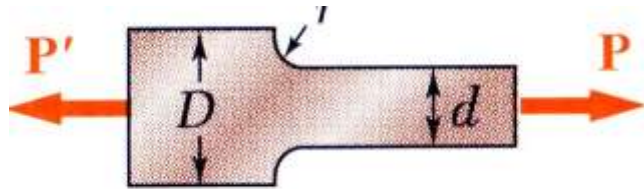
$$K = \frac{\sigma_{\max}}{\sigma_{\text{ave}}}$$

Stress Concentration: Fillet



(b) Flat bars with fillets

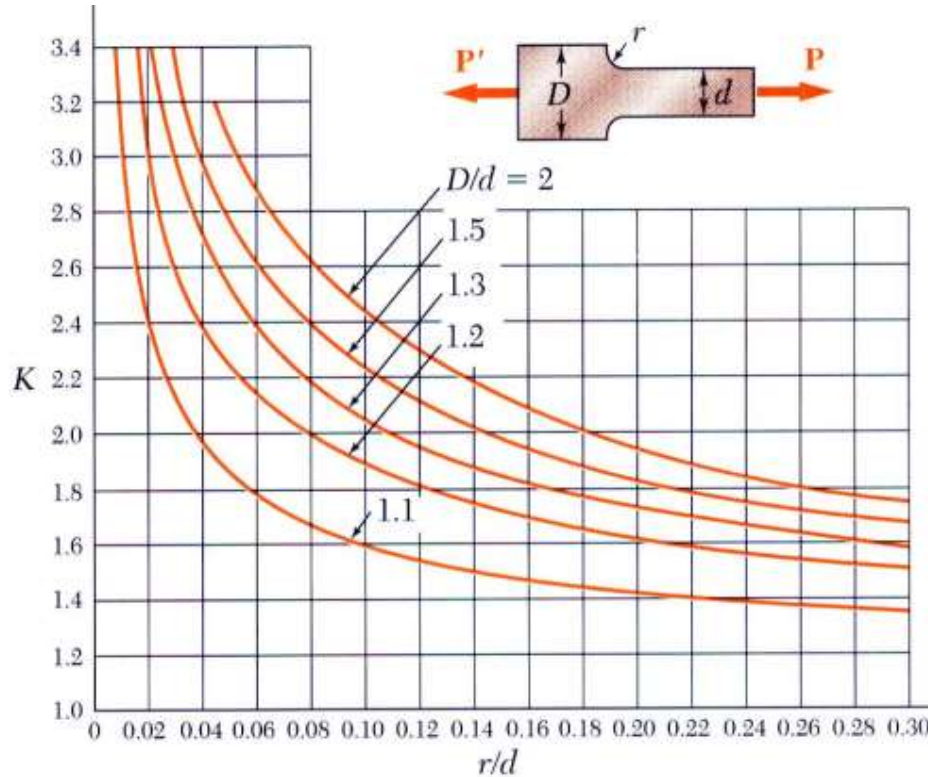
Example 2.12



Determine the largest axial load P that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius $r = 8$ mm. Assume an allowable normal stress of 165 MPa.

SOLUTION:

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64b.
- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.
- Apply the definition of normal stress to find the allowable load.



(b) Flat bars with fillets

- Determine the geometric ratios and find the stress concentration factor from Fig. 2.64b.

$$\frac{D}{d} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.50 \quad \frac{r}{d} = \frac{8 \text{ mm}}{40 \text{ mm}} = 0.20$$

$$K = 1.82$$

- Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

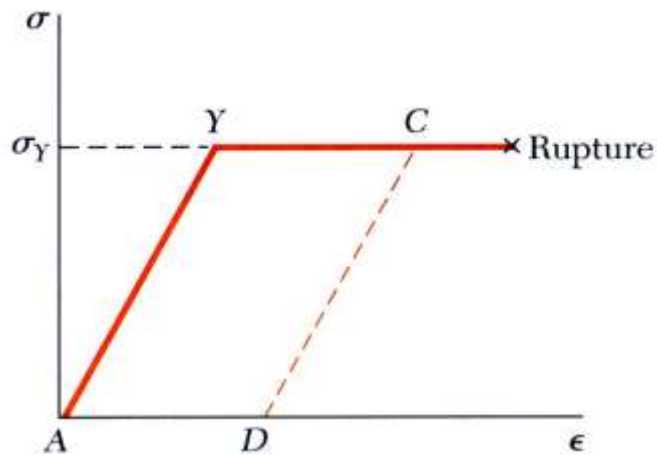
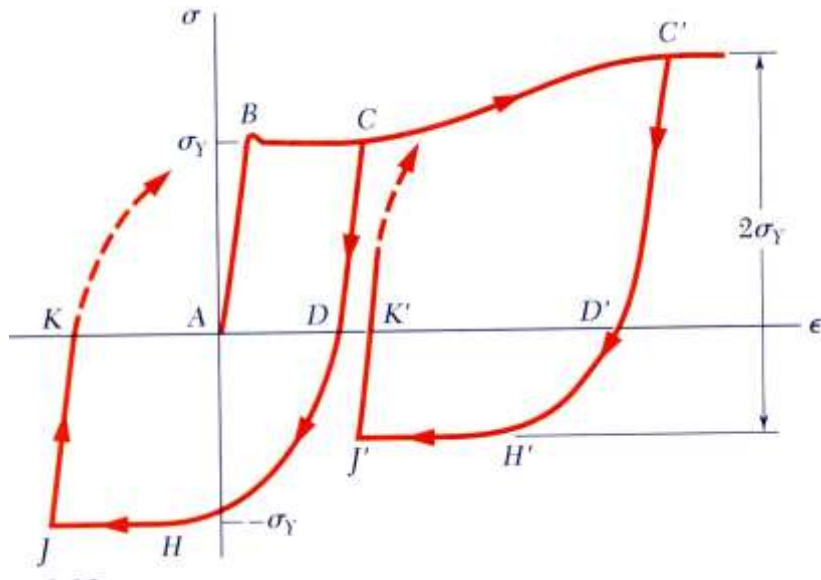
$$\sigma_{\text{ave}} = \frac{\sigma_{\text{max}}}{K} = \frac{165 \text{ MPa}}{1.82} = 90.7 \text{ MPa}$$

- Apply the definition of normal stress to find the allowable load.

$$P = A\sigma_{\text{ave}} = (40 \text{ mm})(10 \text{ mm})(90.7 \text{ MPa}) \\ = 36.3 \times 10^3 \text{ N}$$

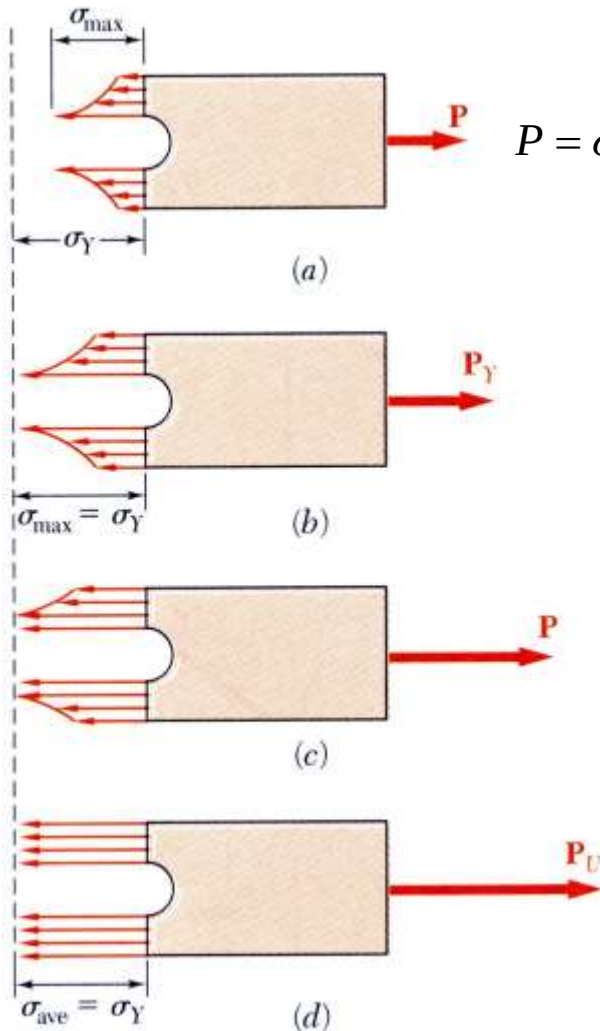
$$P = 36.3 \text{ kN}$$

Elastoplastic Materials



- Previous analyses based on assumption of linear stress-strain relationship, i.e., stresses below the yield stress
- Assumption is good for brittle material which rupture without yielding
- If the yield stress of ductile materials is exceeded, then plastic deformations occur
- Analysis of plastic deformations is simplified by assuming an idealized *elastoplastic material*
- Deformations of an elastoplastic material are divided into elastic and plastic ranges
- Permanent deformations result from loading beyond the yield stress

Plastic Deformations



$$P = \sigma_{\text{ave}} A = \frac{\sigma_{\max} A}{K}$$

- Elastic deformation while maximum stress is less than yield stress

$$P_Y = \frac{\sigma_Y A}{K}$$

- Maximum stress is equal to the yield stress at the maximum elastic loading

- At loadings above the maximum elastic load, a region of plastic deformations develop near the hole

$$P_U = \sigma_Y A = K P_Y$$

- As the loading increases, the plastic region expands until the section is at a uniform stress equal to the yield stress

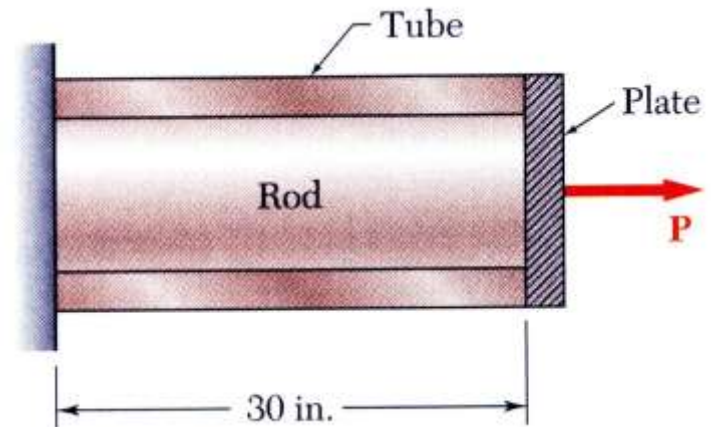
Residual Stresses

- When a single structural element is loaded uniformly beyond its yield stress and then unloaded, it is permanently deformed but all stresses disappear. This is not the general result.
- *Residual stresses* will remain in a structure after loading and unloading if
 - only part of the structure undergoes plastic deformation
 - different parts of the structure undergo different plastic deformations
- Residual stresses also result from the uneven heating or cooling of structures or structural elements



Example 2.14, 2.15, 2.16

A cylindrical rod is placed inside a tube of the same length. The ends of the rod and tube are attached to a rigid support on one side and a rigid plate on the other. The load on the rod-tube assembly is increased from zero to 5.7 kips and decreased back to zero.



- draw a load-deflection diagram for the rod-tube assembly
- determine the maximum elongation
- determine the permanent set
- calculate the residual stresses in the rod and tube.

$$A_r = 0.075 \text{ in.}^2$$

$$A_t = 0.100 \text{ in.}^2$$

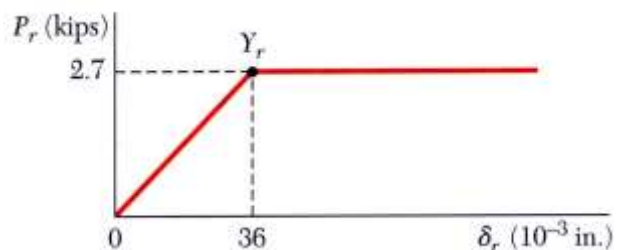
$$E_r = 30 \times 10^6 \text{ psi}$$

$$E_t = 15 \times 10^6 \text{ psi}$$

$$\sigma_{Y,r} = 36 \text{ ksi}$$

$$\sigma_{Y,t} = 45 \text{ ksi}$$

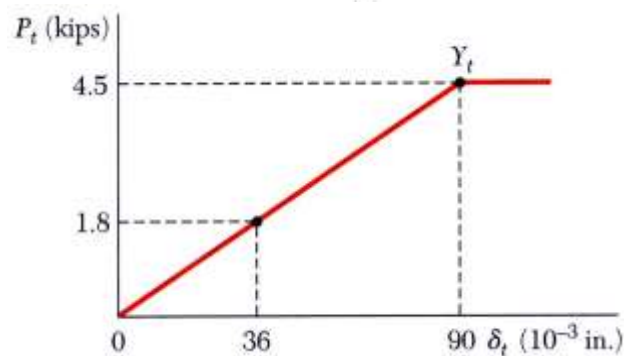
Example 2.14, 2.15, 2.16



- a) draw a load-deflection diagram for the rod-tube assembly

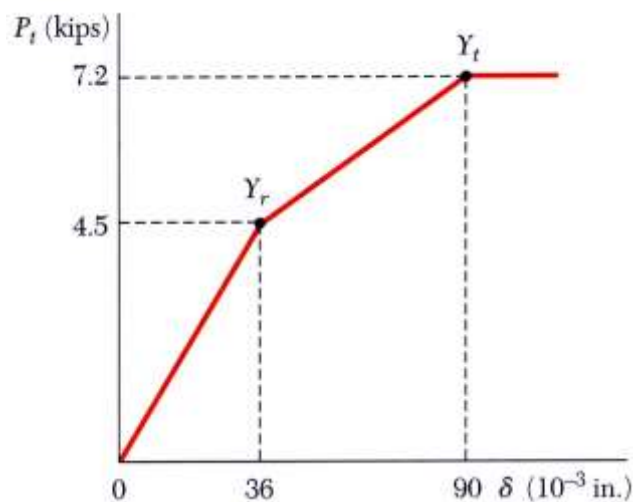
$$P_{Y,r} = \sigma_{Y,r} A_r = (36 \text{ ksi})(0.075 \text{ in}^2) = 2.7 \text{ kips}$$

$$\delta_{Y,r} = \varepsilon_{Y,r} L = \frac{\sigma_{Y,r}}{E_{Y,r}} L = \frac{36 \times 10^3 \text{ psi}}{30 \times 10^6 \text{ psi}} 30 \text{ in.} = 36 \times 10^{-3} \text{ in.}$$



$$P_{Y,t} = \sigma_{Y,t} A_t = (45 \text{ ksi})(0.100 \text{ in}^2) = 4.5 \text{ kips}$$

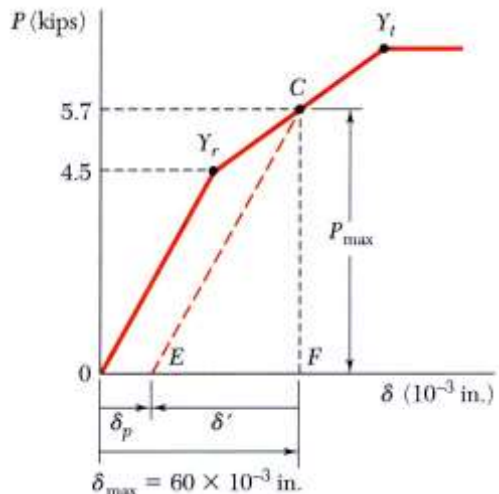
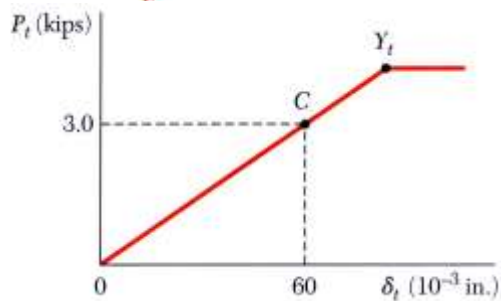
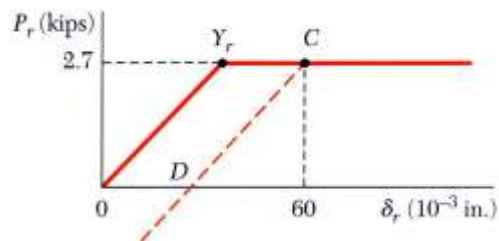
$$\delta_{Y,t} = \varepsilon_{Y,t} L = \frac{\sigma_{Y,t}}{E_{Y,t}} L = \frac{45 \times 10^3 \text{ psi}}{15 \times 10^6 \text{ psi}} 30 \text{ in.} = 90 \times 10^{-3} \text{ in.}$$



$$P = P_r + P_t$$

$$\delta = \delta_r = \delta_t$$

Example 2.14, 2.15, 2.16



- at a load of $P = 5.7$ kips, the rod has reached the plastic range while the tube is still in the elastic range

$$P_r = P_{Y,r} = 2.7 \text{ kips}$$

$$P_t = P - P_r = (5.7 - 2.7) \text{ kips} = 3.0 \text{ kips}$$

$$\sigma_t = \frac{P_t}{A_t} = \frac{3.0 \text{ kips}}{0.1 \text{ in}^2} = 30 \text{ ksi}$$

$$\delta_t = \varepsilon_t L = \frac{\sigma_t}{E_t} L = \frac{30 \times 10^3 \text{ psi}}{15 \times 10^6 \text{ psi}} 30 \text{ in.}$$

$$\delta_{\max} = \delta_t = 60 \times 10^{-3} \text{ in.}$$

- the rod-tube assembly unloads along a line parallel to $0Y_r$

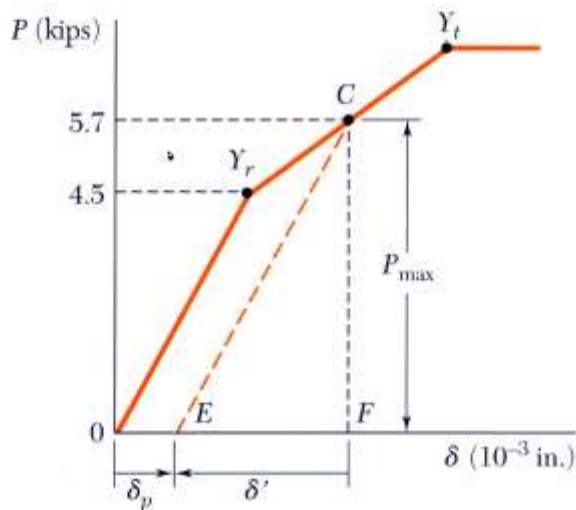
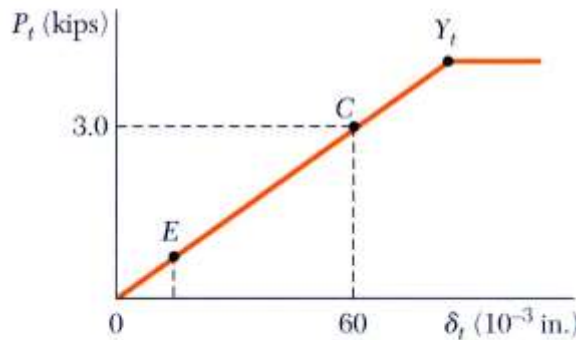
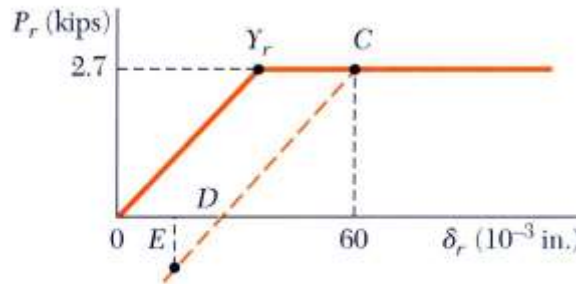
$$m = \frac{4.5 \text{ kips}}{36 \times 10^{-3} \text{ in.}} = 125 \text{ kips/in.} = \text{slope}$$

$$\delta' = -\frac{P_{\max}}{m} = -\frac{5.7 \text{ kips}}{125 \text{ kips/in.}} = -45.6 \times 10^{-3} \text{ in.}$$

$$\delta_p = \delta_{\max} + \delta' = (60 - 45.6) \times 10^{-3} \text{ in.}$$

$$\delta_p = 14.4 \times 10^{-3} \text{ in.}$$

Example 2.14, 2.15, 2.16



- calculate the residual stresses in the rod and tube.
- calculate the reverse stresses in the rod and tube caused by unloading and add them to the maximum stresses.

$$\varepsilon' = \frac{\delta'}{L} = \frac{-45.6 \times 10^{-3} \text{ in.}}{30 \text{ in.}} = -1.52 \times 10^{-3} \text{ in./in.}$$

$$\sigma'_r = \varepsilon' E_r = (-1.52 \times 10^{-3}) (30 \times 10^6 \text{ psi}) = -45.6 \text{ ksi}$$

$$\sigma'_t = \varepsilon' E_t = (-1.52 \times 10^{-3}) (15 \times 10^6 \text{ psi}) = -22.8 \text{ ksi}$$

$$\sigma_{\text{residual},r} = \sigma_r + \sigma'_r = (36 - 45.6) \text{ ksi} = -9.6 \text{ ksi}$$

$$\sigma_{\text{residual},t} = \sigma_t + \sigma'_t = (30 - 22.8) \text{ ksi} = 7.2 \text{ ksi}$$

CHAPTER

3

MECHANICS OF MATERIALS

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Torsion

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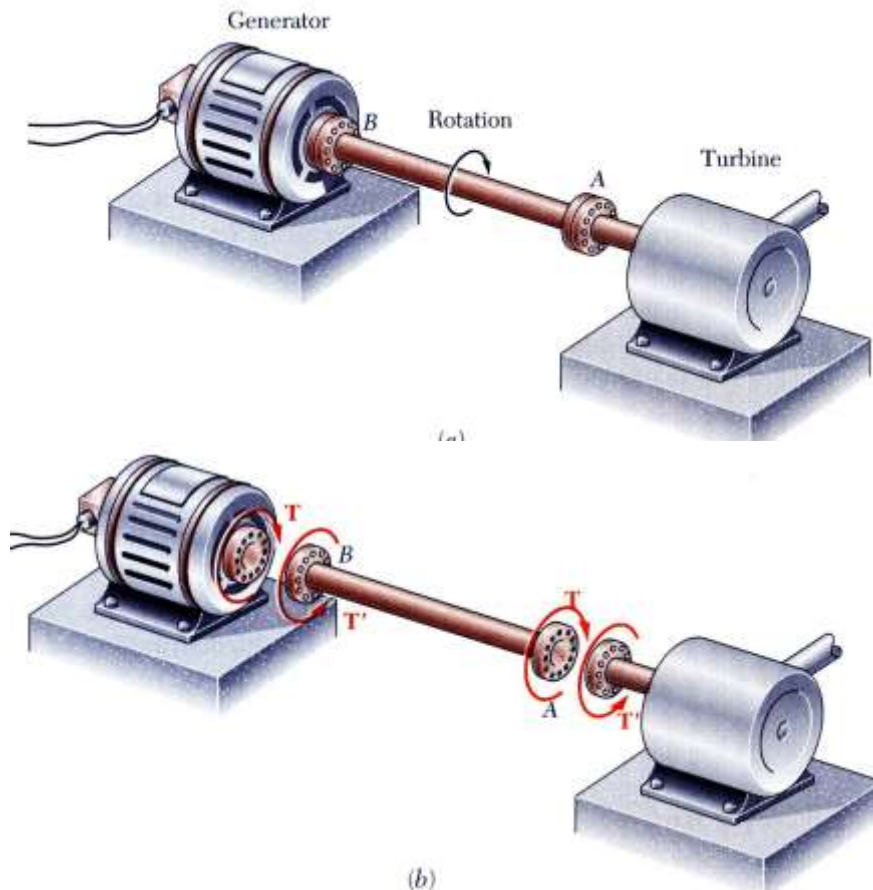
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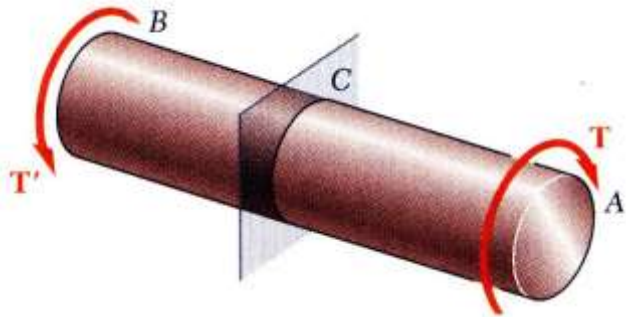


Torsional Loads on Circular Shafts



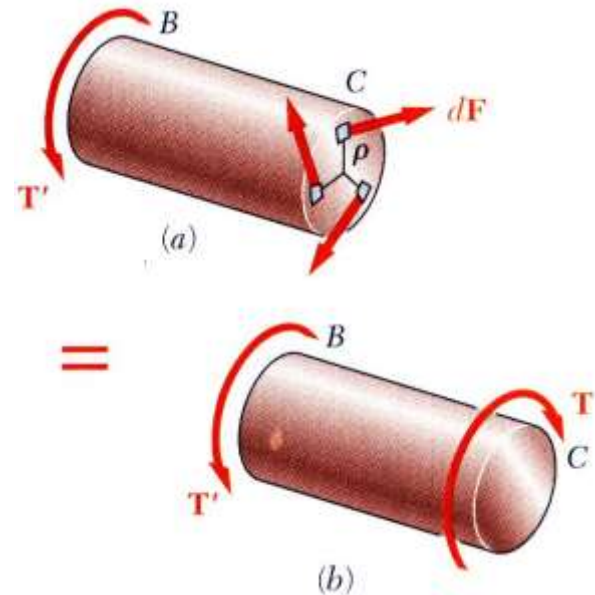
- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque T on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque T'

Net Torque Due to Internal Stresses



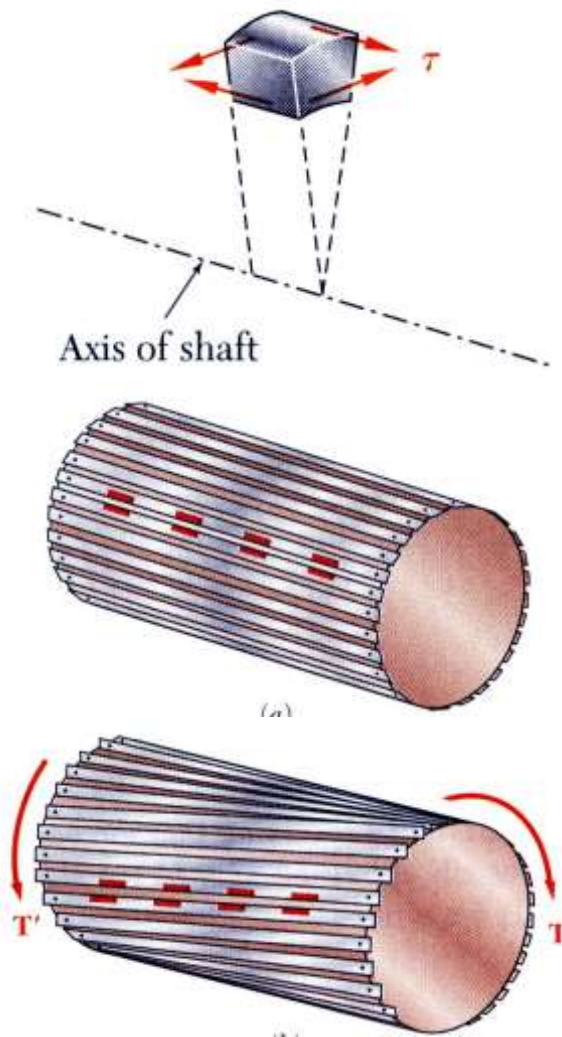
- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$



- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

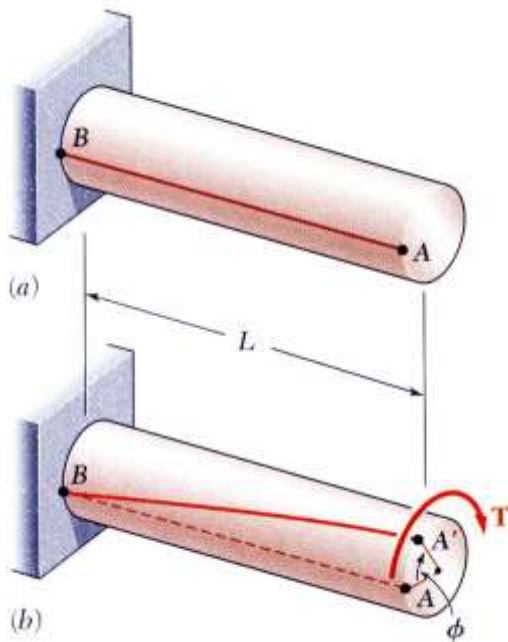
Axial Shear Components



- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations

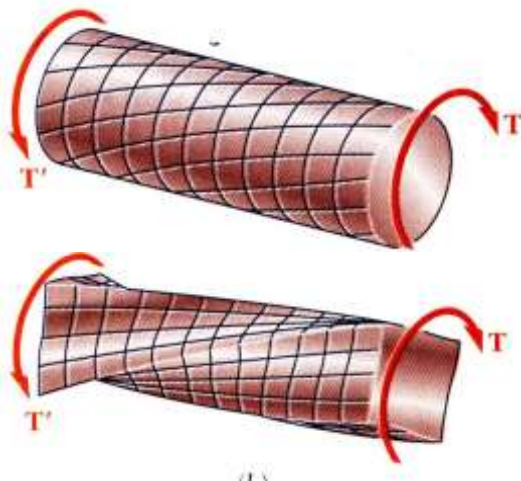


- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

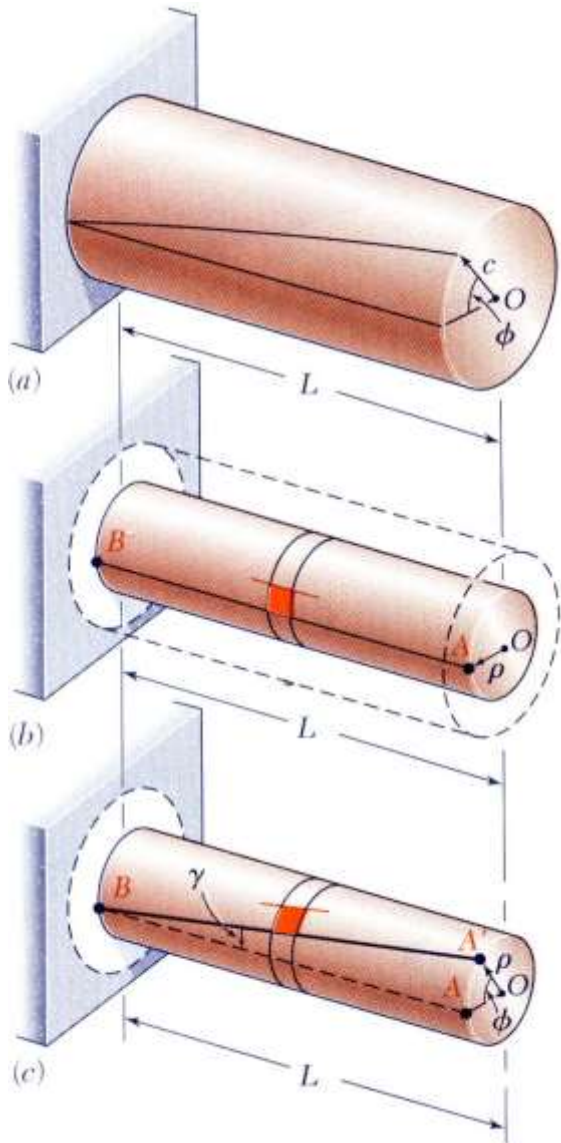
$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.



Shearing Strain



- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

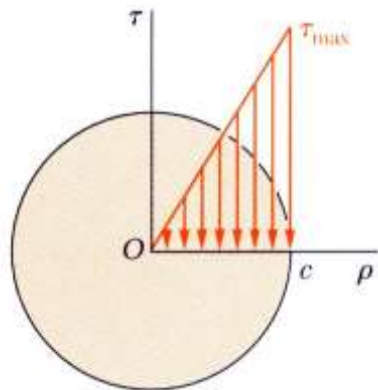
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

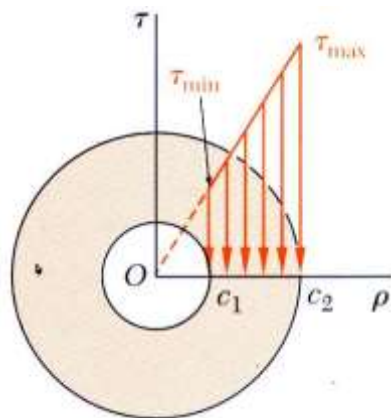
- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range



$$J = \frac{1}{2} \pi c^4$$



$$J = \frac{1}{2} \pi (c_2^4 - c_1^4)$$

- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

From Hooke's Law, $\tau = G\gamma$, so

$$\tau = \frac{\rho}{c} \tau_{\max}$$

The shearing stress varies linearly with the radial position in the section.

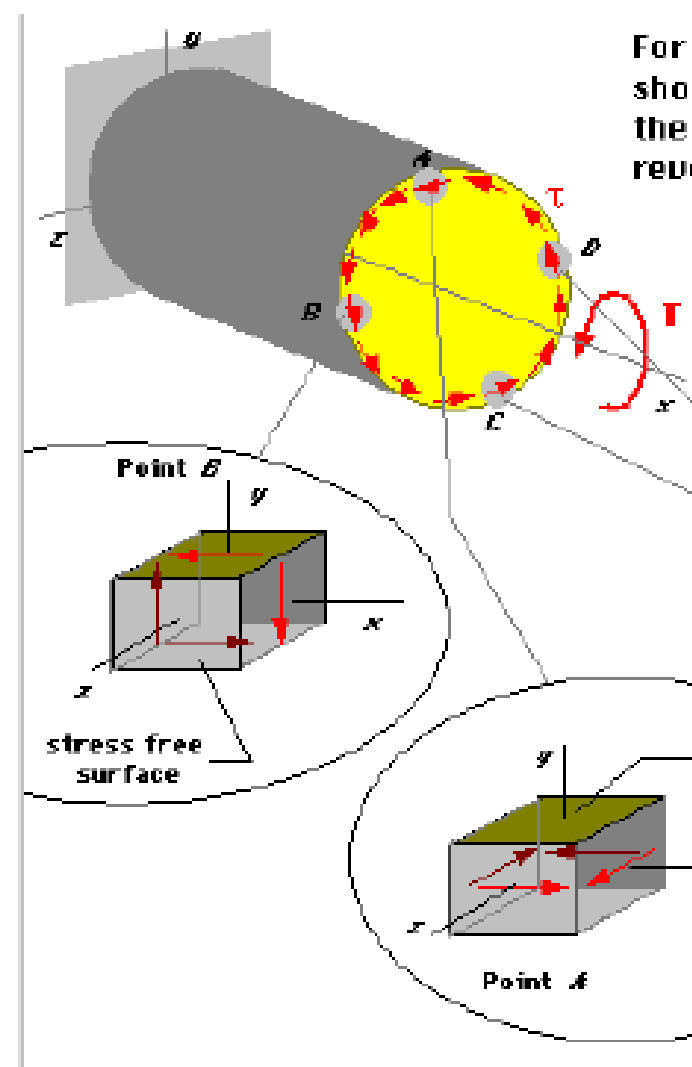
- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho \tau dA = \frac{\tau_{\max}}{c} \int \rho^2 dA = \frac{\tau_{\max}}{c} J$$

- The results are known as the elastic torsion formulas,

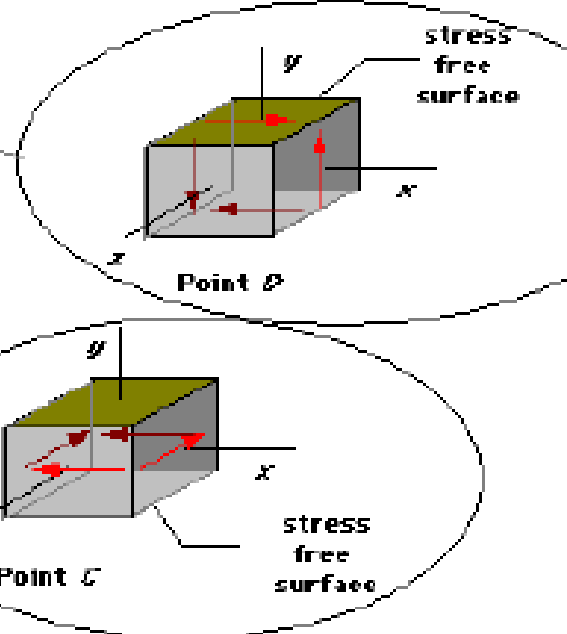
$$\tau_{\max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Stresses in Elastic Range

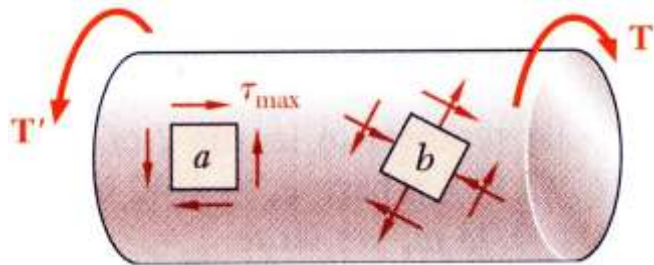


For a circular shaft acted on by the applied torque shown, the shearing stresses around the perimeter of the shaft are as shown. If the direction of the torque is reversed, the shearing stresses are counterclockwise.

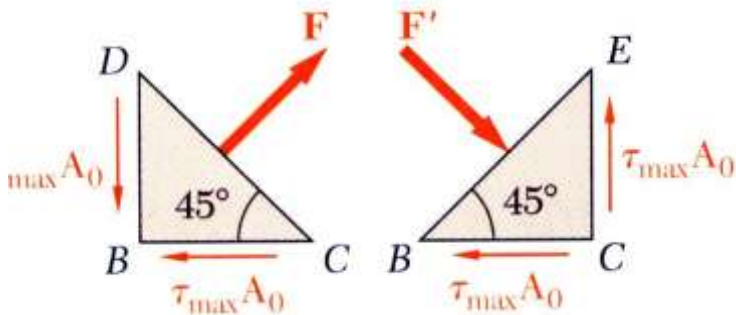
These stresses do not completely describe the stresses on an element representing a specific point on the perimeter of the shaft. **Click on one of the points (A , B , C , and D) to view an element showing stresses on all faces. After all points are seen, return to the menu for this section.**



Normal Stresses



- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.

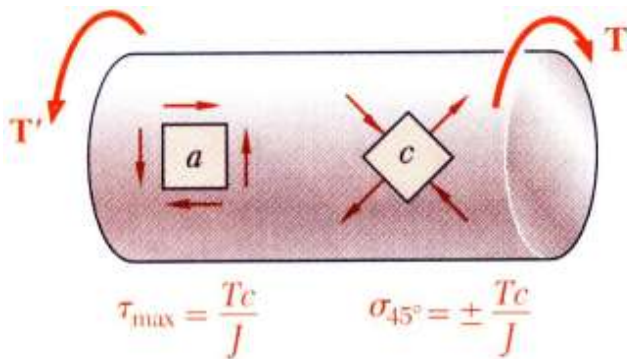


- Consider an element at 45° to the shaft axis,

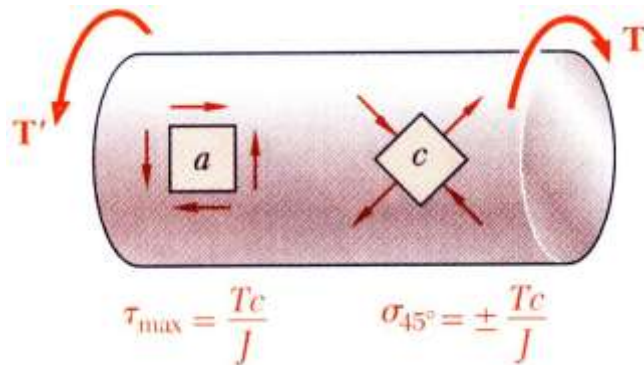
$$F = 2(\tau_{\max} A_0) \cos 45 = \tau_{\max} A_0 \sqrt{2}$$

$$\sigma_{45^\circ} = \frac{F}{A} = \frac{\tau_{\max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{\max}$$

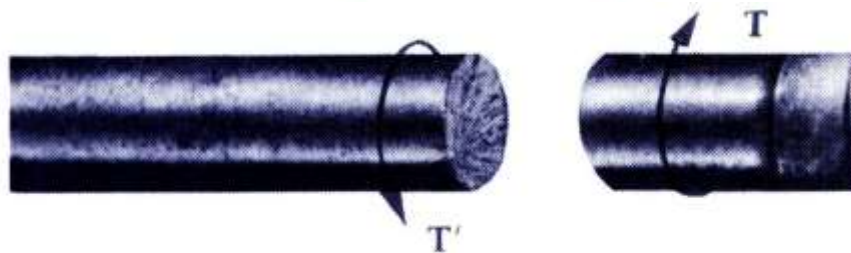
- Element a is in pure shear.
- Element c is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements a and c have the same magnitude



Torsional Failure Modes



- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.

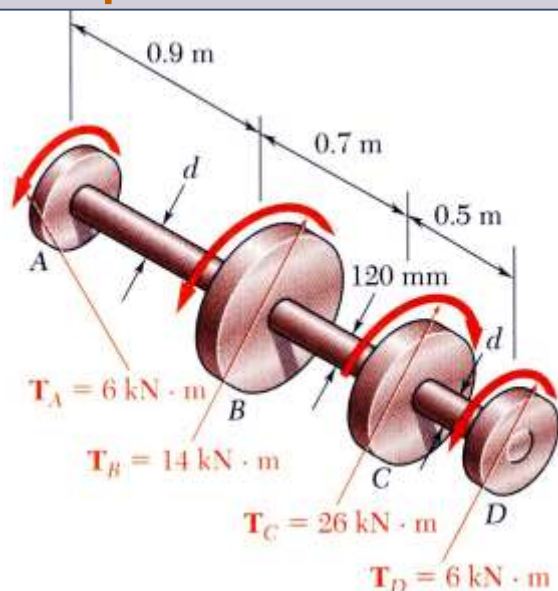


- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.



- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45° to the shaft axis.

Sample Problem 3.1



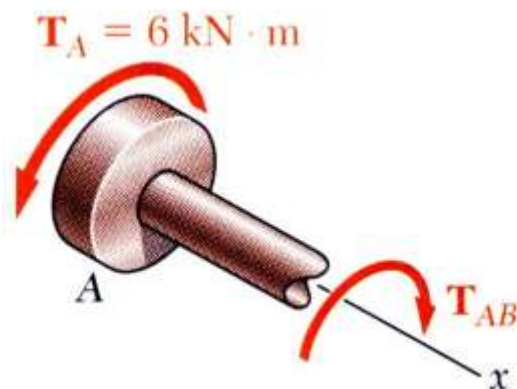
Shaft BC is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts AB and CD are solid of diameter d . For the loading shown, determine (a) the minimum and maximum shearing stress in shaft BC , (b) the required diameter d of shafts AB and CD if the allowable shearing stress in these shafts is 65 MPa.

SOLUTION:

- Cut sections through shafts AB and BC and perform static equilibrium analysis to find torque loadings
- Apply elastic torsion formulas to find minimum and maximum stress on shaft BC
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

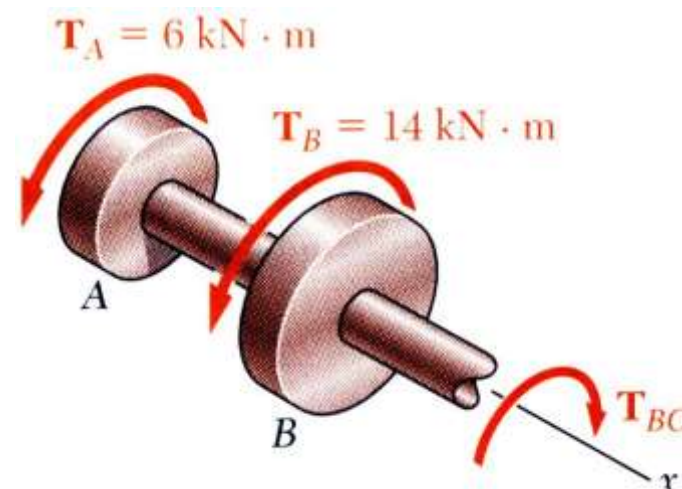
Sample Problem 3.1

- Cut sections through shafts AB and BC and perform static equilibrium analysis to find torque loadings



$$\sum M_x = 0 = (6 \text{ kN} \cdot \text{m}) - T_{AB}$$

$$T_{AB} = 6 \text{ kN} \cdot \text{m} = T_{CD}$$

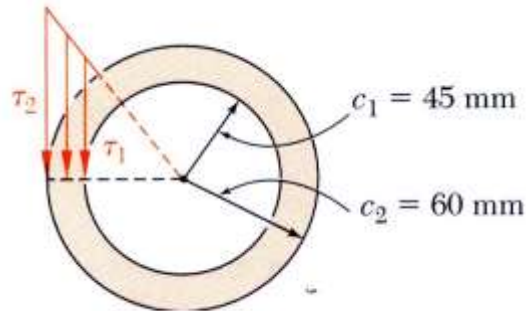


$$\sum M_x = 0 = (6 \text{ kN} \cdot \text{m}) + (14 \text{ kN} \cdot \text{m}) - T_{BC}$$

$$T_{BC} = 20 \text{ kN} \cdot \text{m}$$

Sample Problem 3.1

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter



$$J = \frac{\pi}{2} (c_2^4 - c_1^4) = \frac{\pi}{2} [(0.060)^4 - (0.045)^4]$$

$$= 13.92 \times 10^{-6} \text{ m}^4$$

$$\tau_{\max} = \tau_2 = \frac{T_{BC} c_2}{J} = \frac{(20 \text{ kN} \cdot \text{m})(0.060 \text{ m})}{13.92 \times 10^{-6} \text{ m}^4}$$

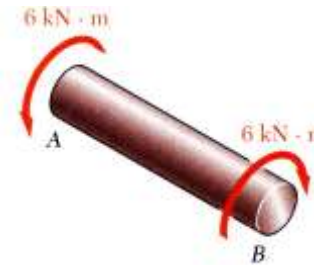
$$= 86.2 \text{ MPa}$$

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \quad \frac{\tau_{\min}}{86.2 \text{ MPa}} = \frac{45 \text{ mm}}{60 \text{ mm}}$$

$$\tau_{\min} = 64.7 \text{ MPa}$$

$$\tau_{\max} = 86.2 \text{ MPa}$$

$$\tau_{\min} = 64.7 \text{ MPa}$$



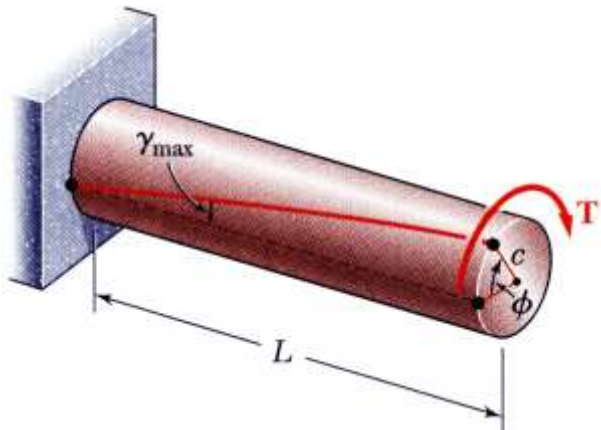
$$\tau_{\max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2} c^4}$$

$$c = 38.9 \times 10^{-3} \text{ m}$$

$$65 \text{ MPa} = \frac{6 \text{ kN} \cdot \text{m}}{\frac{\pi}{2} c^3}$$

$$d = 2c = 77.8 \text{ mm}$$

Angle of Twist in Elastic Range



- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{\max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

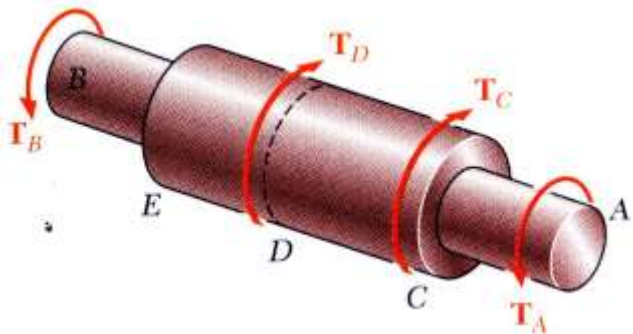
$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

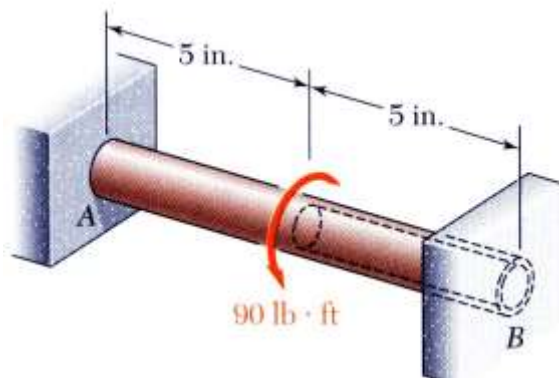
$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$



Statically Indeterminate Shafts



- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at A and B.

- From a free-body analysis of the shaft,

$$T_A + T_B = 90 \text{ lb} \cdot \text{ft}$$

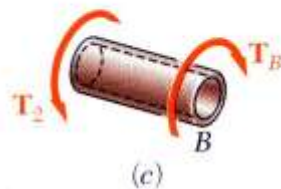
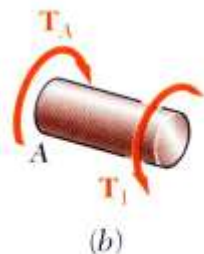
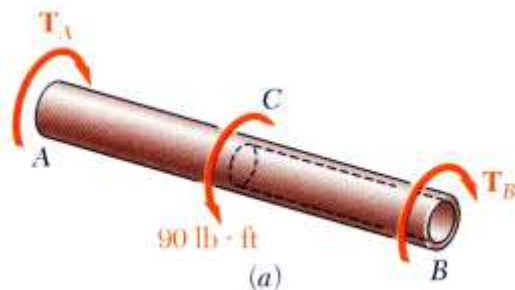
which is not sufficient to find the end torques. The problem is statically indeterminate.

- Divide the shaft into two components which must have compatible deformations,

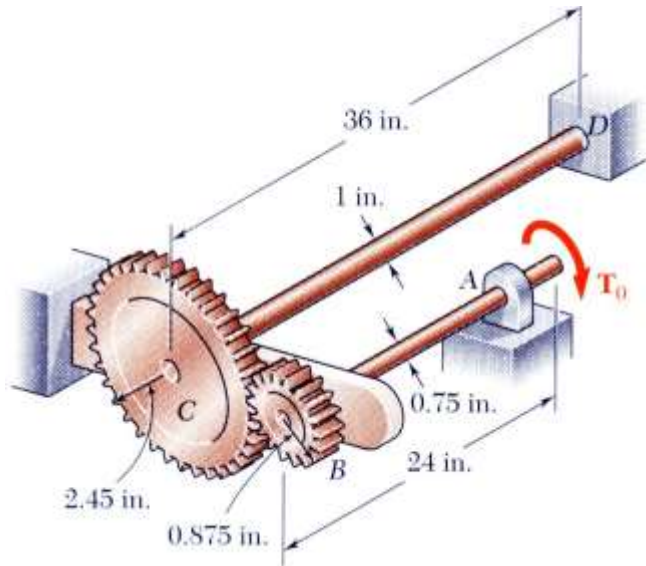
$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \quad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90 \text{ lb} \cdot \text{ft}$$



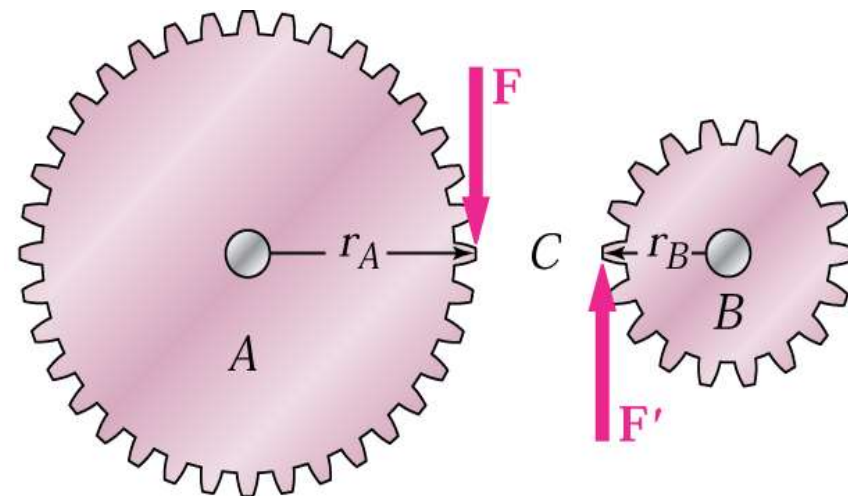
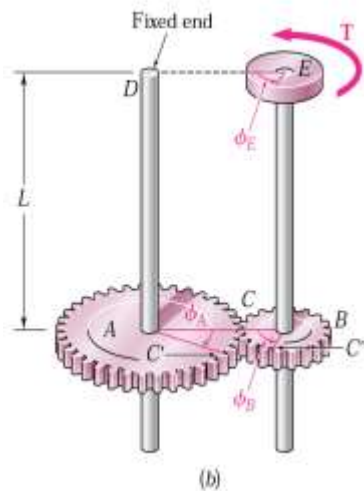
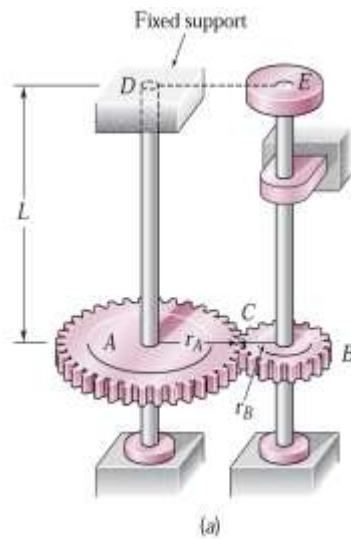
Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (a) the largest torque T_0 that may be applied to the end of shaft AB, (b) the corresponding angle through which end A of shaft AB rotates.

SOLUTION:

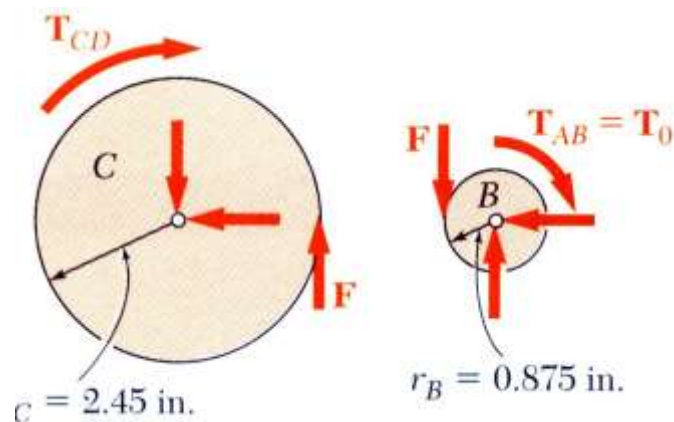
- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0
- Apply a kinematic analysis to relate the angular rotations of the gears
- Find the maximum allowable torque on each shaft – choose the smallest
- Find the corresponding angle of twist for each shaft and the net angular rotation of end A



Sample Problem 3.4

SOLUTION:

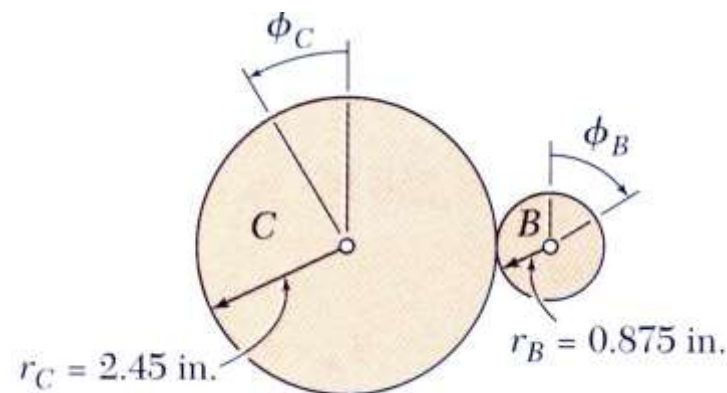
- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0
- Apply a kinematic analysis to relate the angular rotations of the gears



$$\sum M_B = 0 = F(0.875 \text{ in.}) - T_0$$

$$\sum M_C = 0 = F(2.45 \text{ in.}) - T_{CD}$$

$$T_{CD} = 2.8T_0$$



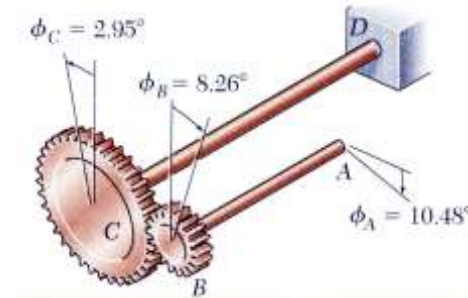
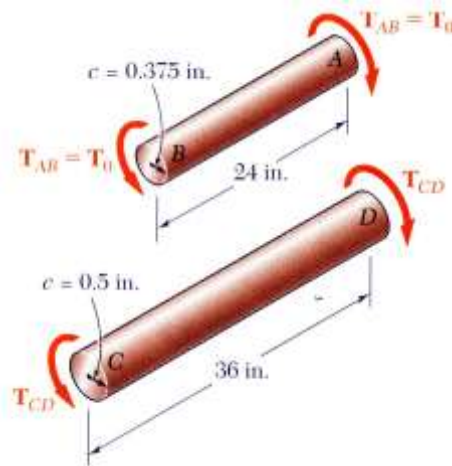
$$r_B \phi_B = r_C \phi_C$$

$$\phi_B = \frac{r_C}{r_B} \phi_C = \frac{2.45 \text{ in.}}{0.875 \text{ in.}} \phi_C$$

$$\phi_B = 2.8 \phi_C$$

Sample Problem 3.4

- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest
- Find the corresponding angle of twist for each shaft and the net angular rotation of end A



$$\phi_{A/B} = \frac{T_{AB}L}{J_{AB}G} = \frac{(561\text{lb} \cdot \text{in.})(24\text{in.})}{\frac{\pi}{2}(0.375\text{in.})^4(11.2 \times 10^6 \text{psi})}$$

$$= 0.387 \text{rad} = 2.22^\circ$$

$$\phi_{C/D} = \frac{T_{CD}L}{J_{CD}G} = \frac{2.8(561\text{lb} \cdot \text{in.})(24\text{in.})}{\frac{\pi}{2}(0.5\text{in.})^4(11.2 \times 10^6 \text{psi})}$$

$$= 0.514 \text{rad} = 2.95^\circ$$

$$\phi_B = 2.8\phi_C = 2.8(2.95^\circ) = 8.26^\circ$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.26^\circ + 2.22^\circ$$

$$\phi_A = 10.48^\circ$$

$$\tau_{\max} = \frac{T_{ABC}}{J_{AB}} \quad 8000 \text{psi} = \frac{T_0(0.375\text{in.})}{\frac{\pi}{2}(0.375\text{in.})^4}$$

$$T_0 = 663 \text{lb} \cdot \text{in.}$$

$$\tau_{\max} = \frac{T_{CDE}}{J_{CD}} \quad 8000 \text{psi} = \frac{2.8T_0(0.5\text{in.})}{\frac{\pi}{2}(0.5\text{in.})^4}$$

$$T_0 = 561 \text{lb} \cdot \text{in.}$$

$$T_0 = 561 \text{lb} \cdot \text{in.}$$

Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
 - power
 - speed
- Designer must select shaft material and cross-section to meet performance specifications without exceeding allowable shearing stress.

- Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi fT$$

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

- Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2}c^3 = \frac{T}{\tau_{\max}} \quad (\text{solid shafts})$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}(c_2^4 - c_1^4) = \frac{T}{\tau_{\max}} \quad (\text{hollow shafts})$$

Stress Concentrations

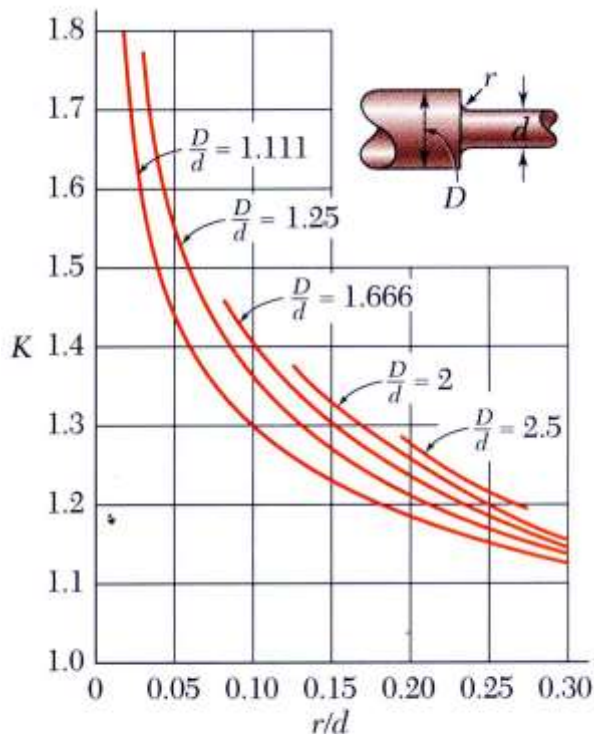
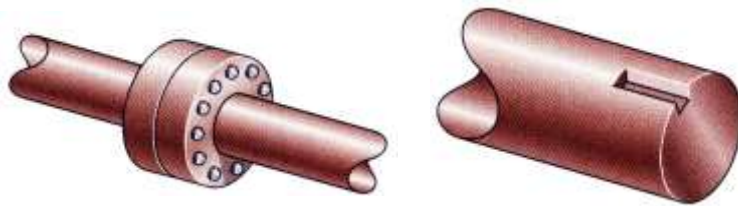


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

assumed a circular shaft with uniform cross-section loaded through rigid end plates.

- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations
- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K \frac{Tc}{J}$$

Plastic Deformations

- With the assumption of a linearly elastic material,

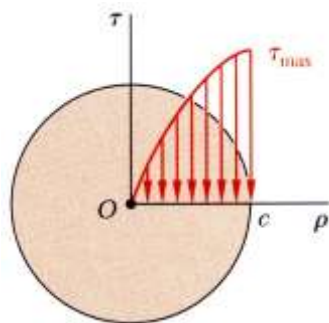
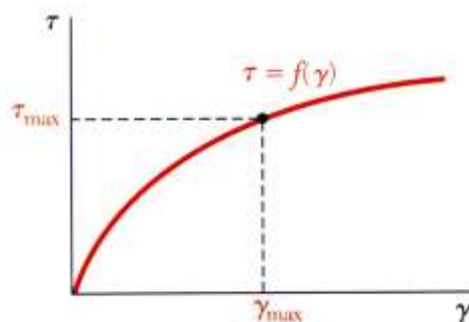
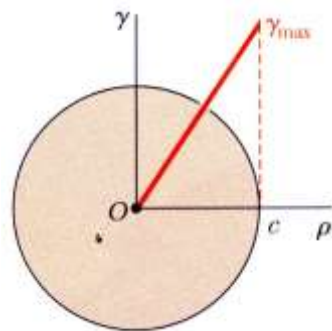
$$\tau_{\max} = \frac{Tc}{J}$$

- If the yield strength is exceeded or the material has a nonlinear shearing-stress-strain curve, this expression does not hold.

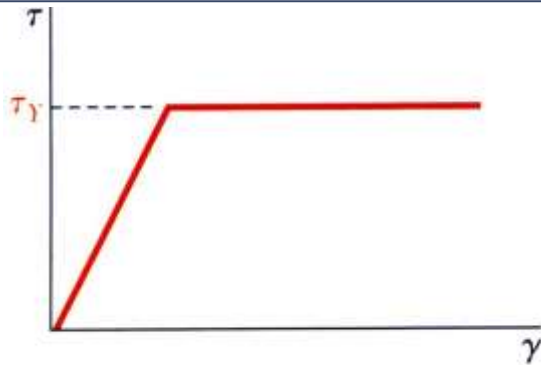
- Shearing strain varies linearly regardless of material properties. Application of shearing-stress-strain curve allows determination of stress distribution.

- The integral of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int_0^c \rho \tau (2\pi \rho d\rho) = 2\pi \int_0^c \rho^2 \tau d\rho$$



Elastoplastic Materials



- At the *maximum elastic torque*,

$$T_Y = \frac{J}{c} \tau_Y = \frac{1}{2} \pi c^3 \tau_Y \quad \phi_Y = \frac{L \gamma_Y}{c}$$

- As the torque is increased, a plastic region ($\tau = \tau_Y$) develops around an elastic core ($\tau = \frac{\rho}{\rho_Y} \tau_Y$)

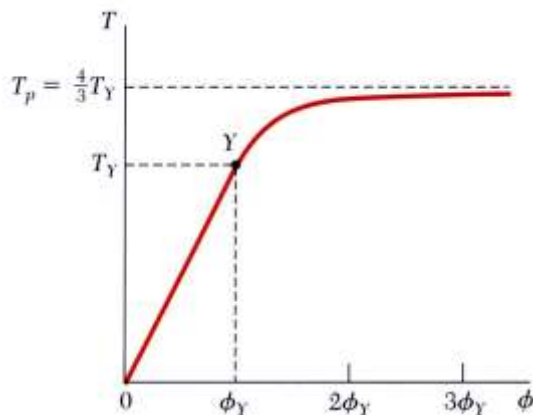
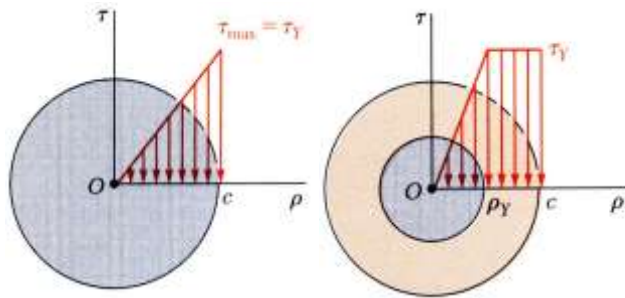
$$\rho_Y = \frac{L \gamma_Y}{\phi}$$

$$T = \frac{2}{3} \pi c^3 \tau_Y \left(1 - \frac{1}{4} \frac{\rho_Y^3}{c^3} \right) = \frac{4}{3} T_Y \left(1 - \frac{1}{4} \frac{\rho_Y^3}{c^3} \right)$$

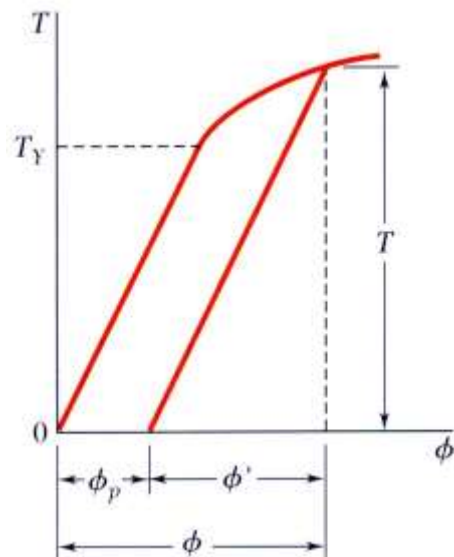
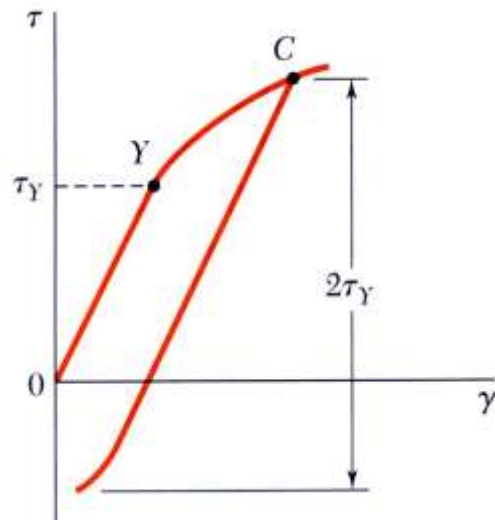
$$T = \frac{4}{3} T_Y \left(1 - \frac{1}{4} \frac{\phi_Y^3}{\phi^3} \right)$$

- As $\rho_Y \rightarrow 0$, the torque approaches a limiting value,

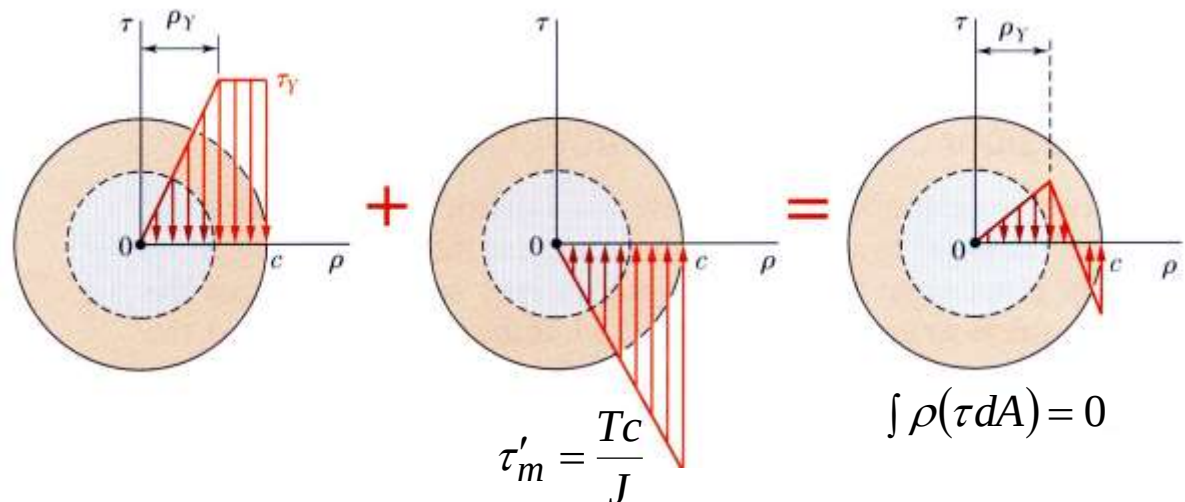
$$T_P = \frac{4}{3} T_Y = \text{plastic torque}$$



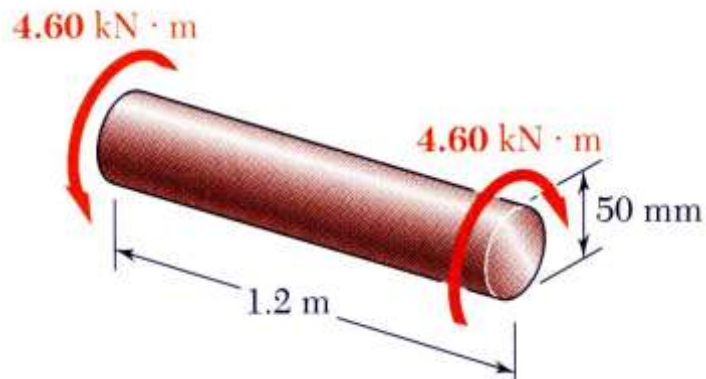
Residual Stresses



- Plastic region develops in a shaft when subjected to a large enough torque
- When the torque is removed, the reduction of stress and strain at each point takes place along a straight line to a generally non-zero residual stress
- On a $T-\phi$ curve, the shaft unloads along a straight line to an angle greater than zero
- Residual stresses found from principle of superposition



Example 3.08/3.09



A solid circular shaft is subjected to a torque $T = 4.6 \text{ kN} \cdot \text{m}$ at each end. Assuming that the shaft is made of an elastoplastic material with $\tau_Y = 150 \text{ MPa}$ and $G = 77 \text{ GPa}$ determine (a) the radius of the elastic core, (b) the angle of twist of the shaft. When the torque is removed, determine (c) the permanent twist, (d) the distribution of residual stresses.

SOLUTION:

- Solve Eq. (3.32) for ρ_Y/c and evaluate the elastic core radius
- Solve Eq. (3.36) for the angle of twist
- Evaluate Eq. (3.16) for the angle which the shaft untwists when the torque is removed. The permanent twist is the difference between the angles of twist and untwist
- Find the residual stress distribution by a superposition of the stress due to twisting and untwisting the shaft

Example 3.08/3.09

- Solve Eq. (3.32) for ρ_Y/c and evaluate the elastic core radius

$$T = \frac{4}{3}T_Y \left(1 - \frac{1}{4} \frac{\rho_Y^3}{c^3}\right) \Rightarrow \frac{\rho_Y}{c} = \left(4 - 3 \frac{T}{T_Y}\right)^{1/3}$$

$$J = \frac{1}{2}\pi c^4 = \frac{1}{2}\pi (25 \times 10^{-3} \text{ m})^4$$

$$= 614 \times 10^{-9} \text{ m}^4$$

$$\tau_Y = \frac{T_Y c}{J} \Rightarrow T_Y = \frac{\tau_Y J}{c}$$

$$T_Y = \frac{(150 \times 10^6 \text{ Pa})(614 \times 10^{-9} \text{ m}^4)}{25 \times 10^{-3} \text{ m}}$$

$$= 3.68 \text{ kN} \cdot \text{m}$$

$$\frac{\rho_Y}{c} = \left(4 - 3 \frac{4.6}{3.68}\right)^{1/3} = 0.630$$

$$\rho_Y = 15.8 \text{ mm}$$

- Solve Eq. (3.36) for the angle of twist

$$\frac{\phi}{\phi_Y} = \frac{\rho_Y}{c} \Rightarrow \phi = \frac{\phi_Y}{\rho_Y/c}$$

$$\phi_Y = \frac{T_Y L}{JG} = \frac{(3.68 \times 10^3 \text{ N})(1.2 \text{ m})}{(614 \times 10^{-9} \text{ m}^4)(77 \times 10 \text{ Pa})}$$

$$\phi_Y = 93.4 \times 10^{-3} \text{ rad}$$

$$\phi = \frac{93.4 \times 10^{-3} \text{ rad}}{0.630} = 148.3 \times 10^{-3} \text{ rad} = 8.50^\circ$$

$$\phi = 8.50^\circ$$

Example 3.08/3.09

- Evaluate Eq. (3.16) for the angle which the shaft untwists when the torque is removed. The permanent twist is the difference between the angles of twist and untwist

$$\phi' = \frac{TL}{JG}$$

$$= \frac{(4.6 \times 10^3 \text{ N} \cdot \text{m})(1.2 \text{ m})}{(6.14 \times 10^9 \text{ m}^4)(77 \times 10^9 \text{ Pa})}$$

$$= 116.8 \times 10^{-3} \text{ rad}$$

$$\phi_p = \phi - \phi'$$

$$= (116.8 \times 10^{-3} - 116.8 \times 10^{-3}) \text{ rad}$$

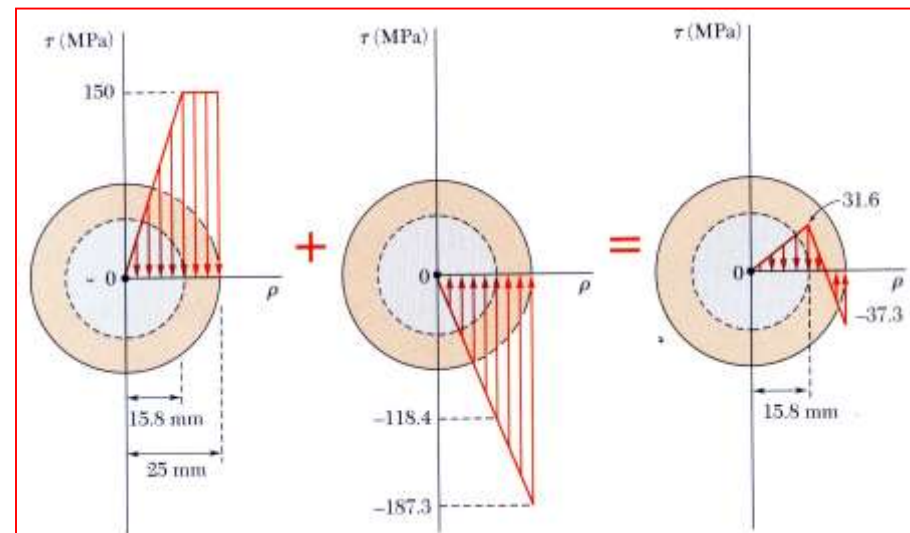
$$= 1.81^\circ$$

$$\phi_p = 1.81^\circ$$

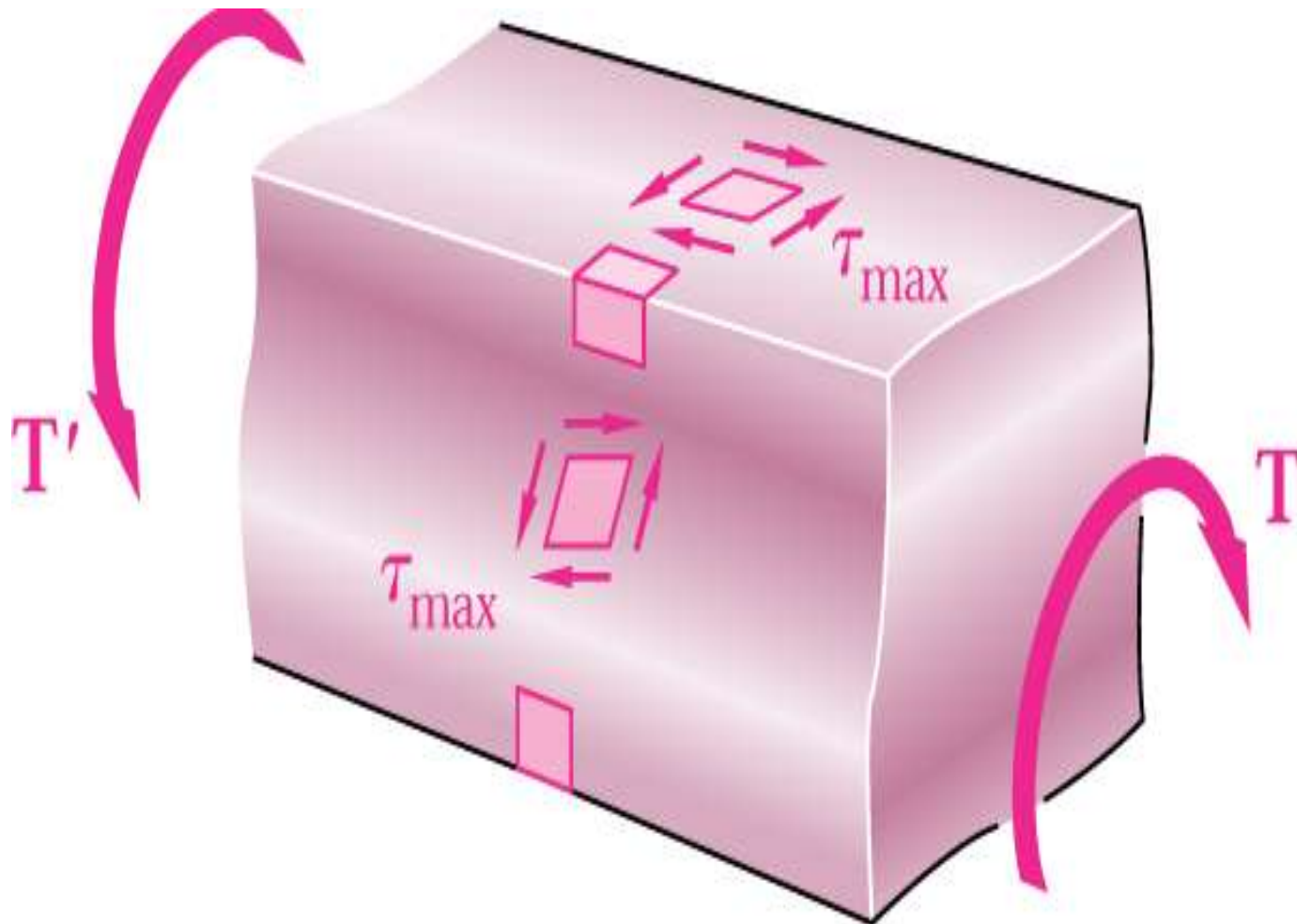
- Find the residual stress distribution by a superposition of the stress due to twisting and untwisting the shaft

$$\tau'_{\max} = \frac{Tc}{J} = \frac{(4.6 \times 10^3 \text{ N} \cdot \text{m})(25 \times 10^{-3} \text{ m})}{614 \times 10^{-9} \text{ m}^4}$$

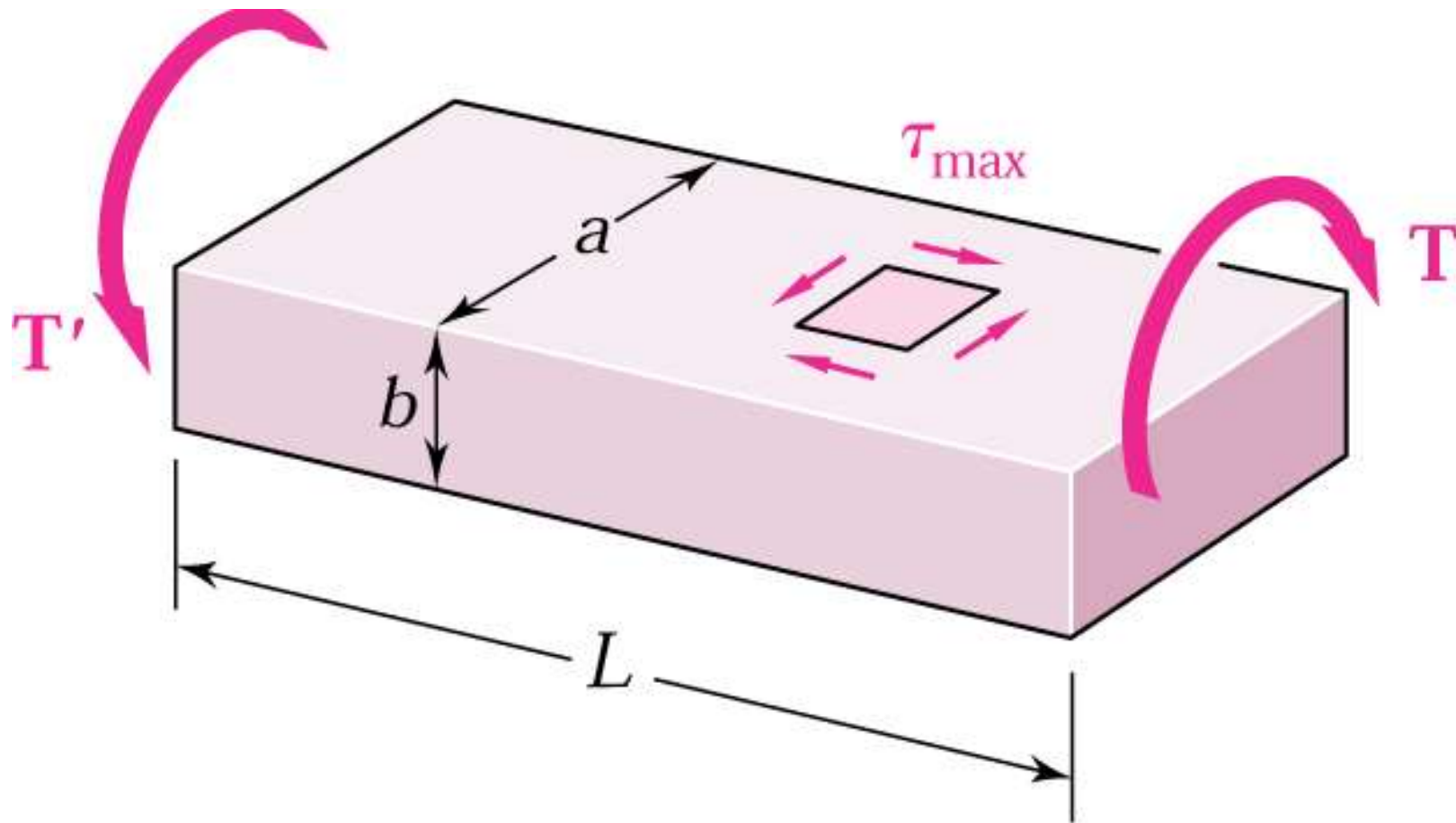
$$= 187.3 \text{ MPa}$$



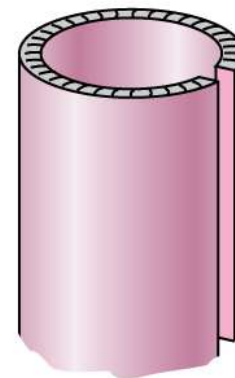
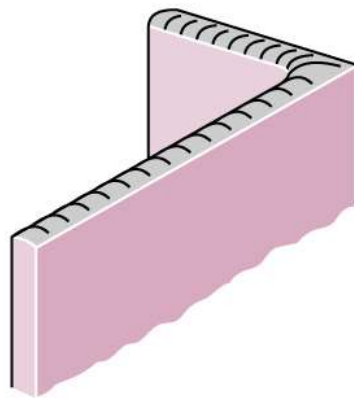
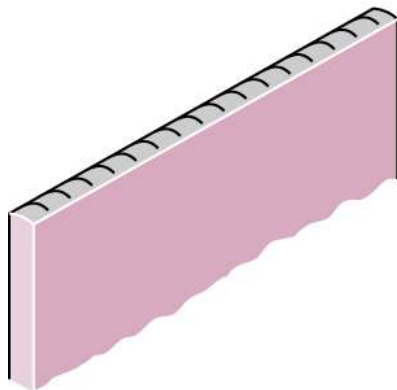
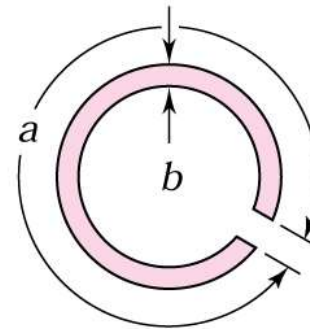
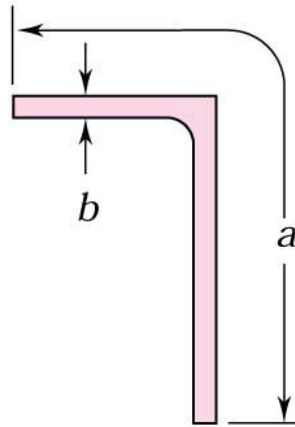
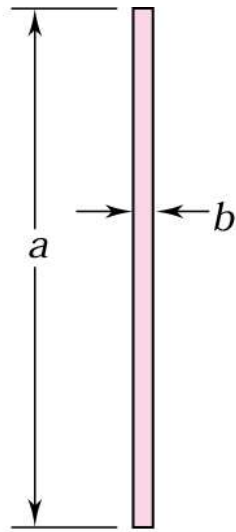
Torsion of Noncircular Members



Torsion of Noncircular Members



Torsion of Noncircular Members



Torsion of Noncircular Members



- Previous torsion formulas are valid for axisymmetric or circular shafts
- Planar cross-sections of noncircular shafts do not remain planar and stress and strain distribution do not vary linearly

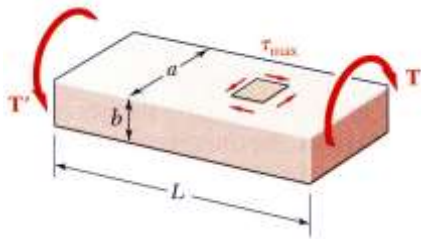


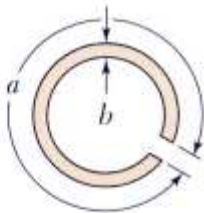
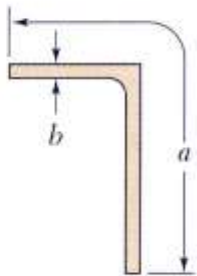
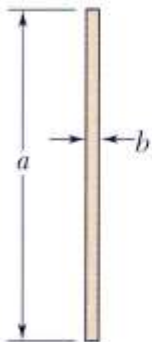
TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

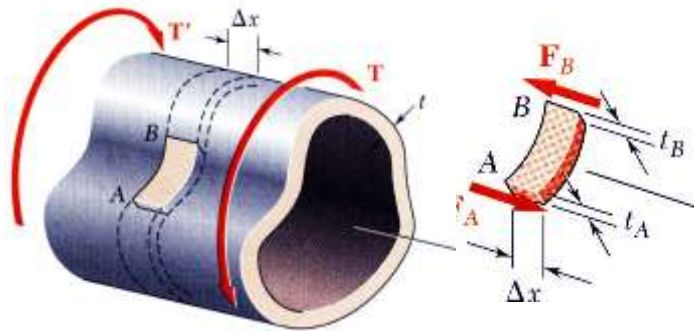
- For uniform rectangular cross-sections,

$$\tau_{\max} = \frac{T}{c_1 a b^2} \quad \phi = \frac{TL}{c_2 a b^3 G}$$

- At large values of a/b , the maximum shear stress and angle of twist for other open sections are the same as a rectangular bar.



Thin-Walled Hollow Shafts

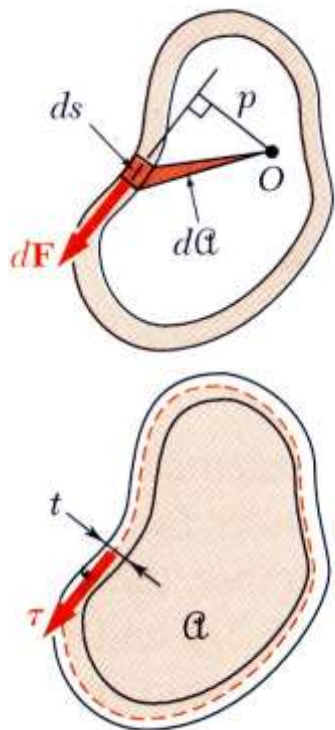


- Summing forces in the x-direction on AB,

$$\sum F_x = 0 = \tau_A(t_A \Delta x) - \tau_B(t_B \Delta x)$$

$$\tau_A t_A = \tau_B t_B = \tau t = q = \text{shear flow}$$

shear stress varies inversely with thickness



- Compute the shaft torque from the integral of the moments due to shear stress

$$dM_0 = p dF = p \tau (t ds) = q (p ds) = 2q dA$$

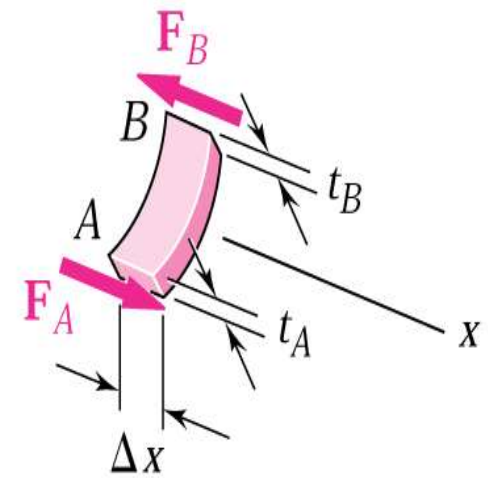
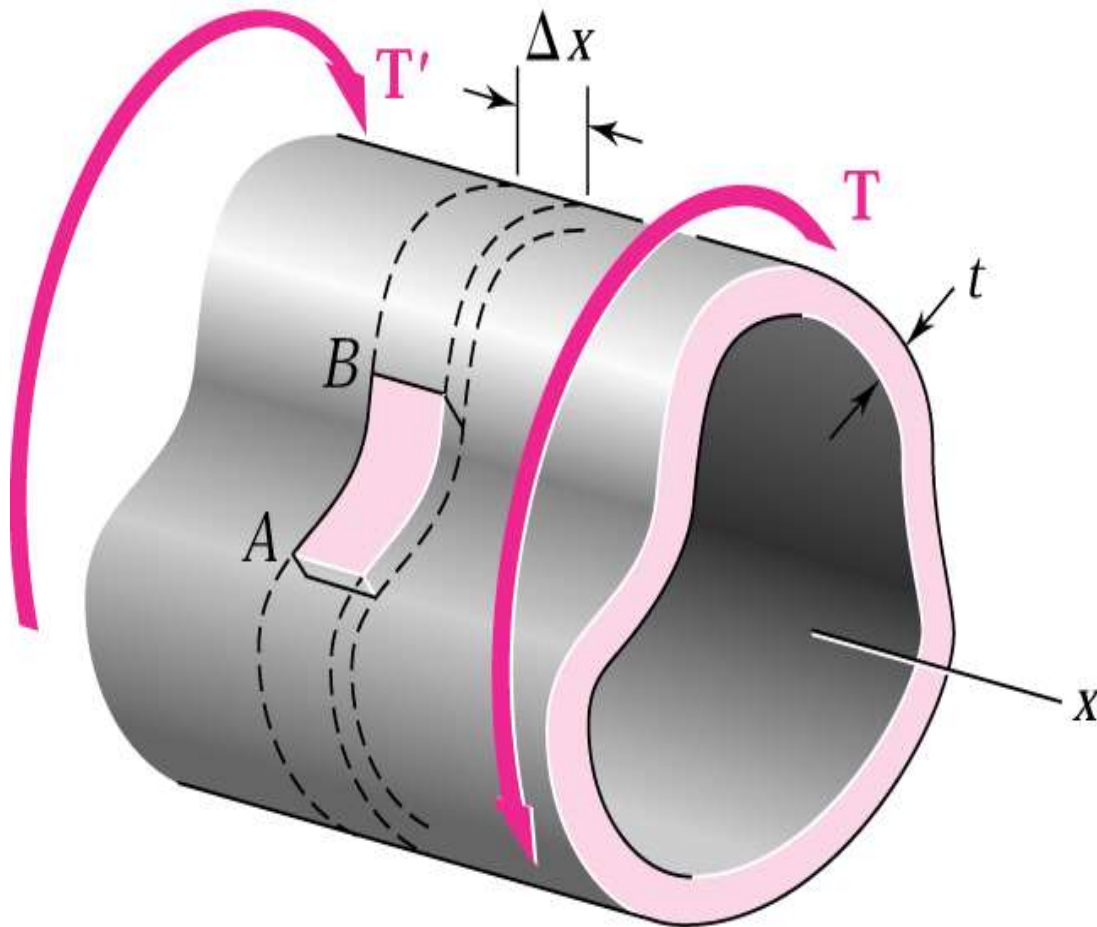
$$T = \oint dM_0 = \oint 2q dA = 2qA$$

$$\tau = \frac{T}{2tA}$$

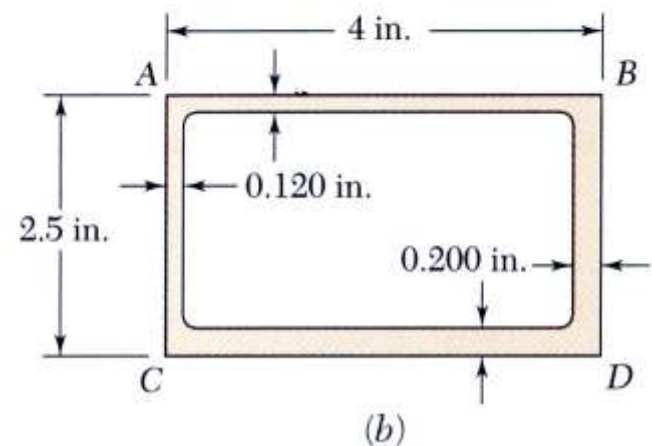
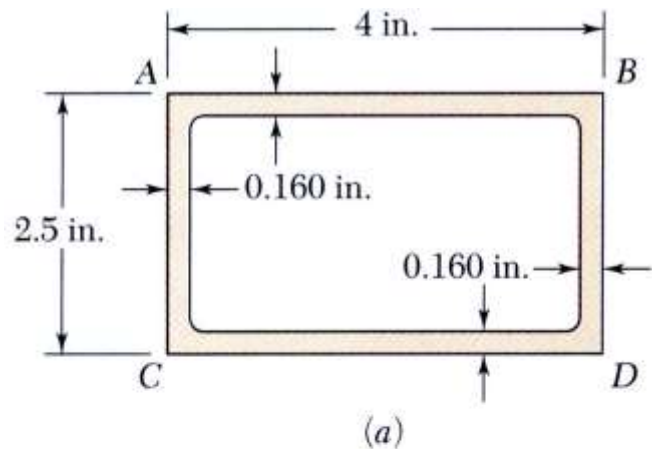
- Angle of twist (from Chapter 11)

$$\phi = \frac{TL}{4A^2 G} \oint \frac{ds}{t}$$

Torsion of Noncircular Members



Example 3.10



Extruded aluminum tubing with a rectangular cross-section has a torque loading of 24 kip-in. Determine the shearing stress in each of the four walls with (a) uniform wall thickness of 0.160 in. and wall thicknesses of (b) 0.120 in. on AB and CD and 0.200 in. on CD and BD .

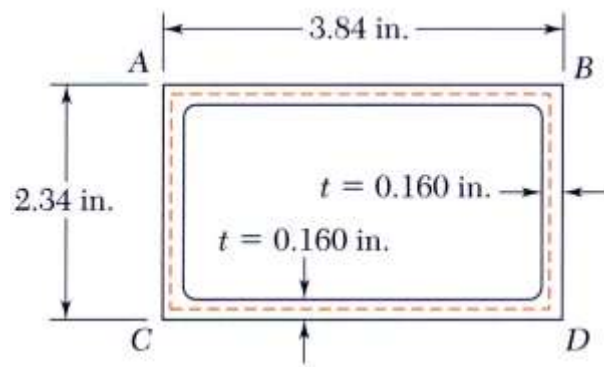
SOLUTION:

- Determine the shear flow through the tubing walls
- Find the corresponding shearing stress with each wall thickness

Example 3.10

SOLUTION:

- Determine the shear flow through the tubing walls



$$A = (3.84 \text{ in.})(2.34 \text{ in.}) = 8.986 \text{ in.}^2$$

$$q = \frac{T}{2A} = \frac{24 \text{ kip} \cdot \text{in.}}{2(8.986 \text{ in.}^2)} = 1.335 \frac{\text{kip}}{\text{in.}}$$

- Find the corresponding shearing stress with each wall thickness

with a uniform wall thickness,

$$\tau = \frac{q}{t} = \frac{1.335 \text{ kip/in.}}{0.160 \text{ in.}}$$

$$\tau = 8.34 \text{ ksi}$$

with a variable wall thickness

$$\tau_{AB} = \tau_{AC} = \frac{1.335 \text{ kip/in.}}{0.120 \text{ in.}}$$

$$\tau_{AB} = \tau_{BC} = 11.13 \text{ ksi}$$

$$\tau_{BD} = \tau_{CD} = \frac{1.335 \text{ kip/in.}}{0.200 \text{ in.}}$$

$$\tau_{BC} = \tau_{CD} = 6.68 \text{ ksi}$$

CHAPTER

4

MECHANICS OF MATERIALS

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Lecture Notes:
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Pure Bending

Pure Bending

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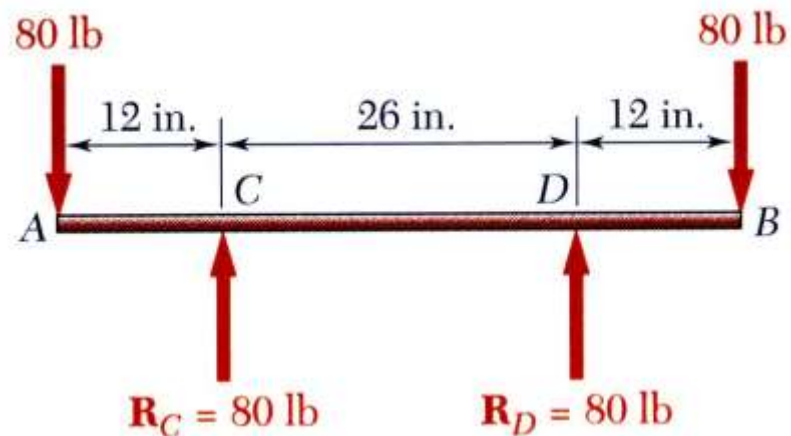
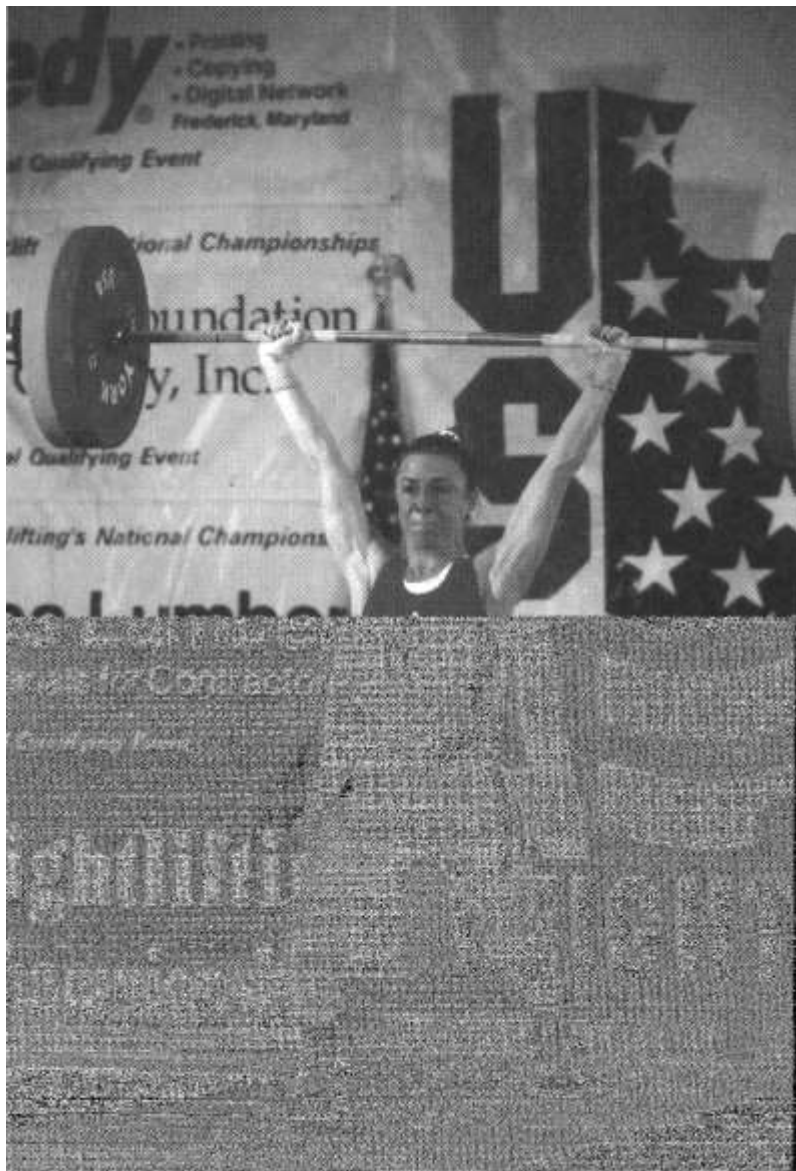
[Unsymmetric Bending](#)

[Example 4.08](#)

[General Case of Eccentric Axial Loading](#)



Pure Bending



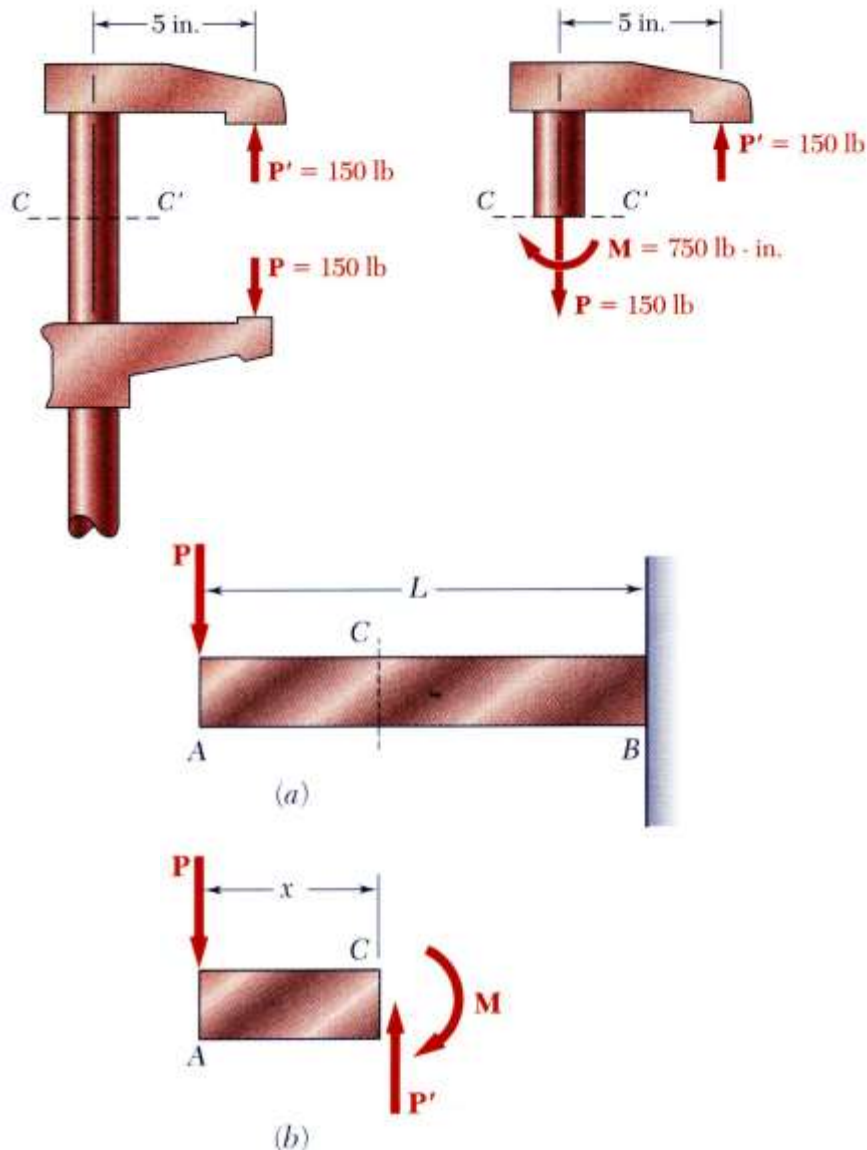
(a)



(b)

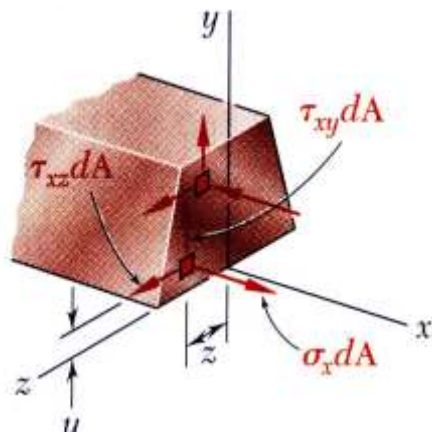
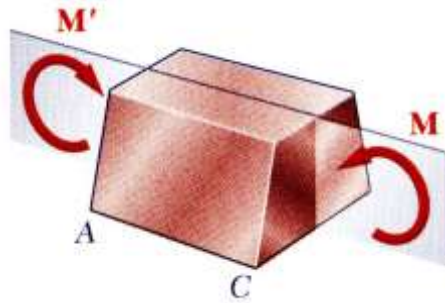
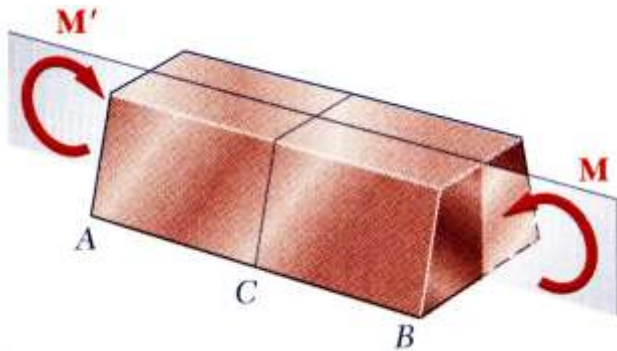
Pure Bending: Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane

Other Loading Types



- *Eccentric Loading:* Axial loading which does not pass through section centroid produces internal forces equivalent to an axial force and a couple
- *Transverse Loading:* Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple
- *Principle of Superposition:* The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.

Symmetric Member in Pure Bending



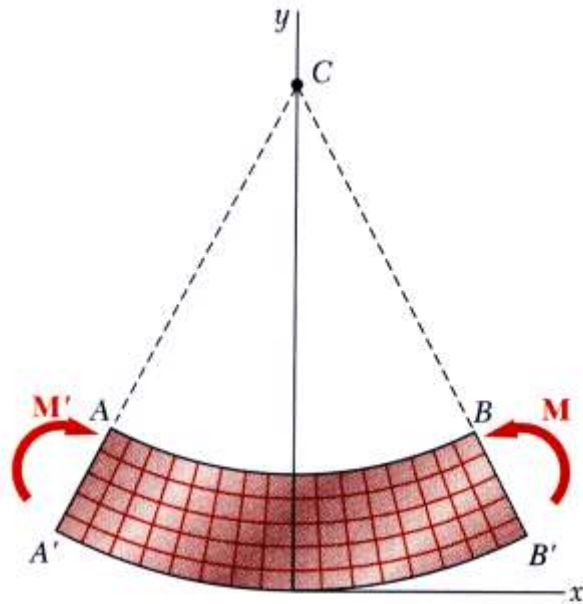
- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple M consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about any axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x dA = 0$$

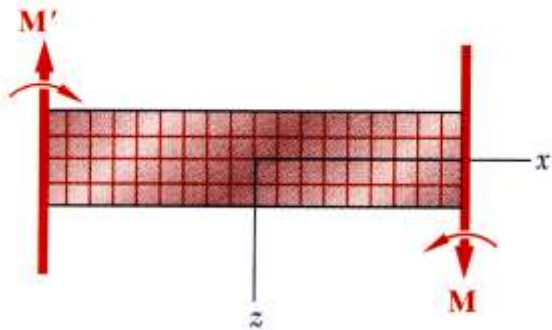
$$M_y = \int z \sigma_x dA = 0$$

$$M_z = \int -y \sigma_x dA = M$$

Bending Deformations



(a) Longitudinal, vertical section
(plane of symmetry)



(b) Longitudinal, horizontal section

Beam with a plane of symmetry in pure bending:

- member remains symmetric
- bends uniformly to form a circular arc
- cross-sectional plane passes through arc center and remains planar
- length of top decreases and length of bottom increases
- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change
- stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it

Strain Due to Bending

Consider a beam segment of length L .

After deformation, the length of the neutral surface remains L . At other sections,

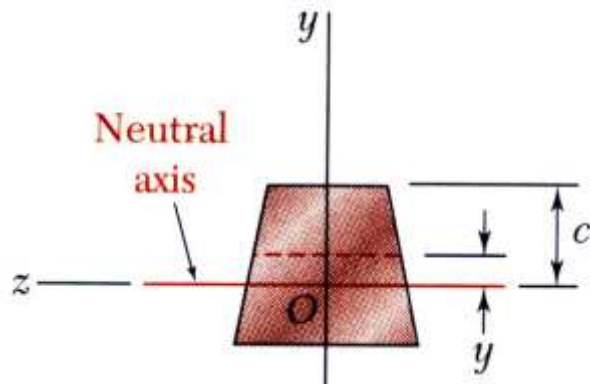
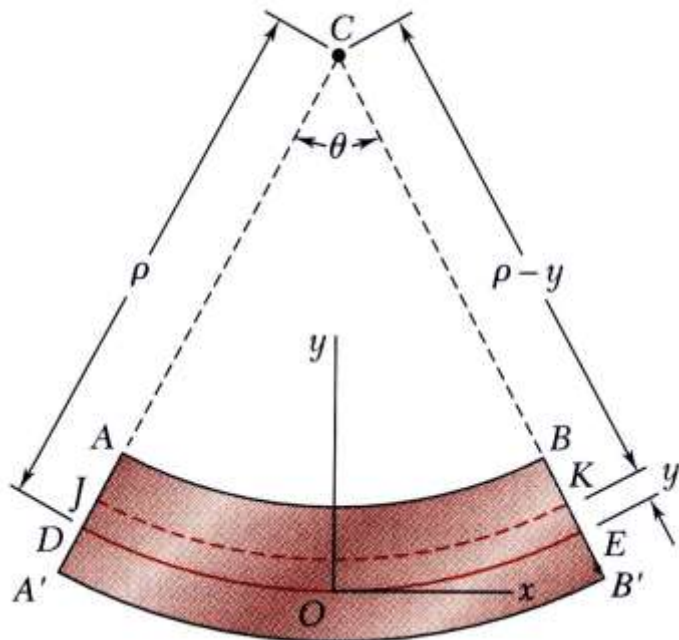
$$L' = (\rho - y)\theta$$

$$\delta = L - L' = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\epsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad (\text{strain varies linearly})$$

$$\epsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\epsilon_m}$$

$$\epsilon_x = -\frac{y}{c}\epsilon_m$$



Stress Due to Bending

- For a linearly elastic material,

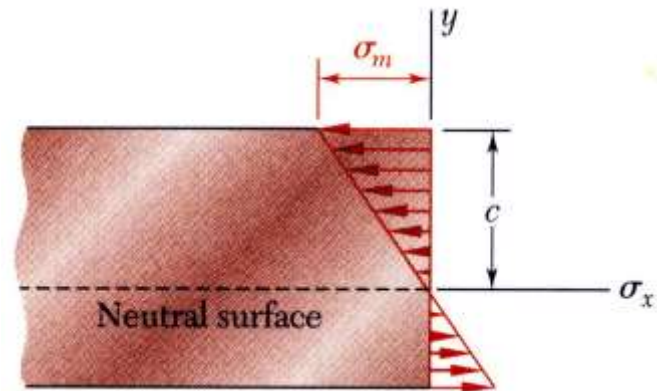
$$\begin{aligned}\sigma_x &= E\varepsilon_x = -\frac{y}{c} E\varepsilon_m \\ &= -\frac{y}{c} \sigma_m \quad (\text{stress varies linearly})\end{aligned}$$

- For static equilibrium,

$$F_x = 0 = \int \sigma_x dA = \int -\frac{y}{c} \sigma_m dA$$

$$0 = -\frac{\sigma_m}{c} \int y dA$$

First moment with respect to neutral plane is zero. Therefore, the neutral surface must pass through the section centroid.



- For static equilibrium,

$$M = \int -y \sigma_x dA = \int -y \left(-\frac{y}{c} \sigma_m \right) dA$$

$$M = \frac{\sigma_m}{c} \int y^2 dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Substituting $\sigma_x = -\frac{y}{c} \sigma_m$

$$\sigma_x = -\frac{My}{I}$$



Beam Section Properties

- The maximum normal stress due to bending,

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

I = section moment of inertia

$$S = \frac{I}{c} = \text{section modulus}$$

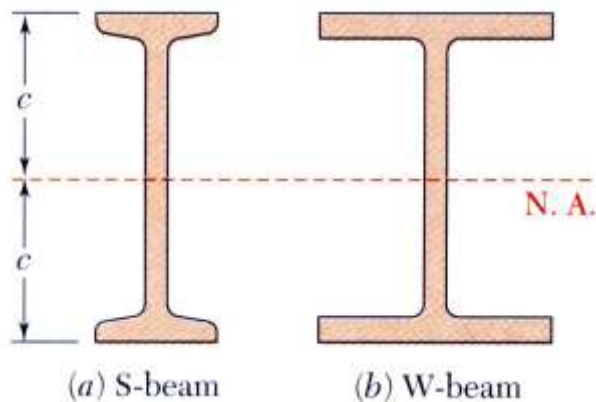
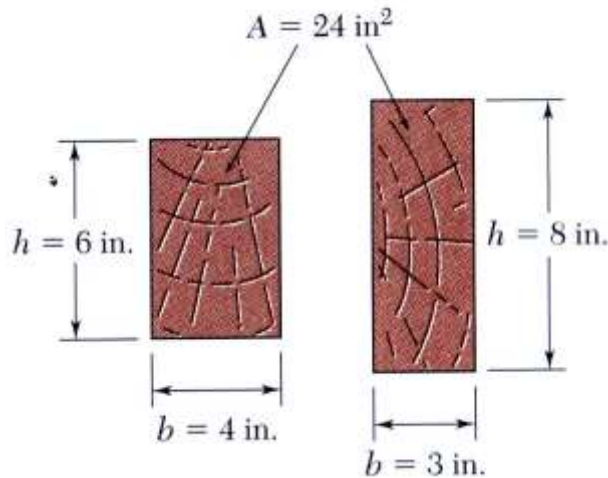
A beam section with a larger section modulus will have a lower maximum stress

- Consider a rectangular beam cross section,

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \frac{1}{6}bh^3 = \frac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the greater depth will be more effective in resisting bending.

- Structural steel beams are designed to have a large section modulus.

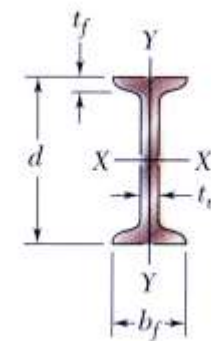


Properties of American Standard Shapes

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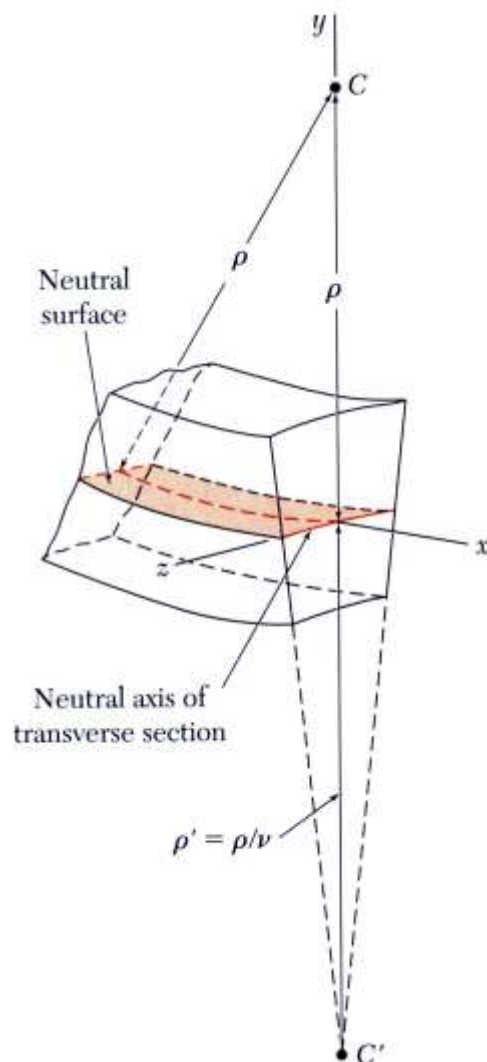
Appendix C. Properties of Rolled-Steel Shapes (SI Units)

S Shapes (American Standard Shapes)



			Flange		Web Thick- ness t_w , mm	Axis X-X			Axis Y-Y		
Designation†	Area A , mm ²	Depth d , mm	Width b_f , mm	Thick- ness t_f , mm		I_x 10 ⁶ mm ⁴	S_x 10 ³ mm ³	r_x mm	I_y 10 ⁶ mm ⁴	S_y 10 ³ mm ³	r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

Deformations in a Transverse Cross Section



- Deformation due to bending moment M is quantified by the curvature of the neutral surface

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec} \frac{Mc}{I} = \frac{M}{EI}$$

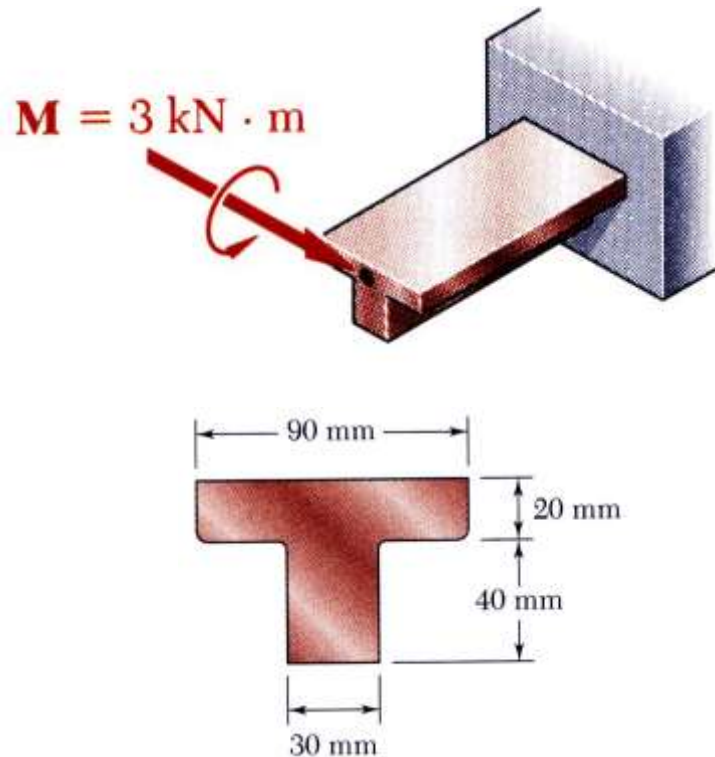
- Although cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero,

$$\varepsilon_y = -\nu \varepsilon_x = \frac{wy}{\rho} \quad \varepsilon_z = -\nu \varepsilon_x = \frac{wz}{\rho}$$

- Expansion above the neutral surface and contraction below it cause an in-plane curvature,

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvature}$$

Sample Problem 4.2



A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing $E = 165 \text{ GPa}$ and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \quad I_{x'} = \sum (\bar{I} + Ad^2)$$

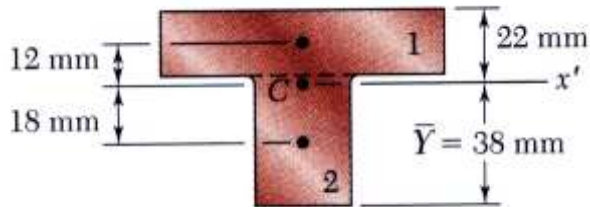
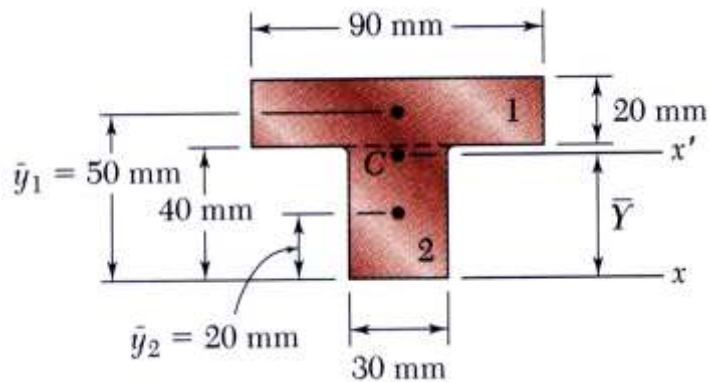
- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

Sample Problem 4.2



SOLUTION:

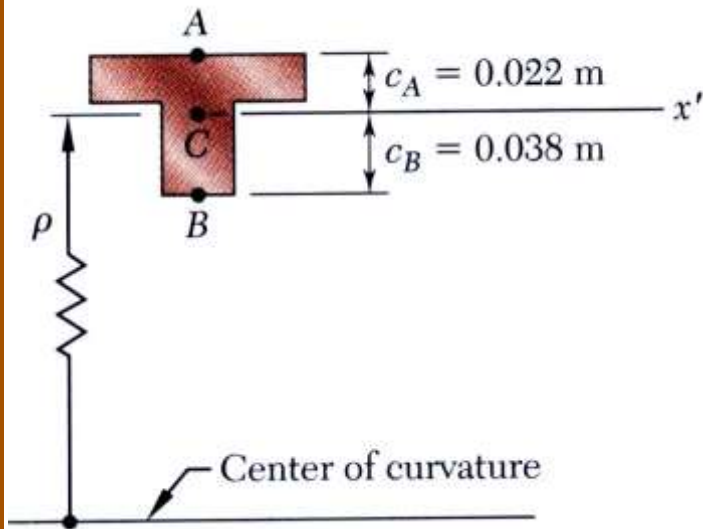
Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm ²	$\bar{y}$, mm	$\bar{y}A$, mm ³
1	$20 \times 90 = 1800$	50	90×10^3
2	$40 \times 30 = 1200$	20	24×10^3
	$\Sigma A = 3000$		$\Sigma \bar{y}A = 114 \times 10^3$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{114 \times 10^3}{3000} = 38 \text{ mm}$$

$$\begin{aligned}
 I_{x'} &= \Sigma (\bar{I} + Ad^2) = \Sigma \left(\frac{1}{12}bh^3 + Ad^2 \right) \\
 &= \left(\frac{1}{12}90 \times 20^3 + 1800 \times 12^2 \right) + \left(\frac{1}{12}30 \times 40^3 + 1200 \times 18^2 \right) \\
 I &= 868 \times 10^3 \text{ mm}^4 = 868 \times 10^{-9} \text{ m}^4
 \end{aligned}$$

Sample Problem 4.2



- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{M c_A}{I} = \frac{3 \text{ kN} \cdot \text{m} \times 0.022 \text{ m}}{868 \times 10^{-9} \text{ m}^4}$$

$$\sigma_A = +76.0 \text{ MPa}$$

$$\sigma_B = -\frac{M c_B}{I} = -\frac{3 \text{ kN} \cdot \text{m} \times 0.038 \text{ m}}{868 \times 10^{-9} \text{ m}^4}$$

$$\sigma_B = -131.3 \text{ MPa}$$

- Calculate the curvature

$$\frac{1}{\rho} = \frac{M}{EI}$$

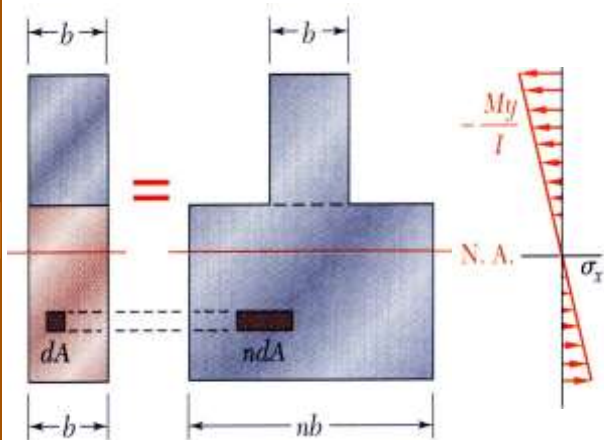
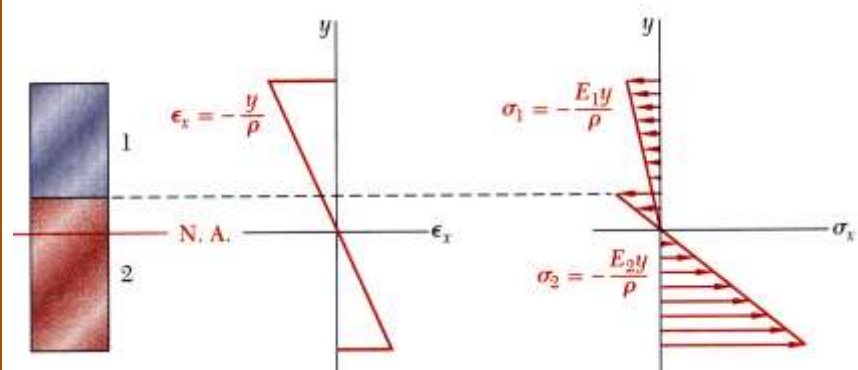
$$= \frac{3 \text{ kN} \cdot \text{m}}{(165 \text{ GPa})(868 \times 10^{-9} \text{ m}^4)}$$

$$\frac{1}{\rho} = 20.95 \times 10^{-3} \text{ m}^{-1}$$

$$\rho = 47.7 \text{ m}$$



Bending of Members Made of Several Materials



$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \quad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2 .

- Normal strain varies linearly.

$$\epsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1 \epsilon_x = -\frac{E_1 y}{\rho} \quad \sigma_2 = E_2 \epsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

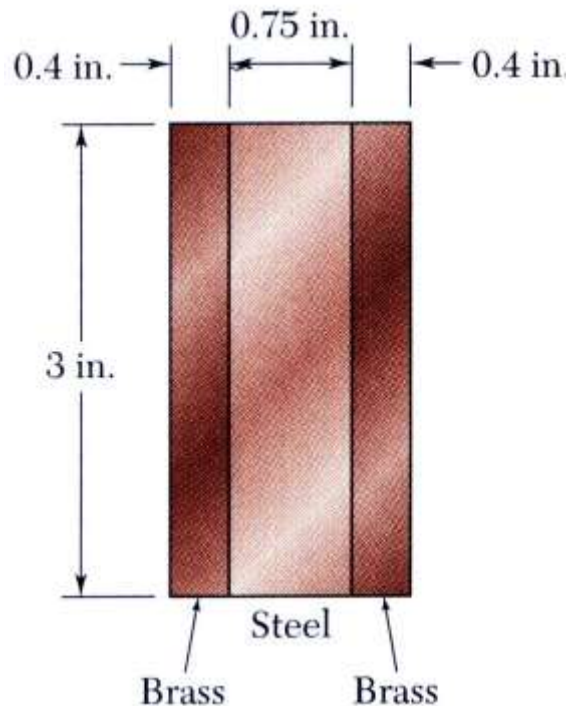
- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho} (n dA) \quad n = \frac{E_2}{E_1}$$

Example 4.03

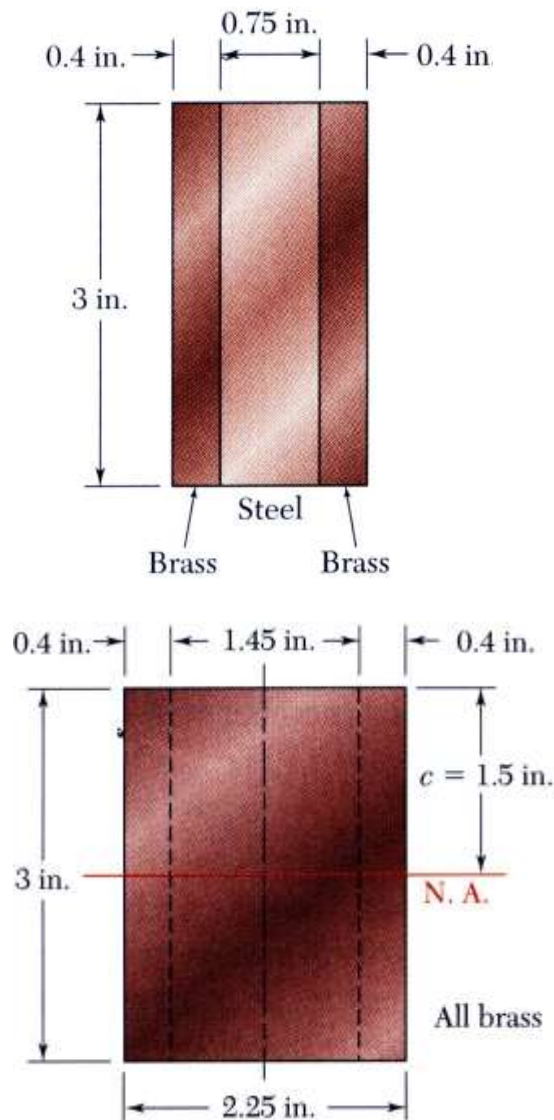


Bar is made from bonded pieces of steel ($E_s = 29 \times 10^6$ psi) and brass ($E_b = 15 \times 10^6$ psi). Determine the maximum stress in the steel and brass when a moment of 40 kip*in is applied.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass
- Evaluate the cross sectional properties of the transformed section
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.

Example 4.03



SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

$$n = \frac{E_s}{E_b} = \frac{29 \times 10^6 \text{ psi}}{15 \times 10^6 \text{ psi}} = 1.933$$

$$b_T = 0.4 \text{ in} + 1.933 \times 0.75 \text{ in} + 0.4 \text{ in} = 2.25 \text{ in}$$

- Evaluate the transformed cross sectional properties

$$I = \frac{1}{12} b_T h^3 = \frac{1}{12} (2.25 \text{ in.})(3 \text{ in})^3 = 5.063 \text{ in}^4$$

- Calculate the maximum stresses

$$\sigma_m = \frac{Mc}{I} = \frac{(40 \text{ kip} \cdot \text{in})(1.5 \text{ in})}{5.063 \text{ in}^4} = 11.85 \text{ ksi}$$

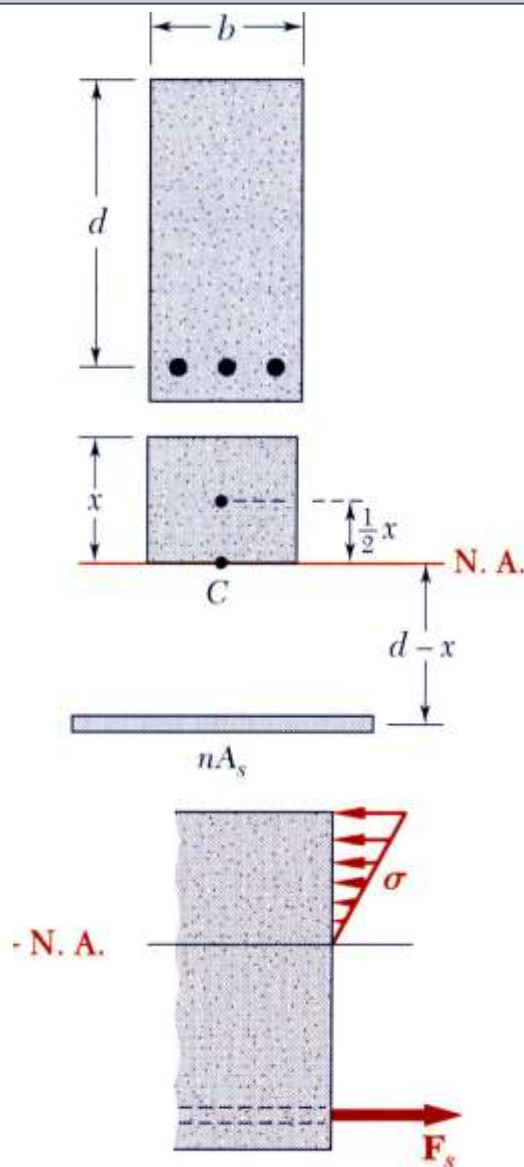
$$(\sigma_b)_{\max} = \sigma_m$$

$$(\sigma_b)_{\max} = 11.85 \text{ ksi}$$

$$(\sigma_s)_{\max} = n \sigma_m = 1.933 \times 11.85 \text{ ksi}$$

$$(\sigma_s)_{\max} = 22.9 \text{ ksi}$$

Reinforced Concrete Beams



- Concrete beams subjected to bending moments are reinforced by steel rods.
- The steel rods carry the entire tensile load below the neutral surface. The upper part of the concrete beam carries the compressive load.
- In the transformed section, the cross sectional area of the steel, A_s , is replaced by the equivalent area nA_s where $n = E_s/E_c$.
- To determine the location of the neutral axis,

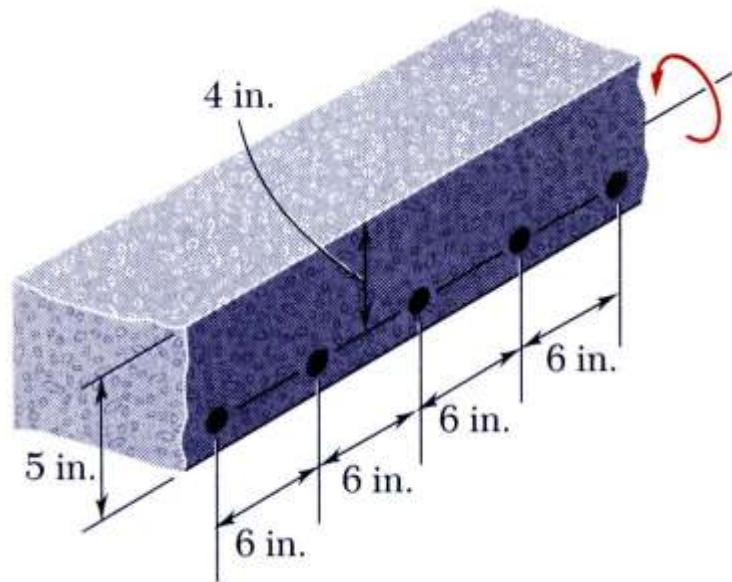
$$(bx)\frac{x}{2} - nA_s(d-x) = 0$$

$$\frac{1}{2}bx^2 + nA_sx - nA_sd = 0$$
- The normal stress in the concrete and steel

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_c = \sigma_x \quad \sigma_s = n\sigma_x$$

Sample Problem 4.4



SOLUTION:

- Transform to a section made entirely of concrete.
- Evaluate geometric properties of transformed section.
- Calculate the maximum stresses in the concrete and steel.

A concrete floor slab is reinforced with 5/8-in.-diameter steel rods. The modulus of elasticity is 29×10^6 psi for steel and 3.6×10^6 psi for concrete. With an applied bending moment of 40 kip*in for 1-ft width of the slab, determine the maximum stress in the concrete and steel.

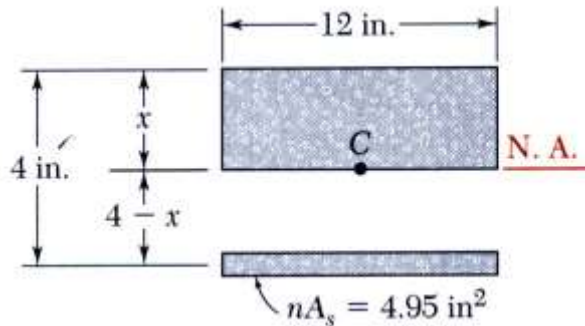
Sample Problem 4.4

SOLUTION:

- Transform to a section made entirely of concrete.

$$n = \frac{E_s}{E_c} = \frac{29 \times 10^6 \text{ psi}}{3.6 \times 10^6 \text{ psi}} = 8.06$$

$$nA_s = 8.06 \times 2 \left[\frac{\pi}{4} \left(\frac{5}{8} \text{ in} \right)^2 \right] = 4.95 \text{ in}^2$$



- Evaluate the geometric properties of the transformed section.

$$12x \left(\frac{x}{2} \right) - 4.95(4 - x) = 0 \quad x = 1.450 \text{ in}$$

$$I = \frac{1}{3} (12 \text{ in}) (1.45 \text{ in})^3 + (4.95 \text{ in}^2) (2.55 \text{ in})^2 = 44.4 \text{ in}^4$$

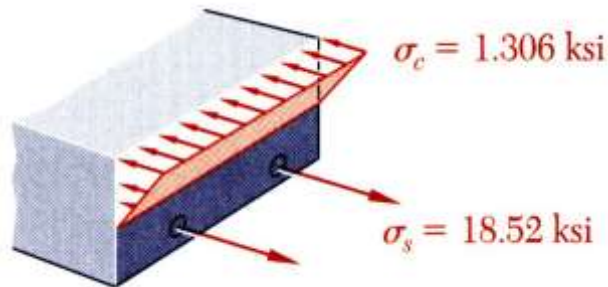
- Calculate the maximum stresses.

$$\sigma_c = \frac{Mc_1}{I} = \frac{40 \text{ kip} \cdot \text{in} \times 1.45 \text{ in}}{44.4 \text{ in}^4}$$

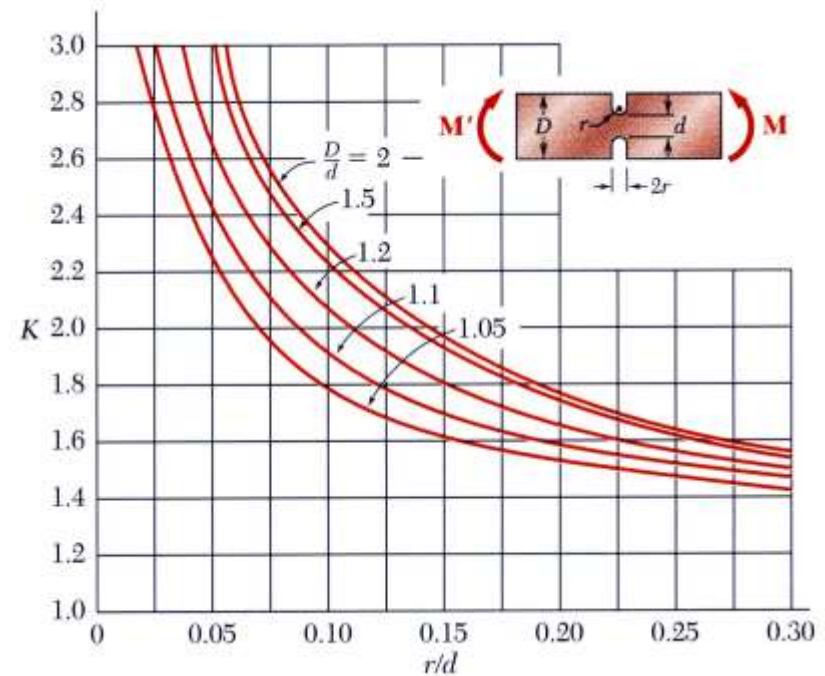
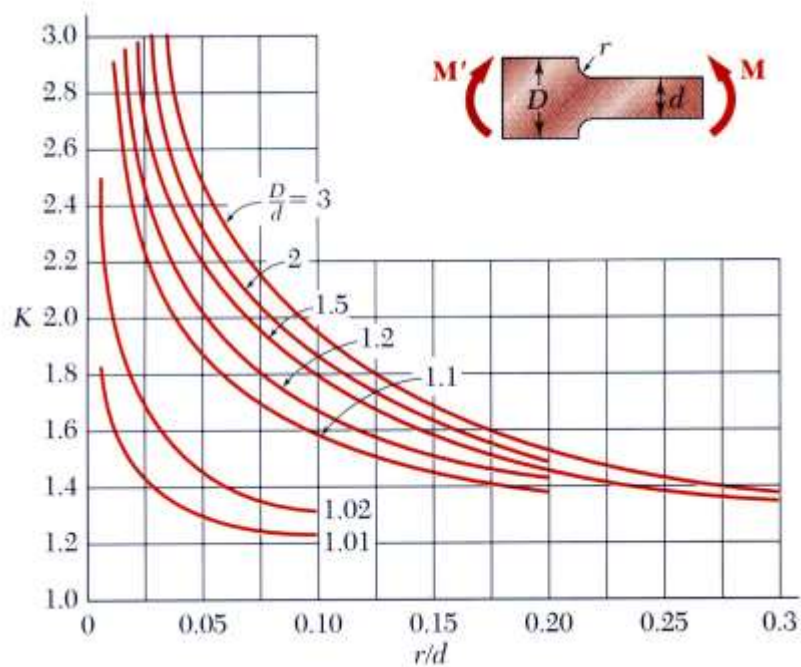
$$\sigma_c = 1.306 \text{ ksi}$$

$$\sigma_s = n \frac{Mc_2}{I} = 8.06 \frac{40 \text{ kip} \cdot \text{in} \times 2.55 \text{ in}}{44.4 \text{ in}^4}$$

$$\sigma_s = 18.52 \text{ ksi}$$



Stress Concentrations

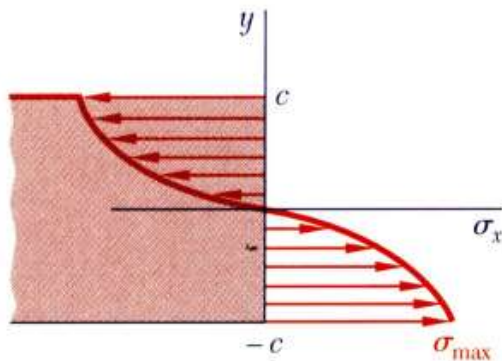
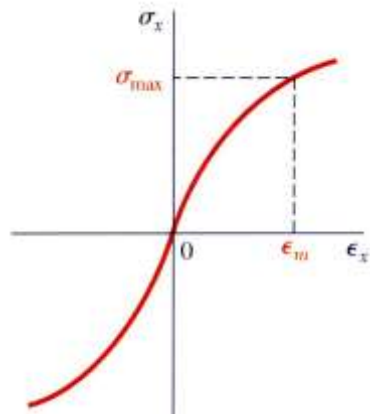
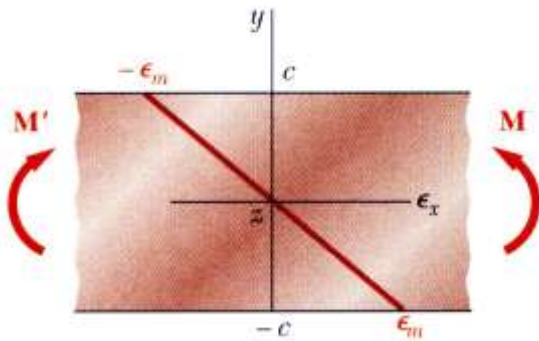


Stress concentrations may occur:

- in the vicinity of points where the loads are applied
- in the vicinity of abrupt changes in cross section

$$\sigma_m = K \frac{Mc}{I}$$

Plastic Deformations



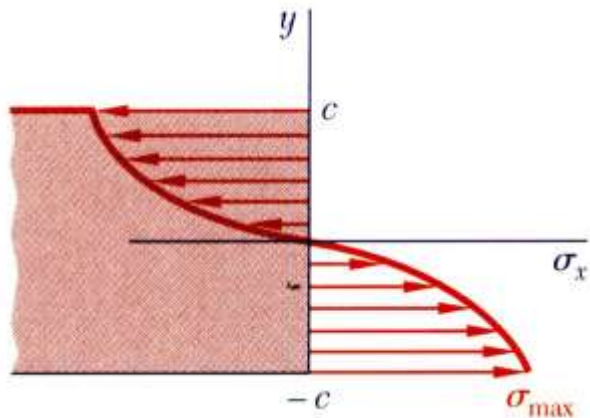
- For any member subjected to pure bending

$$\epsilon_x = -\frac{y}{c} \epsilon_m$$
 strain varies linearly across the section
- If the member is made of a *linearly elastic material*, the neutral axis passes through the section centroid

$$\text{and } \sigma_x = -\frac{My}{I}$$
- For a material with a nonlinear stress-strain curve, the neutral axis location is found by satisfying

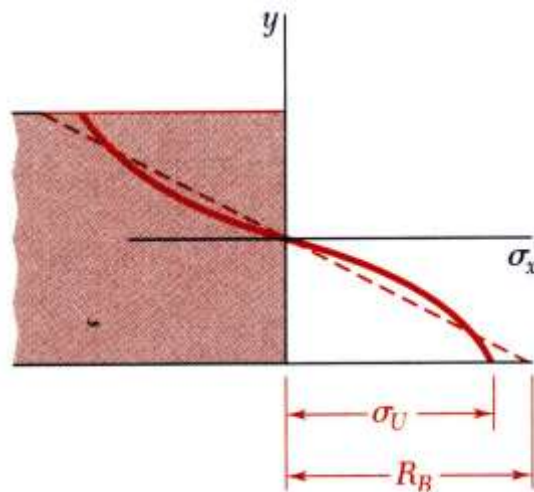
$$F_x = \int \sigma_x dA = 0 \quad M = \int -y \sigma_x dA$$
- For a member with vertical and horizontal planes of symmetry and a material with the same tensile and compressive stress-strain relationship, the neutral axis is located at the section centroid and the stress-strain relationship may be used to map the strain distribution from the stress distribution.

Plastic Deformations



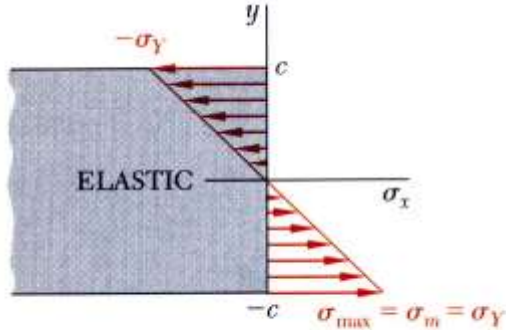
- When the maximum stress is equal to the ultimate strength of the material, failure occurs and the corresponding moment M_U is referred to as the *ultimate bending moment*.
- The *modulus of rupture in bending*, R_B , is found from an experimentally determined value of M_U and a fictitious linear stress distribution.

$$R_B = \frac{M_U c}{I}$$



- R_B may be used to determine M_U of any member made of the same material and with the same cross sectional shape but different dimensions.

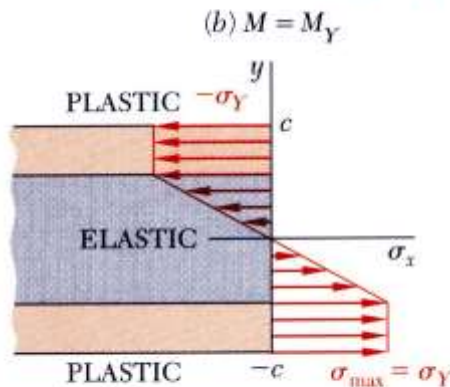
Members Made of an Elastoplastic Material



- Rectangular beam made of an elastoplastic material

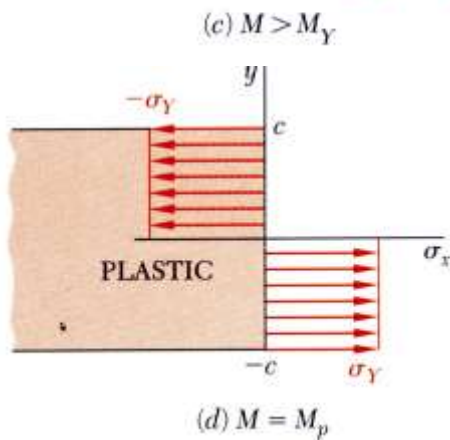
$$\sigma_x \leq \sigma_Y \quad \sigma_m = \frac{Mc}{I}$$

$$\sigma_m = \sigma_Y \quad M_Y = \frac{I}{c} \sigma_Y = \text{maximum elastic moment}$$



- If the moment is increased beyond the maximum elastic moment, plastic zones develop around an elastic core.

$$M = \frac{3}{2} M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right) \quad y_Y = \text{elastic core half - thickness}$$

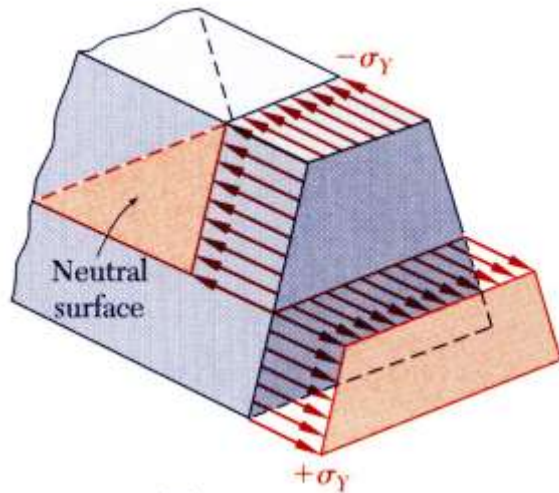


- In the limit as the moment is increased further, the elastic core thickness goes to zero, corresponding to a fully plastic deformation.

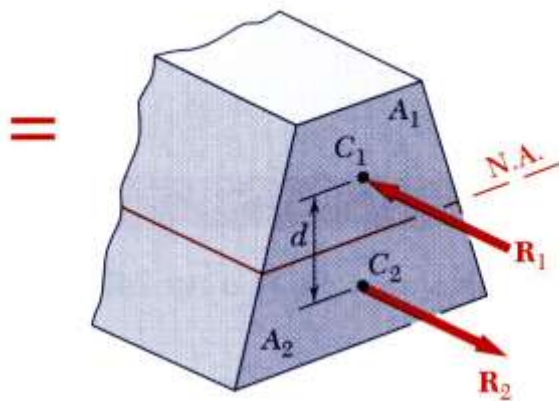
$$M_p = \frac{3}{2} M_Y = \text{plastic moment}$$

$$k = \frac{M_p}{M_Y} = \text{shape factor (depends only on cross section shape)}$$

Plastic Deformations of Members With a Single Plane of Symmetry



(a)



(b)

- Fully plastic deformation of a beam with only a vertical plane of symmetry.
- The neutral axis cannot be assumed to pass through the section centroid.
- Resultants R_1 and R_2 of the elementary compressive and tensile forces form a couple.

$$R_1 = R_2$$

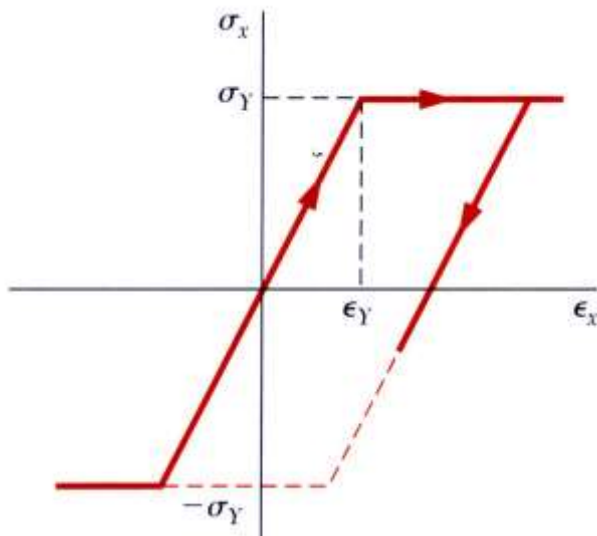
$$A_1 \sigma_Y = A_2 \sigma_Y$$

The neutral axis divides the section into equal areas.

- The plastic moment for the member,

$$M_p = \left(\frac{1}{2} A \sigma_Y \right) d$$

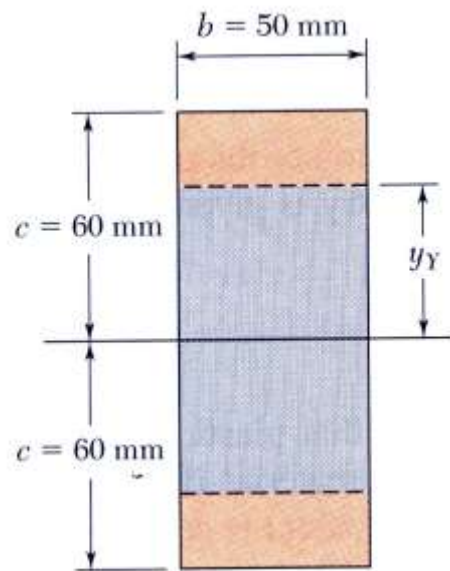
Residual Stresses



- Plastic zones develop in a member made of an elastoplastic material if the bending moment is large enough.
- Since the linear relation between normal stress and strain applies at all points during the unloading phase, it may be handled by assuming the member to be fully elastic.
- Residual stresses are obtained by applying the principle of superposition to combine the stresses due to loading with a moment M (elastoplastic deformation) and unloading with a moment $-M$ (elastic deformation).
- The final value of stress at a point will not, in general, be zero.



Example 4.05, 4.06



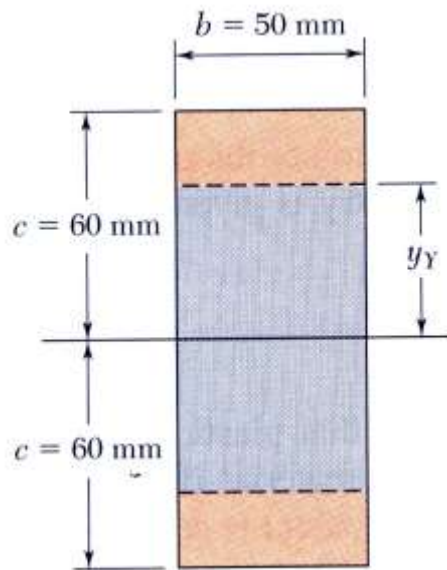
A member of uniform rectangular cross section is subjected to a bending moment $M = 36.8 \text{ kN-m}$. The member is made of an elastoplastic material with a yield strength of 240 MPa and a modulus of elasticity of 200 GPa .

Determine (a) the thickness of the elastic core, (b) the radius of curvature of the neutral surface.

After the loading has been reduced back to zero, determine (c) the distribution of residual stresses, (d) radius of curvature.



Example 4.05, 4.06



- Thickness of elastic core:

$$M = \frac{3}{2} M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right)$$

$$36.8 \text{ kN} \cdot \text{m} = \frac{3}{2} (28.8 \text{ kN} \cdot \text{m}) \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right)$$

$$\frac{y_Y}{c} = \frac{y_Y}{60 \text{ mm}} = 0.666$$

$$2y_Y = 80 \text{ mm}$$

- Radius of curvature:

$$\begin{aligned} \epsilon_Y &= \frac{\sigma_Y}{E} = \frac{240 \times 10^6 \text{ Pa}}{200 \times 10^9 \text{ Pa}} \\ &= 1.2 \times 10^{-3} \end{aligned}$$

$$\epsilon_Y = \frac{y_Y}{\rho}$$

$$\rho = \frac{y_Y}{\epsilon_Y} = \frac{40 \times 10^{-3} \text{ m}}{1.2 \times 10^{-3}}$$

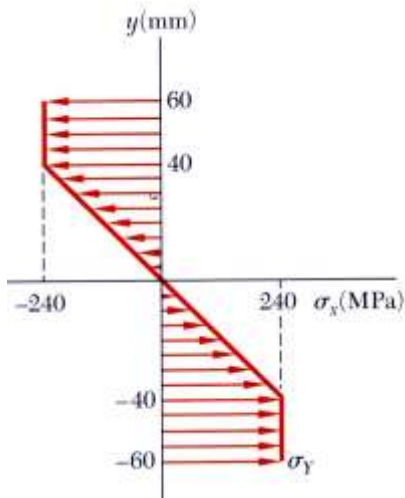
$$\rho = 33.3 \text{ m}$$

- Maximum elastic moment:

$$\begin{aligned} \frac{I}{c} &= \frac{2}{3} b c^2 = \frac{2}{3} (50 \times 10^{-3} \text{ m}) (60 \times 10^{-3} \text{ m})^2 \\ &= 120 \times 10^{-6} \text{ m}^3 \end{aligned}$$

$$\begin{aligned} M_Y &= \frac{I}{c} \sigma_Y = (120 \times 10^{-6} \text{ m}^3) (240 \text{ MPa}) \\ &= 28.8 \text{ kN} \cdot \text{m} \end{aligned}$$

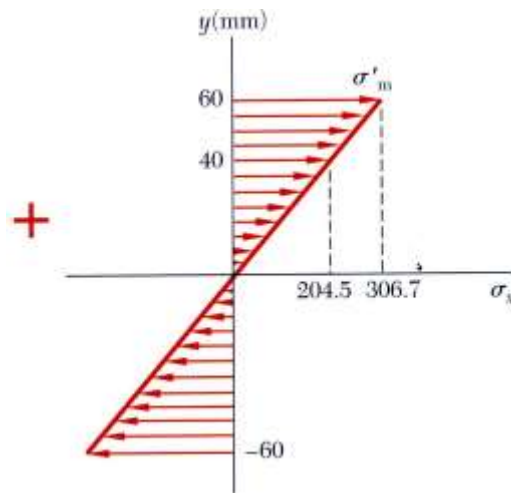
Example 4.05, 4.06



- $M = 36.8 \text{ kN-m}$

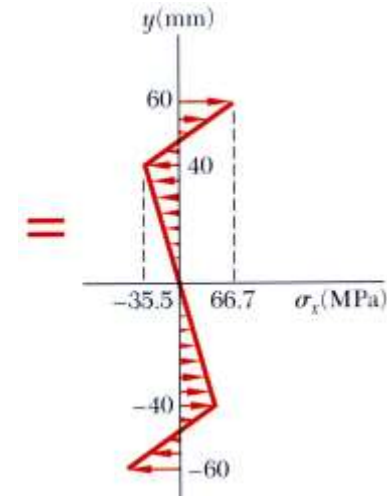
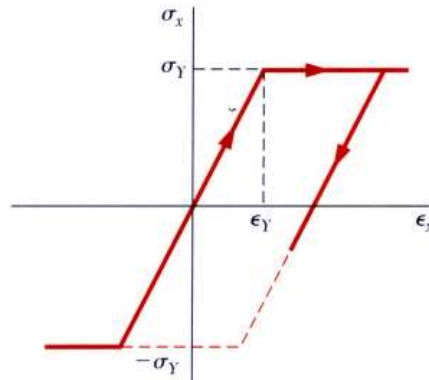
$$y_Y = 40 \text{ mm}$$

$$\sigma_Y = 240 \text{ MPa}$$



- $M = -36.8 \text{ kN-m}$

$$\sigma'_m = \frac{Mc}{I} = \frac{36.8 \text{ kN} \cdot \text{m}}{120 \times 10^6 \text{ m}^3} = 306.7 \text{ MPa} < 2\sigma_Y$$



- $M = 0$

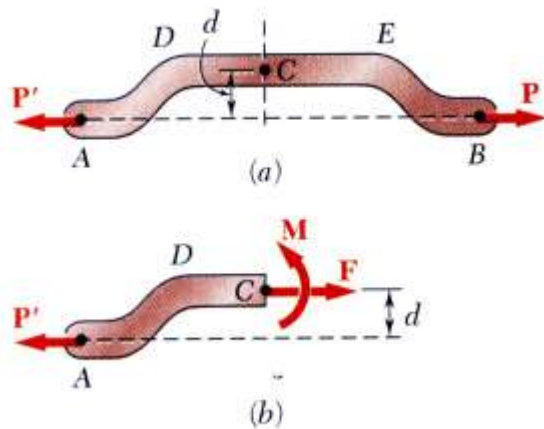
At the edge of the elastic core,

$$\epsilon_x = \frac{\sigma_x}{E} = \frac{-35.5 \times 10^6 \text{ Pa}}{200 \times 10^9 \text{ Pa}} = -177.5 \times 10^{-6}$$

$$\rho = -\frac{y_Y}{\epsilon_x} = \frac{40 \times 10^{-3} \text{ m}}{177.5 \times 10^{-6}}$$

$$\rho = 225 \text{ m}$$

Eccentric Axial Loading in a Plane of Symmetry



- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and linear stress distribution due a pure bending moment

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

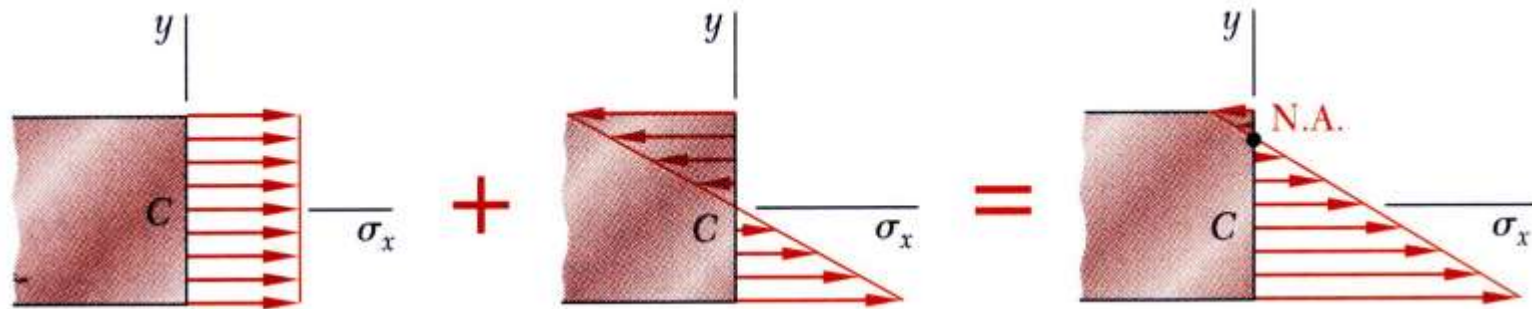
$$= \frac{P}{A} - \frac{My}{I}$$

- Eccentric loading

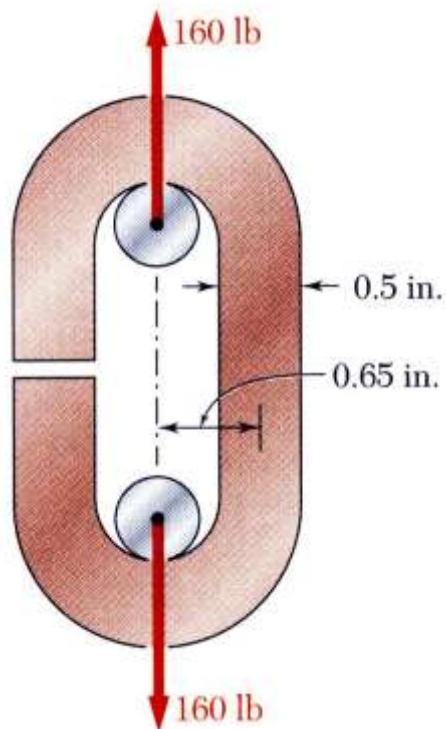
$$F = P$$

$$M = Pd$$

- Validity requires stresses below proportional limit, deformations have negligible effect on geometry, and stresses not evaluated near points of load application.



Example 4.07

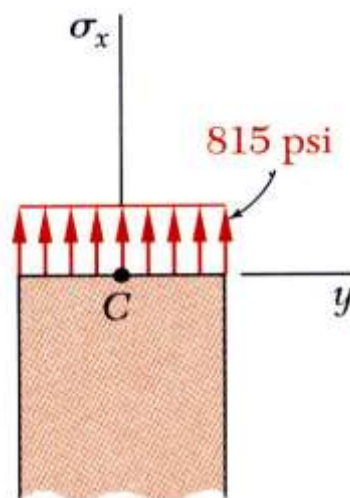
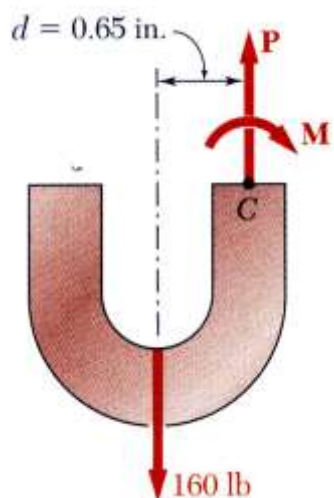


An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For 160 lb load, determine (a) maximum tensile and compressive stresses, (b) distance between section centroid and neutral axis

SOLUTION:

- Find the equivalent centric load and bending moment
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Find the neutral axis by determining the location where the normal stress is zero.

Example 4.07



- Normal stress due to a centric load

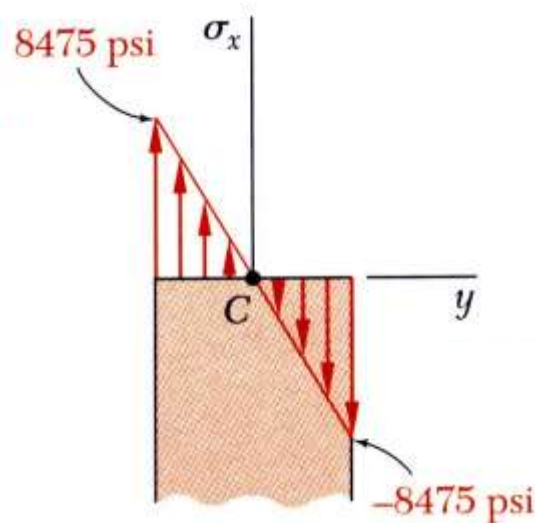
$$A = \pi c^2 = \pi (0.25 \text{ in})^2 = 0.1963 \text{ in}^2$$

$$\sigma_0 = \frac{P}{A} = \frac{160 \text{ lb}}{0.1963 \text{ in}^2} = 815 \text{ psi}$$

- Equivalent centric load and bending moment

$$P = 160 \text{ lb}$$

$$M = Pd = (160 \text{ lb})(0.6 \text{ in}) = 104 \text{ lb} \cdot \text{in}$$

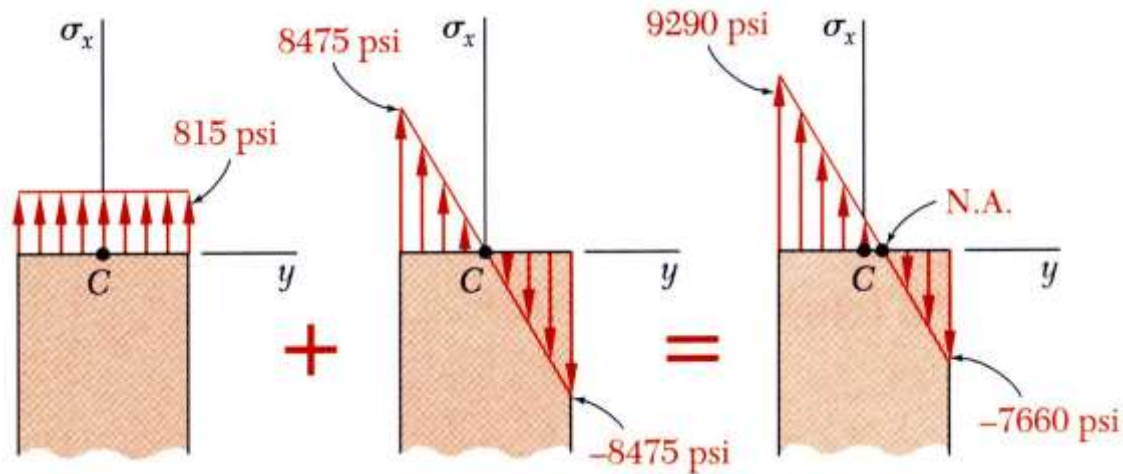


- Normal stress due to bending moment

$$I = \frac{1}{4} \pi c^4 = \frac{1}{4} \pi (0.25)^4 = 3.068 \times 10^{-3} \text{ in}^4$$

$$\sigma_m = \frac{Mc}{I} = \frac{(104 \text{ lb} \cdot \text{in})(0.25 \text{ in})}{3.068 \times 10^{-3} \text{ in}^4} = 8475 \text{ psi}$$

Example 4.07



- Maximum tensile and compressive stresses

$$\begin{aligned}\sigma_t &= \sigma_0 + \sigma_m \\ &= 815 + 8475\end{aligned}$$

$$\sigma_t = 9260 \text{ psi}$$

$$\begin{aligned}\sigma_c &= \sigma_0 - \sigma_m \\ &= 815 - 8475\end{aligned}$$

$$\sigma_c = -7660 \text{ psi}$$

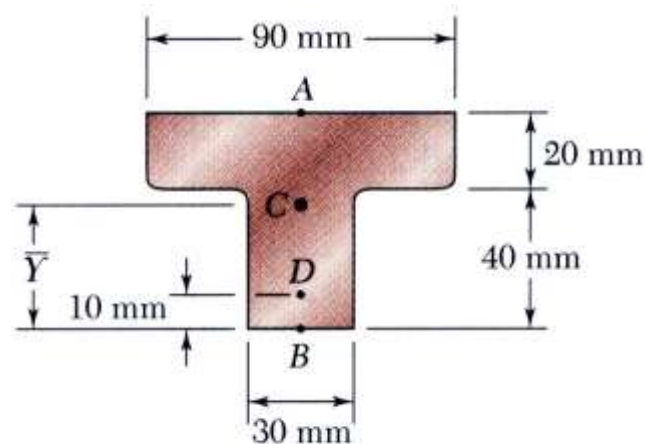
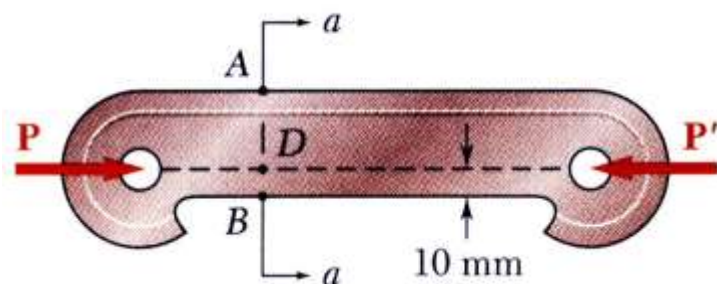
- Neutral axis location

$$0 = \frac{P}{A} - \frac{My_0}{I}$$

$$y_0 = \frac{P}{A} \frac{I}{M} = (815 \text{ psi}) \frac{3.068 \times 10^{-3} \text{ in}^4}{105 \text{ lb} \cdot \text{in}}$$

$$y_0 = 0.0240 \text{ in}$$

Sample Problem 4.8



Section a-a

The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force P which can be applied to the link.

SOLUTION:

- Determine an equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

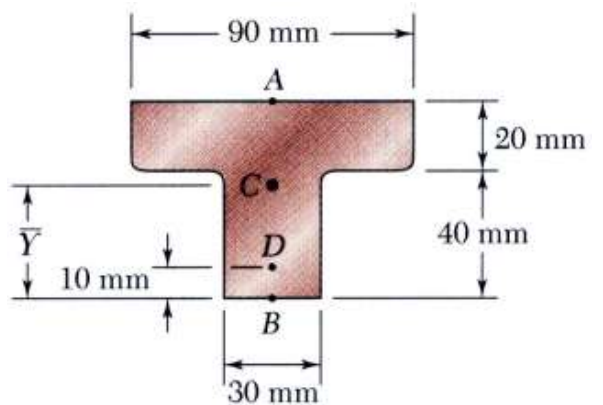
From Sample Problem 2.4,

$$A = 3 \times 10^{-3} \text{ m}^2$$

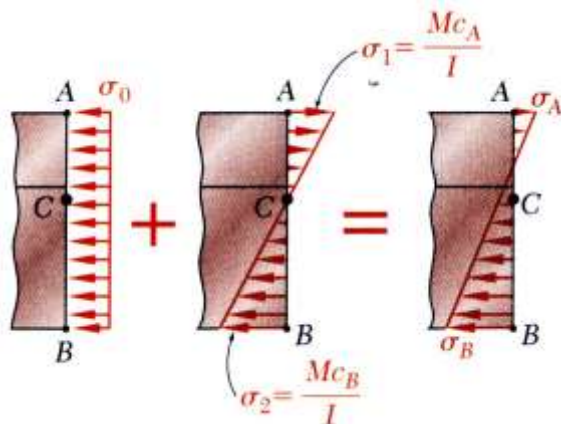
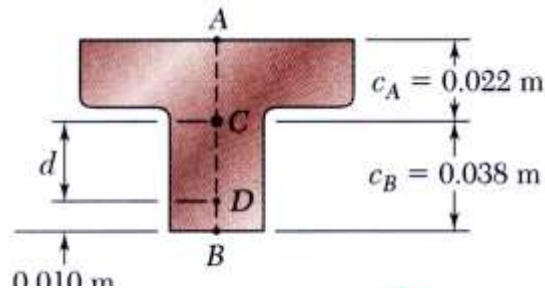
$$\bar{Y} = 0.038 \text{ m}$$

$$I = 868 \times 10^{-9} \text{ m}^4$$

Sample Problem 4.8



Section a-a



- Determine an equivalent centric and bending loads.

$$d = 0.038 - 0.010 = 0.028 \text{ m}$$

P = centric load

$$M = Pd = 0.028 P = \text{bending moment}$$

- Superpose stresses due to centric and bending loads

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3 \times 10^{-3}} + \frac{(0.028 P)(0.022)}{868 \times 10^{-9}} = +377 P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_B}{I} = -\frac{P}{3 \times 10^{-3}} - \frac{(0.028 P)(0.022)}{868 \times 10^{-9}} = -1559 P$$

- Evaluate critical loads for allowable stresses.

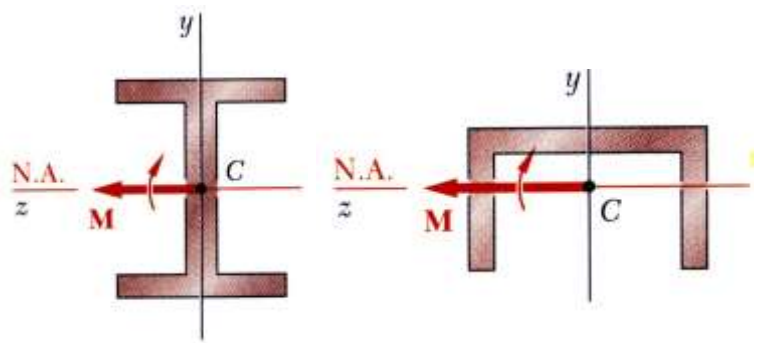
$$\sigma_A = +377 P = 30 \text{ MPa} \quad P = 79.6 \text{ kN}$$

$$\sigma_B = -1559 P = -120 \text{ MPa} \quad P = 79.6 \text{ kN}$$

- The largest allowable load

$$P = 77.0 \text{ kN}$$

Unsymmetric Bending



- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.

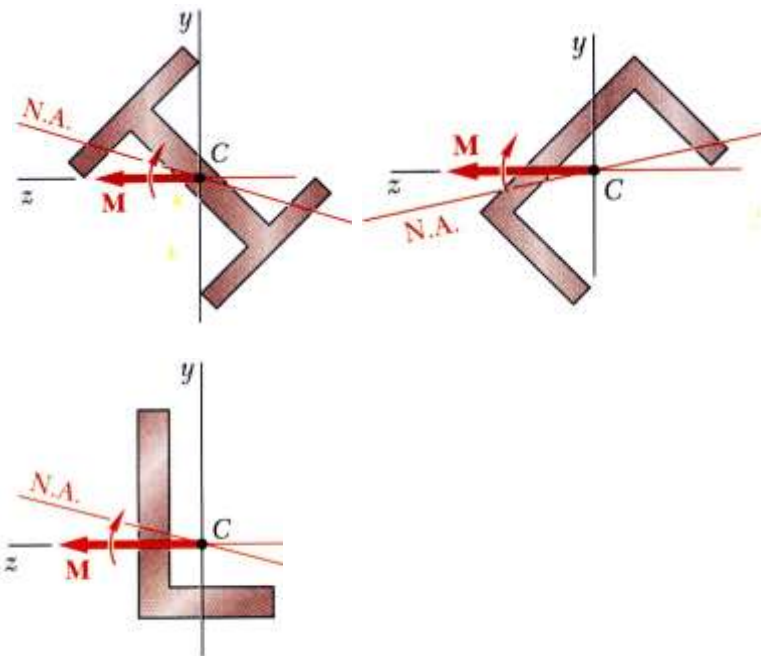
- Members remain symmetric and bend in the plane of symmetry.

- The neutral axis of the cross section coincides with the axis of the couple

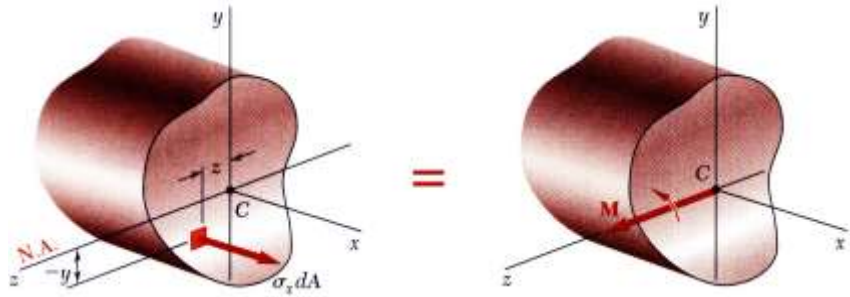
- Will now consider situations in which the bending couples do not act in a plane of symmetry.

- Cannot assume that the member will bend in the plane of the couples.

- In general, the neutral axis of the section will not coincide with the axis of the couple.



Unsymmetric Bending



Wish to determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy

$$F_x = 0 = M_y \quad M_z = M = \text{applied couple}$$

$$0 = F_x = \int \sigma_x dA = \int \left(-\frac{y}{c} \sigma_m \right) dA$$

$$\text{or } 0 = \int y dA$$

neutral axis passes through centroid

$$M = M_z = -\int y \left(-\frac{y}{c} \sigma_m \right) dA$$

$$\text{or } M = \frac{\sigma_m I}{c} \quad I = I_z = \text{moment of inertia}$$

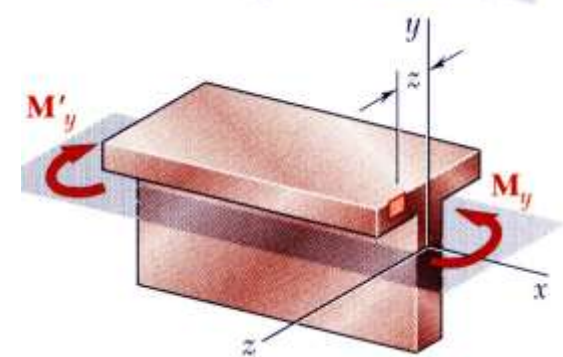
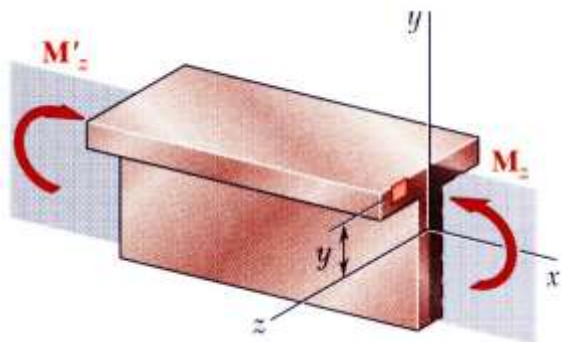
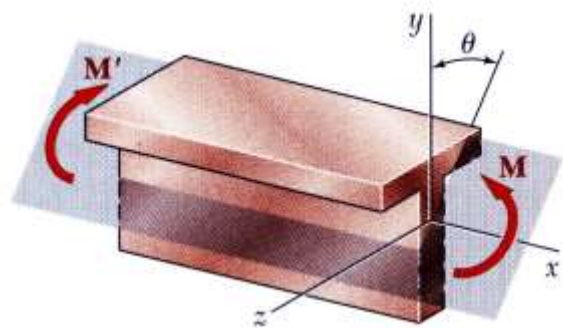
defines stress distribution

$$0 = M_y = \int z \sigma_x dA = \int z \left(-\frac{y}{c} \sigma_m \right) dA$$

$$\text{or } 0 = \int yz dA = I_{yz} = \text{product of inertia}$$

couple vector must be directed along a principal centroidal axis

Unsymmetric Bending



Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.

$$M_z = M \cos \theta \quad M_y = M \sin \theta$$

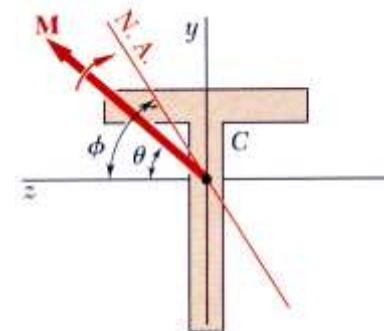
- Superpose the component stress distributions

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

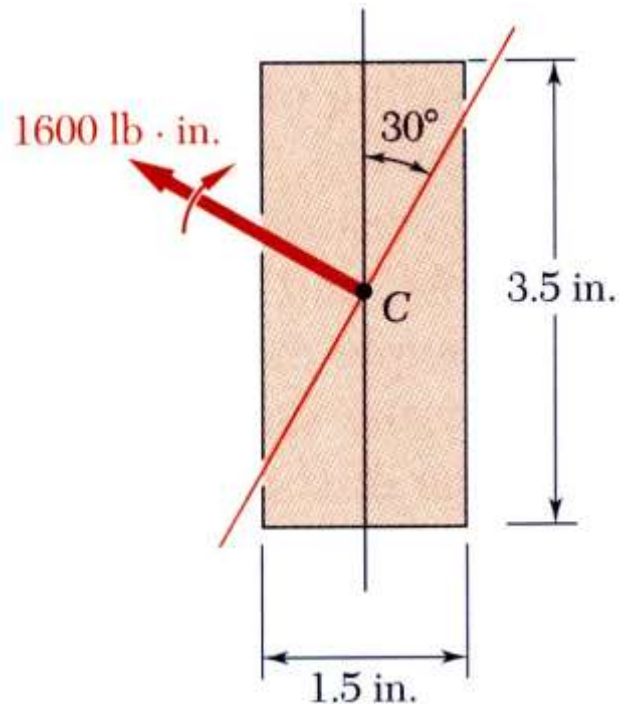
- Along the neutral axis,

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y} = -\frac{(M \cos \theta) y}{I_z} + \frac{(M \sin \theta) y}{I_y}$$

$$\tan \phi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta$$



Example 4.08



A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30 deg. with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.

$$M_z = M \cos \theta \quad M_y = M \sin \theta$$

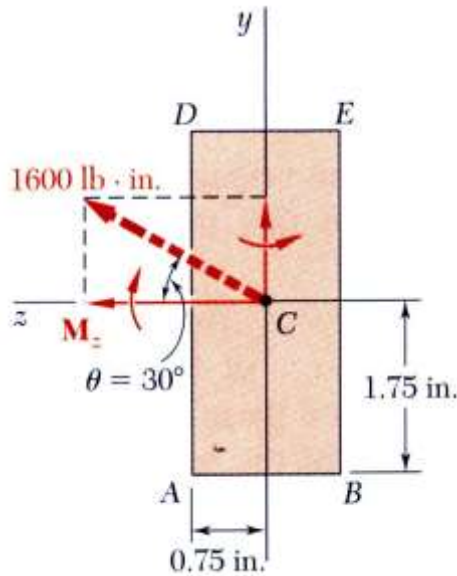
- Combine the stresses from the component stress distributions.

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y y}{I_y}$$

- Determine the angle of the neutral axis.

$$\tan \phi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta$$

Example 4.08



- Resolve the couple vector into components and calculate the corresponding maximum stresses.

$$M_z = (1600 \text{ lb} \cdot \text{in}) \cos 30 = 1386 \text{ lb} \cdot \text{in}$$

$$M_y = (1600 \text{ lb} \cdot \text{in}) \sin 30 = 800 \text{ lb} \cdot \text{in}$$

$$I_z = \frac{1}{12} (1.5 \text{ in}) (3.5 \text{ in})^3 = 5.359 \text{ in}^4$$

$$I_y = \frac{1}{12} (3.5 \text{ in}) (1.5 \text{ in})^3 = 0.9844 \text{ in}^4$$

The largest tensile stress due to M_z occurs along AB

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386 \text{ lb} \cdot \text{in})(1.75 \text{ in})}{5.359 \text{ in}^4} = 452.6 \text{ psi}$$

The largest tensile stress due to M_y occurs along AD

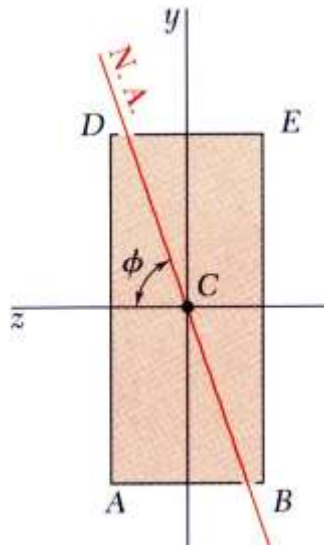
$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800 \text{ lb} \cdot \text{in})(0.75 \text{ in})}{0.9844 \text{ in}^4} = 609.5 \text{ psi}$$

- The largest tensile stress due to the combined loading occurs at A .

$$\sigma_{\max} = \sigma_1 + \sigma_2 = 452.6 + 609.5$$

$$\sigma_{\max} = 1062 \text{ psi}$$

Example 4.08

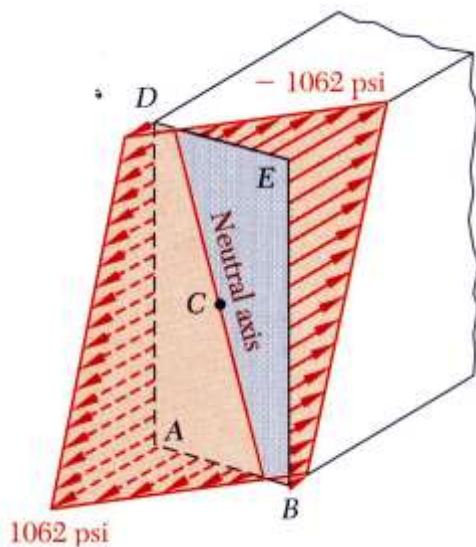


- Determine the angle of the neutral axis.

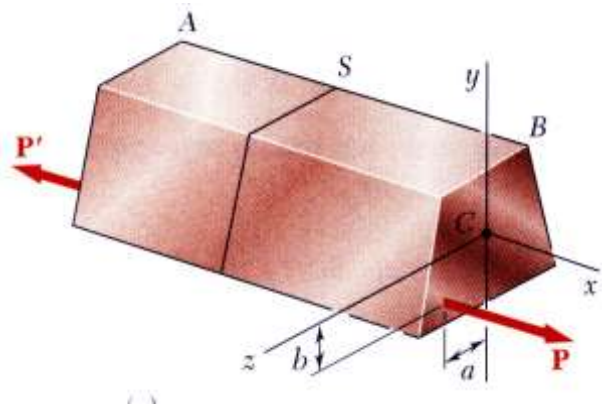
$$\tan \phi = \frac{I_z}{I_y} \tan \theta = \frac{5.359 \text{ in}^4}{0.9844 \text{ in}^4} \tan 30$$

$$= 3.143$$

$$\phi = 72.4^\circ$$



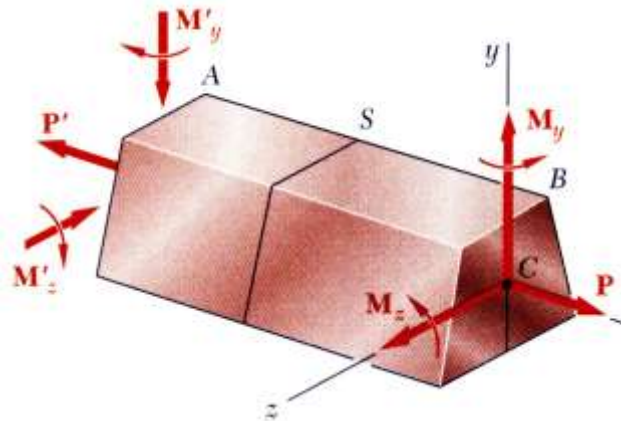
General Case of Eccentric Axial Loading



- Consider a straight member subject to equal and opposite eccentric forces.
- The eccentric force is equivalent to the system of a centric force and two couples.

P = centric force

$$M_y = Pa \quad M_z = Pb$$



- By the principle of superposition, the combined stress distribution is

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- If the neutral axis lies on the section, it may be found from

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$

Third Edition

CHAPTER

5

MECHANICS OF MATERIALS

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Lecture Notes:
J. Walt Oler
Texas Tech University

Analysis and Design of Beams for Bending



Analysis and Design of Beams for Bending

[Introduction](#)

[Shear and Bending Moment Diagrams](#)

[Sample Problem 5.1](#)

[Sample Problem 5.2](#)

[Relations Among Load, Shear, and Bending Moment](#)

[Sample Problem 5.3](#)

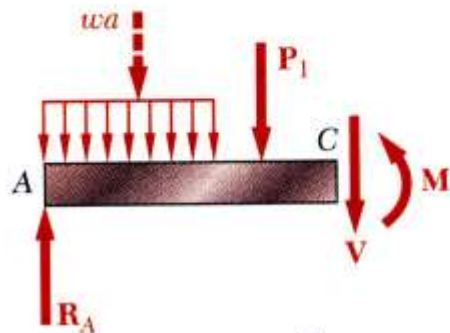
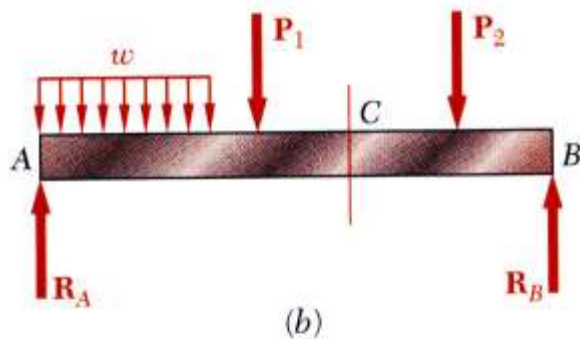
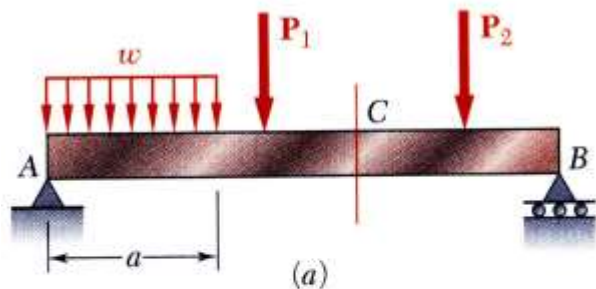
[Sample Problem 5.5](#)

[Design of Prismatic Beams for Bending](#)

[Sample Problem 5.8](#)



Introduction



- Objective - Analysis and design of beams
- *Beams* - structural members supporting loads at various points along the member
- Transverse loadings of beams are classified as *concentrated* loads or *distributed* loads
- Applied loads result in internal forces consisting of a shear force (from the shear stress distribution) and a bending couple (from the normal stress distribution)
- Normal stress is often the critical design criteria

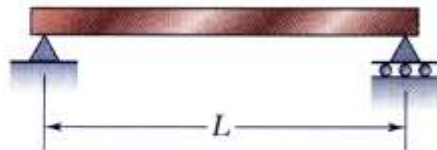
$$\sigma_x = -\frac{My}{I} \quad \sigma_m = \frac{|M|c}{I} = \frac{|M|}{S}$$

Requires determination of the location and magnitude of largest bending moment

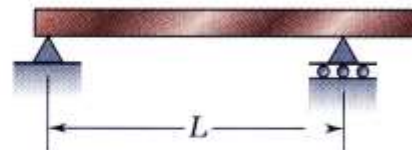
Introduction

Classification of Beam Supports

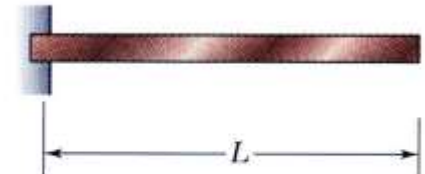
Statically
Determinate
Beams



(a) Simply supported beam

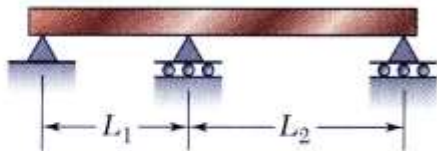


(b) Overhanging beam

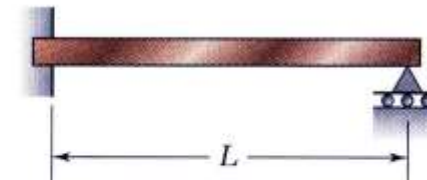


(c) Cantilever beam

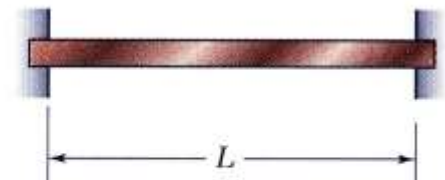
Statically
Indeterminate
Beams



(d) Continuous beam

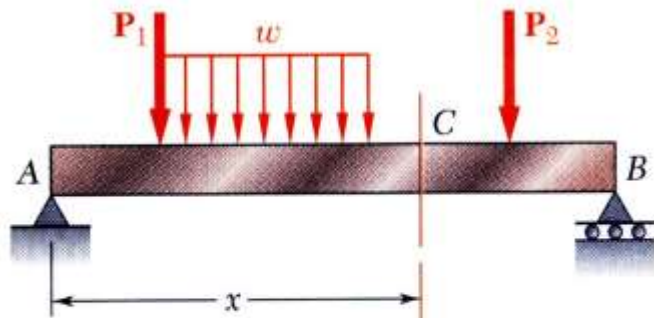


(e) Beam fixed at one end
and simply supported
at the other end

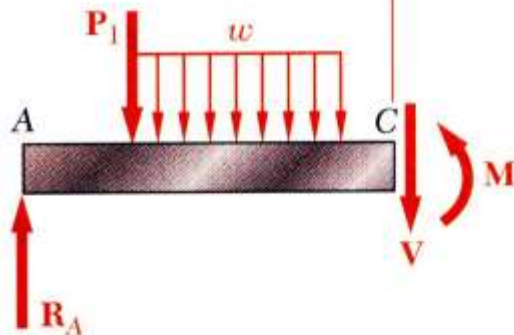


(f) Fixed beam

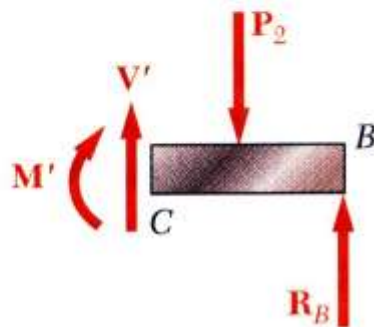
Shear and Bending Moment Diagrams



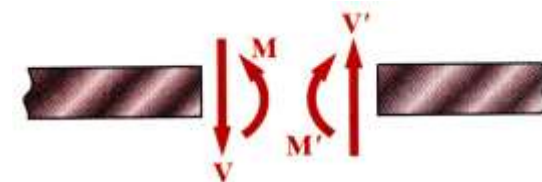
(a)



(b)

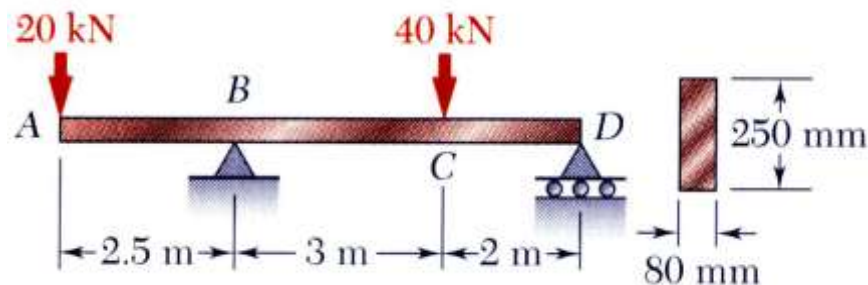


- Determination of maximum normal and shearing stresses requires identification of maximum internal shear force and bending couple.
- Shear force and bending couple at a point are determined by passing a section through the beam and applying an equilibrium analysis on the beam portions on either side of the section.
- Sign conventions for shear forces V and V' and bending couples M and M'



(a) Internal forces
(positive shear and positive bending moment)

Sample Problem 5.1

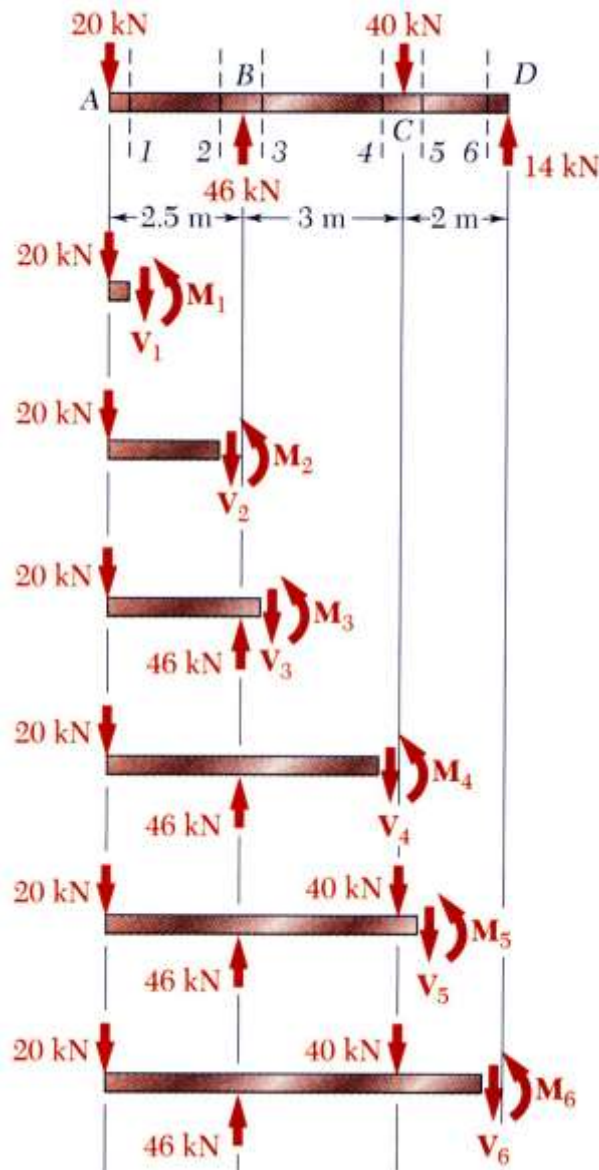


For the timber beam and loading shown, draw the shear and bending-moment diagrams and determine the maximum normal stress due to bending.

SOLUTION:

- Treating the entire beam as a rigid body, determine the reaction forces
- Section the beam at points near supports and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples
- Identify the maximum shear and bending-moment from plots of their distributions.
- Apply the elastic flexure formulas to determine the corresponding maximum normal stress.

Sample Problem 5.1



SOLUTION:

- Treating the entire beam as a rigid body, determine the reaction forces

$$\text{from } \sum F_y = 0 = \sum M_B : \quad R_B = 40 \text{ kN} \quad R_D = 14 \text{ kN}$$

- Section the beam and apply equilibrium analyses on resulting free-bodies

$$\sum F_y = 0 \quad -20 \text{ kN} - V_1 = 0 \quad V_1 = -20 \text{ kN}$$

$$\sum M_1 = 0 \quad (20 \text{ kN})(0 \text{ m}) + M_1 = 0 \quad M_1 = 0$$

$$\sum F_y = 0 \quad -20 \text{ kN} - V_2 = 0 \quad V_2 = -20 \text{ kN}$$

$$\sum M_2 = 0 \quad (20 \text{ kN})(2.5 \text{ m}) + M_2 = 0 \quad M_2 = -50 \text{ kN} \cdot \text{m}$$

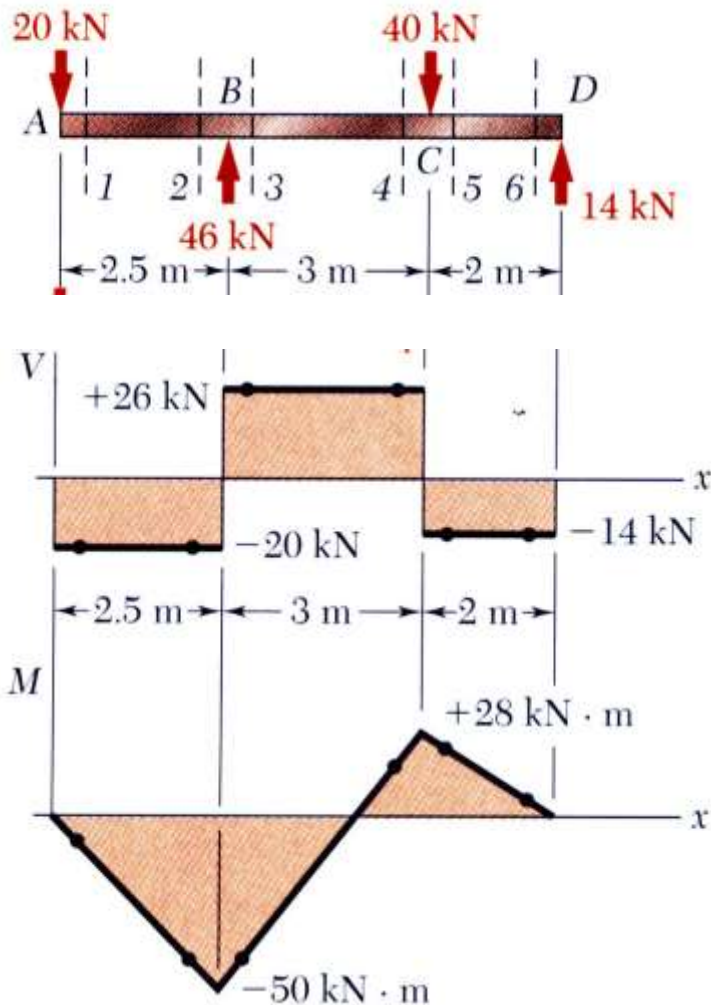
$$V_3 = +26 \text{ kN} \quad M_3 = -50 \text{ kN} \cdot \text{m}$$

$$V_4 = +26 \text{ kN} \quad M_4 = +28 \text{ kN} \cdot \text{m}$$

$$V_5 = -14 \text{ kN} \quad M_5 = +28 \text{ kN} \cdot \text{m}$$

$$V_6 = -14 \text{ kN} \quad M_6 = 0$$

Sample Problem 5.1



- Identify the maximum shear and bending-moment from plots of their distributions.

$$V_m = 26 \text{ kN} \quad M_m = |M_B| = 50 \text{ kN} \cdot \text{m}$$

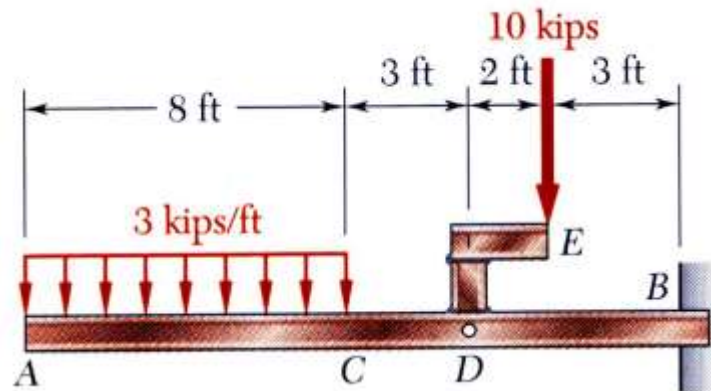
- Apply the elastic flexure formulas to determine the corresponding maximum normal stress.

$$S = \frac{1}{6} b h^2 = \frac{1}{6} (0.080 \text{ m}) (0.250 \text{ m})^2 \\ = 833.33 \times 10^{-6} \text{ m}^3$$

$$\sigma_m = \frac{|M_B|}{S} = \frac{50 \times 10^3 \text{ N} \cdot \text{m}}{833.33 \times 10^{-6} \text{ m}^3}$$

$$\sigma_m = 60.0 \times 10^6 \text{ Pa}$$

Sample Problem 5.2

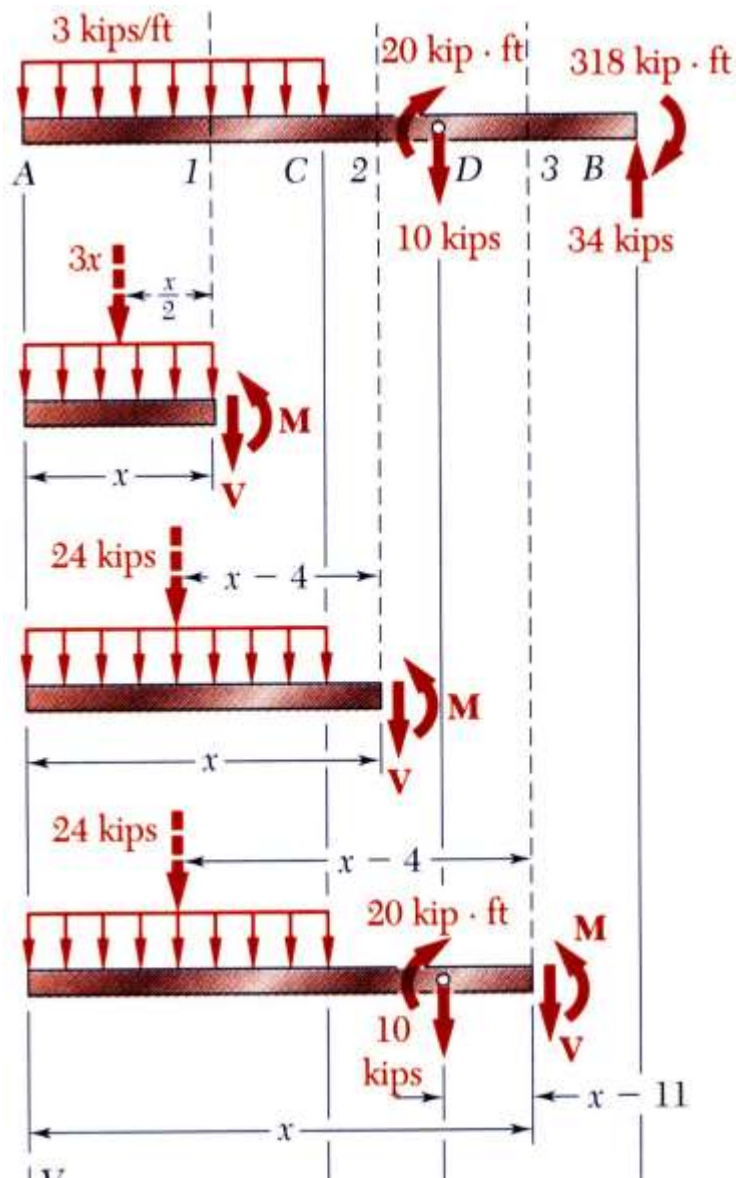


The structure shown is constructed of a W10x112 rolled-steel beam. (a) Draw the shear and bending-moment diagrams for the beam and the given loading. (b) determine normal stress in sections just to the right and left of point D .

SOLUTION:

- Replace the 10 kip load with an equivalent force-couple system at D . Find the reactions at B by considering the beam as a rigid body.
- Section the beam at points near the support and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples.
- Apply the elastic flexure formulas to determine the maximum normal stress to the left and right of point D .

Sample Problem 5.2



SOLUTION:

- Replace the 10 kip load with equivalent force-couple system at *D*. Find reactions at *B*.
- Section the beam and apply equilibrium analyses on resulting free-bodies.

From A to C :

$$\sum F_y = 0 \quad -3x - V = 0 \quad V = -3x \text{ kips}$$

$$\sum M_1 = 0 \quad (3x)\left(\frac{1}{2}x\right) + M = 0 \quad M = -1.5x^2 \text{ kip} \cdot \text{ft}$$

From C to D :

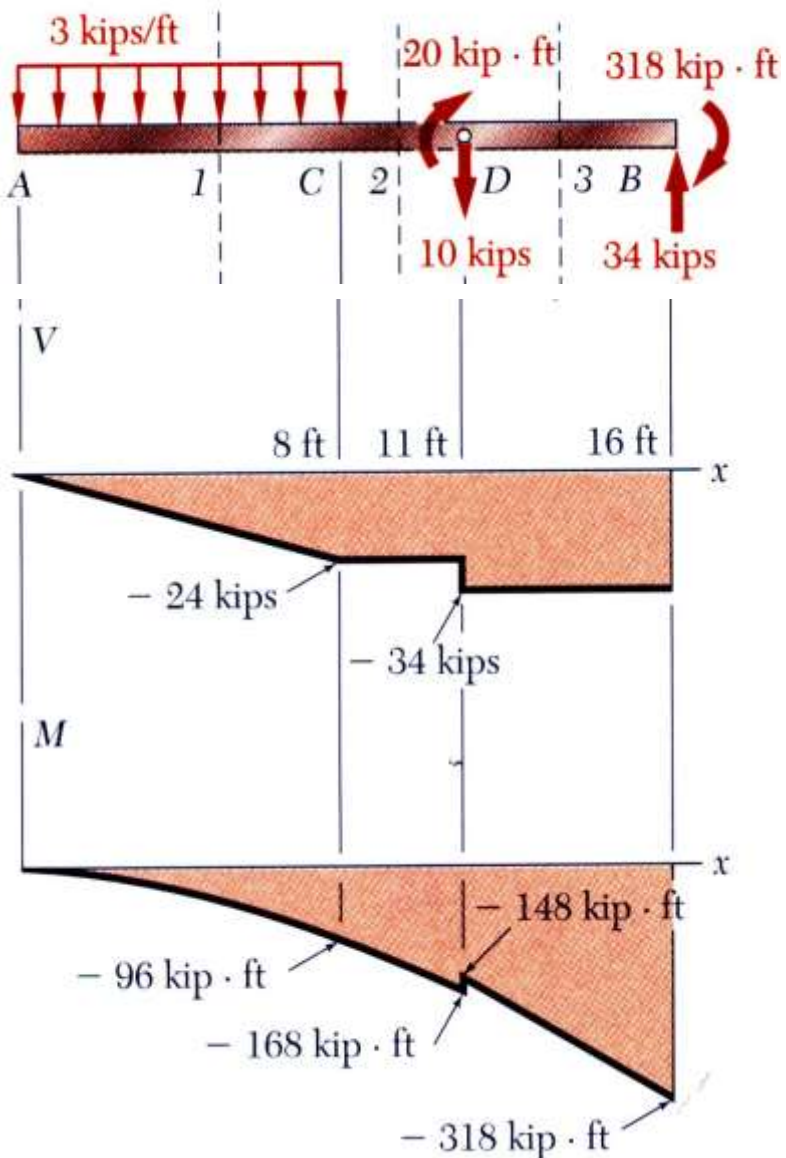
$$\sum F_y = 0 \quad -24 - V = 0 \quad V = -24 \text{ kips}$$

$$\sum M_2 = 0 \quad 24(x - 4) + M = 0 \quad M = (96 - 24x) \text{ kip} \cdot \text{ft}$$

From D to B :

$$V = -34 \text{ kips} \quad M = (226 - 34x) \text{ kip} \cdot \text{ft}$$

Sample Problem 5.2



- Apply the elastic flexure formulas to determine the maximum normal stress to the left and right of point D .

From Appendix C for a W10x112 rolled steel shape, $S = 126 \text{ in}^3$ about the $X-X$ axis.

To the left of D :

$$\sigma_m = \frac{|M|}{S} = \frac{2016 \text{ kip} \cdot \text{in}}{126 \text{ in}^3}$$

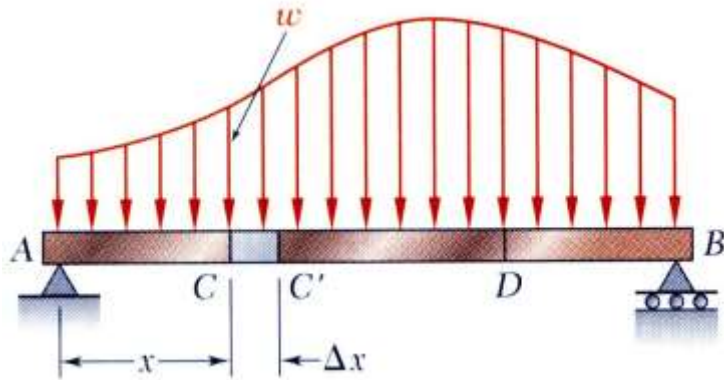
$$\sigma_m = 16.0 \text{ ksi}$$

To the right of D :

$$\sigma_m = \frac{|M|}{S} = \frac{1776 \text{ kip} \cdot \text{in}}{126 \text{ in}^3}$$

$$\sigma_m = 14.1 \text{ ksi}$$

Relations Among Load, Shear, and Bending Moment



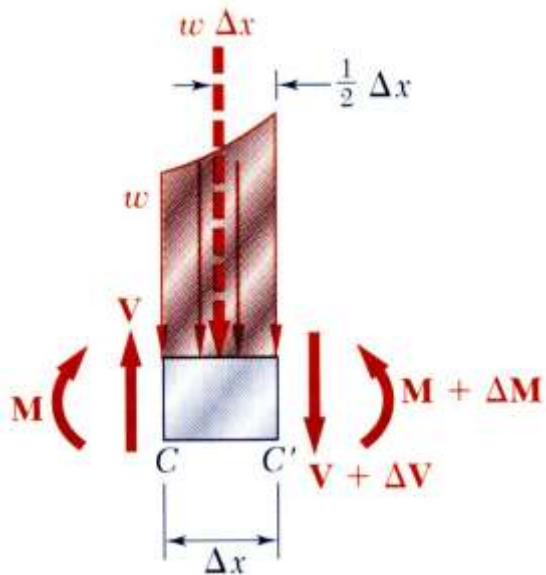
- Relationship between load and shear:

$$\sum F_y = 0: V - (V + \Delta V) - w \Delta x = 0$$

$$\Delta V = -w \Delta x$$

$$\frac{dV}{dx} = -w$$

$$V_D - V_C = - \int_{x_C}^{x_D} w \, dx$$



- Relationship between shear and bending moment:

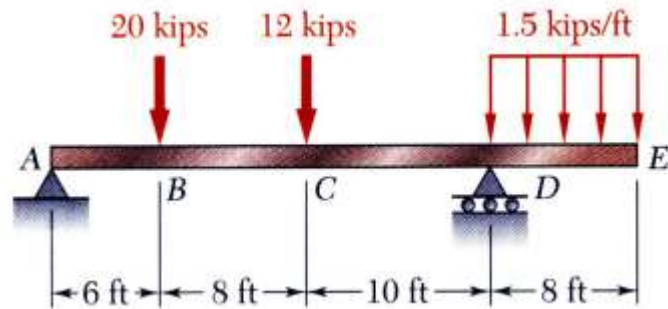
$$\sum M_{C'} = 0: (M + \Delta M) - M - V \Delta x + w \Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V \Delta x - \frac{1}{2} w (\Delta x)^2$$

$$\frac{dM}{dx} = V$$

$$M_D - M_C = \int_{x_C}^{x_D} V \, dx$$

Sample Problem 5.3

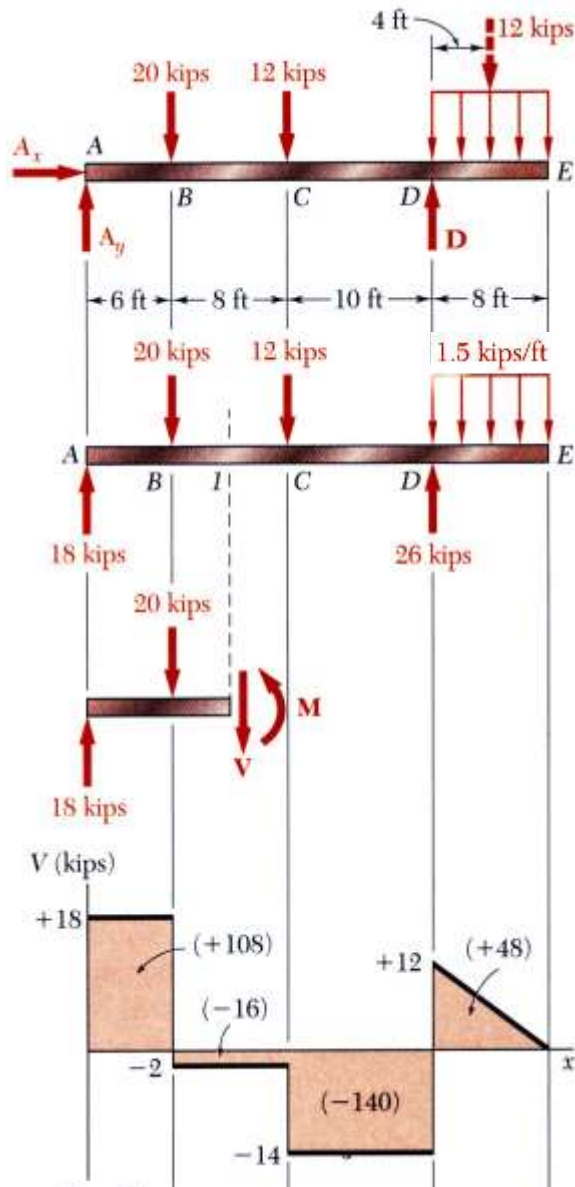


Draw the shear and bending moment diagrams for the beam and loading shown.

SOLUTION:

- Taking the entire beam as a free body, determine the reactions at A and D.
- Apply the relationship between shear and load to develop the shear diagram.
- Apply the relationship between bending moment and shear to develop the bending moment diagram.

Sample Problem 5.3



SOLUTION:

- Taking the entire beam as a free body, determine the reactions at A and D.

$$\sum M_A = 0$$

$$0 = D(24 \text{ ft}) - (20 \text{ kips})(6 \text{ ft}) - (12 \text{ kips})(14 \text{ ft}) - (12 \text{ kips})(28 \text{ ft})$$

$$D = 26 \text{ kips}$$

$$\sum F_y = 0$$

$$0 = A_y - 20 \text{ kips} - 12 \text{ kips} + 26 \text{ kips} - 12 \text{ kips}$$

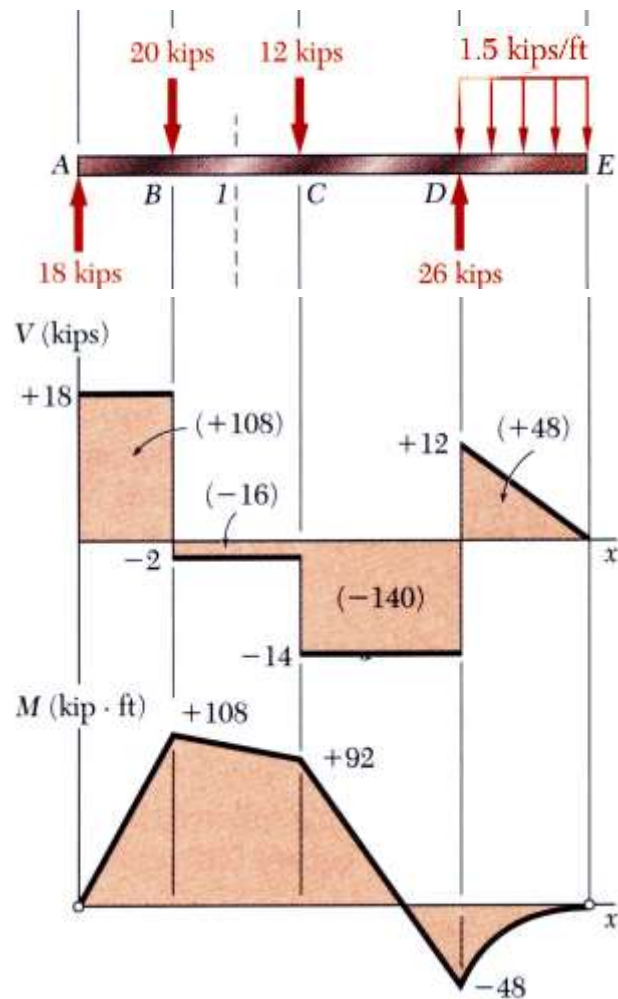
$$A_y = 18 \text{ kips}$$

- Apply the relationship between shear and load to develop the shear diagram.

$$\frac{dV}{dx} = -w \quad dV = -w dx$$

- zero slope between concentrated loads
- linear variation over uniform load segment

Sample Problem 5.3

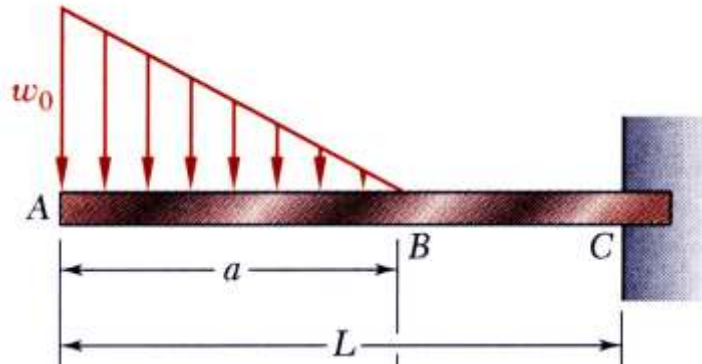


- Apply the relationship between bending moment and shear to develop the bending moment diagram.

$$\frac{dM}{dx} = V \quad dM = V dx$$

- bending moment at A and E is zero
- bending moment variation between A, B, C and D is linear
- bending moment variation between D and E is quadratic
- net change in bending moment is equal to areas under shear distribution segments
- total of all bending moment changes across the beam should be zero

Sample Problem 5.5



SOLUTION:

- Taking the entire beam as a free body, determine the reactions at C .
- Apply the relationship between shear and load to develop the shear diagram.
- Apply the relationship between bending moment and shear to develop the bending moment diagram.

Draw the shear and bending moment diagrams for the beam and loading shown.

Sample Problem 5.5

SOLUTION:

- Taking the entire beam as a free body, determine the reactions at C.

$$\Sigma F_y = 0 = -\frac{1}{2}w_0a + R_C \quad R_C = \frac{1}{2}w_0a$$

$$\Sigma M_C = 0 = \frac{1}{2}w_0a\left(L - \frac{a}{3}\right) + M_C \quad M_C = -\frac{1}{2}w_0a\left(L - \frac{a}{3}\right)$$

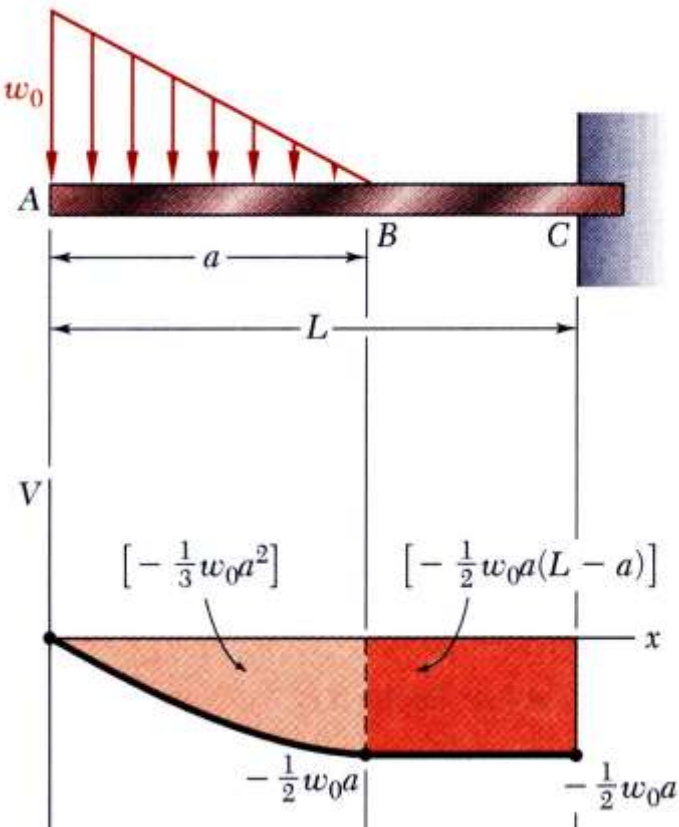
Results from integration of the load and shear distributions should be equivalent.

- Apply the relationship between shear and load to develop the shear diagram.

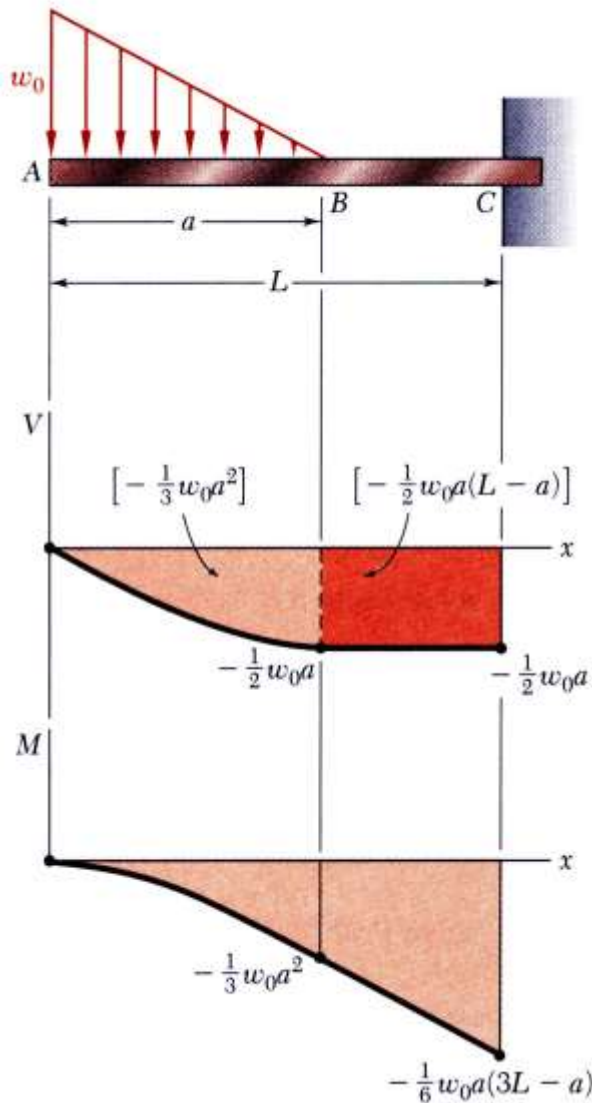
$$V_B - V_A = -\int_0^a w_0\left(1 - \frac{x}{a}\right)dx = -\left[w_0\left(x - \frac{x^2}{2a}\right)\right]_0^a$$

$$V_B = -\frac{1}{2}w_0a = -(\text{area under load curve})$$

- No change in shear between B and C.
- Compatible with free body analysis



Sample Problem 5.5



- Apply the relationship between bending moment and shear to develop the bending moment diagram.

$$M_B - M_A = \int_0^a \left(-w_0 \left(x - \frac{x^2}{2a} \right) \right) dx = \left[-w_0 \left(\frac{x^2}{2} - \frac{x^3}{6a} \right) \right]_0^a$$

$$M_B = -\frac{1}{3} w_0 a^2$$

$$M_B - M_C = \int_a^L \left(-\frac{1}{2} w_0 a \right) dx = -\frac{1}{2} w_0 a (L - a)$$

$$M_C = -\frac{1}{6} w_0 a (3L - a) = \frac{a w_0}{2} \left(L - \frac{a}{3} \right)$$

Results at C are compatible with free-body analysis

Design of Prismatic Beams for Bending

- The largest normal stress is found at the surface where the maximum bending moment occurs.

$$\sigma_m = \frac{|M|_{\max} c}{I} = \frac{|M|_{\max}}{S}$$

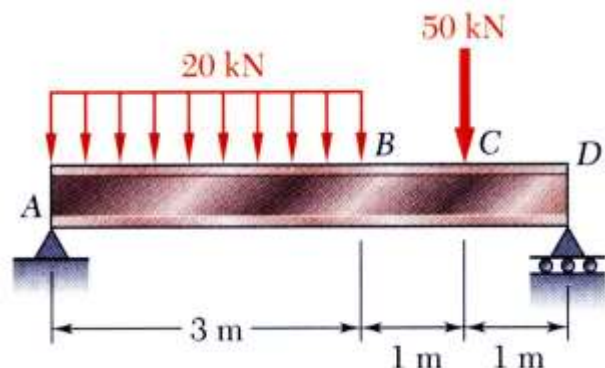
- A safe design requires that the maximum normal stress be less than the allowable stress for the material used. This criteria leads to the determination of the minimum acceptable section modulus.

$$\sigma_m \leq \sigma_{all}$$

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}}$$

- Among beam section choices which have an acceptable section modulus, the one with the smallest weight per unit length or cross sectional area will be the least expensive and the best choice.

Sample Problem 5.8

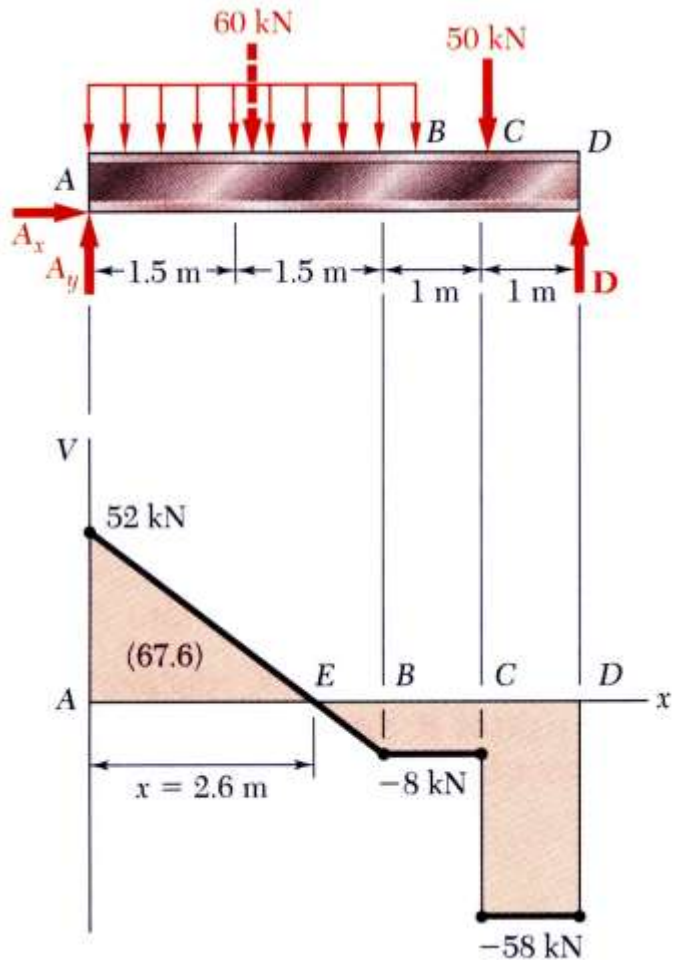


A simply supported steel beam is to carry the distributed and concentrated loads shown. Knowing that the allowable normal stress for the grade of steel to be used is 160 MPa, select the wide-flange shape that should be used.

SOLUTION:

- Considering the entire beam as a free-body, determine the reactions at A and D.
- Develop the shear diagram for the beam and load distribution. From the diagram, determine the maximum bending moment.
- Determine the minimum acceptable beam section modulus. Choose the best standard section which meets this criteria.

Sample Problem 5.8



- Considering the entire beam as a free-body, determine the reactions at A and D.

$$\sum M_A = 0 = D(5\text{ m}) - (60\text{ kN})(1.5\text{ m}) - (50\text{ kN})(4\text{ m})$$

$$D = 58.0\text{ kN}$$

$$\sum F_y = 0 = A_y + 58.0\text{ kN} - 60\text{ kN} - 50\text{ kN}$$

$$A_y = 52.0\text{ kN}$$

- Develop the shear diagram and determine the maximum bending moment.

$$V_A = A_y = 52.0\text{ kN}$$

$$V_B - V_A = -(\text{area under load curve}) = -60\text{ kN}$$

$$V_B = -8\text{ kN}$$

- Maximum bending moment occurs at $V = 0$ or $x = 2.6\text{ m}$.

$$\begin{aligned} |M|_{\max} &= (\text{area under shear curve, A to E}) \\ &= 67.6\text{ kN} \end{aligned}$$

Sample Problem 5.8

- Determine the minimum acceptable beam section modulus.

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}} = \frac{67.6 \text{ kN} \cdot \text{m}}{160 \text{ MPa}}$$

$$= 422.5 \times 10^{-6} \text{ m}^3 = 422.5 \times 10^3 \text{ mm}^3$$

- Choose the best standard section which meets this criteria.

Shape	S, mm^3
W410 × 38.8	637
W360 × 32.9	474
W310 × 38.7	549
W250 × 44.8	535
W200 × 46.1	448

W360 × 32.9



CHAPTER

6

MECHANICS OF MATERIALS

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Shearing Stresses in Beams and Thin- Walled Members

Shearing Stresses in Beams and Thin-Walled Members

[Introduction](#)

[Shear on the Horizontal Face of a Beam Element](#)

[Example 6.01](#)

[Determination of the Shearing Stress in a Beam](#)

[Shearing Stresses \$\tau_{xy}\$ in Common Types of Beams](#)

[Further Discussion of the Distribution of Stresses in a ...](#)

[Sample Problem 6.2](#)

[Longitudinal Shear on a Beam Element of Arbitrary Shape](#)

[Example 6.04](#)

[Shearing Stresses in Thin-Walled Members](#)

[Plastic Deformations](#)

[Sample Problem 6.3](#)

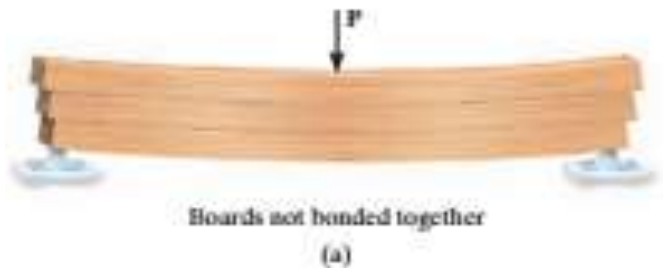
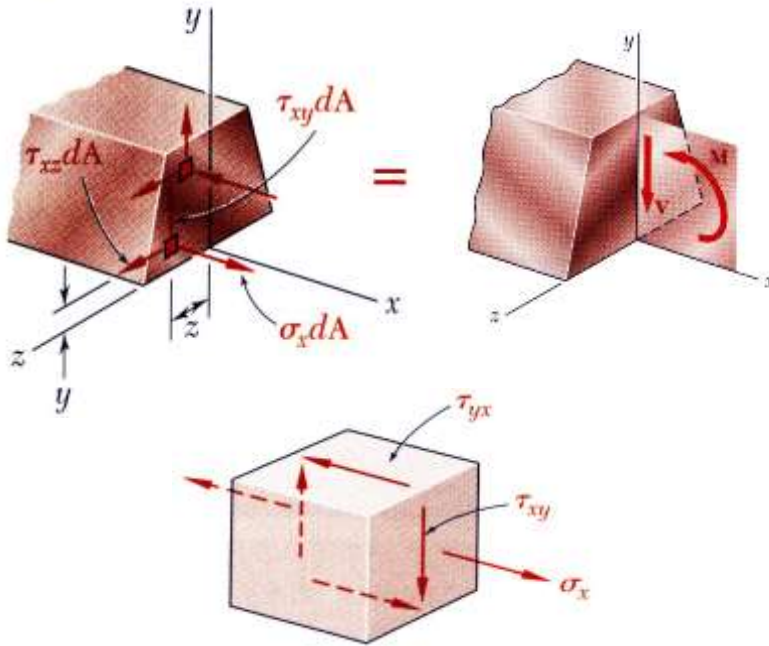
[Unsymmetric Loading of Thin-Walled Members](#)

[Example 6.05](#)

[Example 6.06](#)



Introduction



- Transverse loading applied to a beam results in normal and shearing stresses in transverse sections.

- Distribution of normal and shearing stresses satisfies

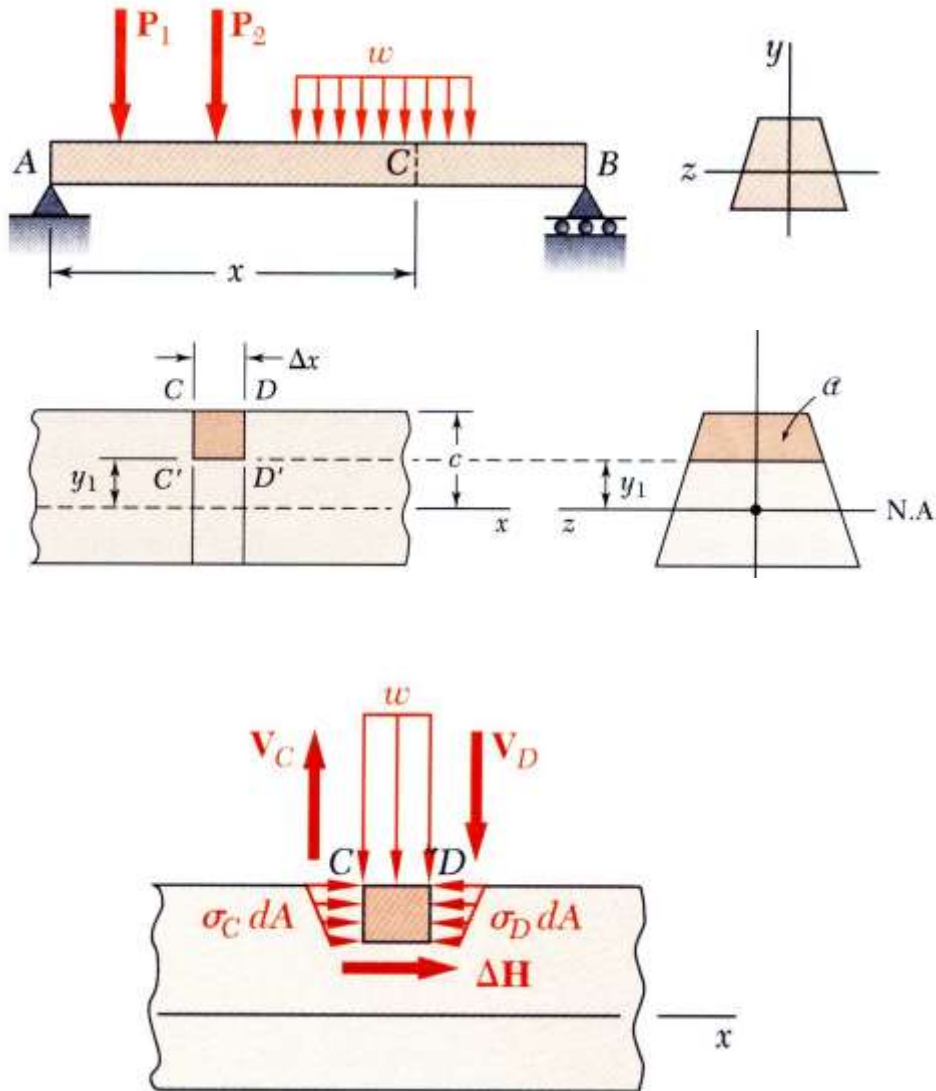
$$F_x = \int \sigma_x dA = 0 \quad M_x = \int (y \tau_{xz} - z \tau_{xy}) dA = 0$$

$$F_y = \int \tau_{xy} dA = -V \quad M_y = \int z \sigma_x dA = 0$$

$$F_z = \int \tau_{xz} dA = 0 \quad M_z = \int (-y \sigma_x) dA = 0$$

- When shearing stresses are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces
- Longitudinal shearing stresses must exist in any member subjected to transverse loading.

Shear on the Horizontal Face of a Beam Element



- Consider prismatic beam
- For equilibrium of beam element

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_C) dA$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y dA$$

- Note,

$$Q = \int_A y dA$$

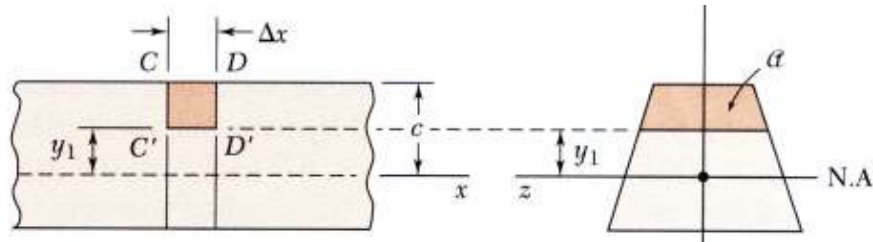
$$M_D - M_C = \frac{dM}{dx} \Delta x = V \Delta x$$

- Substituting,

$$\Delta H = \frac{VQ}{I} \Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow}$$

Shear on the Horizontal Face of a Beam Element



- Shear flow,

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = \text{shear flow}$$

- where

$$Q = \int_A y dA$$

= first moment of area above y_1

$$I = \int_{A+A'} y^2 dA$$

= second moment of full cross section

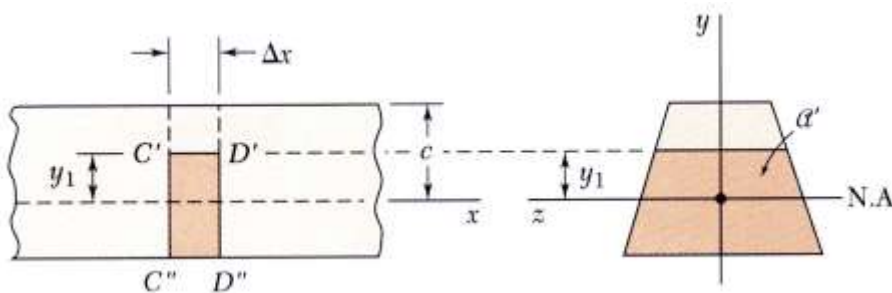
- Same result found for lower area

$$q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = -q'$$

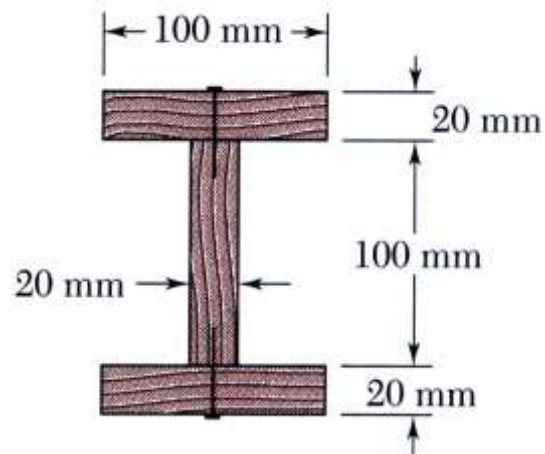
$$Q + Q' = 0$$

= first moment with respect to neutral axis

$$\Delta H' = -\Delta H$$



Example 6.01



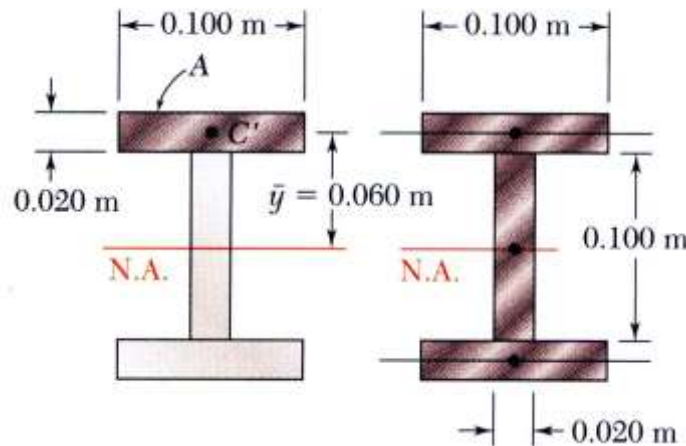
SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.
- Calculate the corresponding shear force in each nail.

A beam is made of three planks, nailed together. Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is $V = 500$ N, determine the shear force in each nail.



Example 6.01



$$\begin{aligned}
 Q &= A\bar{y} \\
 &= (0.020\text{ m} \times 0.100\text{ m})(0.060\text{ m}) \\
 &= 120 \times 10^{-6}\text{ m}^3 \\
 I &= \frac{1}{12}(0.020\text{ m})(0.100\text{ m})^3 \\
 &\quad + 2\left[\frac{1}{12}(0.100\text{ m})(0.020\text{ m})^3\right. \\
 &\quad \left.+ (0.020\text{ m} \times 0.100\text{ m})(0.060\text{ m})^2\right] \\
 &= 16.20 \times 10^{-6}\text{ m}^4
 \end{aligned}$$

SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.

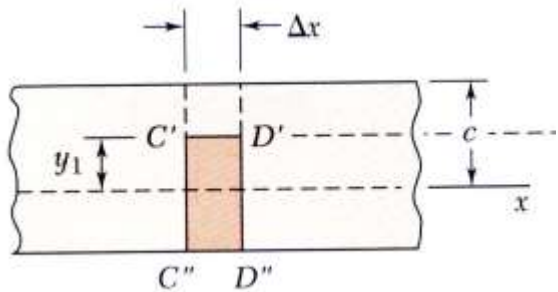
$$\begin{aligned}
 q &= \frac{VQ}{I} = \frac{(500\text{ N})(120 \times 10^{-6}\text{ m}^3)}{16.20 \times 10^{-6}\text{ m}^4} \\
 &= 3704\text{ N/m}
 \end{aligned}$$

- Calculate the corresponding shear force in each nail for a nail spacing of 25 mm.

$$F = (0.025\text{ m})q = (0.025\text{ m})(3704\text{ N/m})$$

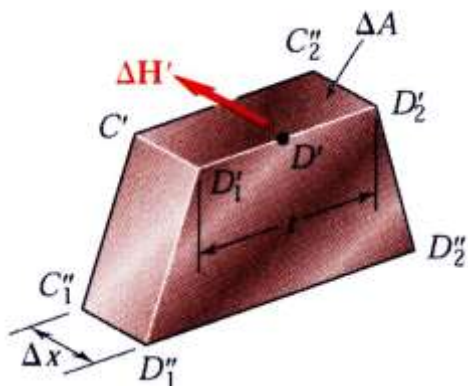
$$F = 92.6\text{ N}$$

Determination of the Shearing Stress in a Beam

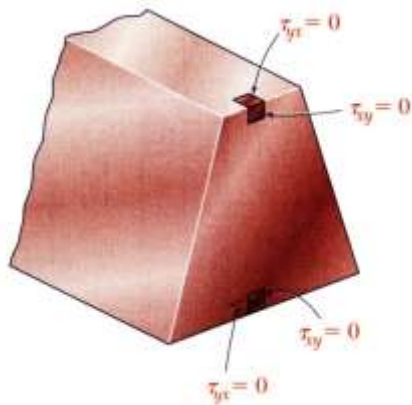


- The *average* shearing stress on the horizontal face of the element is obtained by dividing the shearing force on the element by the area of the face.

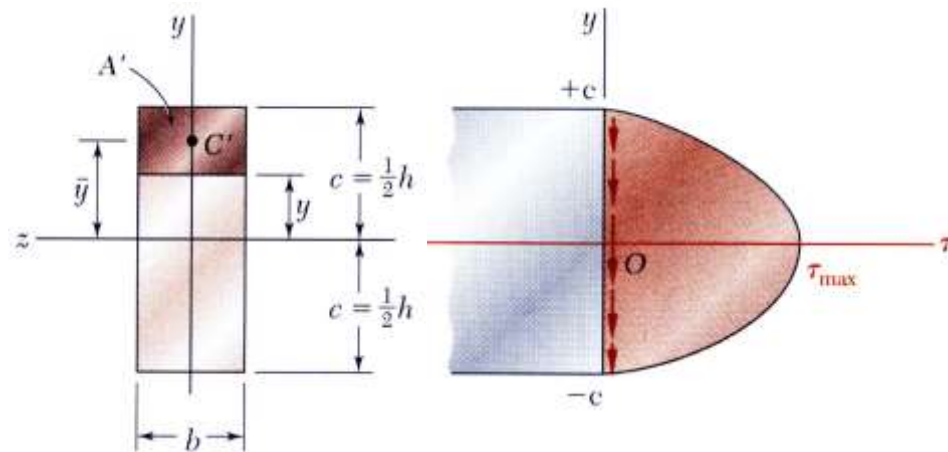
$$\begin{aligned}\tau_{ave} &= \frac{\Delta H}{\Delta A} = \frac{q \Delta x}{\Delta A} = \frac{VQ}{I} \frac{\Delta x}{t \Delta x} \\ &= \frac{VQ}{It}\end{aligned}$$



- On the upper and lower surfaces of the beam, $\tau_{yx} = 0$. It follows that $\tau_{xy} = 0$ on the upper and lower edges of the transverse sections.
- If the width of the beam is comparable or large relative to its depth, the shearing stresses at D_1 and D_2 are significantly higher than at D .



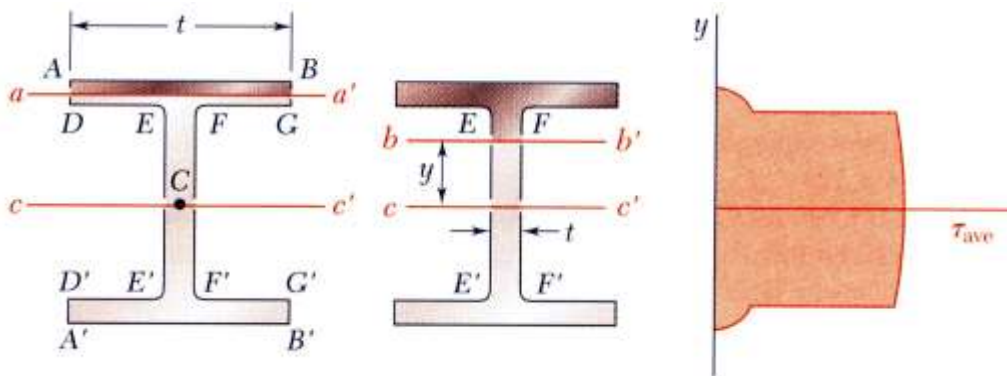
Shearing Stresses τ_{xy} in Common Types of Beams



- For a narrow rectangular beam,

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3V}{2A} \left(1 - \frac{y^2}{c^2} \right)$$

$$\tau_{max} = \frac{3V}{2A}$$

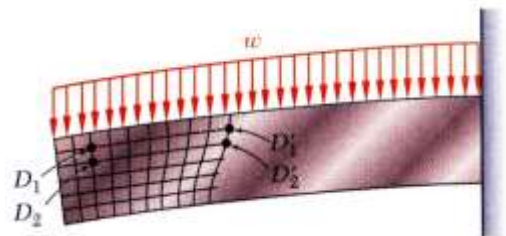
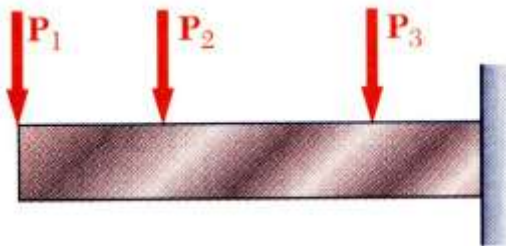
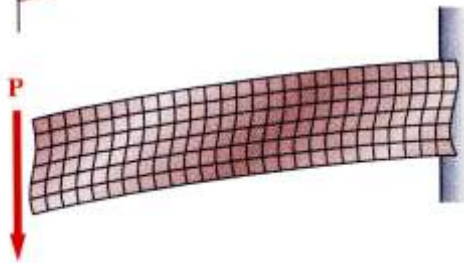
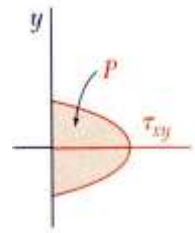


- For American Standard (S-beam) and wide-flange (W-beam) beams

$$\tau_{ave} = \frac{VQ}{It}$$

$$\tau_{max} = \frac{V}{A_{web}}$$

Further Discussion of the Distribution of Stresses in a Narrow Rectangular Beam

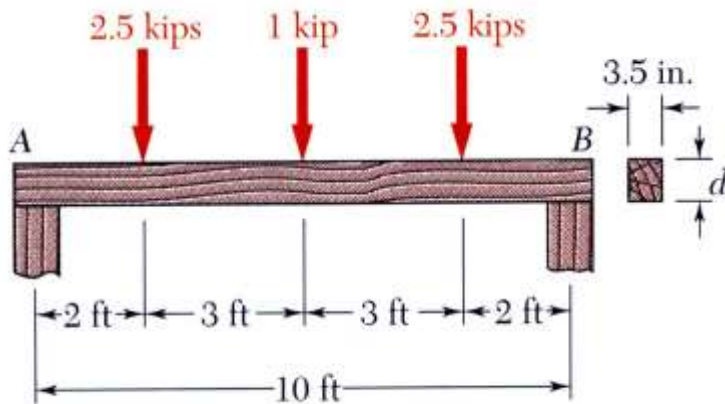


- Consider a narrow rectangular cantilever beam subjected to load P at its free end:

$$\tau_{xy} = \frac{3P}{2A} \left(1 - \frac{y^2}{c^2} \right) \quad \sigma_x = + \frac{Pxy}{I}$$

- Shearing stresses are independent of the distance from the point of application of the load.
- Normal strains and normal stresses are unaffected by the shearing stresses.
- From Saint-Venant's principle, effects of the load application mode are negligible except in immediate vicinity of load application points.
- Stress/strain deviations for distributed loads are negligible for typical beam sections of interest.

Sample Problem 6.2



A timber beam is to support the three concentrated loads shown. Knowing that for the grade of timber used,

$$\sigma_{all} = 1800 \text{ psi} \quad \tau_{all} = 120 \text{ psi}$$

determine the minimum required depth d of the beam.

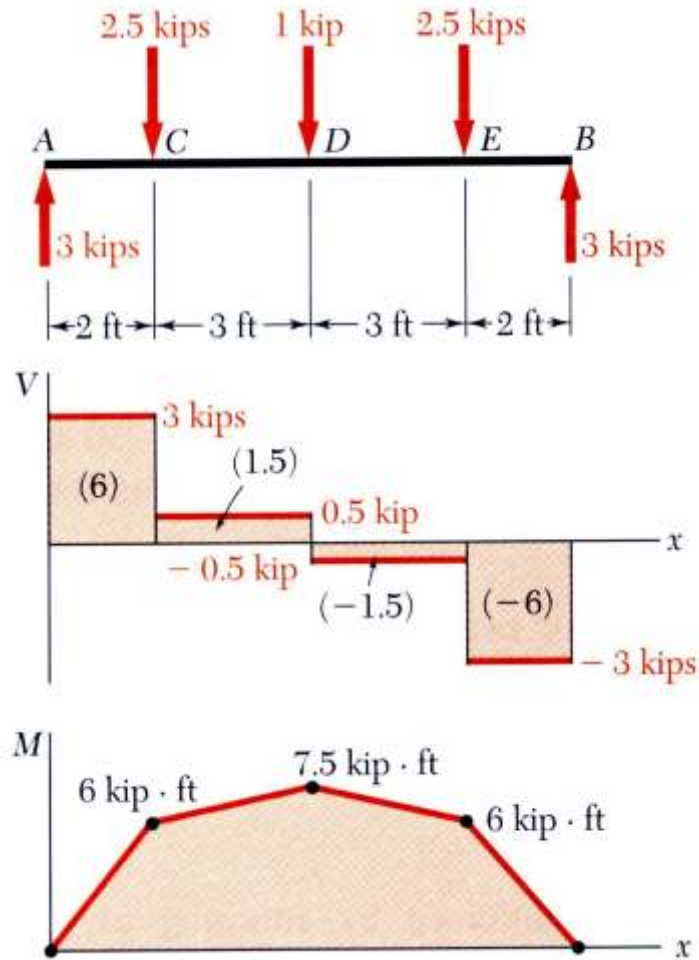
SOLUTION:

- Develop shear and bending moment diagrams. Identify the maximums.
- Determine the beam depth based on allowable normal stress.
- Determine the beam depth based on allowable shear stress.
- Required beam depth is equal to the larger of the two depths found.

Sample Problem 6.2

SOLUTION:

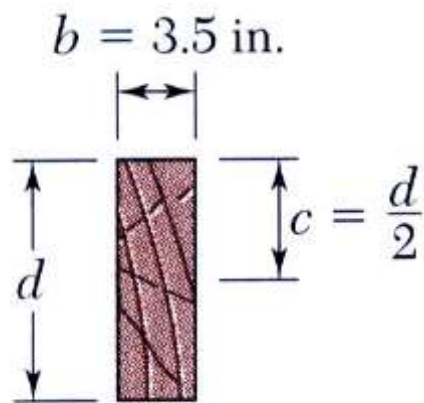
Develop shear and bending moment diagrams. Identify the maximums.



$$V_{\max} = 3 \text{ kips}$$

$$M_{\max} = 7.5 \text{ kip} \cdot \text{ft} = 90 \text{ kip} \cdot \text{in}$$

Sample Problem 6.2



- Determine the beam depth based on allowable normal stress.

$$\sigma_{all} = \frac{M_{max}}{S}$$

$$1800 \text{ psi} = \frac{90 \times 10^3 \text{ lb} \cdot \text{in.}}{(0.5833 \text{ in.})d^2}$$

$$d = 9.26 \text{ in.}$$

- Determine the beam depth based on allowable shear stress.

$$\tau_{all} = \frac{3}{2} \frac{V_{max}}{A}$$

$$120 \text{ psi} = \frac{3}{2} \frac{3000 \text{ lb}}{(3.5 \text{ in.})d}$$

$$d = 10.71 \text{ in.}$$

- Required beam depth is equal to the larger of the two.

$$d = 10.71 \text{ in.}$$

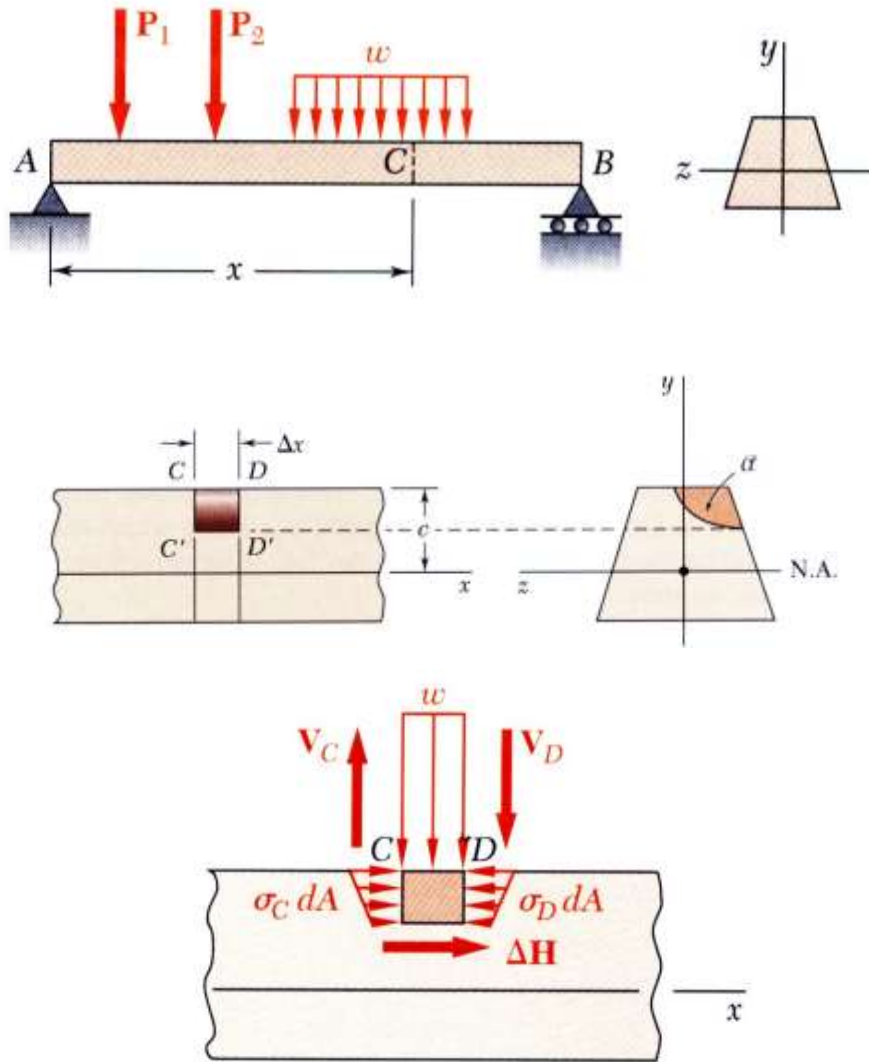
$$I = \frac{1}{12} b d^3$$

$$S = \frac{I}{c} = \frac{1}{6} b d^2$$

$$= \frac{1}{6} (3.5 \text{ in.}) d^2$$

$$= (0.5833 \text{ in.}) d^2$$

Longitudinal Shear on a Beam Element of Arbitrary Shape



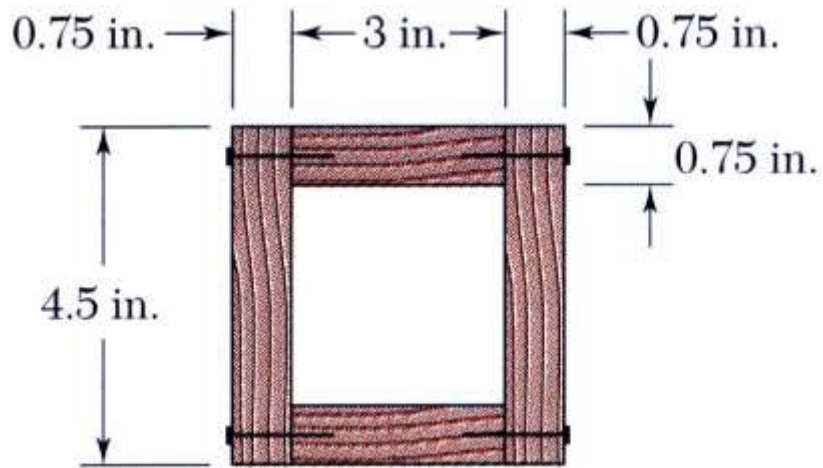
- We have examined the distribution of the vertical components τ_{xy} on a transverse section of a beam. We now wish to consider the horizontal components τ_{xz} of the stresses.
- Consider prismatic beam with an element defined by the curved surface CDD'C'.

$$\sum F_x = 0 = \Delta H + \int_a (\sigma_D - \sigma_C) dA$$

- Except for the differences in integration areas, this is the same result obtained before which led to

$$\Delta H = \frac{VQ}{I} \Delta x \quad q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I}$$

Example 6.04

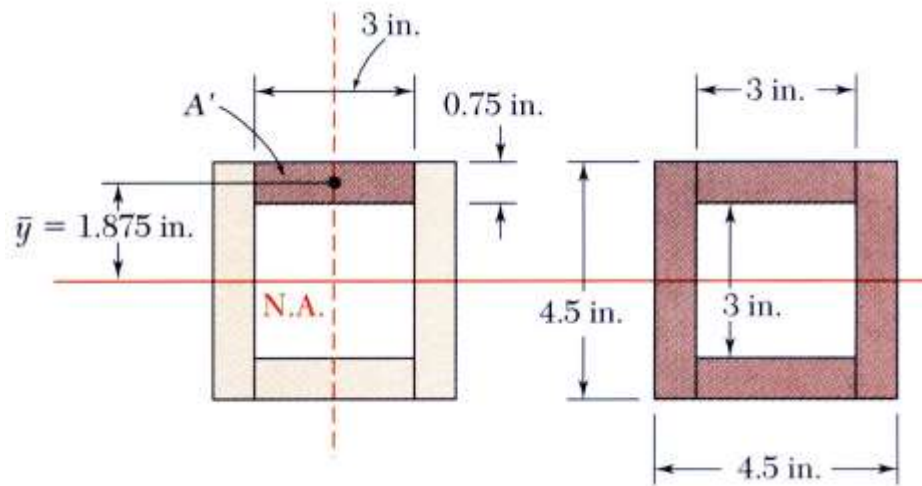


SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.
- Based on the spacing between nails, determine the shear force in each nail.

A square box beam is constructed from four planks as shown. Knowing that the spacing between nails is 1.5 in. and the beam is subjected to a vertical shear of magnitude $V = 600$ lb, determine the shearing force in each nail.

Example 6.04



For the upper plank,

$$Q = A'y = (0.75\text{ in.})(3\text{ in.})(1.875\text{ in.}) = 4.22\text{ in}^3$$

For the overall beam cross-section,

$$I = \frac{1}{12}(4.5\text{ in})^3 - \frac{1}{12}(3\text{ in})^3 = 27.42\text{ in}^4$$

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.

$$q = \frac{VQ}{I} = \frac{(600\text{ lb})(4.22\text{ in}^3)}{27.42\text{ in}^4} = 92.3 \frac{\text{lb}}{\text{in}}$$

$$f = \frac{q}{2} = 46.15 \frac{\text{lb}}{\text{in}}$$

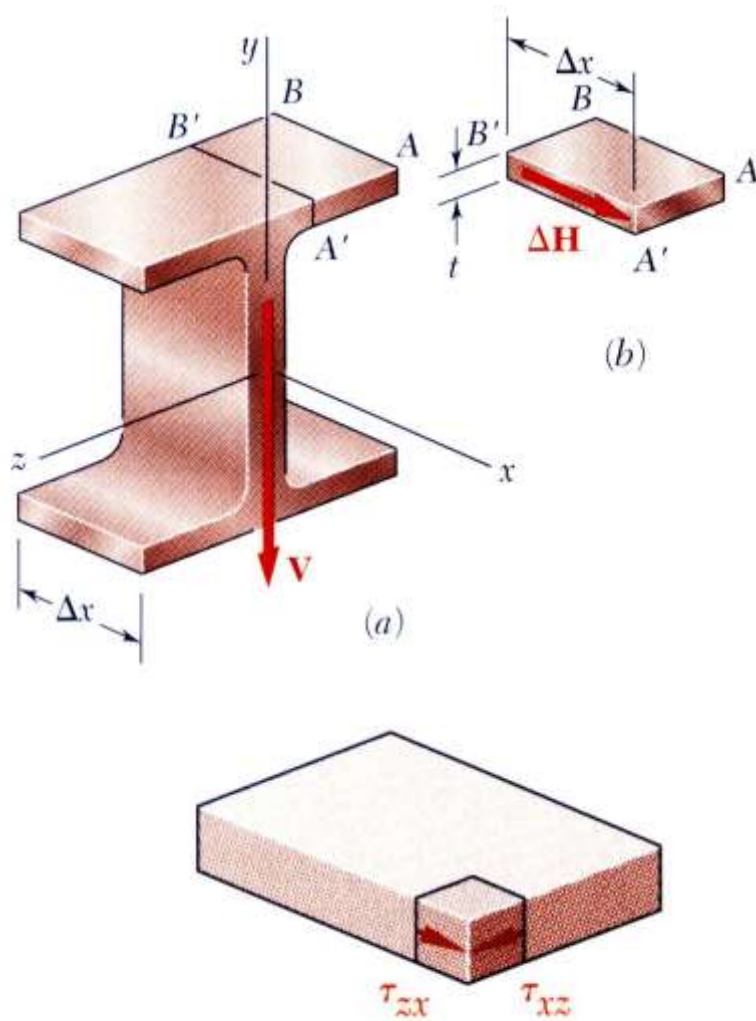
= edge force per unit length

- Based on the spacing between nails, determine the shear force in each nail.

$$F = f \ell = \left(46.15 \frac{\text{lb}}{\text{in}} \right) (1.75\text{ in})$$

$$F = 80.8\text{ lb}$$

Shearing Stresses in Thin-Walled Members



- Consider a segment of a wide-flange beam subjected to the vertical shear V .
- The longitudinal shear force on the element is

$$\Delta H = \frac{VQ}{I} \Delta x$$

- The corresponding shear stress is

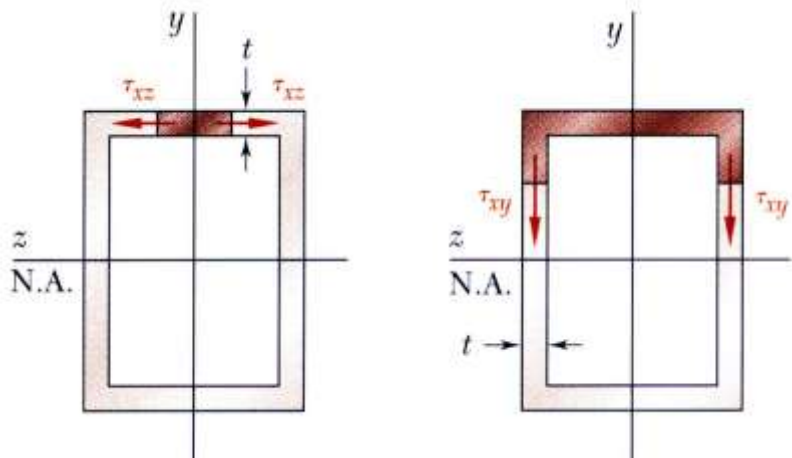
$$\tau_{zx} = \tau_{xz} \approx \frac{\Delta H}{t \Delta x} = \frac{VQ}{It}$$

- Previously found a similar expression for the shearing stress in the web

$$\tau_{xy} = \frac{VQ}{It}$$

- NOTE: $\tau_{xy} \approx 0$ in the flanges
 $\tau_{xz} \approx 0$ in the web

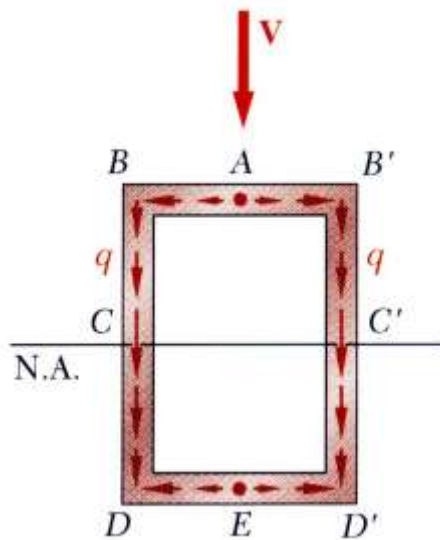
Shearing Stresses in Thin-Walled Members



- The variation of shear flow across the section depends only on the variation of the first moment.

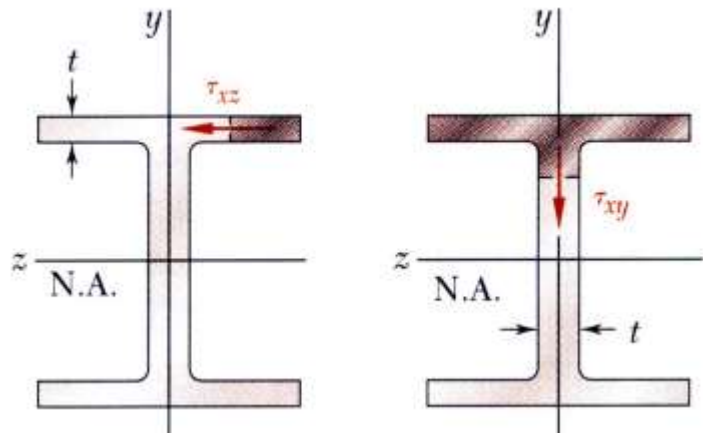
$$q = \tau t = \frac{VQ}{I}$$

- For a box beam, q grows smoothly from zero at A to a maximum at C and C' and then decreases back to zero at E .

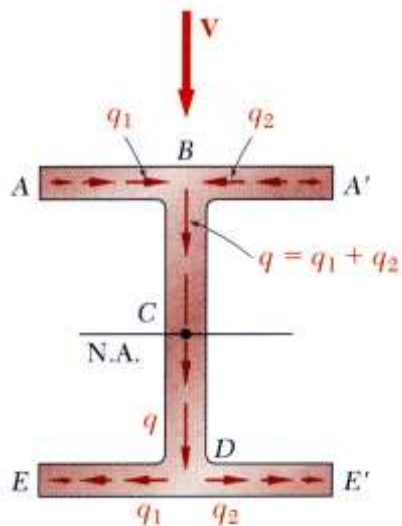


- The sense of q in the horizontal portions of the section may be deduced from the sense in the vertical portions or the sense of the shear V .

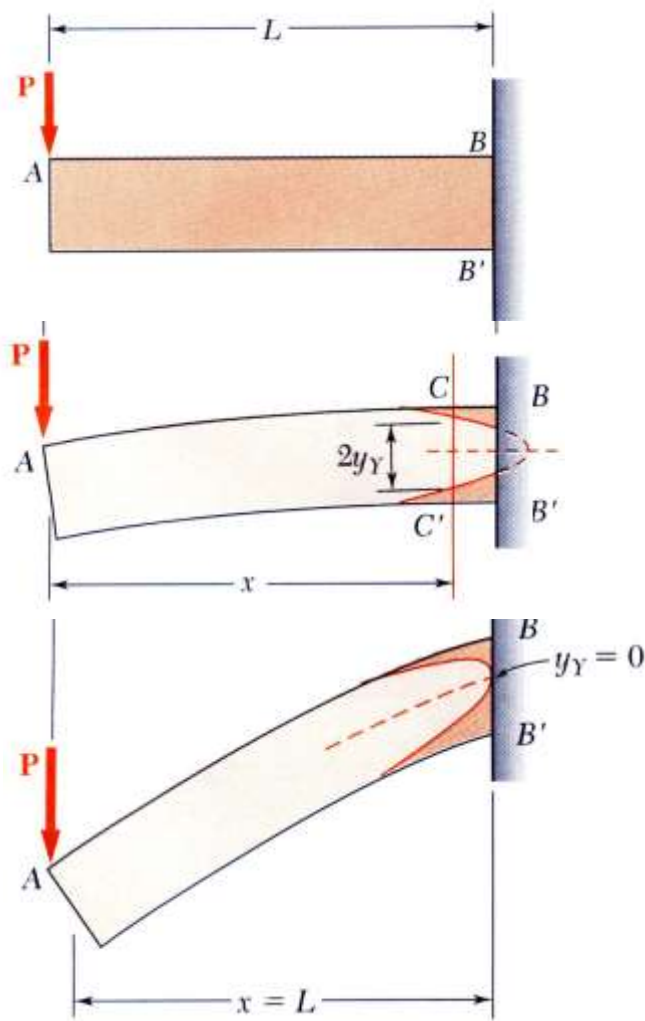
Shearing Stresses in Thin-Walled Members



- For a wide-flange beam, the shear flow increases symmetrically from zero at A and A' , reaches a maximum at C and the decreases to zero at E and E' .
- The continuity of the variation in q and the merging of q from section branches suggests an analogy to fluid flow.



Plastic Deformations



- Recall: $M_Y = \frac{I}{c} \sigma_Y = \text{maximum elastic moment}$
- For $M = PL < M_Y$, the normal stress does not exceed the yield stress anywhere along the beam.
- For $PL > M_Y$, yield is initiated at B and B' . For an elastoplastic material, the half-thickness of the elastic core is found from

$$Px = \frac{3}{2} M_Y \left(1 - \frac{1}{3} \frac{y_Y^2}{c^2} \right)$$

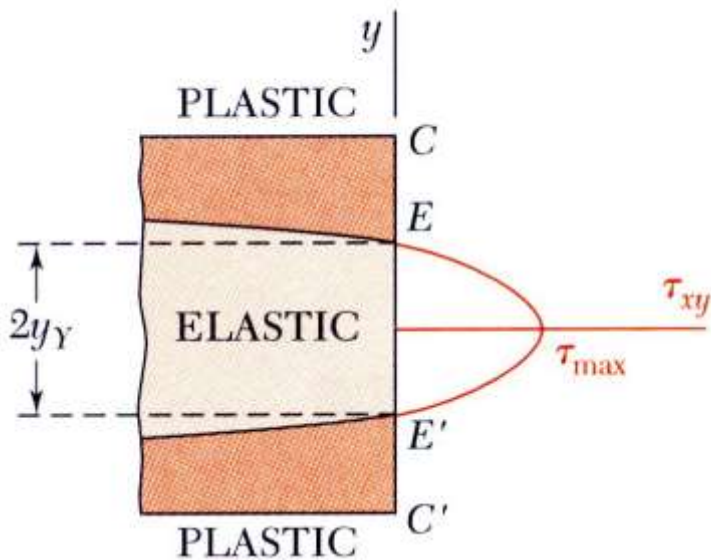
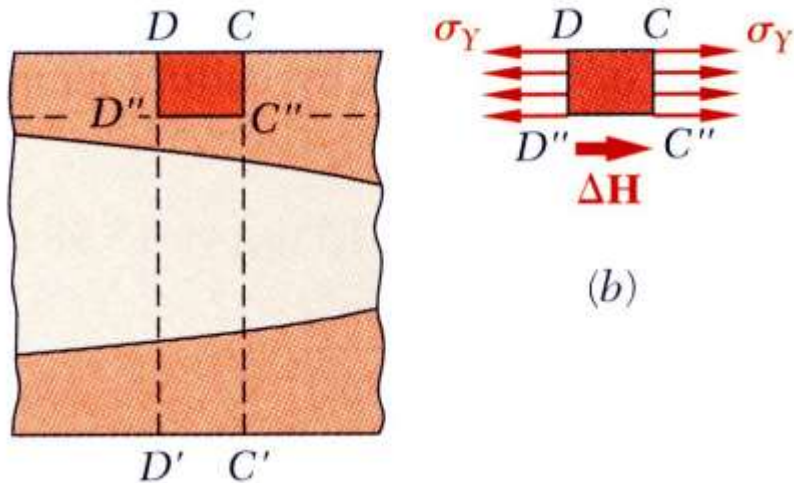
- The section becomes fully plastic ($y_Y = 0$) at the wall when

$$PL = \frac{3}{2} M_Y = M_p$$

- Maximum load which the beam can support is

$$P_{\max} = \frac{M_p}{L}$$

Plastic Deformations



- Preceding discussion was based on normal stresses only
- Consider horizontal shear force on an element within the plastic zone,

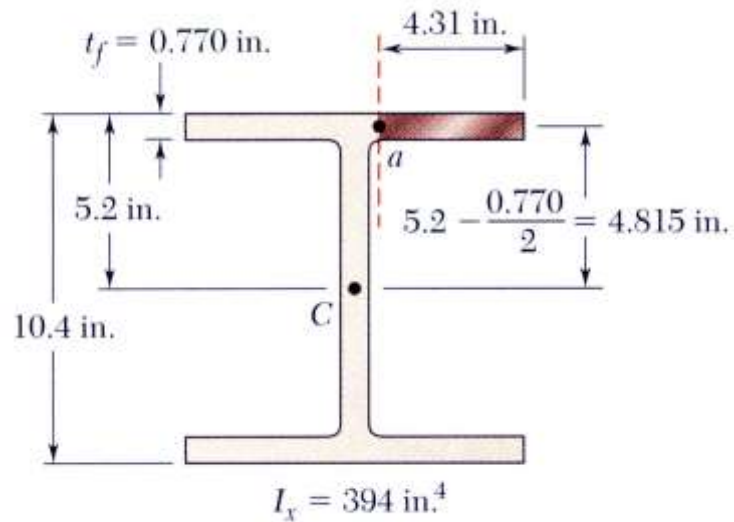
$$\Delta H = -(\sigma_C - \sigma_D)dA = -(\sigma_Y - \sigma_Y)dA = 0$$

Therefore, the shear stress is zero in the plastic zone.
- Shear load is carried by the elastic core,

$$\tau_{xy} = \frac{3}{2} \frac{P}{A'} \left(1 - \frac{y^2}{y_Y^2} \right) \quad \text{where } A' = 2by_Y$$

$$\tau_{\max} = \frac{3}{2} \frac{P}{A'}$$
- As A' decreases, $\tau_{\max}$ increases and may exceed τ_Y

Sample Problem 6.3



Knowing that the vertical shear is 50 kips in a W10x68 rolled-steel beam, determine the horizontal shearing stress in the top flange at the point a .

SOLUTION:

- For the shaded area,

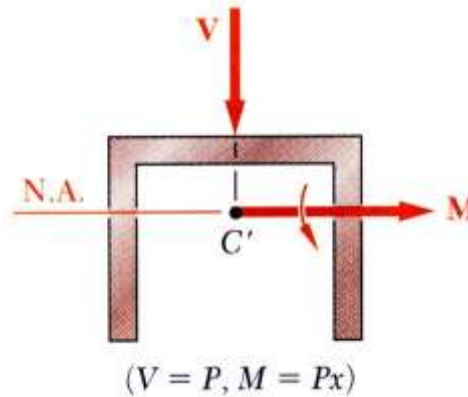
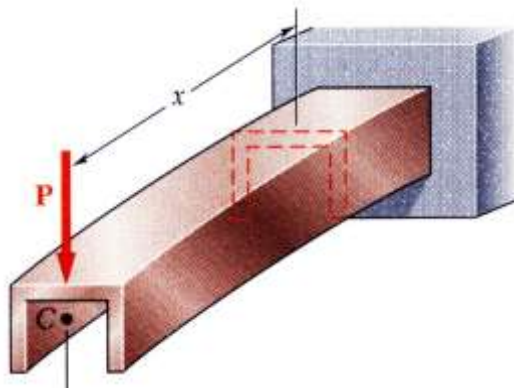
$$Q = (4.31 \text{ in})(0.770 \text{ in})(4.815 \text{ in}) = 15.98 \text{ in}^3$$

- The shear stress at a ,

$$\tau = \frac{VQ}{It} = \frac{(50 \text{ kips})(15.98 \text{ in}^3)}{(394 \text{ in}^4)(0.770 \text{ in})}$$

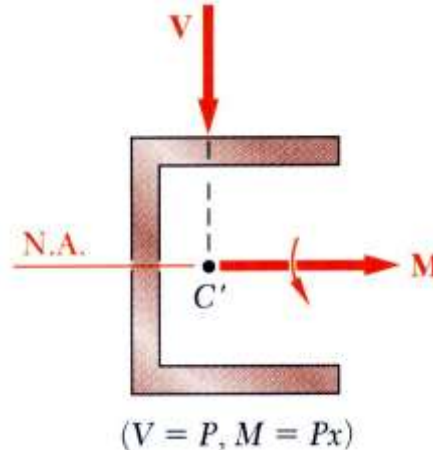
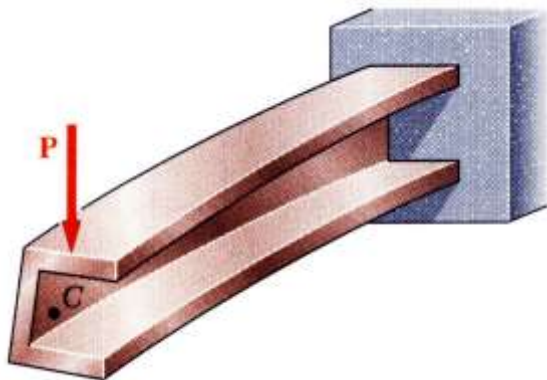
$$\tau = 2.63 \text{ ksi}$$

Unsymmetric Loading of Thin-Walled Members



- Beam loaded in a vertical plane of symmetry deforms in the symmetry plane without twisting.

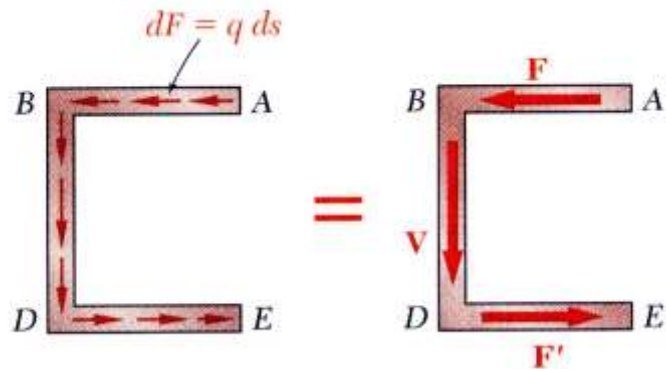
$$\sigma_x = -\frac{My}{I} \quad \tau_{ave} = \frac{VQ}{It}$$



- Beam without a vertical plane of symmetry bends and twists under loading.

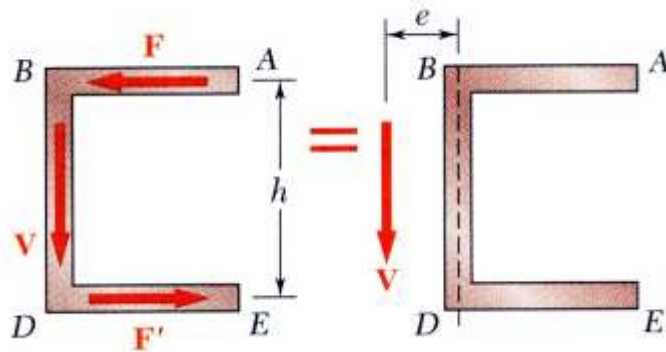
$$\sigma_x = -\frac{My}{I} \quad \tau_{ave} \neq \frac{VQ}{It}$$

Unsymymmetric Loading of Thin-Walled Members



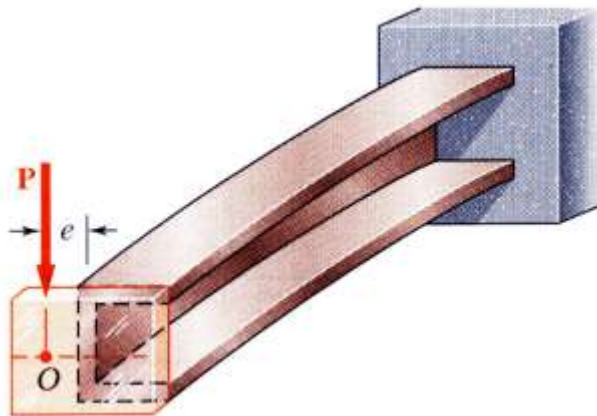
- If the shear load is applied such that the beam does not twist, then the shear stress distribution satisfies

$$\tau_{ave} = \frac{VQ}{It} \quad V = \int_B^D q ds \quad F = \int_A^B q ds = - \int_D^E q ds = -F'$$



- F and F' indicate a couple Fh and the need for the application of a torque as well as the shear load.

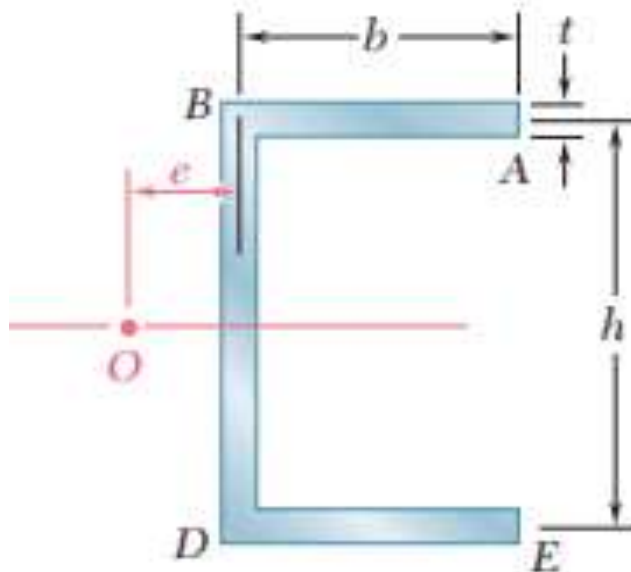
$$Fh = Ve$$



- When the force P is applied at a distance e to the left of the web centerline, the member bends in a vertical plane without twisting.

Example 6.05

Determine the shear center O of a channel section of uniform thickness (Fig. 6.48), knowing that $b = 4$ in., $h = 6$ in., and $t = 0.15$ in.



Example 6.05

Solution

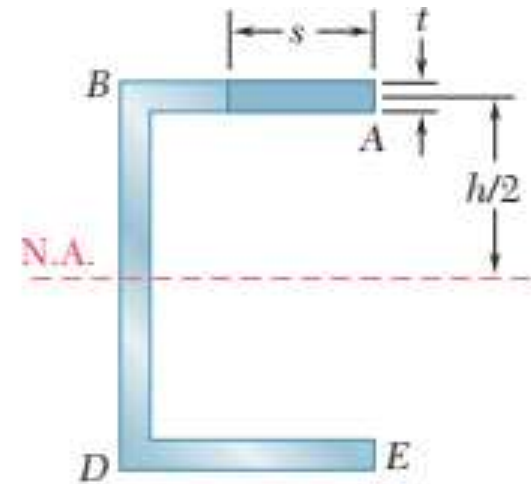
Assuming that the member does not twist, we first determine the shear flow q in flange AB at a distance s from A. Recalling Eq. (6.5) and observing that the first moment Q of the shaded area with respect to the neutral axis is

$$Q = (st)(h/2),$$

$$q = \frac{VQ}{I} = \frac{Vsth}{2I}$$

$$F = \int_0^b q \, ds = \int_0^b \frac{Vsth}{2I} \, ds = \frac{Vth}{2I} \int_0^b s \, ds$$

$$F = \frac{Vthb^2}{4I}$$



Example 6.05

$$e = \frac{Fh}{V} = \frac{Vthb^2}{4I} \frac{h}{V} = \frac{th^2b^2}{4I}$$

$$\begin{aligned} I &= I_{\text{web}} + 2I_{\text{flange}} \\ &= \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right] \end{aligned}$$

Neglecting the term containing t^3 , which is very small, we have

$$\begin{aligned} I &= \frac{1}{12}th^3 + \frac{1}{2}tbh^2 = \frac{1}{12}th^2(6b + h) \\ e &= \frac{3b^2}{6b + h} = \frac{b}{2 + \frac{h}{3b}} \end{aligned}$$

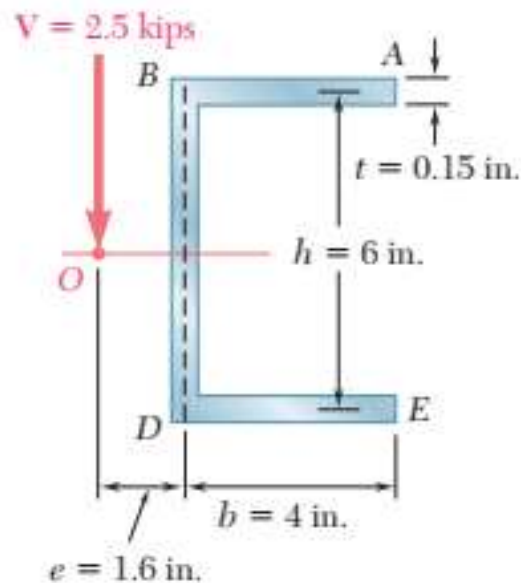
We note that the distance e does not depend upon t and can vary from 0 to $b/2$, depending upon the value of the ratio $h/3b$

$$\frac{h}{3b} = \frac{6 \text{ in.}}{3(4 \text{ in.})} = 0.5$$

$$e = \frac{4 \text{ in.}}{2 + 0.5} = 1.6 \text{ in.}$$

Example 6.06

For the channel section of Example 6.05 determine the distribution of the shearing stresses caused by a 2.5-kip vertical shear V applied at the shear center O



Example 6.06

Solution

A) Shearing stresses in flanges.

$$\tau = \frac{q}{t} = \frac{VQ}{It} = \frac{Vh}{2I} s$$

Note that the stress distribution in flange is linear.

$$\tau_B = \frac{Vhb}{2(\frac{1}{12}th^2)(6b + h)} = \frac{6Vb}{th(6b + h)}$$

$$\begin{aligned}\tau_B &= \frac{6(2.5 \text{ kips})(4 \text{ in.})}{(0.15 \text{ in.})(6 \text{ in.})(6 \times 4 \text{ in.} + 6 \text{ in.})} \\ &= 2.22 \text{ ksi}\end{aligned}$$

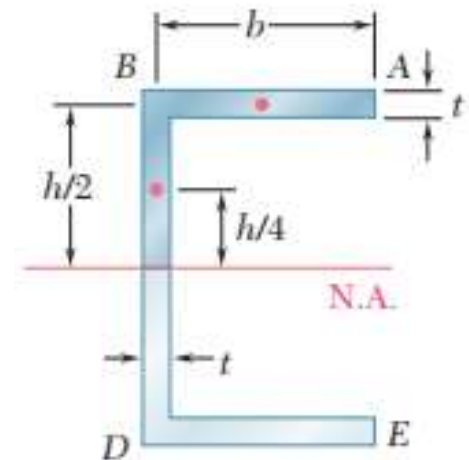


Example 6.06

B) Shearing stresses in web

The distribution of the shearing stresses in the web BD is parabolic, as in the case of a W-beam, and the maximum stress occurs at the neutral axis.

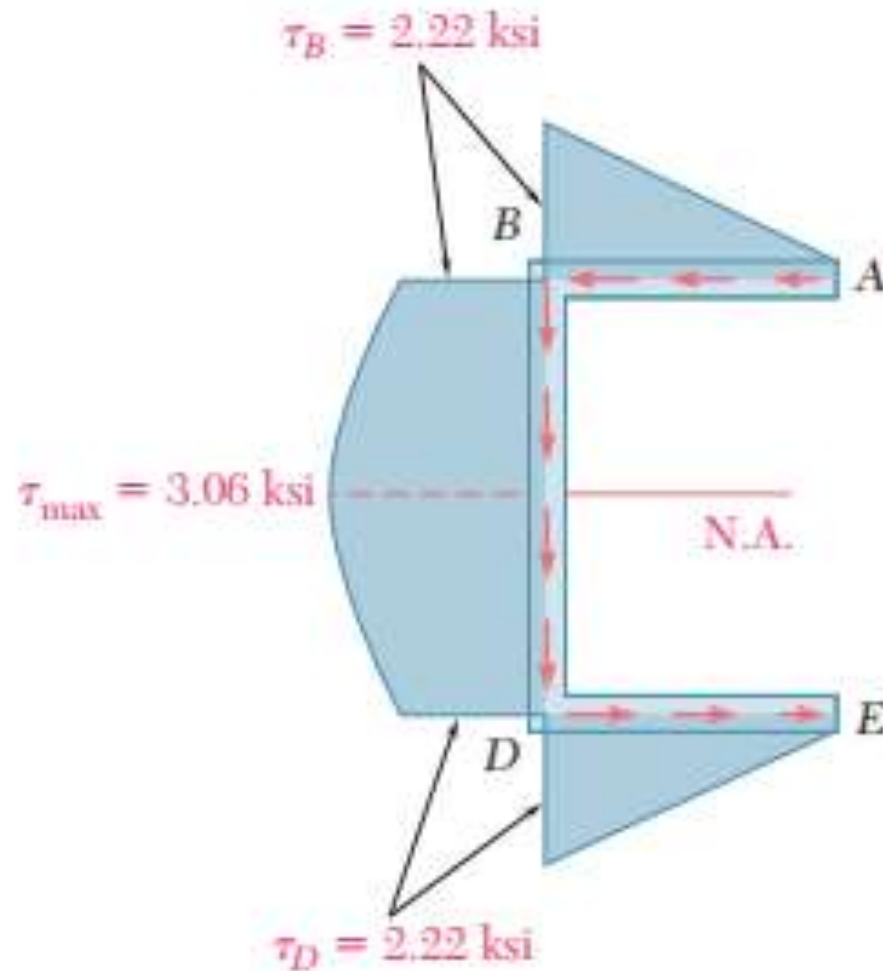
$$Q = bt\left(\frac{1}{2}h\right) + \frac{1}{2}ht\left(\frac{1}{4}h\right) = \frac{1}{8}ht(4b + h)$$



$$\tau_{\max} = \frac{VQ}{It} = \frac{V\left(\frac{1}{8}ht\right)(4b + h)}{\frac{1}{12}th^2(6b + h)t} = \frac{3V(4b + h)}{2th(6b + h)}$$

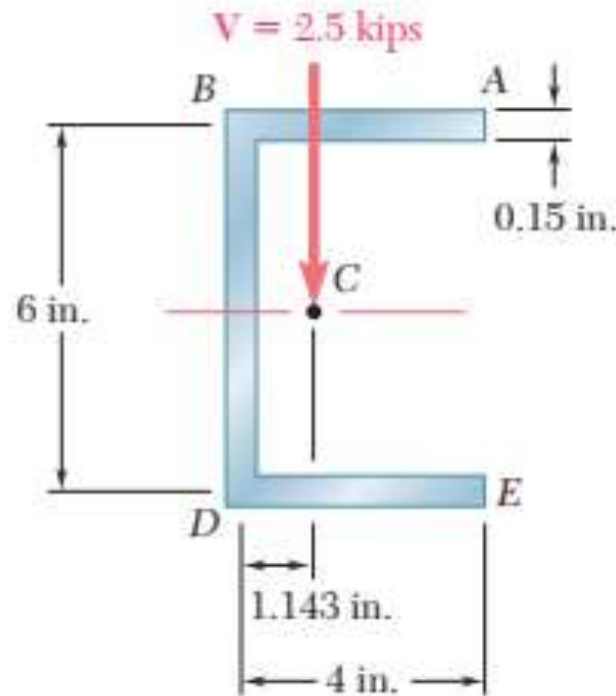
$$\begin{aligned}\tau_{\max} &= \frac{3(2.5 \text{ kips})(4 \times 4 \text{ in.} + 6 \text{ in.})}{2(0.15 \text{ in.})(6 \text{ in.})(6 \times 4 \text{ in.} + 6 \text{ in.})} \\ &= 3.06 \text{ ksi}\end{aligned}$$

Example 6.06



Example 6.07

For the channel section of Example 6.05, and neglecting stress concentrations, determine the maximum shearing stress caused by a 2.5-kip vertical shear V applied at the centroid C of the section, which is located 1.143 in. to the right of the center line of the web BD

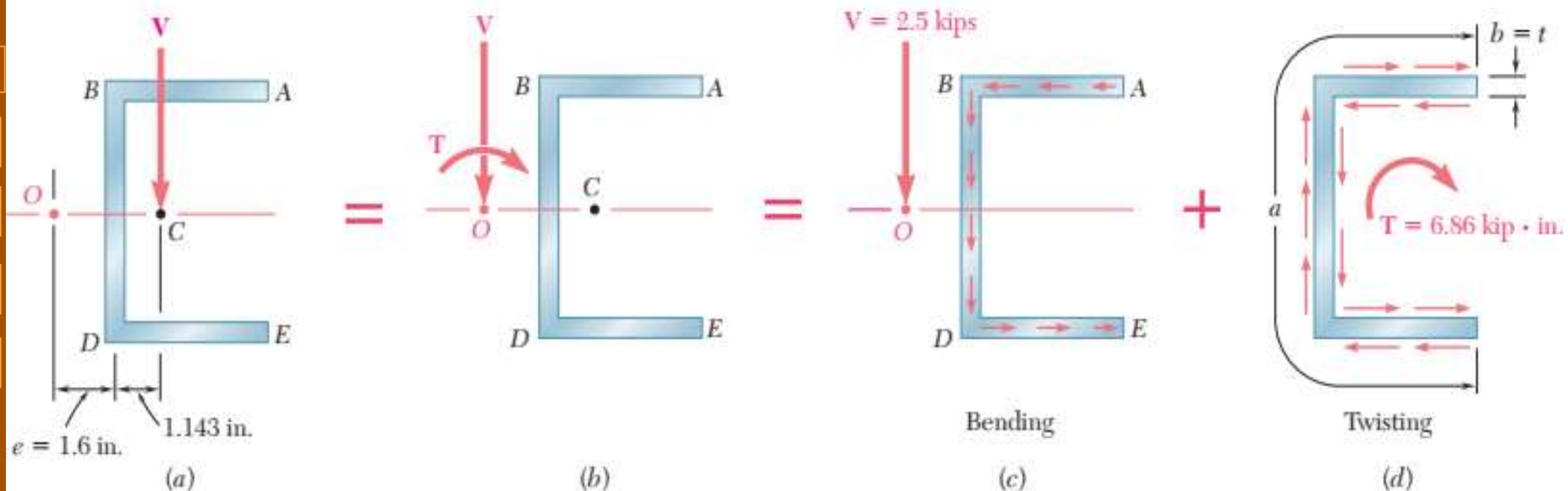


Example 6.07

Solution

The shear center O of the cross section was determined in Example 6.05 and found to be at a distance $e = 1.6$ in. to the left of the center line of the web BD .

We replace the shear $\mathbf{V}$ (Fig. 6.54a) by an equivalent force-couple system at the shear center O . This system consists of a 2.5-kip force $\mathbf{V}$ and of a torque $\mathbf{T}$ of magnitude



Example 6.07

A) Stresses due to bending.

From example 6.06

$$(\tau_{\max})_{\text{bending}} = 3.06 \text{ ksi}$$

B) Stresses due to twisting.

$$a = 4 \text{ in.} + 6 \text{ in.} + 4 \text{ in.} = 14 \text{ in.}$$

$$b = t = 0.15 \text{ in.} \quad b/a = 0.0107$$

$$c_1 = \frac{1}{3}(1 - 0.630b/a) = \frac{1}{3}(1 - 0.630 \times 0.0107) = 0.331$$

$$(\tau_{\max})_{\text{twisting}} = \frac{T}{c_1 a b^2} = \frac{6.86 \text{ kip} \cdot \text{in.}}{(0.331)(14 \text{ in.})(0.15 \text{ in.})^2} = 65.8 \text{ ksi}$$

$$\tau_{\max} = 3.06 \text{ ksi} + 65.8 \text{ ksi} = 68.9 \text{ ksi}$$

CHAPTER

7

MECHANICS OF MATERIALS

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Lecture Notes:
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Transformations of Stress and Strain

Transformations of Stress and Strain

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[Transformation of Plane Stress](#)

[Principal Stresses](#)

[Maximum Shearing Stress](#)

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[Sample Problem 7.1](#)

[Mohr's Circle for Plane Stress](#)

[Example 7.02](#)

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[General State of Stress](#)

[Application of Mohr's Circle to the Three- Dimensional Analysis of Stress](#)

[Yield Criteria for Ductile Materials Under Plane Stress](#)

[Fracture Criteria for Brittle Materials Under Plane Stress](#)

[Stresses in Thin-Walled Pressure Vessels](#)



Introduction

- The most general state of stress at a point may be represented by 6 components,

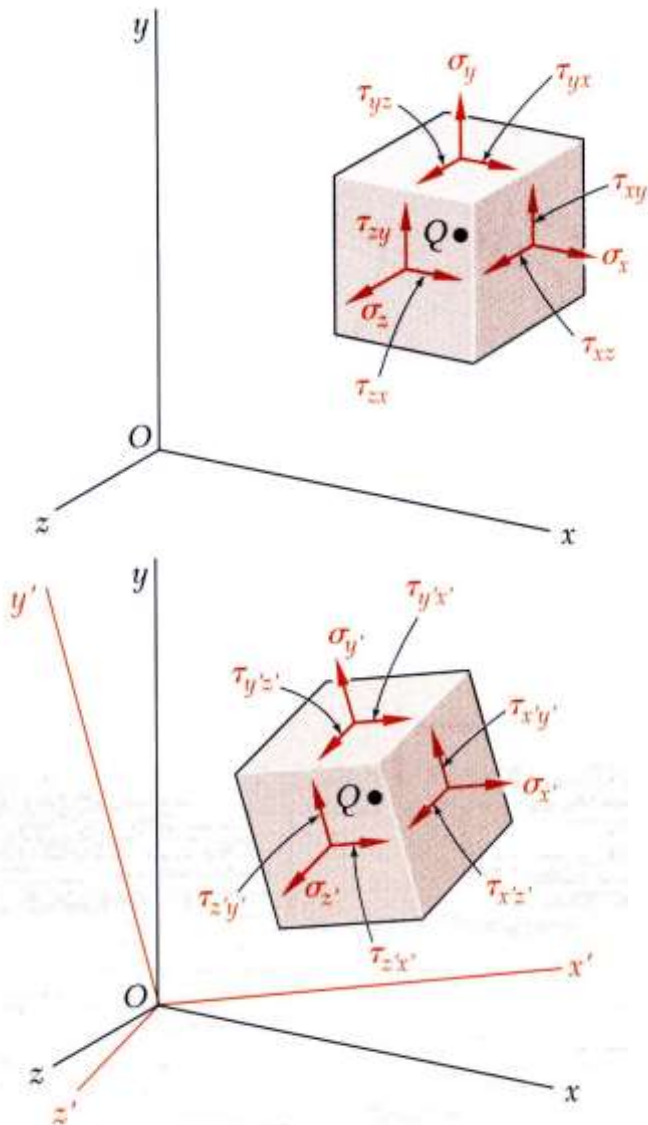
$\sigma_x, \sigma_y, \sigma_z$ normal stresses

$\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

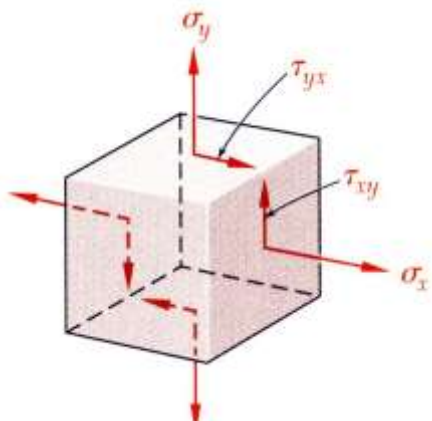
(Note: $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$)

- Same state of stress is represented by a different set of components if axes are rotated.

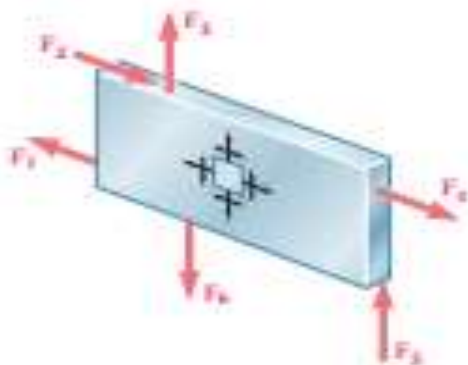
- The first part of the chapter is concerned with how the components of stress are transformed under a rotation of the coordinate axes. The second part of the chapter is devoted to a similar analysis of the transformation of the components of strain.



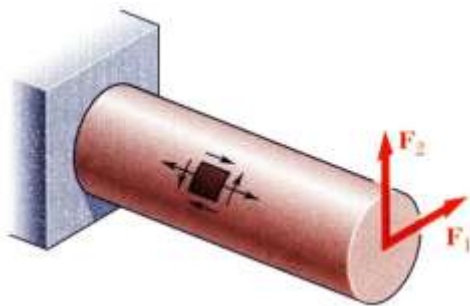
Introduction



- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by $\sigma_x, \sigma_y, \tau_{xy}$ and $\sigma_z = \tau_{zx} = \tau_{zy} = 0$.

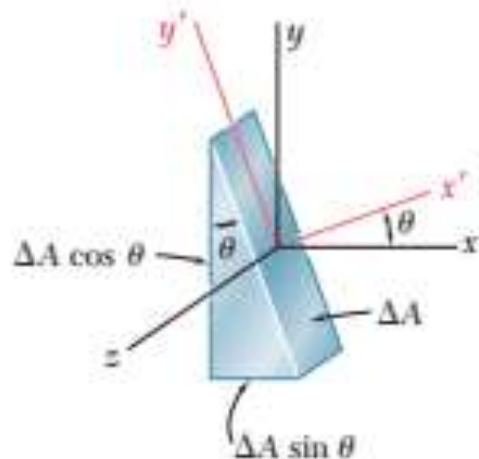


- State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.



- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

Transformation of Plane Stress



- Consider the conditions for equilibrium of a prismatic element with faces perpendicular to the x , y , and x' axes.

$$\sum F_{x'} = 0 = \sigma_{x'} \Delta A - \sigma_x (\Delta A \cos \theta) \cos \theta - \tau_{xy} (\Delta A \cos \theta) \sin \theta - \sigma_y (\Delta A \sin \theta) \sin \theta - \tau_{xy} (\Delta A \sin \theta) \cos \theta$$

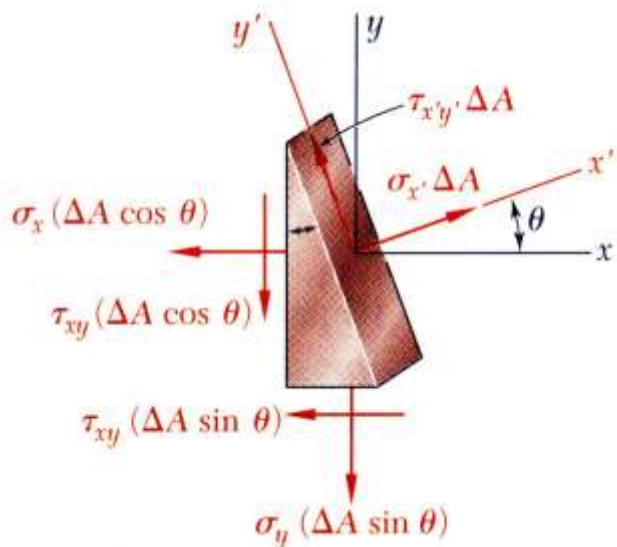
$$\sum F_{y'} = 0 = \tau_{x'y'} \Delta A + \sigma_x (\Delta A \cos \theta) \sin \theta - \tau_{xy} (\Delta A \cos \theta) \cos \theta - \sigma_y (\Delta A \sin \theta) \cos \theta + \tau_{xy} (\Delta A \sin \theta) \sin \theta$$

- The equations may be rewritten to yield

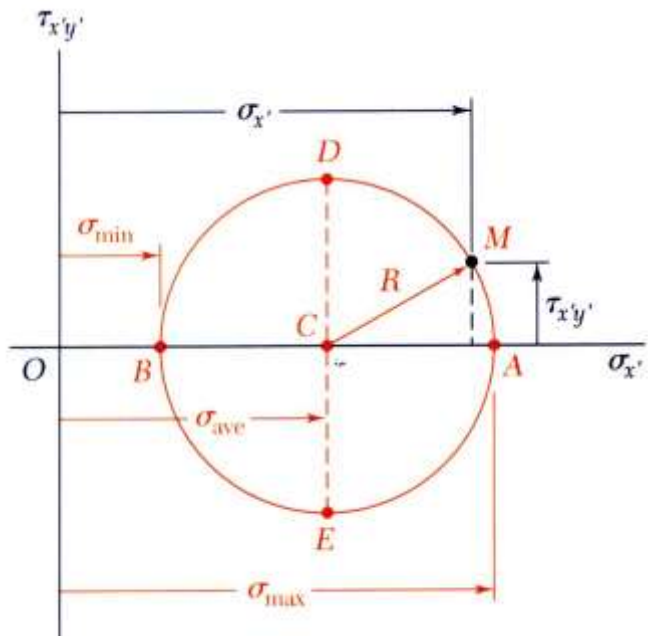
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$



Principal Stresses



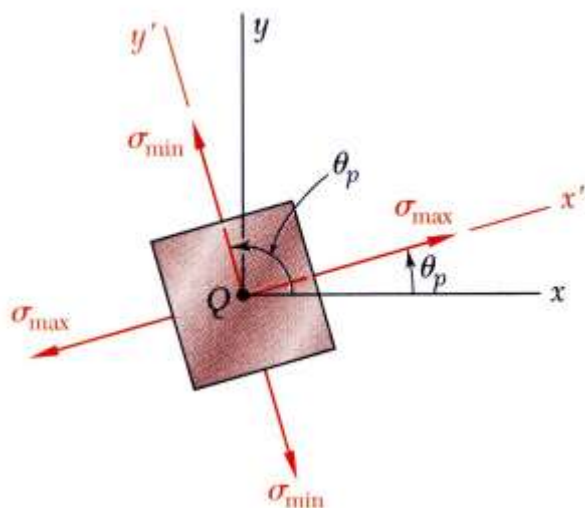
- The previous equations are combined to yield parametric equations for a circle,

$$(\sigma_{x'} - \sigma_{ave})^2 + \tau_{x'y'}^2 = R^2$$

where

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Principal stresses occur on the principal planes of stress with zero shearing stresses.*

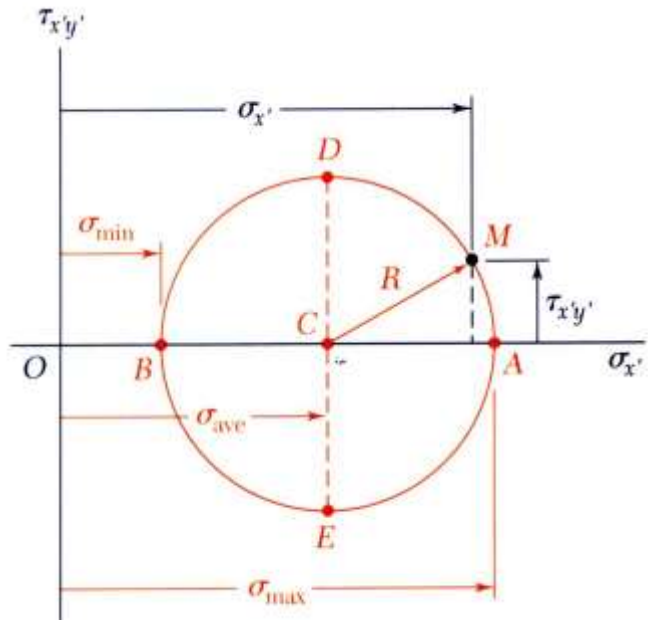


$$\sigma_{\max, \min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Note : defines two angles separated by 90°

Maximum Shearing Stress



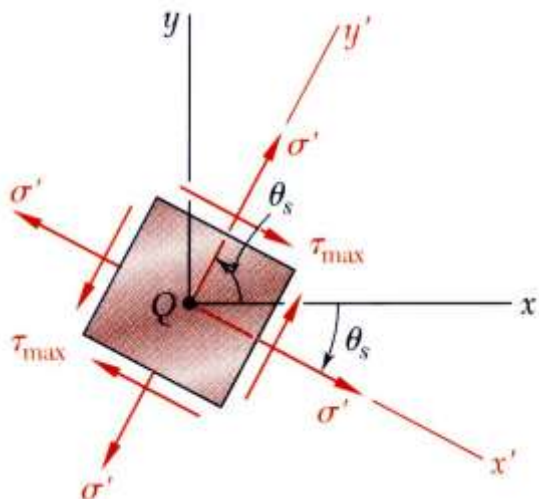
Maximum shearing stress occurs for $\sigma_{x'} = \sigma_{ave}$

$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Note : defines two angles separated by 90° and offset from θ_p by 45°

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$



Example 7.01

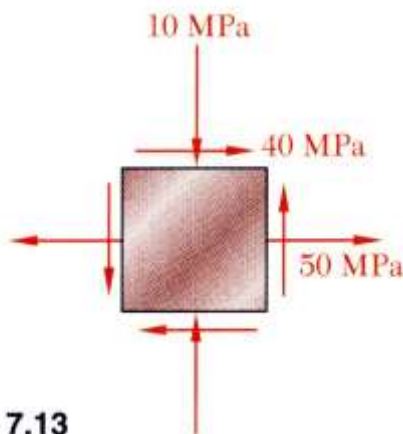


Fig. 7.13

For the state of plane stress shown, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Determine the principal stresses from

$$\sigma_{\max, \min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Calculate the maximum shearing stress with

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma' = \frac{\sigma_x + \sigma_y}{2}$$

Example 7.01

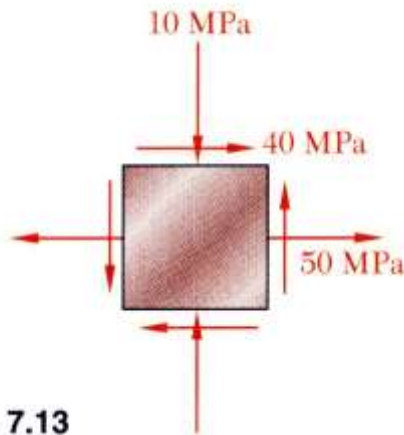


Fig. 7.13

$$\sigma_x = +50 \text{ MPa} \quad \tau_{xy} = +40 \text{ MPa}$$

$$\sigma_y = -10 \text{ MPa}$$

SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50 - (-10)} = 1.333$$

$$2\theta_p = 53.1^\circ, 233.1^\circ$$

$$\theta_p = 26.6^\circ, 116.6^\circ$$

- Determine the principal stresses from

$$\begin{aligned} \sigma_{\max, \min} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= 20 \pm \sqrt{(30)^2 + (40)^2} \end{aligned}$$

$$\sigma_{\max} = 70 \text{ MPa}$$

$$\sigma_{\min} = -30 \text{ MPa}$$

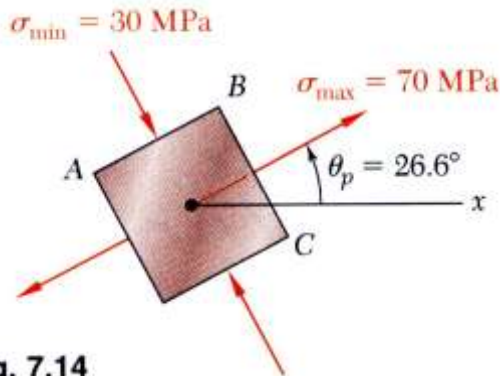


Fig. 7.14

Example 7.01

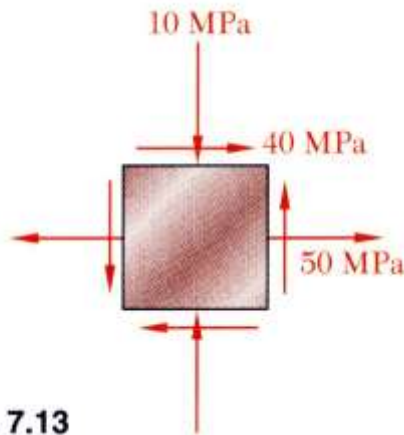


Fig. 7.13

$$\sigma_x = +50 \text{ MPa} \quad \tau_{xy} = +40 \text{ MPa}$$

$$\sigma_y = -10 \text{ MPa}$$

- Calculate the maximum shearing stress with

$$\begin{aligned} \tau_{\max} &= \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \sqrt{(30)^2 + (40)^2} \end{aligned}$$

$$\tau_{\max} = 50 \text{ MPa}$$

$$\theta_s = \theta_p - 45^\circ$$

$$\theta_s = -18.4^\circ, 71.6^\circ$$

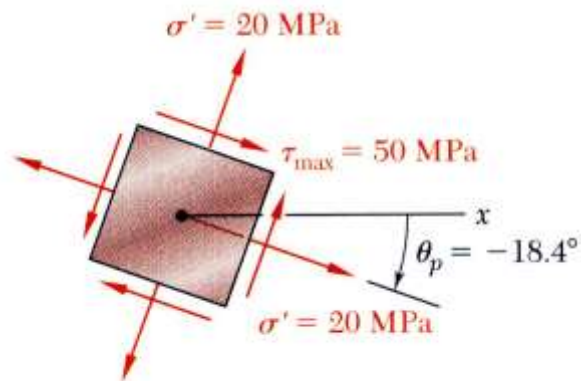


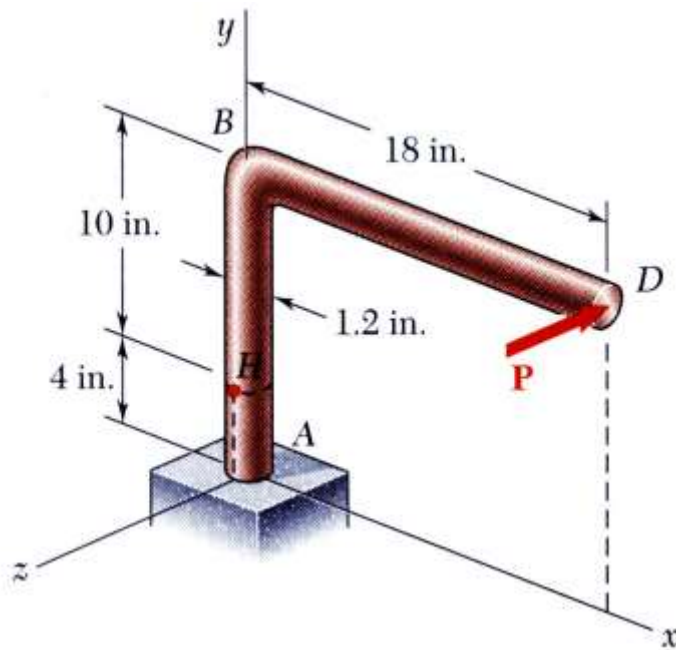
Fig. 7.16

- The corresponding normal stress is

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2}$$

$$\sigma' = 20 \text{ MPa}$$

Sample Problem 7.1

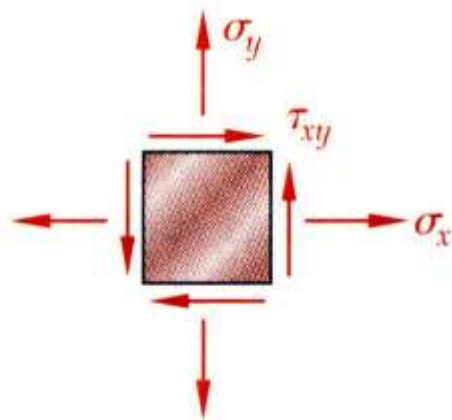
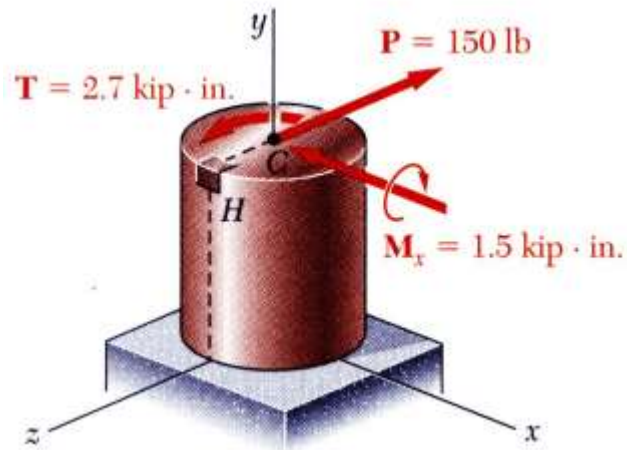


SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through H .
- Evaluate the normal and shearing stresses at H .
- Determine the principal planes and calculate the principal stresses.

A single horizontal force P of 150 lb magnitude is applied to end D of lever ABD . Determine (a) the normal and shearing stresses on an element at point H having sides parallel to the x and y axes, (b) the principal planes and principal stresses at the point H .

Sample Problem 7.1



SOLUTION:

- Determine an equivalent force-couple system at the center of the transverse section passing through H .

$$P = 150 \text{ lb}$$

$$T = (150 \text{ lb})(18 \text{ in}) = 2.7 \text{ kip} \cdot \text{in}$$

$$M_x = (150 \text{ lb})(10 \text{ in}) = 1.5 \text{ kip} \cdot \text{in}$$

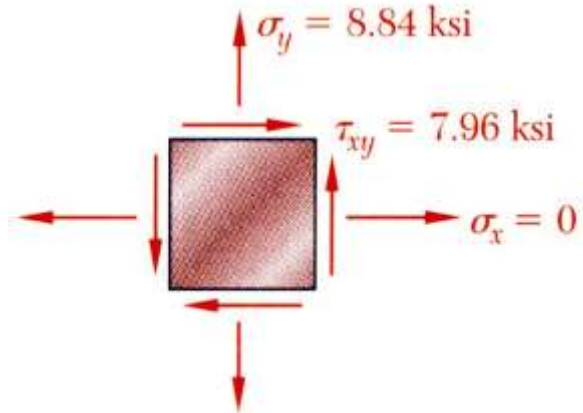
- Evaluate the normal and shearing stresses at H .

$$\sigma_y = +\frac{Mc}{I} = +\frac{(1.5 \text{ kip} \cdot \text{in})(0.6 \text{ in})}{\frac{1}{4}\pi(0.6 \text{ in})^4}$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7 \text{ kip} \cdot \text{in})(0.6 \text{ in})}{\frac{1}{2}\pi(0.6 \text{ in})^4}$$

$$\sigma_x = 0 \quad \sigma_y = +8.84 \text{ ksi} \quad \tau_y = +7.96 \text{ ksi}$$

Sample Problem 7.1

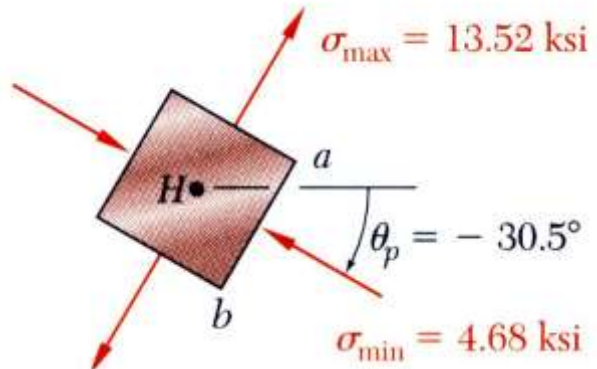


- Determine the principal planes and calculate the principal stresses.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.8$$

$$2\theta_p = -61.0^\circ, 119^\circ$$

$$\theta_p = -30.5^\circ, 59.5^\circ$$



$$\begin{aligned}\sigma_{\max, \min} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \frac{0 + 8.84}{2} \pm \sqrt{\left(\frac{0 - 8.84}{2}\right)^2 + (7.96)^2}\end{aligned}$$

$$\sigma_{\max} = +13.52 \text{ ksi}$$

$$\sigma_{\min} = -4.68 \text{ ksi}$$

Mohr's Circle for Plane Stress

- With the physical significance of Mohr's circle for plane stress established, it may be applied with simple geometric considerations. Critical values are estimated graphically or calculated.
- For a known state of plane stress $\sigma_x, \sigma_y, \tau_{xy}$ plot the points X and Y and construct the circle centered at C .

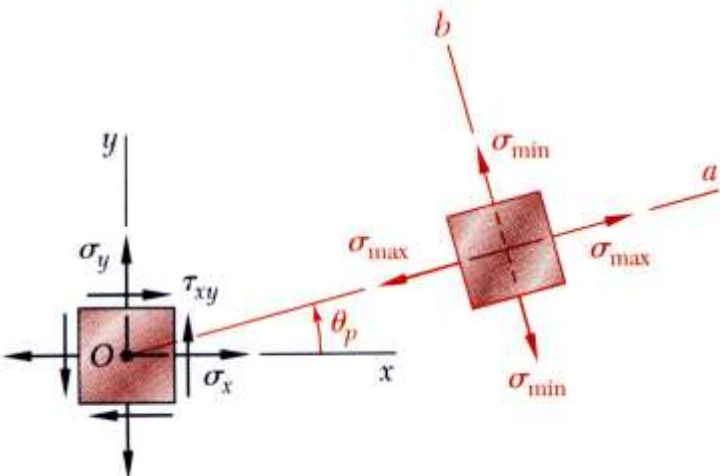
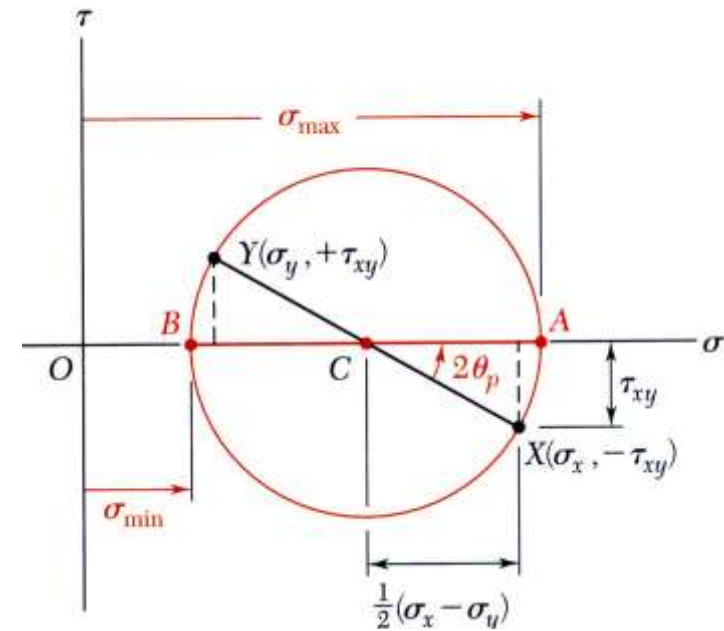
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- The principal stresses are obtained at A and B .

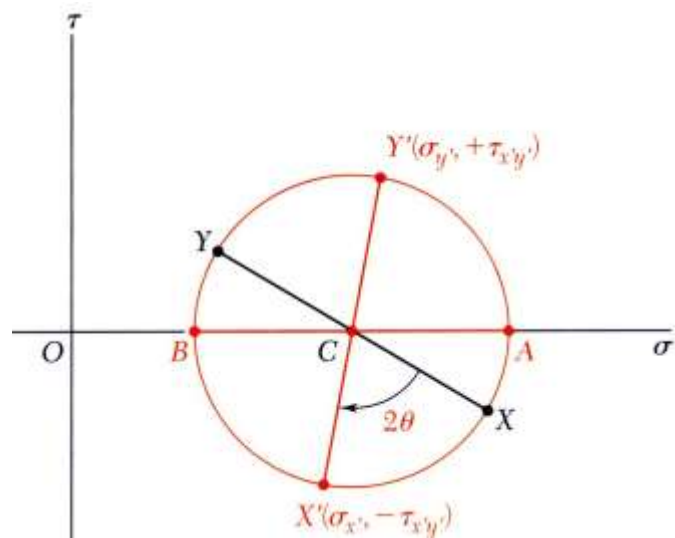
$$\sigma_{max,min} = \sigma_{ave} \pm R$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

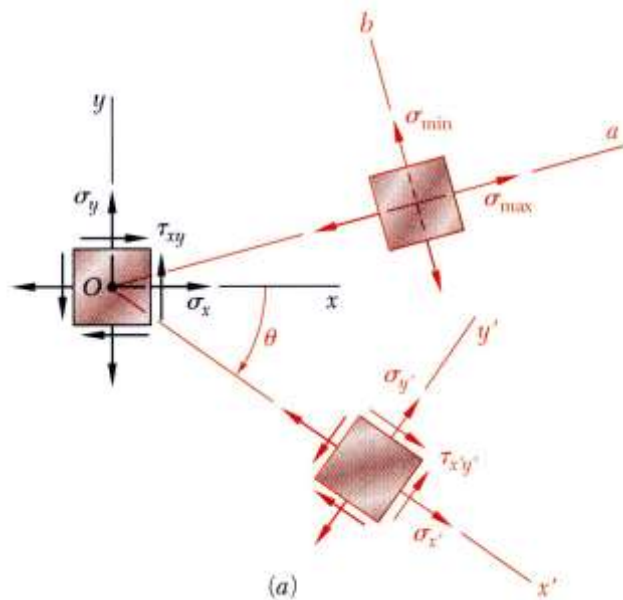
The direction of rotation of Ox to Oa is the same as CX to CA .



Mohr's Circle for Plane Stress

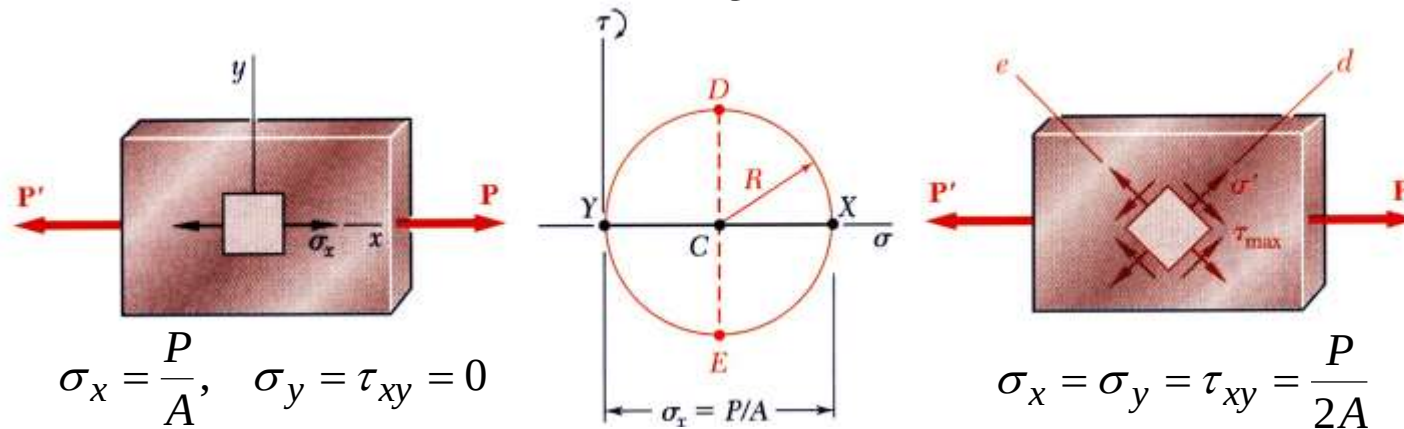


- With Mohr's circle uniquely defined, the state of stress at other axes orientations may be depicted.
- For the state of stress at an angle θ with respect to the xy axes, construct a new diameter $X'Y'$ at an angle 2θ with respect to XY .
- Normal and shear stresses are obtained from the coordinates $X'Y'$.

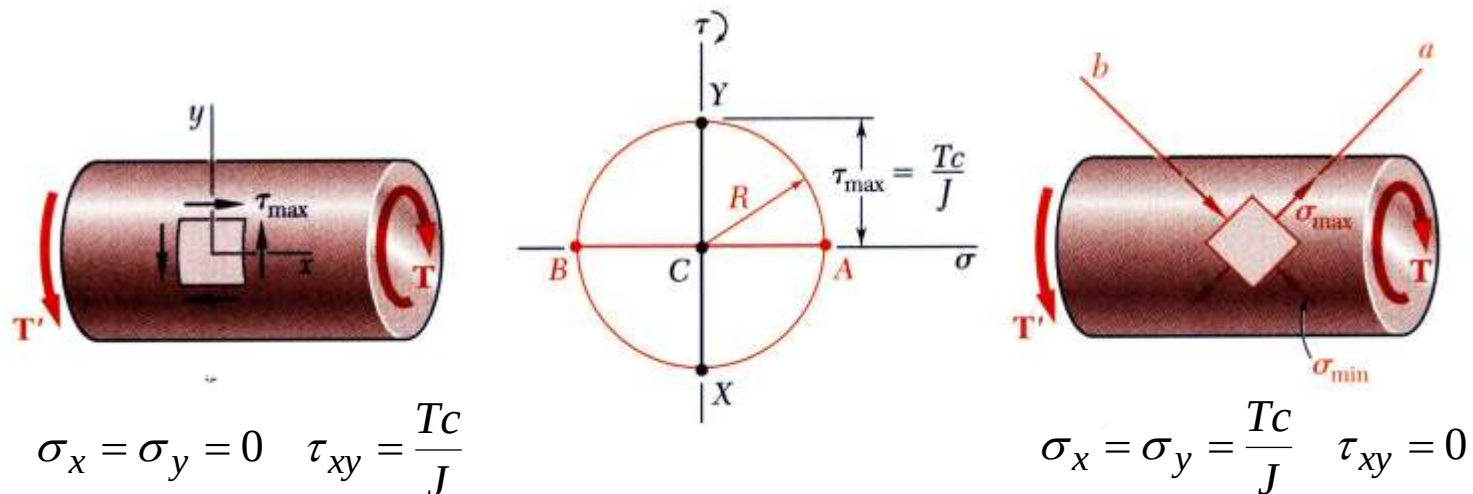


Mohr's Circle for Plane Stress

- Mohr's circle for centric axial loading:



- Mohr's circle for torsional loading:



Example 7.02

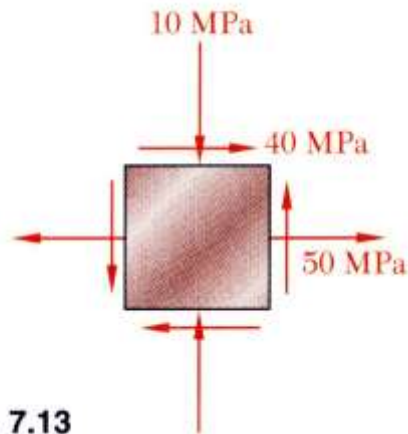
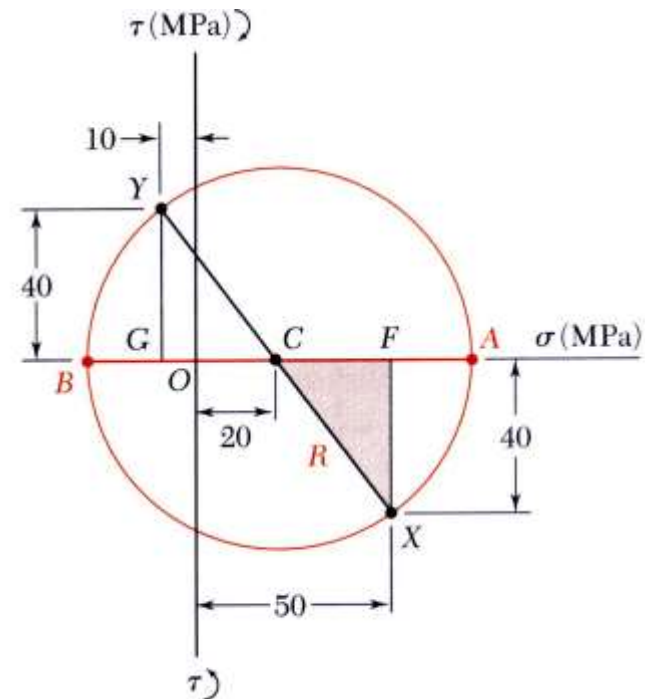


Fig. 7.13

For the state of plane stress shown,
 (a) construct Mohr's circle, determine
 (b) the principal planes, (c) the
 principal stresses, (d) the maximum
 shearing stress and the corresponding
 normal stress.



SOLUTION:

- Construction of Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50) + (-10)}{2} = 20 \text{ MPa}$$

$$CF = 50 - 20 = 30 \text{ MPa} \quad FX = 40 \text{ MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50 \text{ MPa}$$

Example 7.02

- Principal planes and stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50$$

$$\sigma_{\max} = 70 \text{ MPa}$$

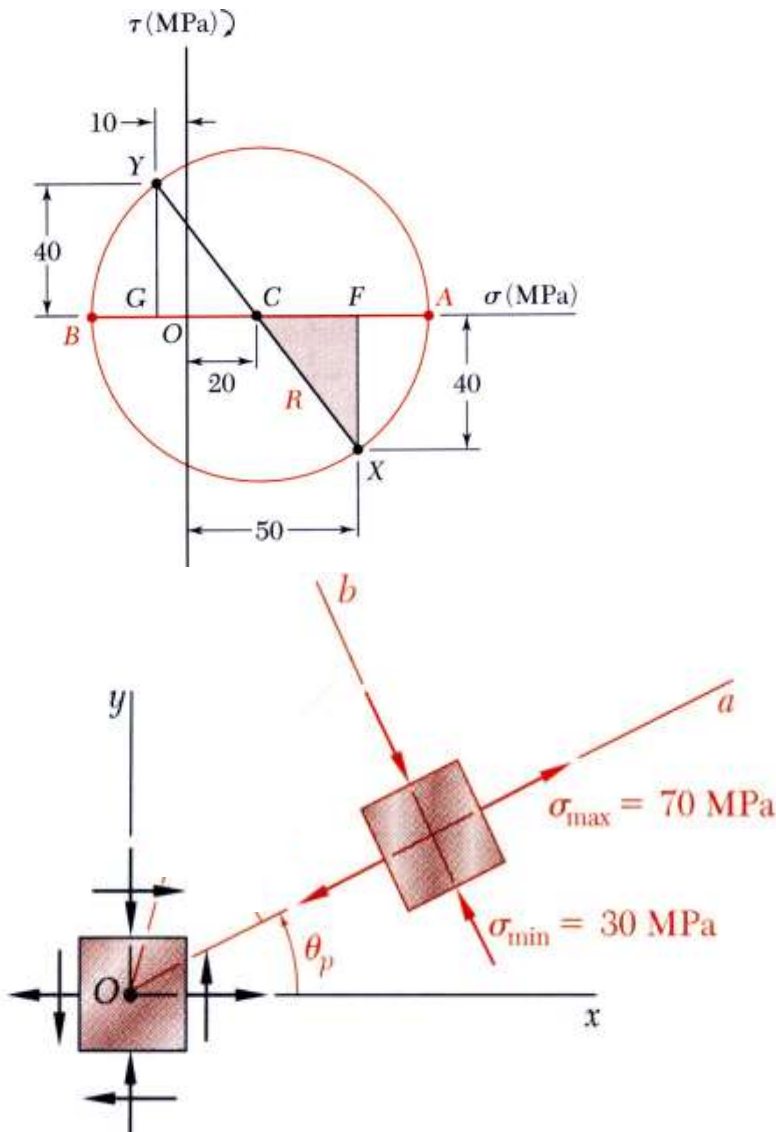
$$\sigma_{\min} = OB = OC - BC = 20 - 50$$

$$\sigma_{\min} = -30 \text{ MPa}$$

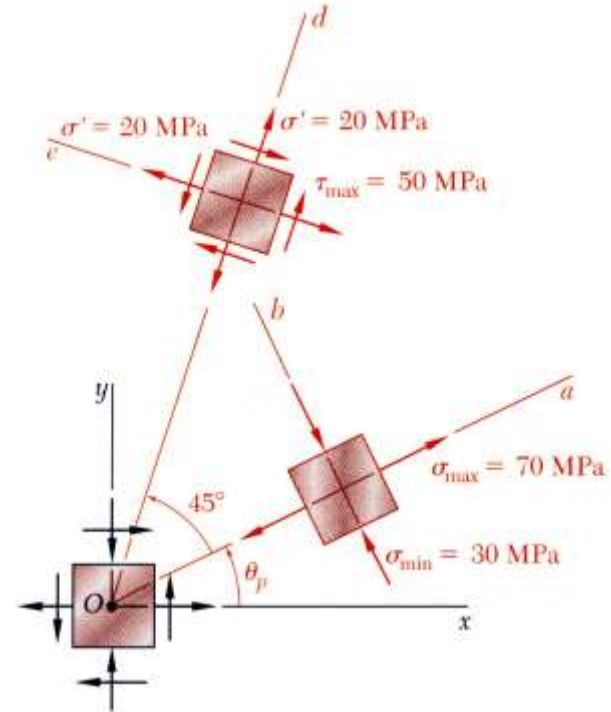
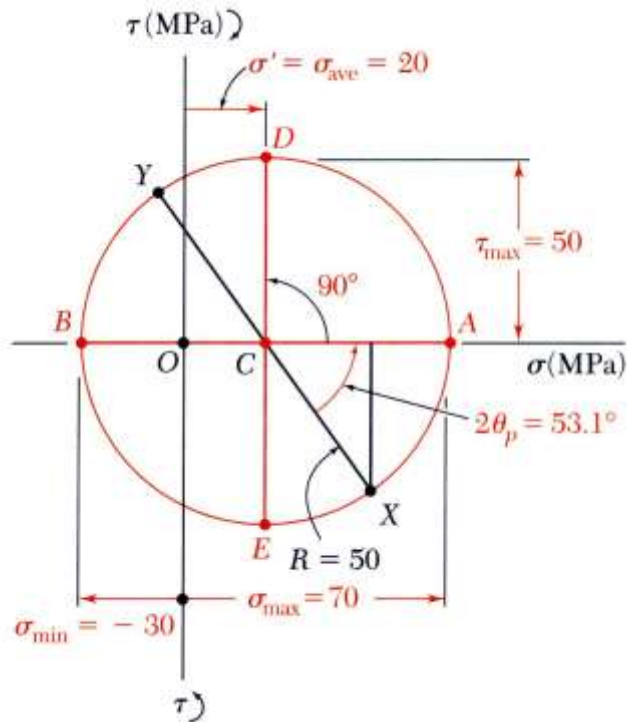
$$\tan 2\theta_p = \frac{FX}{CP} = \frac{40}{30}$$

$$2\theta_p = 53.1^\circ$$

$$\theta_p = 26.6^\circ$$



Example 7.02



- Maximum shear stress

$$\theta_s = \theta_p + 45^\circ$$

$$\theta_s = 71.6^\circ$$

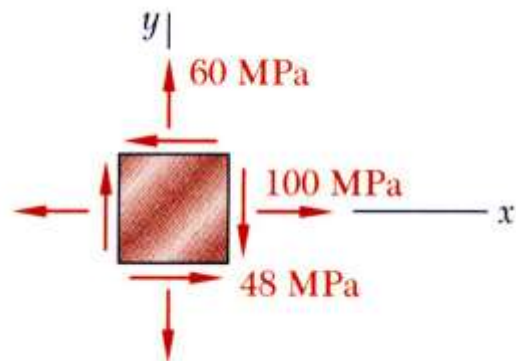
$$\tau_{\max} = R$$

$$\tau_{\max} = 50 \text{ MPa}$$

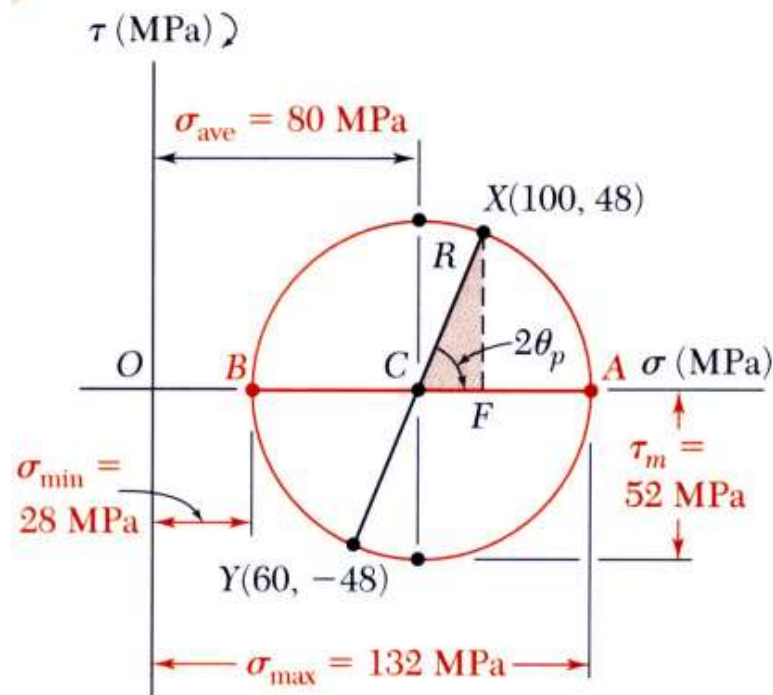
$$\sigma' = \sigma_{\text{ave}}$$

$$\sigma' = 20 \text{ MPa}$$

Sample Problem 7.2



For the state of stress shown, determine (a) the principal planes and the principal stresses, (b) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30 degrees.



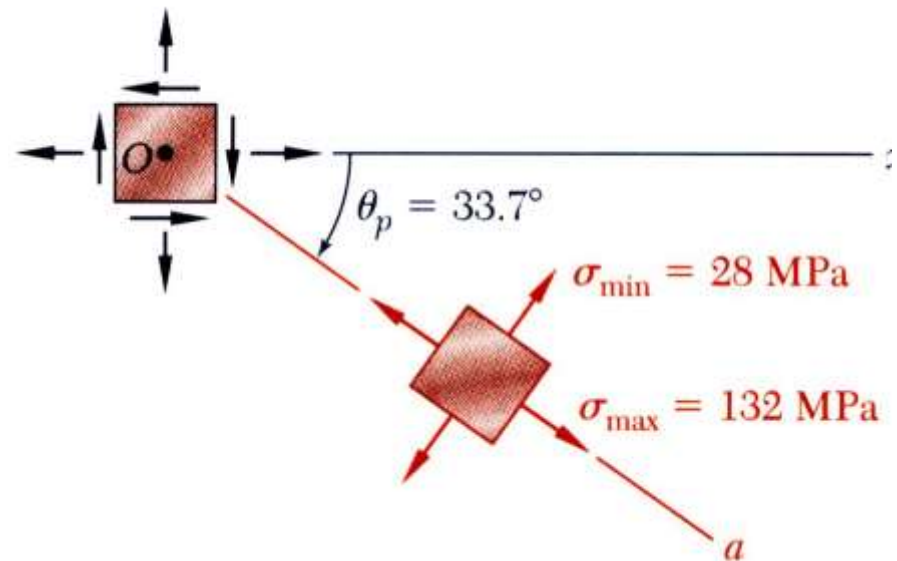
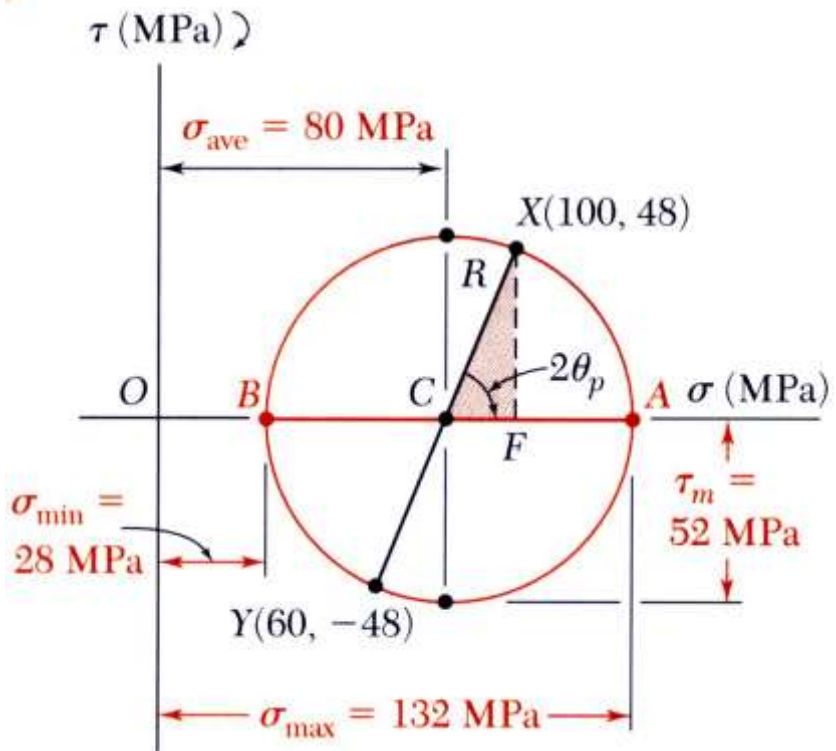
SOLUTION:

- Construct Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{100 + 60}{2} = 80 \text{ MPa}$$

$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52 \text{ MPa}$$

Sample Problem 7.2



- Principal planes and stresses

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4$$

$$2\theta_p = 67.4^\circ$$

$$\theta_p = 33.7^\circ \text{ clockwise}$$

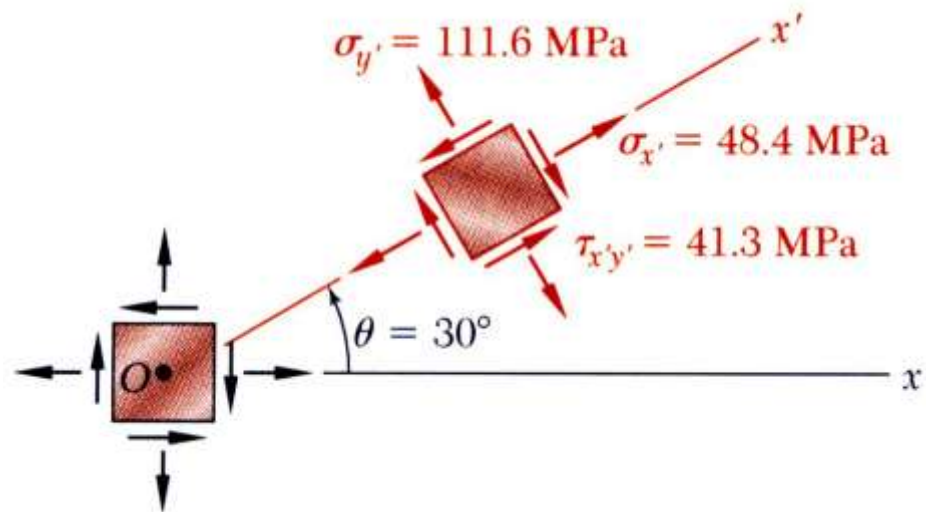
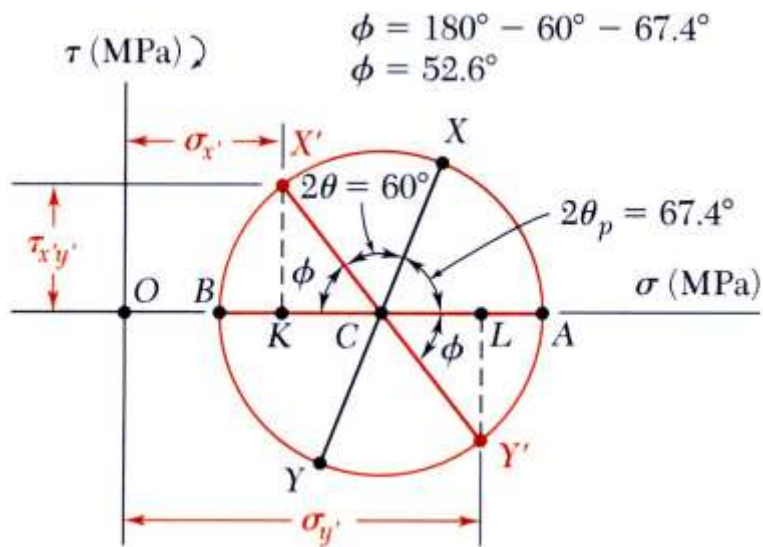
$$\begin{aligned}\sigma_{\max} &= OA = OC + CA \\ &= 80 + 52\end{aligned}$$

$$\sigma_{\max} = +132 \text{ MPa}$$

$$\begin{aligned}\sigma_{\min} &= OA = OC - BC \\ &= 80 - 52\end{aligned}$$

$$\sigma_{\min} = +28 \text{ MPa}$$

Sample Problem 7.2



- Stress components after rotation by 30°
- Points X' and Y' on Mohr's circle that correspond to stress components on the rotated element are obtained by rotating XY counterclockwise through $2\theta = 60^\circ$

$$\phi = 180^\circ - 60^\circ - 67.4^\circ = 52.6^\circ$$

$$\sigma_{x'} = OK = OC - KC = 80 - 52 \cos 52.6^\circ$$

$$\sigma_{y'} = OL = OC + CL = 80 + 52 \cos 52.6^\circ$$

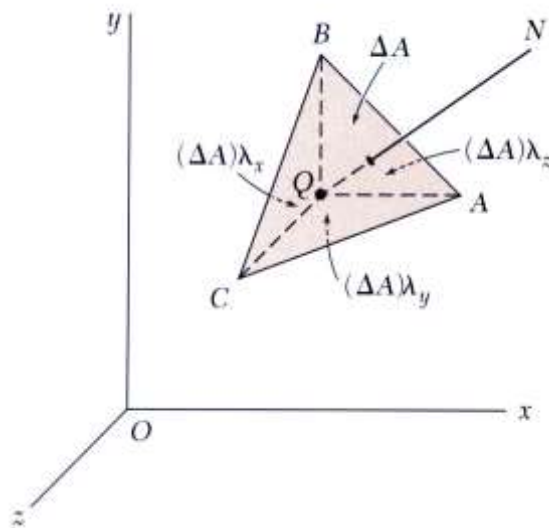
$$\tau_{x'y'} = KX' = 52 \sin 52.6^\circ$$

$$\sigma_{x'} = +48.4 \text{ MPa}$$

$$\sigma_{y'} = +111.6 \text{ MPa}$$

$$\tau_{x'y'} = 41.3 \text{ MPa}$$

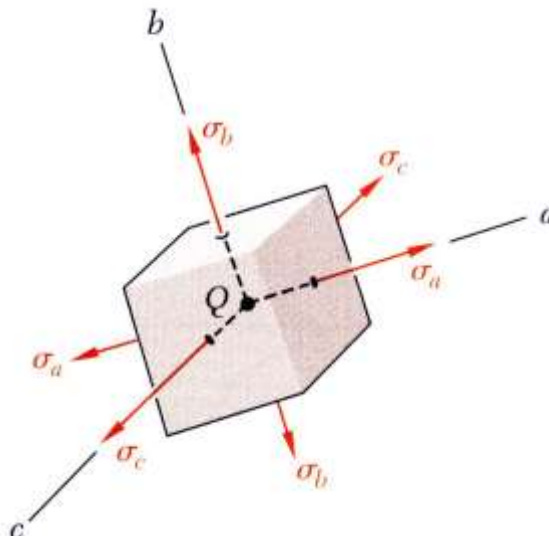
General State of Stress



- Consider the general 3D state of stress at a point and the transformation of stress from element rotation
- State of stress at Q defined by: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- Consider tetrahedron with face perpendicular to the line QN with direction cosines: $\lambda_x, \lambda_y, \lambda_z$

- The requirement $\sum F_n = 0$ leads to,

$$\sigma_n = \sigma_x \lambda_x^2 + \sigma_y \lambda_y^2 + \sigma_z \lambda_z^2 + 2\tau_{xy} \lambda_x \lambda_y + 2\tau_{yz} \lambda_y \lambda_z + 2\tau_{zx} \lambda_z \lambda_x$$

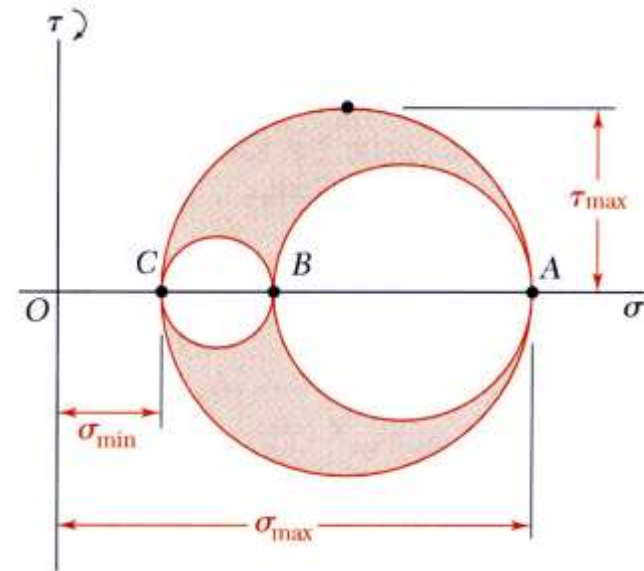
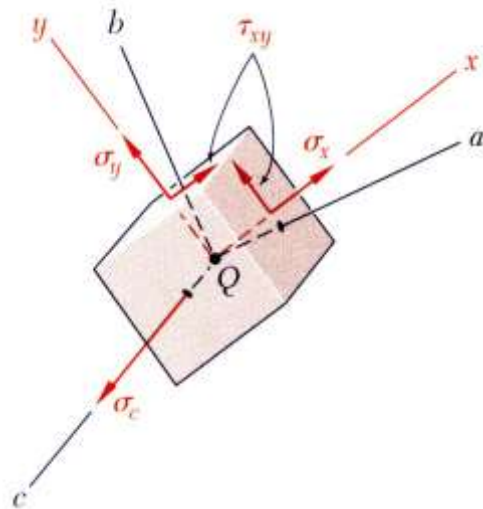


- Form of equation guarantees that an element orientation can be found such that

$$\sigma_n = \sigma_a \lambda_a^2 + \sigma_b \lambda_b^2 + \sigma_c \lambda_c^2$$

These are the principal axes and principal planes and the normal stresses are the principal stresses.

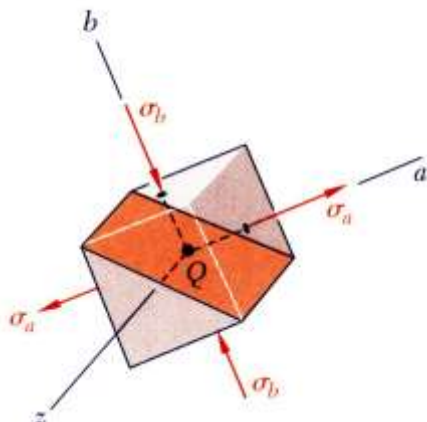
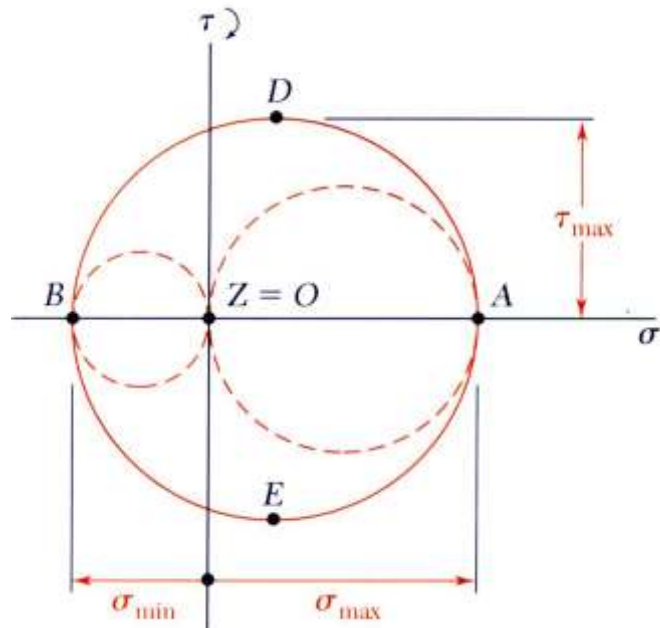
Application of Mohr's Circle to the Three-Dimensional Analysis of Stress



- Transformation of stress for an element rotated around a principal axis may be represented by Mohr's circle.
- Points A, B, and C represent the principal stresses on the principal planes (shearing stress is zero)
- The three circles represent the normal and shearing stresses for rotation around each principal axis.
- Radius of the largest circle yields the maximum shearing stress.

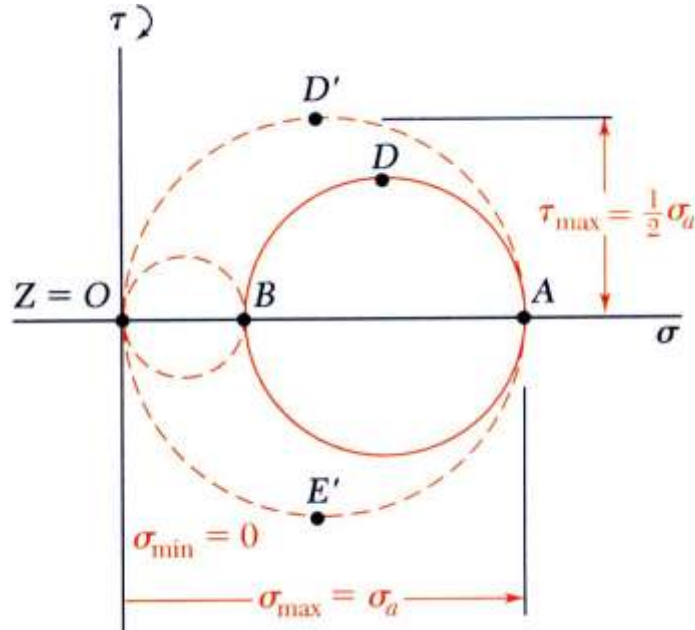
$$\tau_{\max} = \frac{1}{2} |\sigma_{\max} - \sigma_{\min}|$$

Application of Mohr's Circle to the Three-Dimensional Analysis of Stress

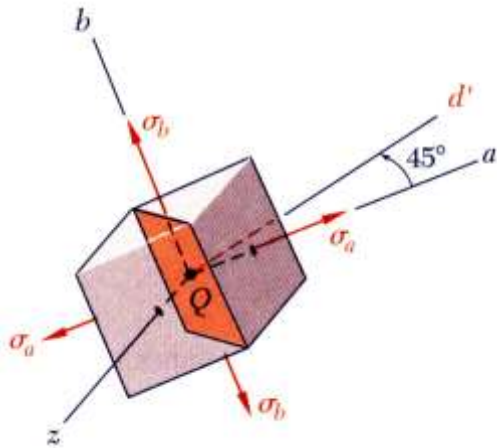


- In the case of plane stress, the axis perpendicular to the plane of stress is a principal axis (shearing stress equal zero).
- If the points A and B (representing the principal planes) are on opposite sides of the origin, then
 - a) the corresponding principal stresses are the maximum and minimum normal stresses for the element
 - b) the maximum shearing stress for the element is equal to the maximum “in-plane” shearing stress
 - c) planes of maximum shearing stress are at 45° to the principal planes.

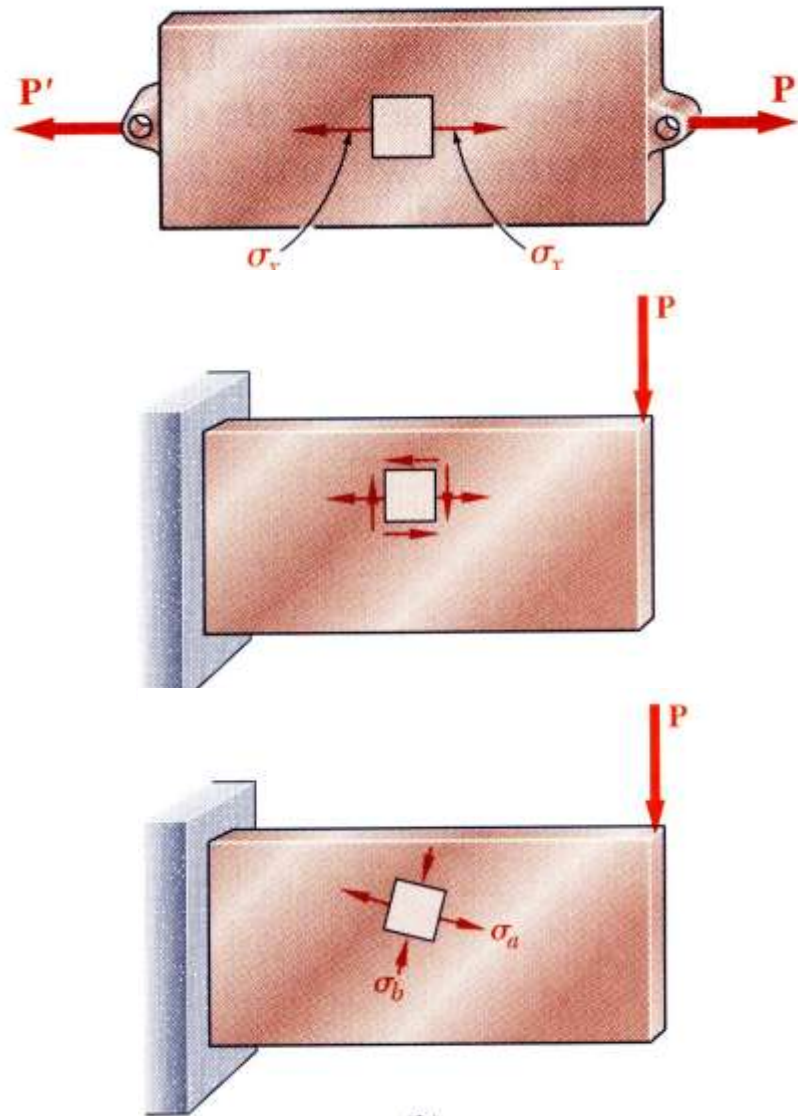
Application of Mohr's Circle to the Three-Dimensional Analysis of Stress



- If A and B are on the same side of the origin (i.e., have the same sign), then
 - a) the circle defining $\sigma_{\max}$, $\sigma_{\min}$, and $\tau_{\max}$ for the element is not the circle corresponding to transformations within the plane of stress
 - b) maximum shearing stress for the element is equal to half of the maximum stress
 - c) planes of maximum shearing stress are at 45 degrees to the plane of stress

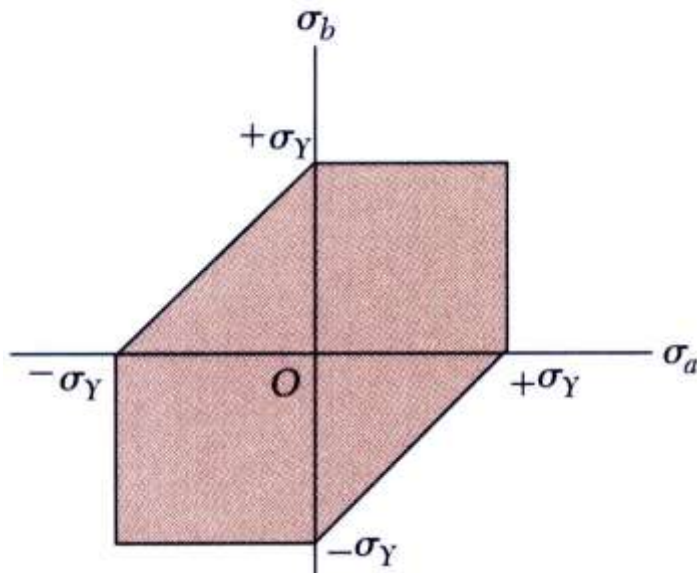


Yield Criteria for Ductile Materials Under Plane Stress



- Failure of a machine component subjected to uniaxial stress is directly predicted from an equivalent tensile test
- Failure of a machine component subjected to plane stress cannot be directly predicted from the uniaxial state of stress in a tensile test specimen
- It is convenient to determine the principal stresses and to base the failure criteria on the corresponding biaxial stress state
- Failure criteria are based on the mechanism of failure. Allows comparison of the failure conditions for a uniaxial stress test and biaxial component loading

Yield Criteria for Ductile Materials Under Plane Stress



Maximum shearing stress criteria:

Structural component is safe as long as the maximum shearing stress is less than the maximum shearing stress in a tensile test specimen at yield, i.e.,

$$\tau_{\max} < \tau_Y = \frac{\sigma_Y}{2}$$

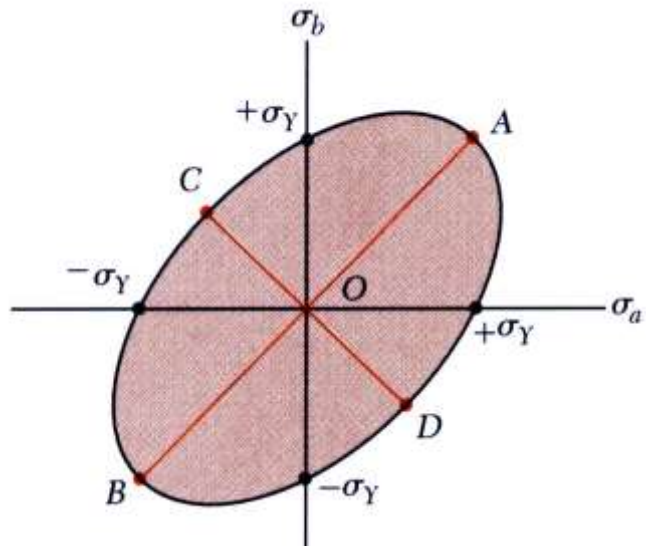
For σ_a and σ_b with the same sign,

$$\tau_{\max} = \frac{|\sigma_a|}{2} \text{ or } \frac{|\sigma_b|}{2} < \frac{\sigma_Y}{2}$$

For σ_a and σ_b with opposite signs,

$$\tau_{\max} = \frac{|\sigma_a - \sigma_b|}{2} < \frac{\sigma_Y}{2}$$

Yield Criteria for Ductile Materials Under Plane Stress



Maximum distortion energy criteria:

Structural component is safe as long as the distortion energy per unit volume is less than that occurring in a tensile test specimen at yield.

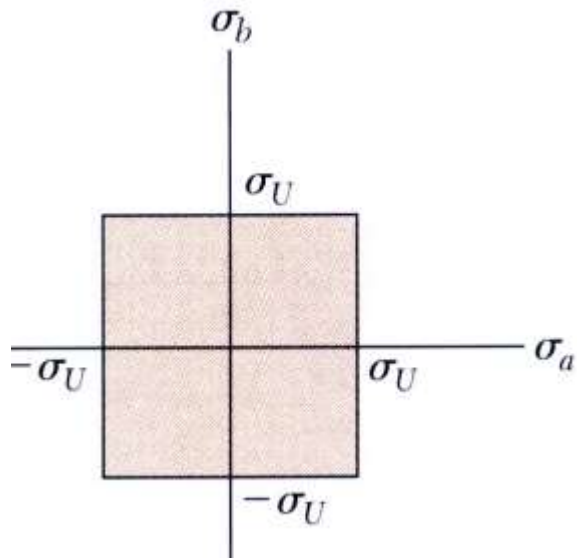
$$u_d < u_Y$$

$$\frac{1}{6G}(\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2) < \frac{1}{6G}(\sigma_Y^2 - \sigma_Y \times 0 + 0^2)$$

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_Y^2$$



Fracture Criteria for Brittle Materials Under Plane Stress



Brittle materials fail suddenly through rupture or fracture in a tensile test. The failure condition is characterized by the ultimate strength σ_U .

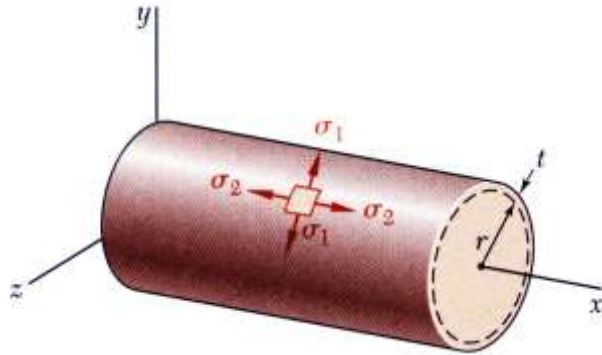
Maximum normal stress criteria:

Structural component is safe as long as the maximum normal stress is less than the ultimate strength of a tensile test specimen.

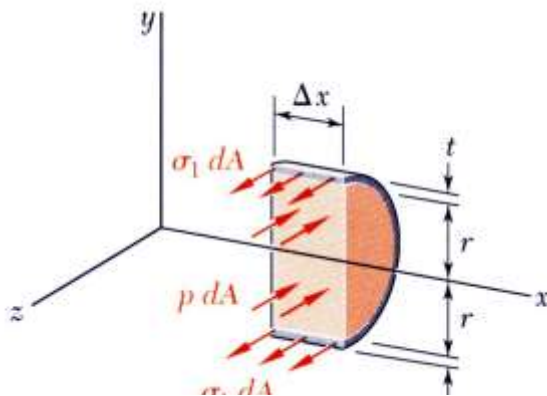
$$|\sigma_a| < \sigma_U$$

$$|\sigma_b| < \sigma_U$$

Stresses in Thin-Walled Pressure Vessels



- Cylindrical vessel with principal stresses
 σ_1 = hoop stress
 σ_2 = longitudinal stress



- Hoop stress:

$$\sum F_z = 0 = \sigma_1(2t \Delta x) - p(2r \Delta x)$$

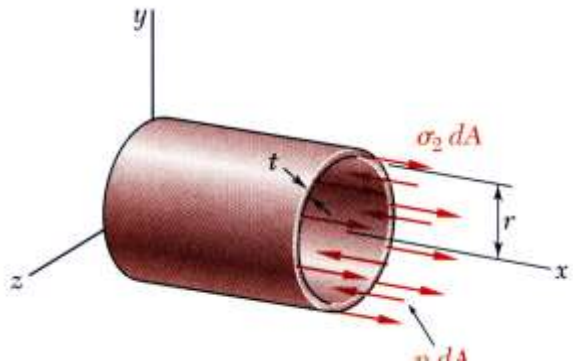
$$\sigma_1 = \frac{pr}{t}$$

- Longitudinal stress:

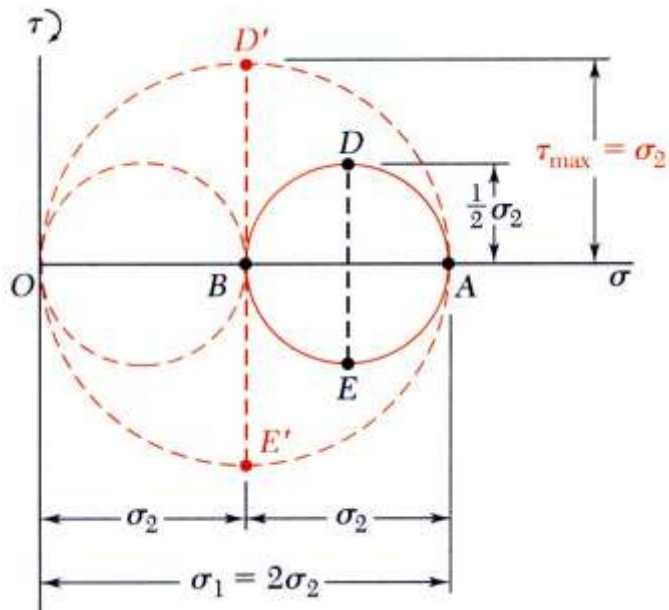
$$\sum F_x = 0 = \sigma_2(2\pi r t) - p(\pi r^2)$$

$$\sigma_2 = \frac{pr}{2t}$$

$$\sigma_1 = 2\sigma_2$$



Stresses in Thin-Walled Pressure Vessels



- Points A and B correspond to hoop stress, σ_1 , and longitudinal stress, σ_2

- Maximum in-plane shearing stress:

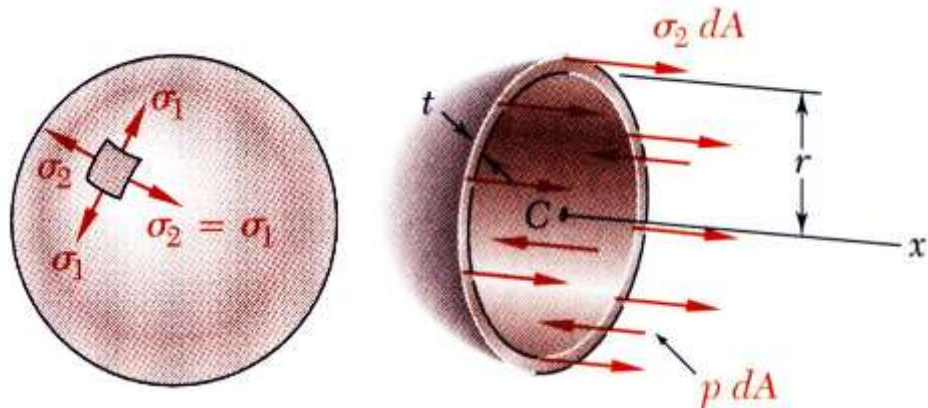
$$\tau_{\max(\text{in-plane})} = \frac{1}{2} \sigma_2 = \frac{pr}{4t}$$

- Maximum out-of-plane shearing stress corresponds to a 45° rotation of the plane stress element around a longitudinal axis

$$\tau_{\max} = \sigma_2 = \frac{pr}{2t}$$



Stresses in Thin-Walled Pressure Vessels



- Spherical pressure vessel:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

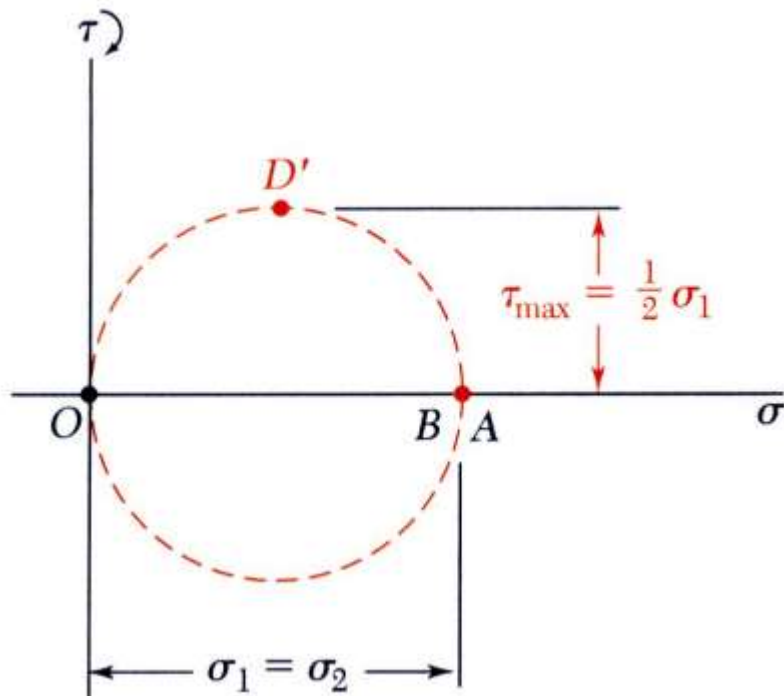
- Mohr's circle for in-plane transformations reduces to a point

$$\sigma = \sigma_1 = \sigma_2 = \text{constant}$$

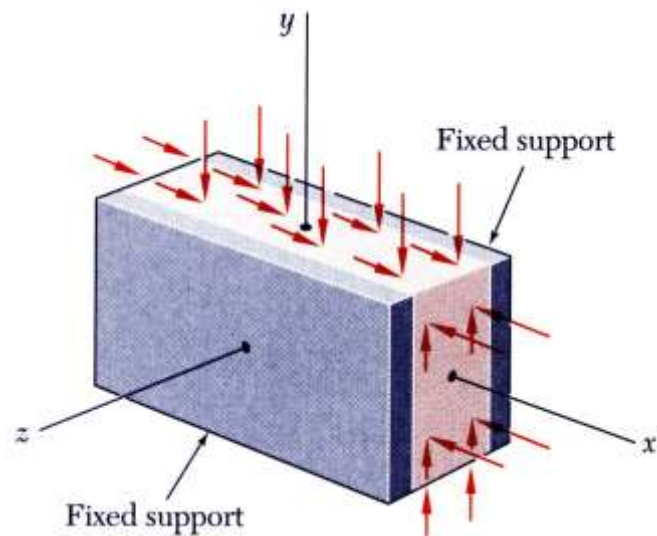
$$\tau_{\max(\text{in-plane})} = 0$$

- Maximum out-of-plane shearing stress

$$\tau_{\max} = \frac{1}{2}\sigma_1 = \frac{pr}{4t}$$



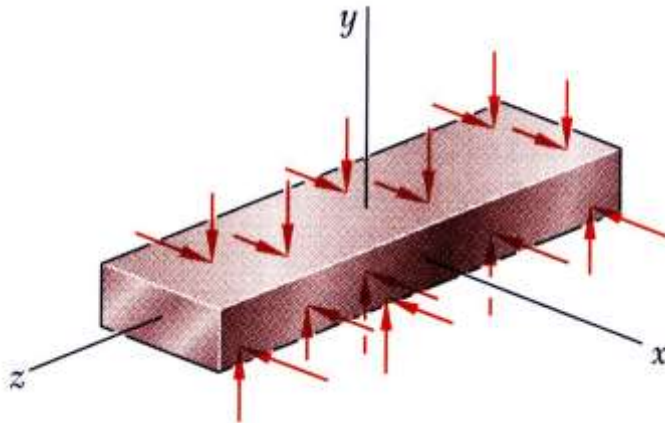
Transformation of Plane Strain



- *Plane strain* - deformations of the material take place in parallel planes and are the same in each of those planes.
- Plane strain occurs in a plate subjected along its edges to a uniformly distributed load and restrained from expanding or contracting laterally by smooth, rigid and fixed supports

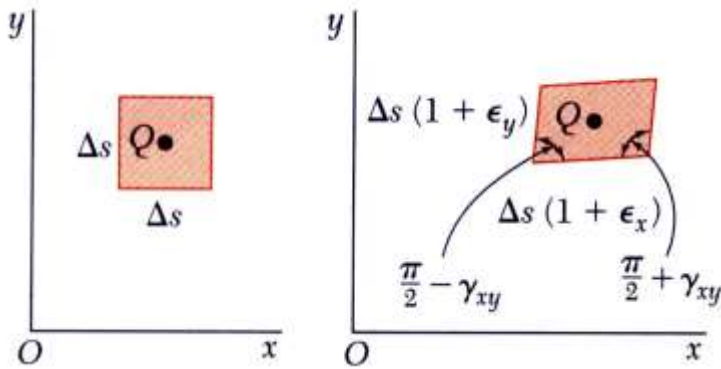
components of strain :

$$\epsilon_x \quad \epsilon_y \quad \gamma_{xy} \quad (\epsilon_z = \gamma_{zx} = \gamma_{zy} = 0)$$



- Example: Consider a long bar subjected to uniformly distributed transverse loads. State of plane stress exists in any transverse section not located too close to the ends of the bar.

Transformation of Plane Strain



- State of strain at the point Q results in different strain components with respect to the xy and $x'y'$ reference frames.

$$\varepsilon(\theta) = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$$

$$\varepsilon_{OB} = \varepsilon(45^\circ) = \frac{1}{2}(\varepsilon_x + \varepsilon_y + \gamma_{xy})$$

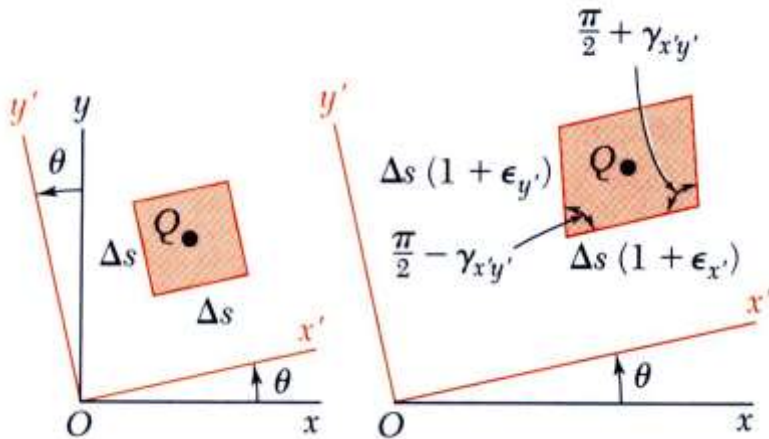
$$\gamma_{xy} = 2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)$$

- Applying the trigonometric relations used for the transformation of stress,

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$



Mohr's Circle for Plane Strain

- The equations for the transformation of plane strain are of the same form as the equations for the transformation of plane stress - *Mohr's circle techniques apply.*

- Abscissa for the center C and radius R ,

$$\epsilon_{ave} = \frac{\epsilon_x + \epsilon_y}{2} \quad R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

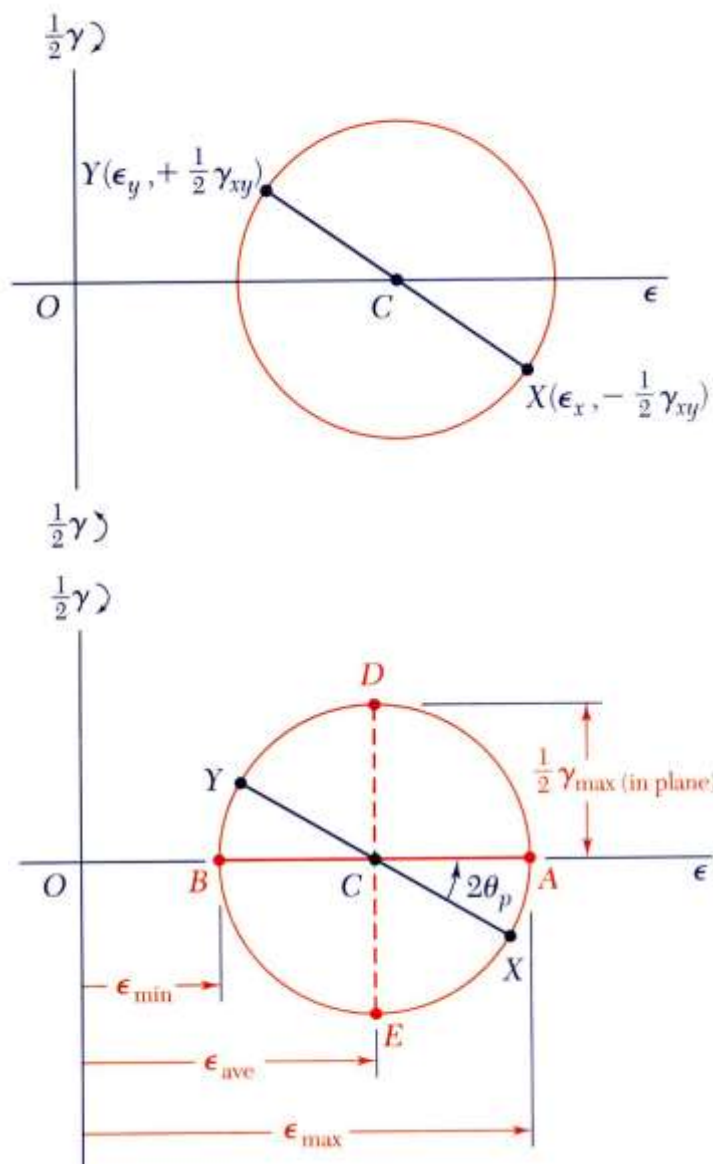
- Principal axes of strain and principal strains,

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

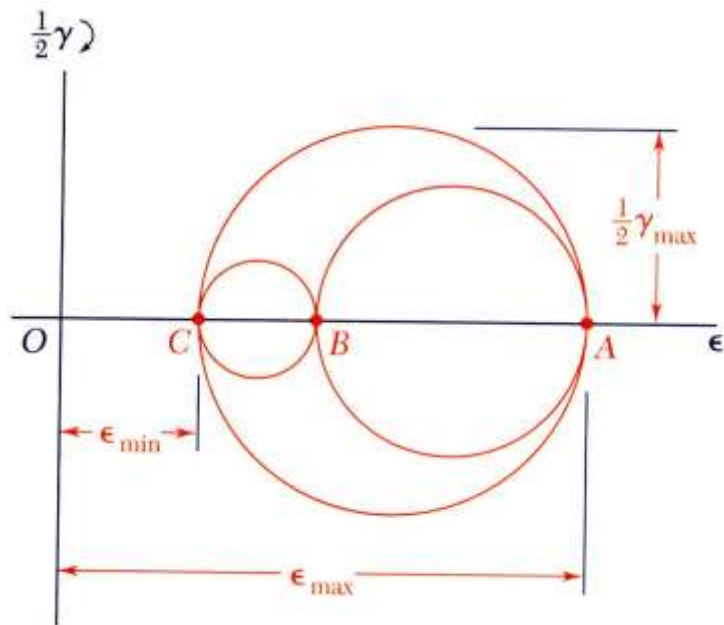
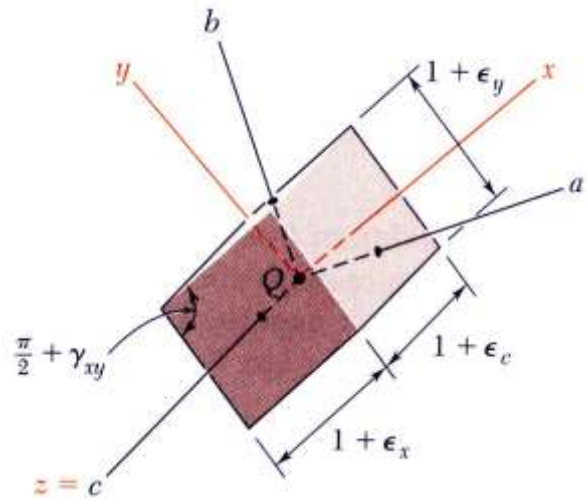
$$\epsilon_{max} = \epsilon_{ave} + R \quad \epsilon_{min} = \epsilon_{ave} - R$$

- Maximum in-plane shearing strain,

$$\gamma_{max} = 2R = \sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2}$$

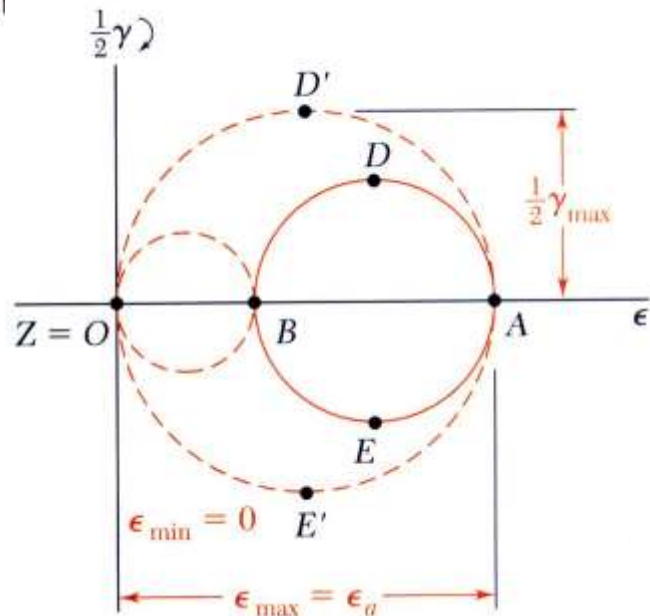
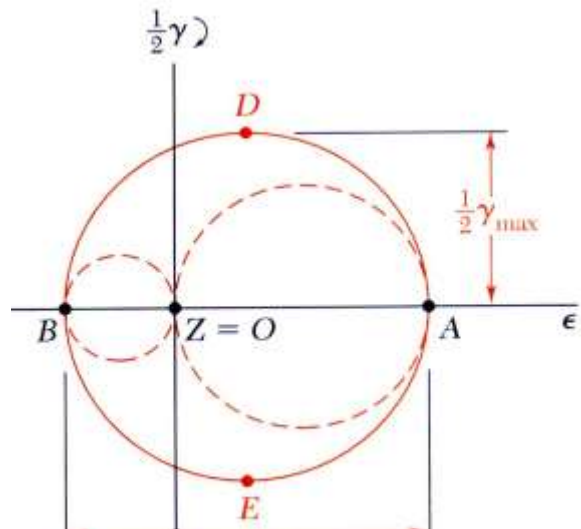


Three-Dimensional Analysis of Strain



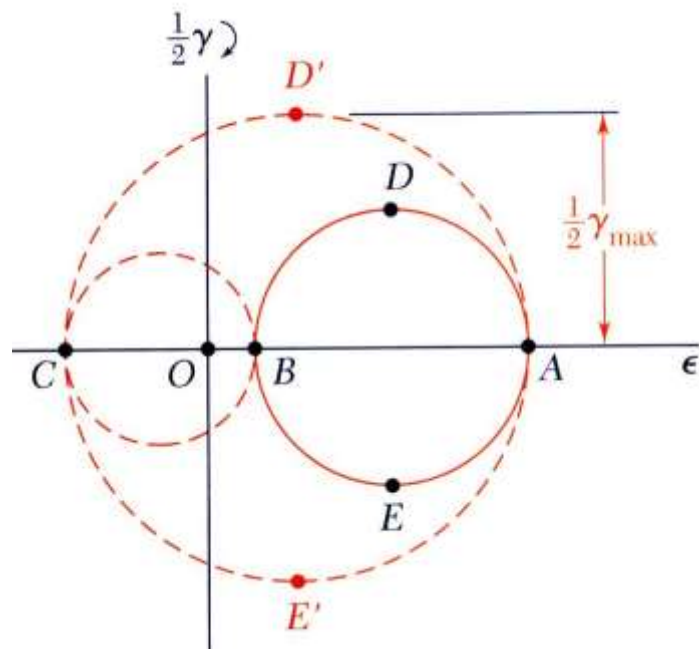
- Previously demonstrated that three principal axes exist such that the perpendicular element faces are free of shearing stresses.
- By Hooke's Law, it follows that the shearing strains are zero as well and that the principal planes of stress are also the principal planes of strain.
- Rotation about the principal axes may be represented by Mohr's circles.

Three-Dimensional Analysis of Strain



- For the case of plane strain where the x and y axes are in the plane of strain,
 - the z axis is also a principal axis
 - the corresponding principal normal strain is represented by the point $Z = 0$ or the origin.
- If the points A and B lie on opposite sides of the origin, the maximum shearing strain is the maximum in-plane shearing strain, D and E .
- If the points A and B lie on the same side of the origin, the maximum shearing strain is out of the plane of strain and is represented by the points D' and E' .

Three-Dimensional Analysis of Strain



- Consider the case of plane stress,

$$\sigma_x = \sigma_a \quad \sigma_y = \sigma_b \quad \sigma_z = 0$$

- Corresponding normal strains,

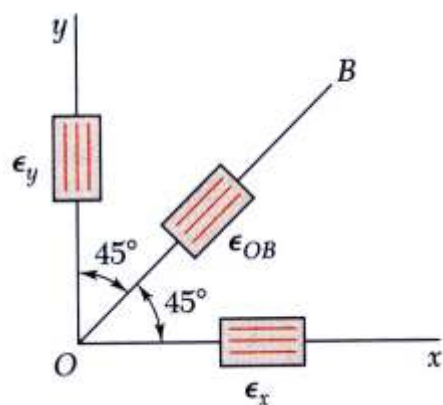
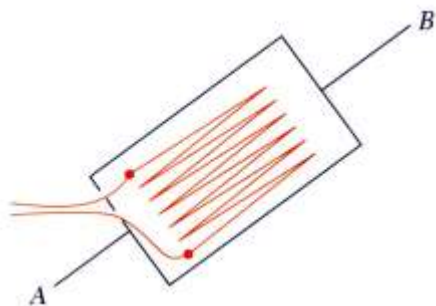
$$\varepsilon_a = \frac{\sigma_a}{E} - \frac{\nu \sigma_b}{E}$$

$$\varepsilon_b = -\frac{\nu \sigma_a}{E} + \frac{\sigma_b}{E}$$

$$\varepsilon_c = -\frac{\nu}{E}(\sigma_a + \sigma_b) = -\frac{\nu}{1-\nu}(\varepsilon_a + \varepsilon_b)$$

- Strain perpendicular to the plane of stress is not zero.
- If B is located between A and C on the Mohr-circle diagram, the maximum shearing strain is equal to the diameter CA .

Measurements of Strain: Strain Rosette



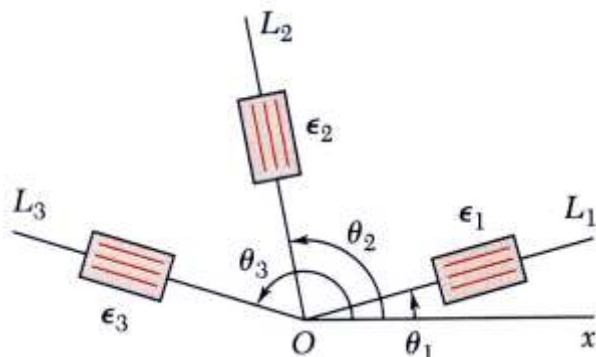
- Strain gages indicate normal strain through changes in resistance.
- With a 45° rosette, ϵ_x and ϵ_y are measured directly. γ_{xy} is obtained indirectly with,

$$\gamma_{xy} = 2\epsilon_{OB} - (\epsilon_x + \epsilon_y)$$
- Normal and shearing strains may be obtained from normal strains in any three directions,

$$\epsilon_1 = \epsilon_x \cos^2 \theta_1 + \epsilon_y \sin^2 \theta_1 + \gamma_{xy} \sin \theta_1 \cos \theta_1$$

$$\epsilon_2 = \epsilon_x \cos^2 \theta_2 + \epsilon_y \sin^2 \theta_2 + \gamma_{xy} \sin \theta_2 \cos \theta_2$$

$$\epsilon_3 = \epsilon_x \cos^2 \theta_3 + \epsilon_y \sin^2 \theta_3 + \gamma_{xy} \sin \theta_3 \cos \theta_3$$





The cracks in this concrete beam were caused by tension stress, even though the beam was subjected to both an internal moment and shear. The stress transformation equations can be used to predict the direction of the cracks, and the principal normal stresses that caused them.



IMPORTANT POINTS

- The *principal stresses* represent the maximum and minimum normal stress at the point.
- When the state of stress is represented by the principal stresses, *no shear stress* will act on the element.
- The state of stress at the point can also be represented in terms of the *maximum in-plane shear stress*. In this case an *average normal stress* will also act on the element.
- The element representing the maximum in-plane shear stress with the associated average normal stresses is oriented 45° from the element representing the principal stresses.



CHAPTER

8

MECHANICS OF MATERIALS

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Principle Stresses Under a Given Loading

Principle Stresses Under a Given Loading

[Introduction](#)

[Principle Stresses in a Beam](#)

[Sample Problem 8.1](#)

[Sample Problem 8.2](#)

[Design of a Transmission Shaft](#)

[Sample Problem 8.3](#)

[Stresses Under Combined Loadings](#)

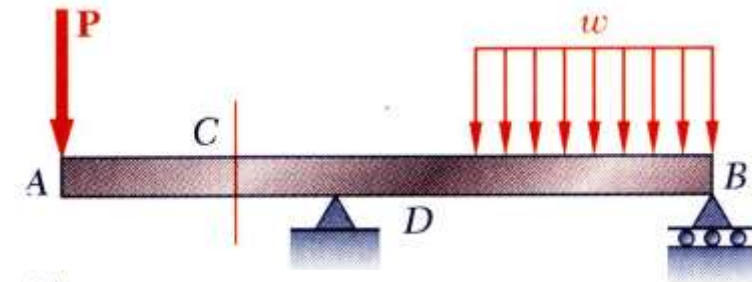
[Sample Problem 8.5](#)



Introduction

- In Chaps. 1 and 2, you learned how to determine the normal stress due to centric loads
In Chap. 3, you analyzed the distribution of shearing stresses in a circular member due to a twisting couple
In Chap. 4, you determined the normal stresses caused by bending couples
In Chaps. 5 and 6, you evaluated the shearing stresses due to transverse loads
In Chap. 7, you learned how the components of stress are transformed by a rotation of the coordinate axes and how to determine the principal planes, principal stresses, and maximum shearing stress at a point.
- In Chapter 8, you will learn how to determine the stress in a structural member or machine element due to a combination of loads and how to find the corresponding principal stresses and maximum shearing stress

Principle Stresses in a Beam



- Prismatic beam subjected to transverse loading

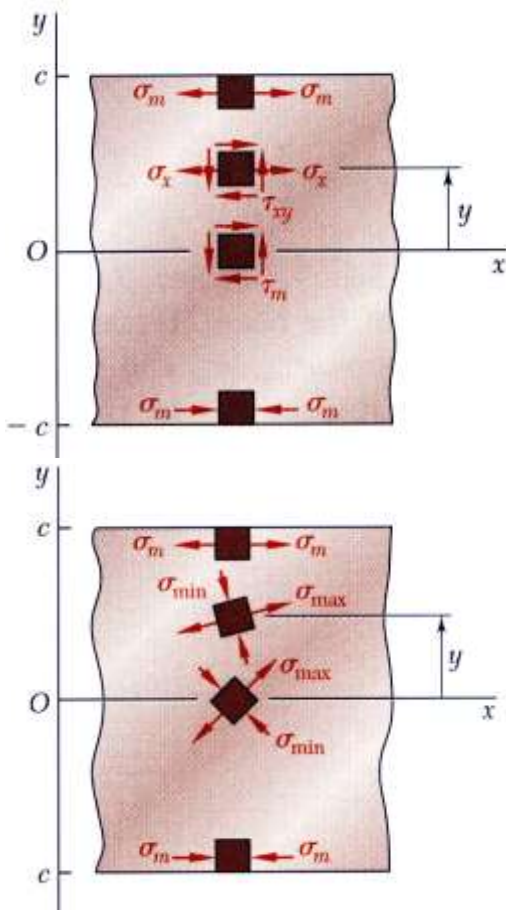
$$\sigma_x = -\frac{My}{I} \quad \sigma_m = \frac{Mc}{I}$$

$$\tau_{xy} = -\frac{VQ}{It} \quad \tau_m = \frac{VQ}{It}$$

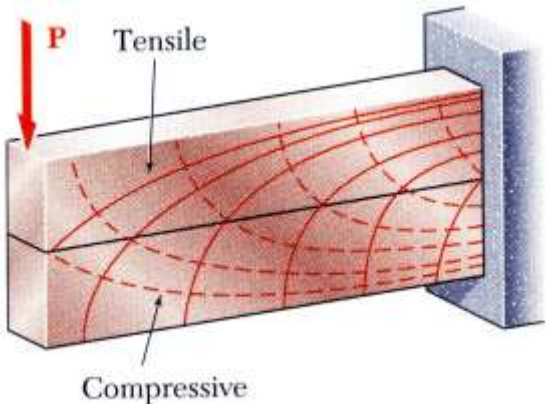
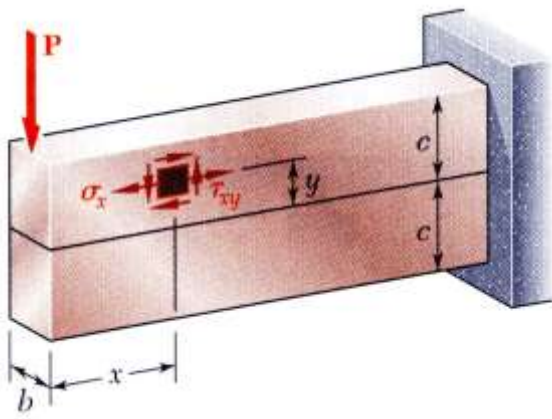
- Principal stresses determined from methods of Chapter 7

- Can the maximum normal stress within the cross-section be larger than

$$\sigma_m = \frac{Mc}{I}$$

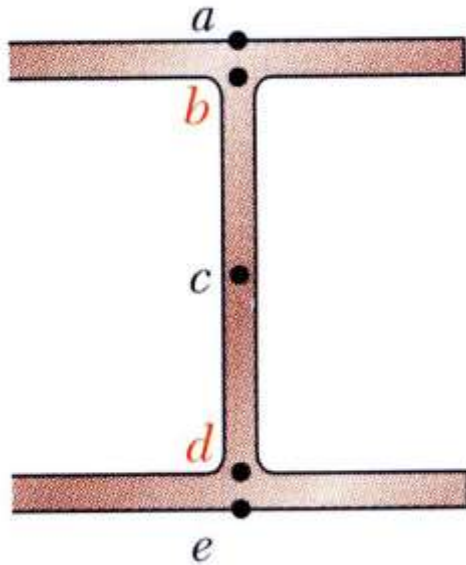


Principle Stresses in a Beam



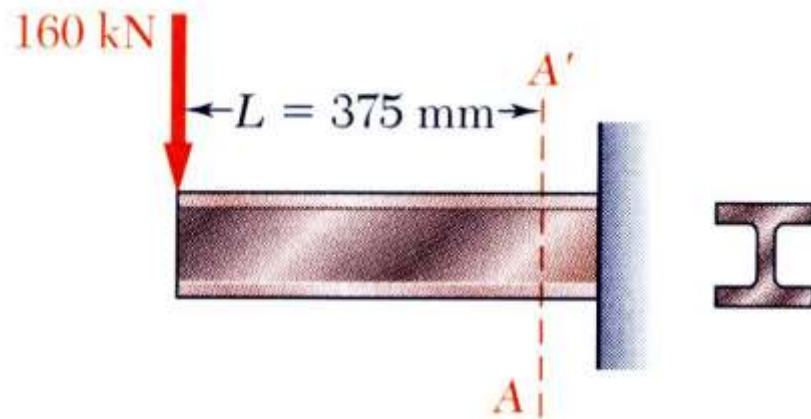
y/c	$x = 2c$		$x = 8c$	
	$\sigma_{\min}/\sigma_m$	$\sigma_{\max}/\sigma_m$	$\sigma_{\min}/\sigma_m$	$\sigma_{\max}/\sigma_m$
1.0	0	1.000	0	1.000
0.8	-0.010	0.810	-0.001	0.801
0.6	-0.040	0.640	-0.003	0.603
0.4	-0.090	0.490	-0.007	0.407
0.2	-0.160	0.360	-0.017	0.217
0	-0.250	0.250	-0.063	0.063
-0.2	-0.360	0.160	-0.217	0.017
-0.4	-0.490	0.090	-0.407	0.007
-0.6	-0.640	0.040	-0.603	0.003
-0.8	-0.810	0.010	-0.801	0.001
-1.0	-1.000	0	-1.000	0

Principle Stresses in a Beam



- Cross-section shape results in large values of τ_{xy} near the surface where σ_x is also large.
- σ_{max} may be greater than σ_m

Sample Problem 8.1



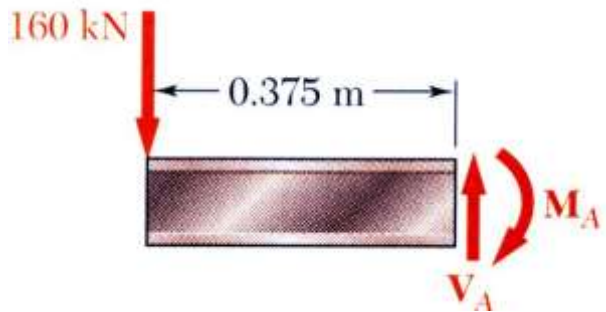
A 160-kN force is applied at the end of a W200x52 rolled-steel beam.

Neglecting the effects of fillets and of stress concentrations, determine whether the normal stresses satisfy a design specification that they be equal to or less than 150 MPa at section A-A'.

SOLUTION:

- Determine shear and bending moment in Section A-A'
- Calculate the normal stress at top surface and at flange-web junction.
- Evaluate the shear stress at flange-web junction.
- Calculate the principal stress at flange-web junction

Sample Problem 8.1

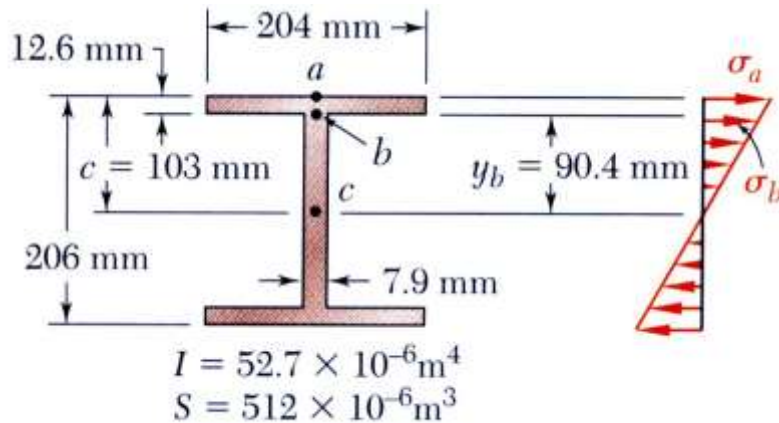


SOLUTION:

- Determine shear and bending moment in Section A-A'

$$M_A = (160 \text{ kN})(0.375 \text{ m}) = 60 \text{ kN} \cdot \text{m}$$

$$V_A = 160 \text{ kN}$$

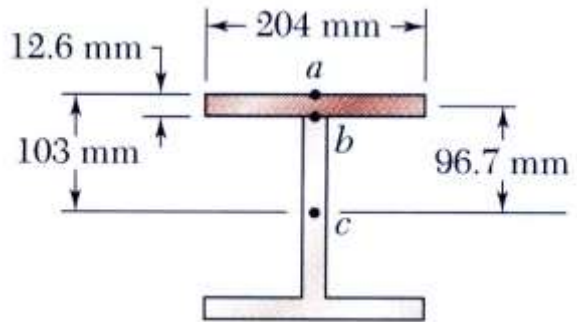


- Calculate the normal stress at top surface and at flange-web junction.

$$\sigma_a = \frac{M_A}{S} = \frac{60 \text{ kN} \cdot \text{m}}{512 \times 10^{-6} \text{ m}^3} = 117.2 \text{ MPa}$$

$$\sigma_b = \sigma_a \frac{y_b}{c} = (117.2 \text{ MPa}) \frac{90.4 \text{ mm}}{103 \text{ mm}} = 102.9 \text{ MPa}$$

Sample Problem 8.1



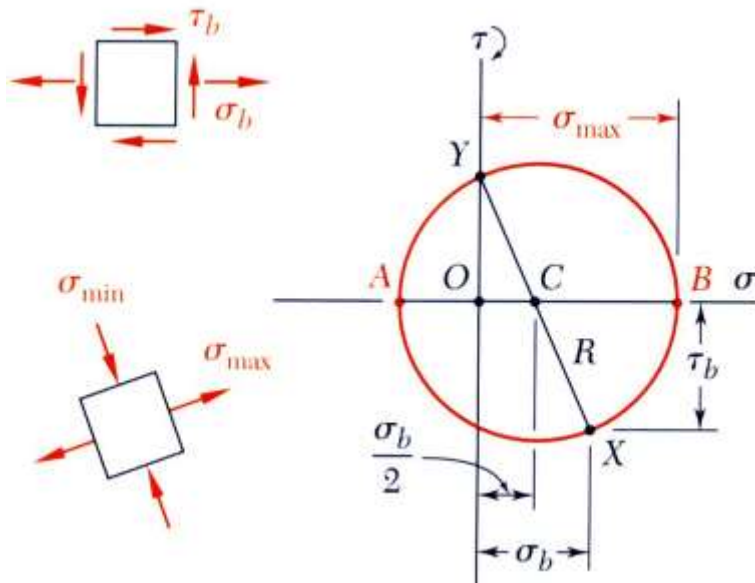
- Evaluate shear stress at flange-web junction.

$$Q = (204 \times 12.6)96.7 = 248.6 \times 10^3 \text{ mm}^3$$

$$= 248.6 \times 10^{-6} \text{ m}^3$$

$$\tau_b = \frac{V_A Q}{I t} = \frac{(160 \text{ kN})(248.6 \times 10^{-6} \text{ m}^3)}{(52.7 \times 10^{-6} \text{ m}^4)(0.0079 \text{ m})}$$

$$= 95.5 \text{ MPa}$$



- Calculate the principal stress at flange-web junction

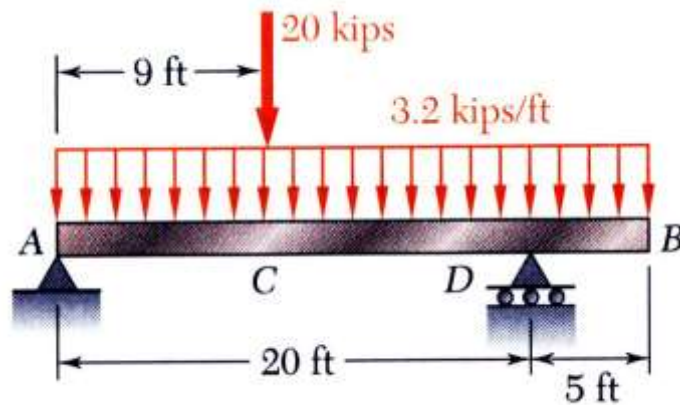
$$\sigma_{\max} = \frac{1}{2} \sigma_b + \sqrt{\left(\frac{1}{2} \sigma_b\right)^2 + \tau_b^2}$$

$$= \frac{102.9}{2} + \sqrt{\left(\frac{102.9}{2}\right)^2 + (95.5)^2}$$

$$= 169.9 \text{ MPa} \quad (> 150 \text{ MPa})$$

Design specification is not satisfied.

Sample Problem 8.2

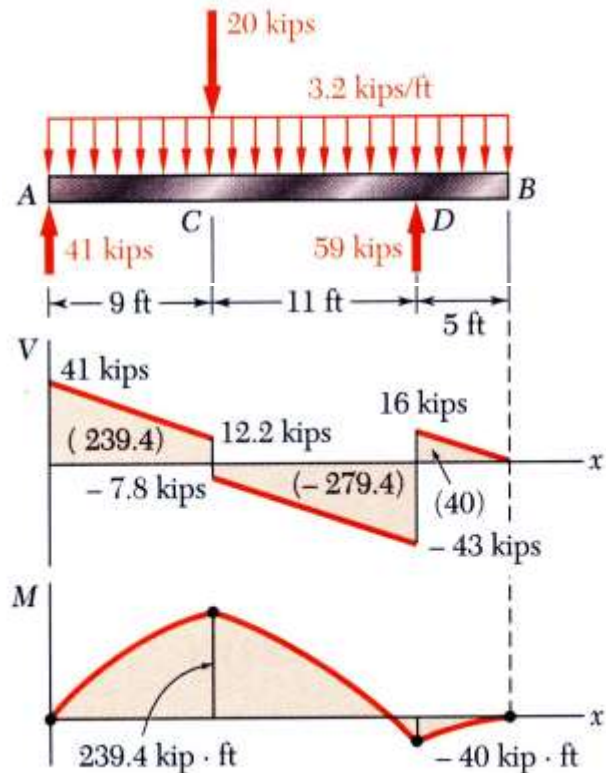


The overhanging beam supports a uniformly distributed load and a concentrated load. Knowing that for the grade of steel to be used $\sigma_{all} = 24$ ksi and $\tau_{all} = 14.5$ ksi, select the wide-flange beam which should be used.

SOLUTION:

- Determine reactions at A and D.
- Determine maximum shear and bending moment from shear and bending moment diagrams.
- Calculate required section modulus and select appropriate beam section.
- Find maximum normal stress.
- Find maximum shearing stress.

Sample Problem 8.2



SOLUTION:

- Determine reactions at A and D.

$$\sum M_A = 0 \Rightarrow R_D = 59 \text{ kips}$$

$$\sum M_D = 0 \Rightarrow R_A = 41 \text{ kips}$$

- Determine maximum shear and bending moment from shear and bending moment diagrams.

$$|M|_{\max} = 239.4 \text{ kip} \cdot \text{ft} \quad \text{with} \quad V = 12.2 \text{ kips}$$

$$|V|_{\max} = 43 \text{ kips}$$

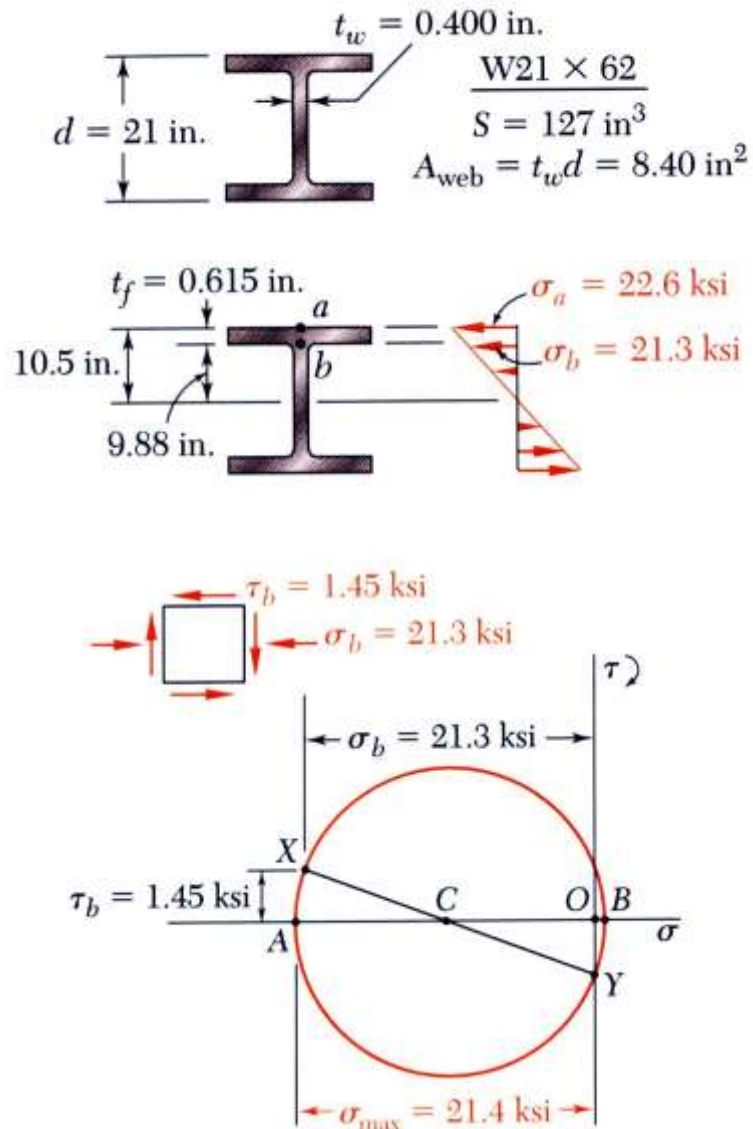
- Calculate required section modulus and select appropriate beam section.

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{\text{all}}} = \frac{239.4 \text{ kip} \cdot \text{ft}}{24 \text{ ksi}} = 119.7 \text{ in}^3$$

select W21×62 beam section

Shape	S (in ³)
W24 × 68	154
W21 × 62	127
W18 × 76	146
W16 × 77	134
W14 × 82	123
W12 × 96	131

Sample Problem 8.2



- Find maximum shearing stress.

Assuming uniform shearing stress in web,

$$\tau_{\text{max}} = \frac{V_{\text{max}}}{A_{\text{web}}} = \frac{43 \text{ kips}}{8.40 \text{ in}^2} = 5.12 \text{ ksi} < 14.5 \text{ ksi}$$

- Find maximum normal stress.

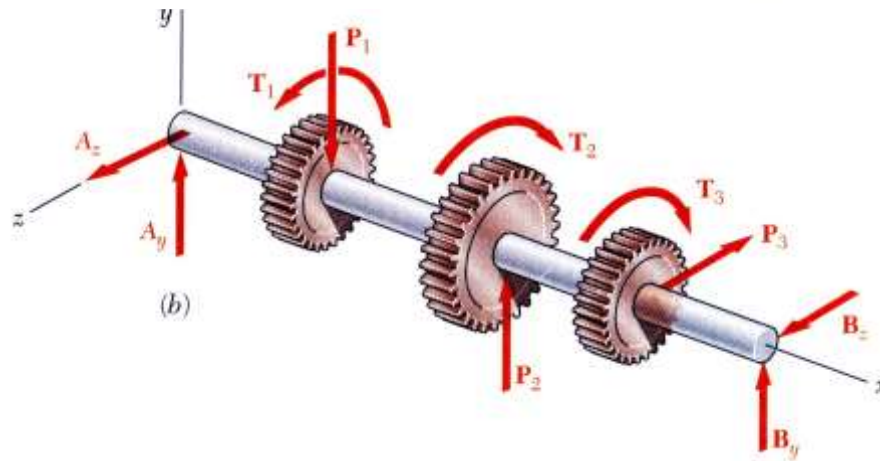
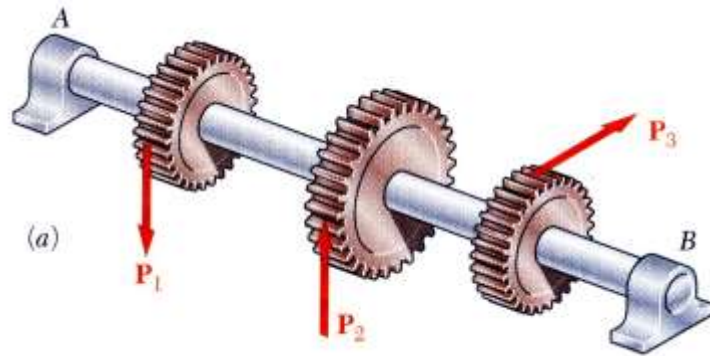
$$\sigma_a = \frac{M_{\text{max}}}{S} = 2873 \frac{60 \text{ kip} \cdot \text{in}}{127 \text{ in}^3} = 22.6 \text{ ksi}$$

$$\sigma_b = \sigma_a \frac{y_b}{c} = (22.6 \text{ ksi}) \frac{9.88}{10.5} = 21.3 \text{ ksi}$$

$$\tau_b = \frac{V}{A_{\text{web}}} = \frac{12.2 \text{ kips}}{8.40 \text{ in}^2} = 1.45 \text{ ksi}$$

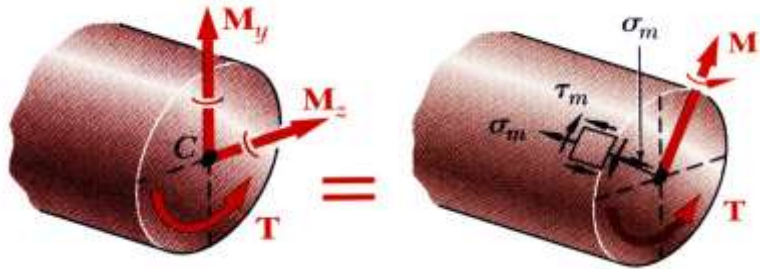
$$\begin{aligned} \sigma_{\text{max}} &= \frac{21.3 \text{ ksi}}{2} + \sqrt{\left(\frac{21.3 \text{ ksi}}{2}\right)^2 + (1.45 \text{ ksi})^2} \\ &= 21.4 \text{ ksi} < 24 \text{ ksi} \end{aligned}$$

Design of a Transmission Shaft



- If power is transferred to and from the shaft by gears or sprocket wheels, the shaft is subjected to transverse loading as well as shear loading.
- Normal stresses due to transverse loads may be large and should be included in determination of maximum shearing stress.
- Shearing stresses due to transverse loads are usually small and contribution to maximum shear stress may be neglected.

Design of a Transmission Shaft



- At any section,

$$\sigma_m = \frac{Mc}{I} \quad \text{where} \quad M^2 = M_y^2 + M_z^2$$

$$\tau_m = \frac{Tc}{J}$$

- Maximum shearing stress,

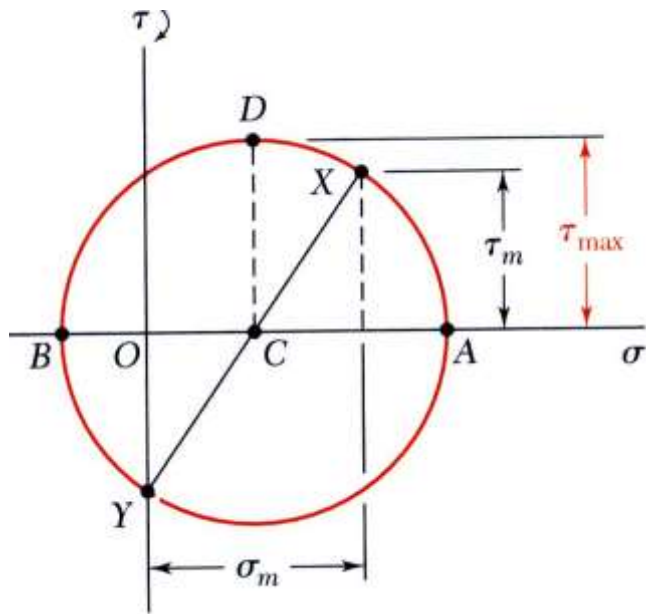
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_m}{2}\right)^2 + (\tau_m)^2} = \sqrt{\left(\frac{Mc}{2I}\right)^2 + \left(\frac{Tc}{J}\right)^2}$$

for a circular or annular cross - section, $2I = J$

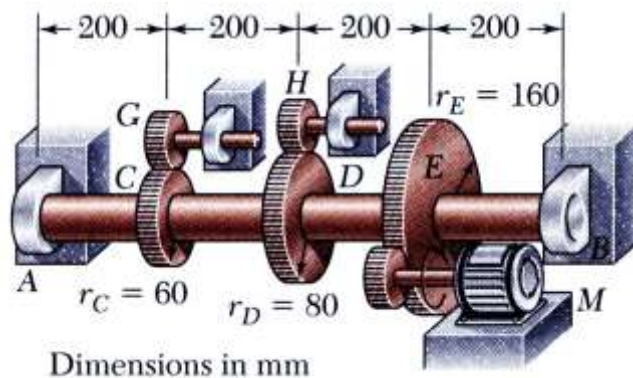
$$\tau_{\max} = \frac{c}{J} \sqrt{M^2 + T^2}$$

- Shaft section requirement,

$$\left(\frac{J}{c}\right)_{\min} = \frac{\left(\sqrt{M^2 + T^2}\right)_{\max}}{\tau_{all}}$$



Sample Problem 8.3

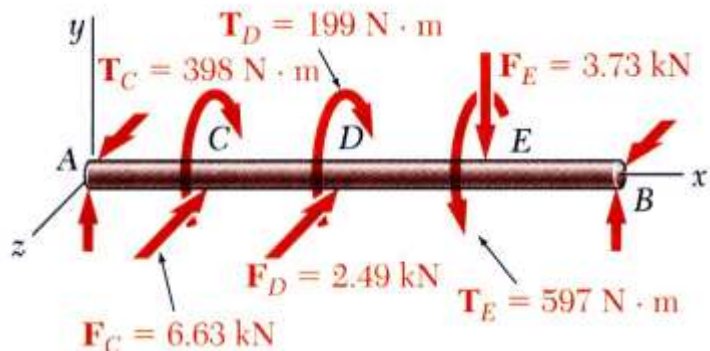
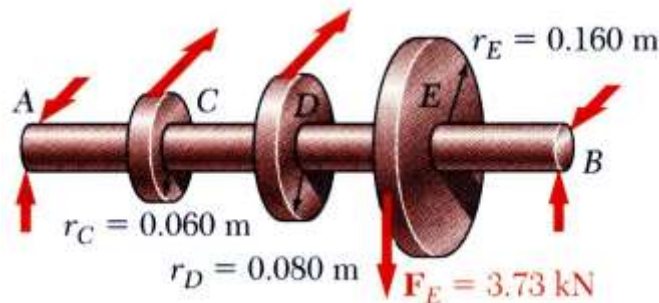
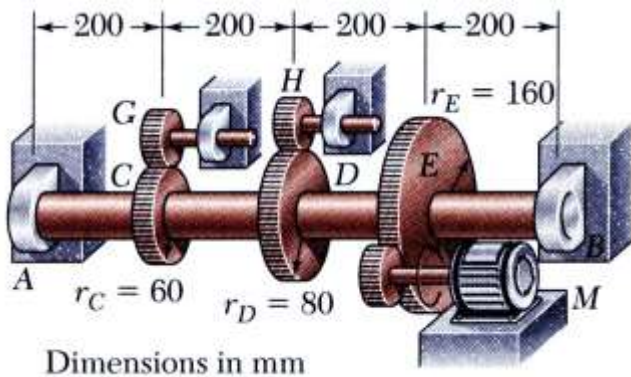


SOLUTION:

- Determine the gear torques and corresponding tangential forces.
- Find reactions at A and B.
- Identify critical shaft section from torque and bending moment diagrams.
- Calculate minimum allowable shaft diameter.

Solid shaft rotates at 480 rpm and transmits 30 kW from the motor to gears G and H; 20 kW is taken off at gear G and 10 kW at gear H. Knowing that $\sigma_{all} = 50$ MPa, determine the smallest permissible diameter for the shaft.

Sample Problem 8.3



SOLUTION:

- Determine the gear torques and corresponding tangential forces.

$$T_E = \frac{P}{2\pi f} = \frac{30 \text{ kW}}{2\pi(80 \text{ Hz})} = 597 \text{ N} \cdot \text{m}$$

$$F_E = \frac{T_E}{r_E} = \frac{597 \text{ N} \cdot \text{m}}{0.16 \text{ m}} = 3.73 \text{ kN}$$

$$T_C = \frac{20 \text{ kW}}{2\pi(80 \text{ Hz})} = 398 \text{ N} \cdot \text{m} \quad F_C = 6.63 \text{ kN}$$

$$T_D = \frac{10 \text{ kW}}{2\pi(80 \text{ Hz})} = 199 \text{ N} \cdot \text{m} \quad F_D = 2.49 \text{ kN}$$

- Find reactions at A and B.

$$A_y = 0.932 \text{ kN} \quad A_z = 6.22 \text{ kN}$$

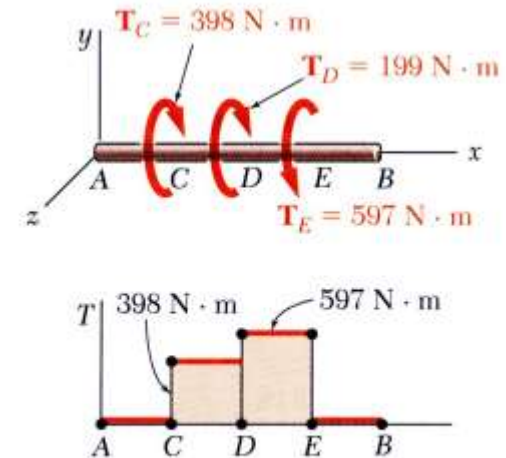
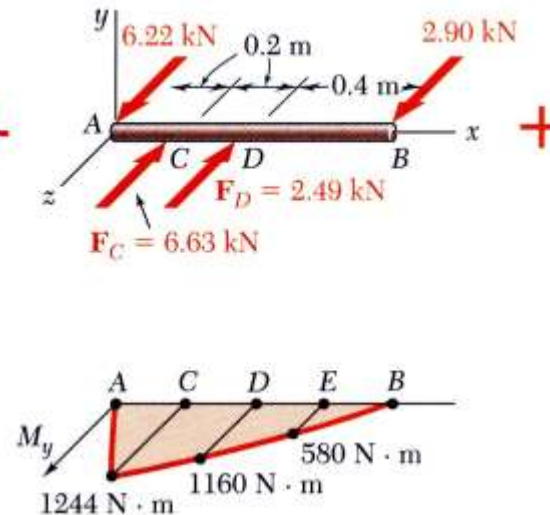
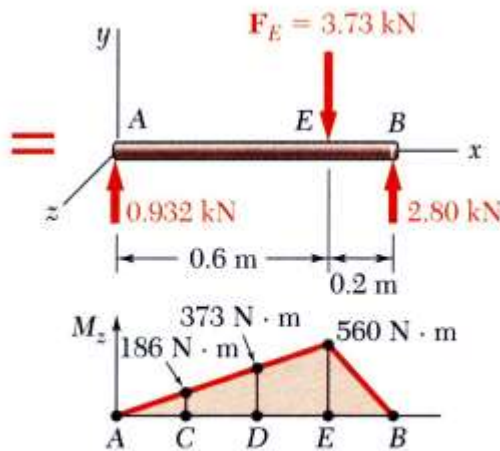
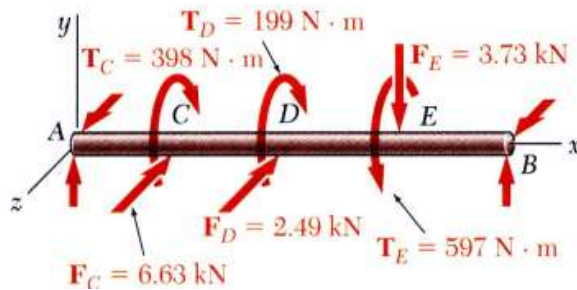
$$B_y = 2.80 \text{ kN} \quad B_z = 2.90 \text{ kN}$$

Sample Problem 8.3

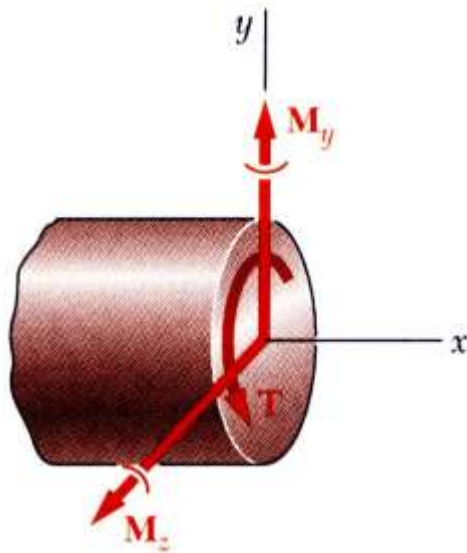
- Identify critical shaft section from torque and bending moment diagrams.

$$\left(\sqrt{M^2 + T^2} \right)_{\max} = \sqrt{(1160^2 + 373^2) + 597^2}$$

$$= 1357 \text{ N} \cdot \text{m}$$



Sample Problem 8.3



- Calculate minimum allowable shaft diameter.

$$\frac{J}{c} = \frac{\sqrt{M^2 + T^2}}{\tau_{all}} = \frac{1357 \text{ N} \cdot \text{m}}{50 \text{ MPa}} = 27.14 \times 10^{-6} \text{ m}^3$$

For a solid circular shaft,

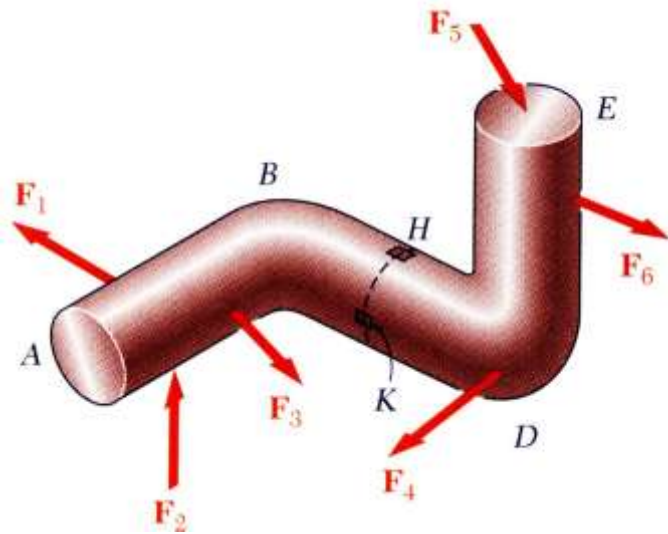
$$\frac{J}{c} = \frac{\pi}{2} c^3 = 27.14 \times 10^{-6} \text{ m}^3$$

$$c = 0.02585 \text{ m} = 25.85 \text{ mm}$$

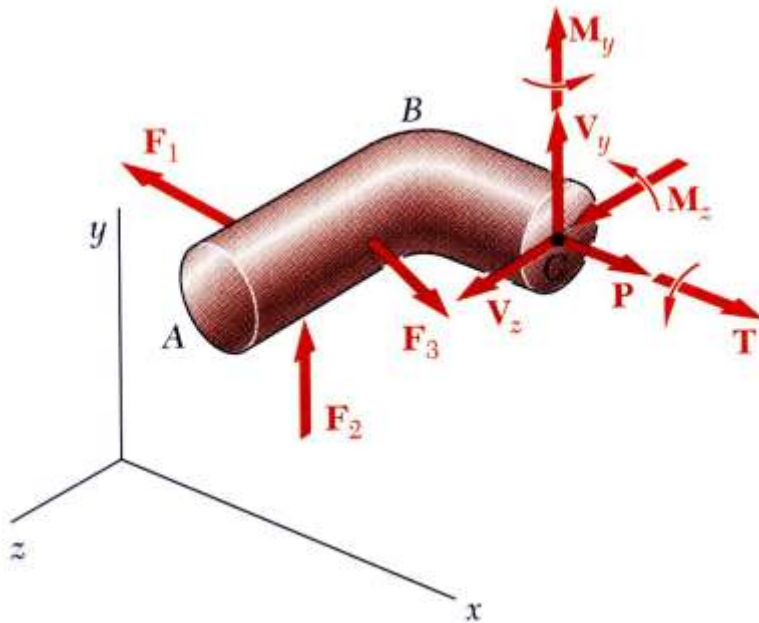
$$d = 2c = 51.7 \text{ mm}$$



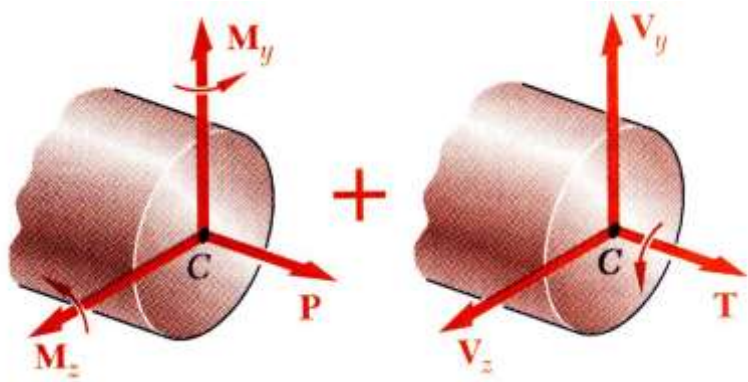
Stresses Under Combined Loadings



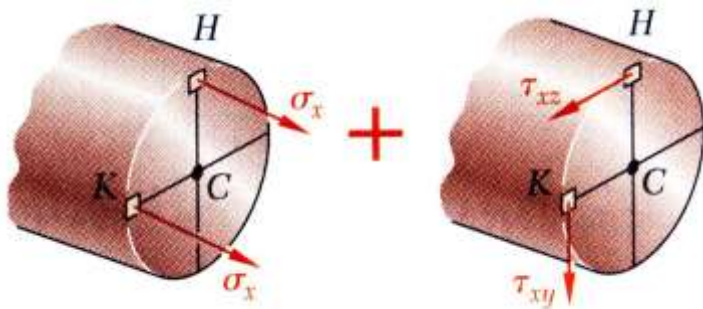
- Wish to determine stresses in slender structural members subjected to arbitrary loadings.
- Pass section through points of interest. Determine force-couple system at centroid of section required to maintain equilibrium.
- System of internal forces consist of three force components and three couple vectors.
- Determine stress distribution by applying the superposition principle.



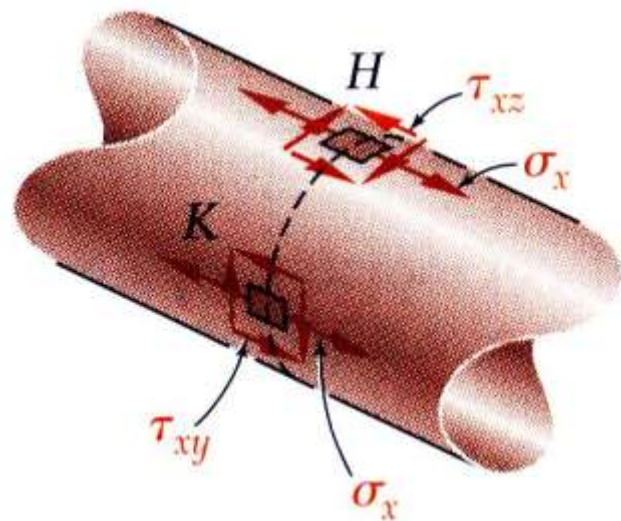
Stresses Under Combined Loadings



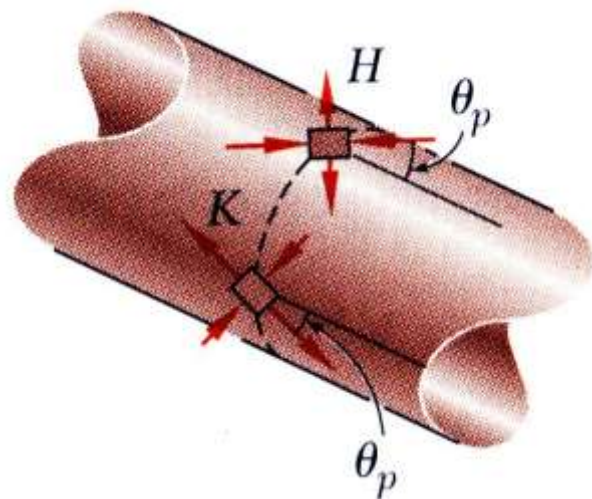
- Axial force and in-plane couple vectors contribute to normal stress distribution in the section.
- Shear force components and twisting couple contribute to shearing stress distribution in the section.



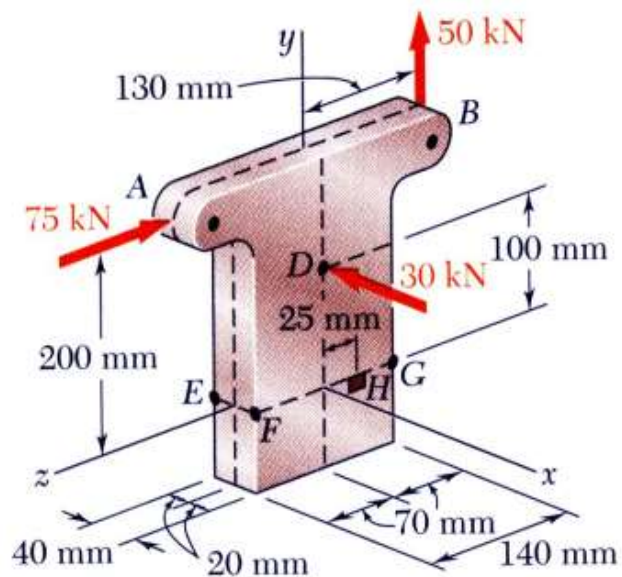
Stresses Under Combined Loadings



- Normal and shearing stresses are used to determine principal stresses, maximum shearing stress and orientation of principal planes.
- Analysis is valid only to extent that conditions of applicability of superposition principle and Saint-Venant's principle are met.



Sample Problem 8.5

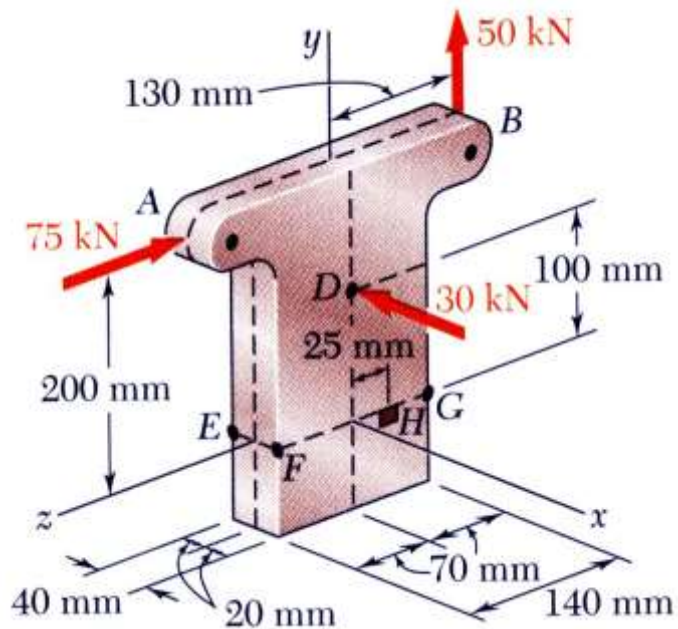


SOLUTION:

- Determine internal forces in Section EFG .
- Evaluate normal stress at H .
- Evaluate shearing stress at H .
- Calculate principal stresses and maximum shearing stress. Determine principal planes.

Three forces are applied to a short steel post as shown. Determine the principle stresses, principal planes and maximum shearing stress at point H .

Sample Problem 8.5



SOLUTION:

- Determine internal forces in Section *EFG*.

$$V_x = -30 \text{ kN} \quad P = 50 \text{ kN} \quad V_z = -75 \text{ kN}$$

$$M_x = (50 \text{ kN})(0.130 \text{ m}) - (75 \text{ kN})(0.200 \text{ m}) \\ = -8.5 \text{ kN} \cdot \text{m}$$

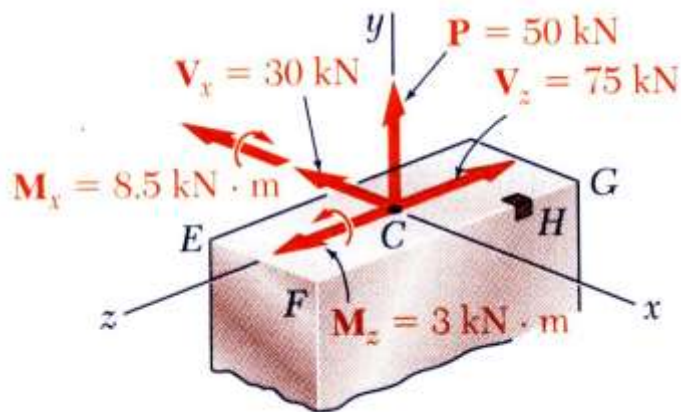
$$M_y = 0 \quad M_z = (30 \text{ kN})(0.100 \text{ m}) = 3 \text{ kN} \cdot \text{m}$$

Note: Section properties,

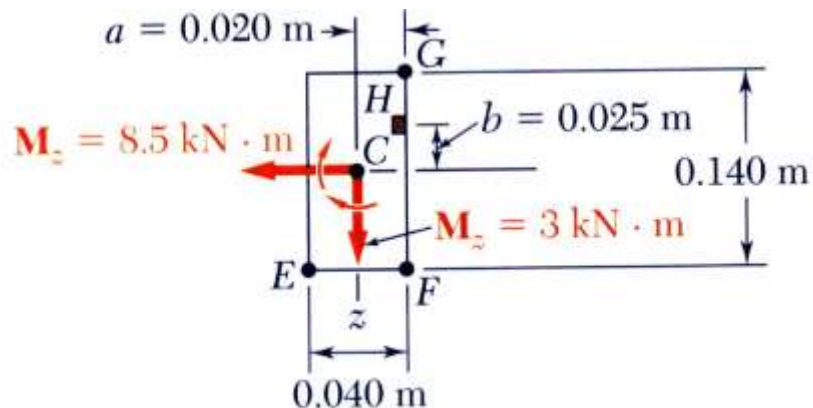
$$A = (0.040 \text{ m})(0.140 \text{ m}) = 5.6 \times 10^{-3} \text{ m}^2$$

$$I_x = \frac{1}{12} (0.040 \text{ m})(0.140 \text{ m})^3 = 9.15 \times 10^{-6} \text{ m}^4$$

$$I_z = \frac{1}{12} (0.140 \text{ m})(0.040 \text{ m})^3 = 0.747 \times 10^{-6} \text{ m}^4$$

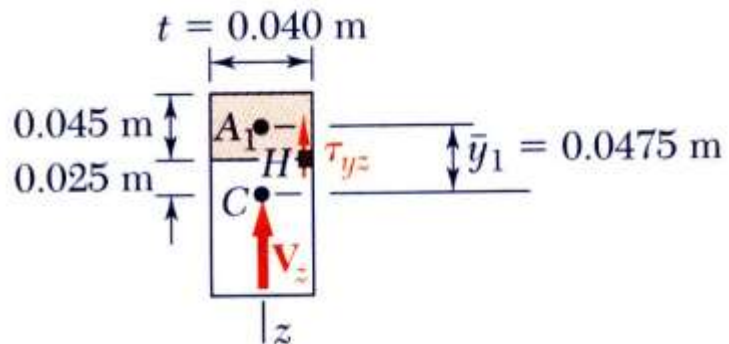


Sample Problem 8.5



- Evaluate normal stress at H .

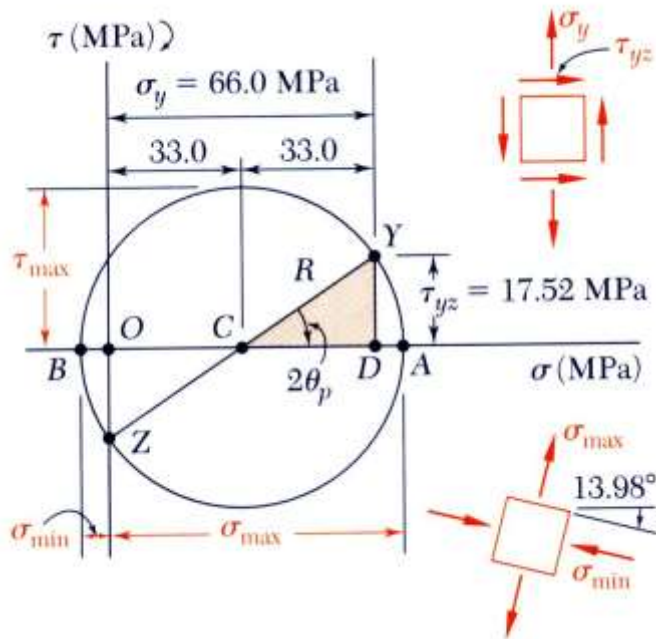
$$\begin{aligned}\sigma_y &= +\frac{P}{A} + \frac{|M_z|a}{I_z} - \frac{|M_x|b}{I_x} \\ &= \frac{50 \text{ kN}}{5.6 \times 10^{-3} \text{ m}^2} + \frac{(3 \text{ kN} \cdot \text{m})(0.020 \text{ m})}{0.747 \times 10^{-6} \text{ m}^4} \\ &\quad - \frac{(8.5 \text{ kN} \cdot \text{m})(0.025 \text{ m})}{9.15 \times 10^{-6} \text{ m}^4} \\ &= (8.93 + 80.3 - 23.2) \text{ MPa} = 66.0 \text{ MPa}\end{aligned}$$



- Evaluate shearing stress at H .

$$\begin{aligned}Q &= A_1 \bar{y}_1 = [(0.040 \text{ m})(0.045 \text{ m})](0.0475 \text{ m}) \\ &= 85.5 \times 10^{-6} \text{ m}^3 \\ \tau_{yz} &= \frac{V_z Q}{I_x t} = \frac{(75 \text{ kN})(85.5 \times 10^{-6} \text{ m}^3)}{(9.15 \times 10^{-6} \text{ m}^4)(0.040 \text{ m})} \\ &= 17.52 \text{ MPa}\end{aligned}$$

Sample Problem 8.5



- Calculate principal stresses and maximum shearing stress.
Determine principal planes.

$$\tau_{\max} = R = \sqrt{33.0^2 + 17.52^2} = 37.4 \text{ MPa}$$

$$\sigma_{\max} = OC + R = 33.0 + 37.4 = 70.4 \text{ MPa}$$

$$\sigma_{\min} = OC - R = 33.0 - 37.4 = -7.4 \text{ MPa}$$

$$\tan 2\theta_p = \frac{CY}{CD} = \frac{17.52}{33.0} \quad 2\theta_p = 27.96^\circ$$

$$\theta_p = 13.98^\circ$$

$$\tau_{\max} = 37.4 \text{ MPa}$$

$$\sigma_{\max} = 70.4 \text{ MPa}$$

$$\sigma_{\min} = -7.4 \text{ MPa}$$

$$\theta_p = 13.98^\circ$$

MECHANICS OF MATERIALS

CHAPTER

10

Ferdinand P. Beer
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Lecture Notes:
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Columns

Columns

[Stability of Structures](#)

[Euler's Formula for Pin-Ended Beams](#)

[Extension of Euler's Formula](#)

[Sample Problem 10.1](#)

[Eccentric Loading; The Secant Formula](#)

[Sample Problem 10.2](#)

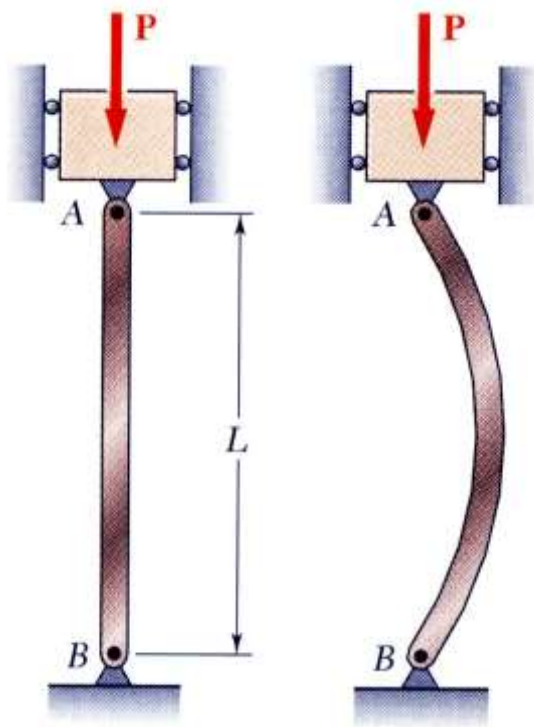
[Design of Columns Under Centric Load](#)

[Sample Problem 10.4](#)

[Design of Columns Under an Eccentric Load](#)



Stability of Structures



- In the design of columns, cross-sectional area is selected such that

- allowable stress is not exceeded

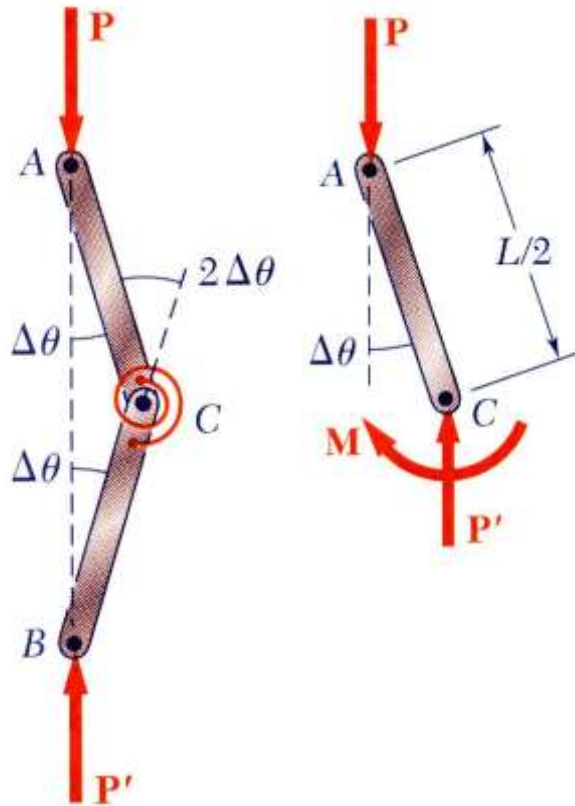
$$\sigma = \frac{P}{A} \leq \sigma_{all}$$

- deformation falls within specifications

$$\delta = \frac{PL}{AE} \leq \delta_{spec}$$

- After these design calculations, may discover that the column is unstable under loading and that it suddenly becomes sharply curved or buckles.

Stability of Structures



- Consider model with two rods and torsional spring. After a small perturbation,

$$K(2\Delta\theta) = \text{restoring moment}$$

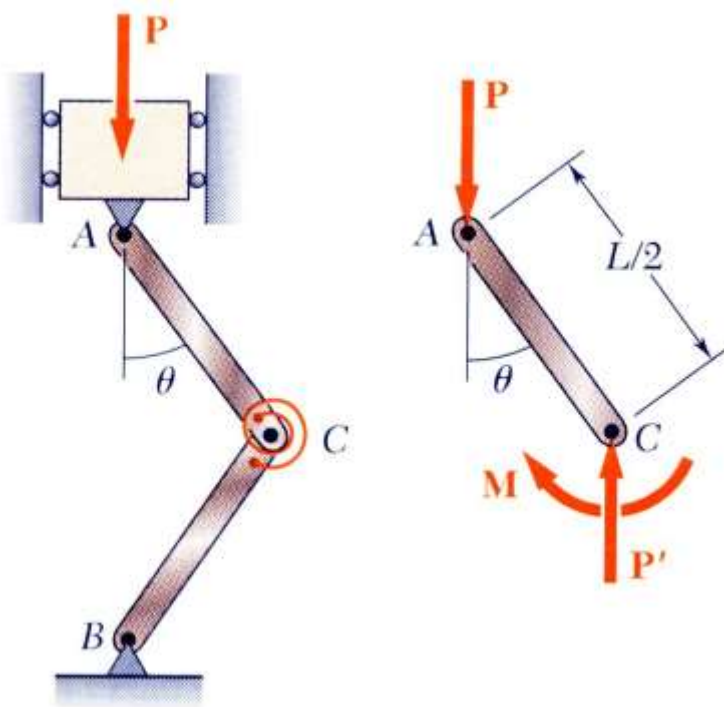
$$P \frac{L}{2} \sin \Delta\theta = P \frac{L}{2} \Delta\theta = \text{destabilizing moment}$$

- Column is stable (tends to return to aligned orientation) if

$$P \frac{L}{2} \Delta\theta < K(2\Delta\theta)$$

$$P < P_{cr} = \frac{4K}{L}$$

Stability of Structures



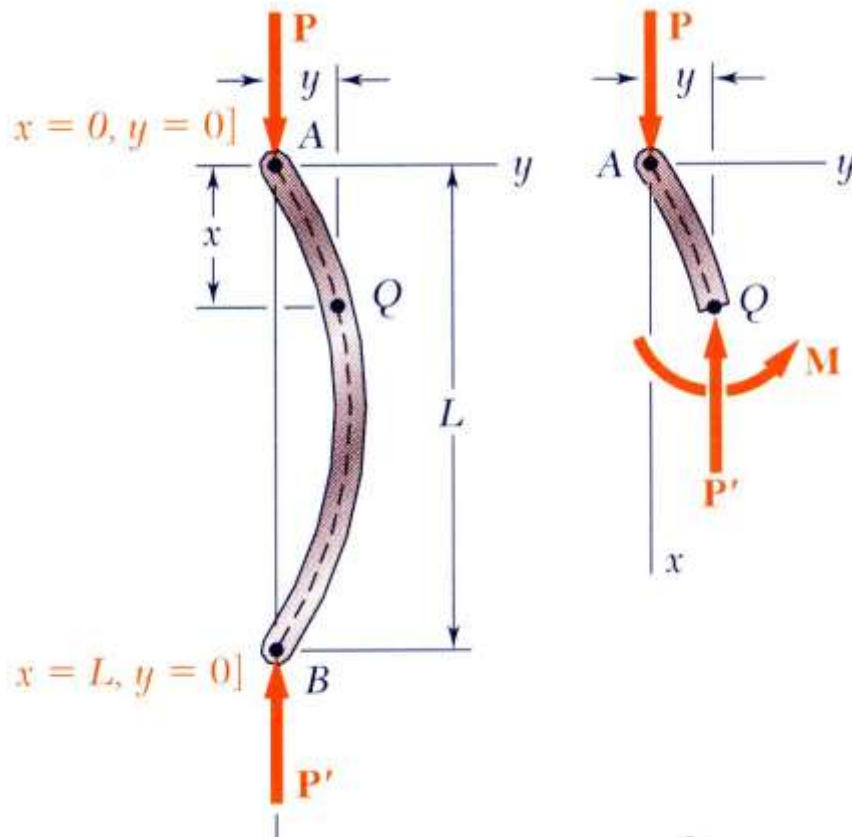
- Assume that a load P is applied. After a perturbation, the system settles to a new equilibrium configuration at a finite deflection angle.

$$P \frac{L}{2} \sin \theta = K(2\theta)$$

$$\frac{PL}{4K} = \frac{P}{P_{cr}} = \frac{\theta}{\sin \theta}$$

- Noting that $\sin \theta < \theta$, the assumed configuration is only possible if $P > P_{cr}$.

Euler's Formula for Pin-Ended Beams



- Consider an axially loaded beam. After a small perturbation, the system reaches an equilibrium configuration such that

$$\frac{d^2 y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI} y$$

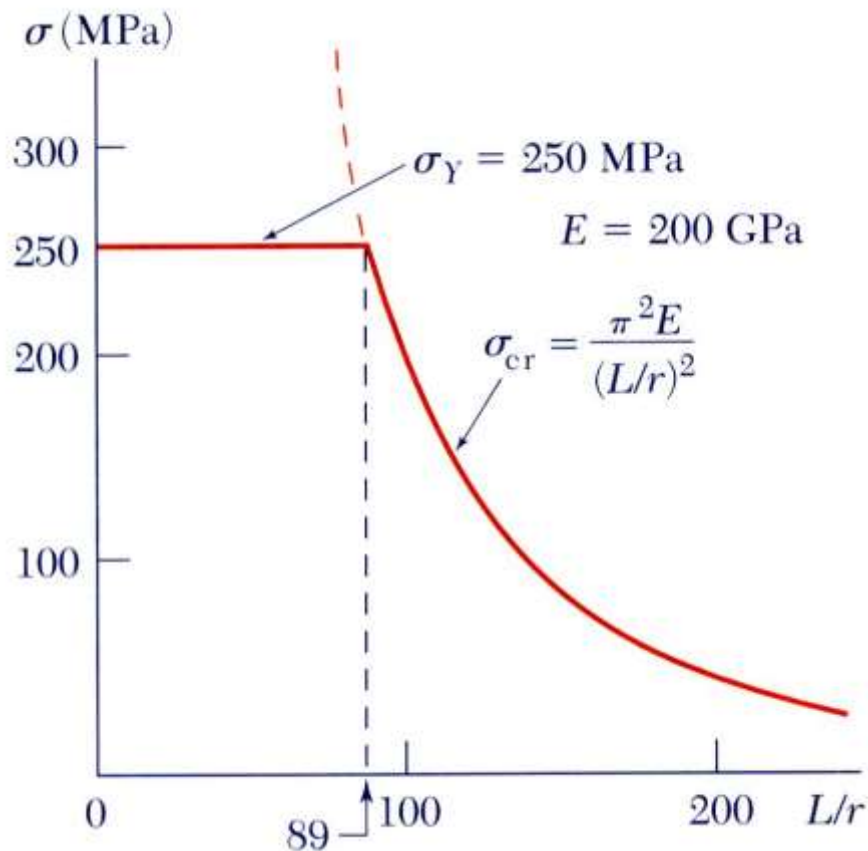
$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = 0$$

- Solution with assumed configuration can only be obtained if

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{\pi^2 E (Ar^2)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2}$$

Euler's Formula for Pin-Ended Beams



- The value of stress corresponding to the critical load,

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{P_{cr}}{A}$$

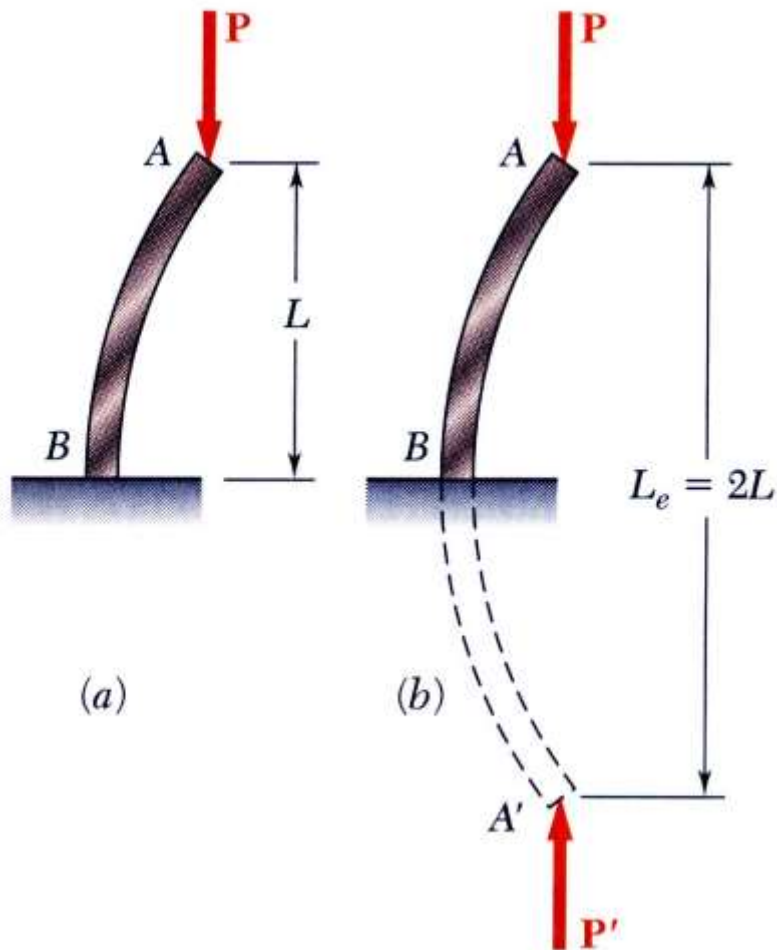
$$\sigma_{cr} = \frac{\pi^2 E (Ar^2)}{L^2 A}$$

$$= \frac{\pi^2 E}{(L/r)^2} = \text{critical stress}$$

$$\frac{L}{r} = \text{slenderness ratio}$$

- Preceding analysis is limited to centric loadings.

Extension of Euler's Formula



- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.
- The critical loading is calculated from Euler's formula,

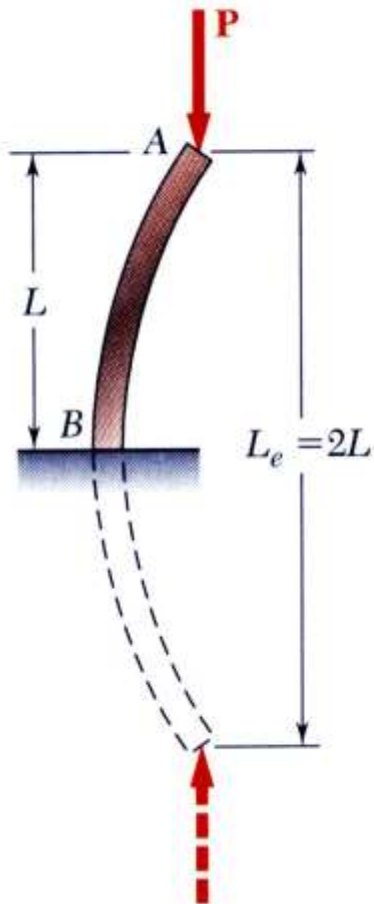
$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

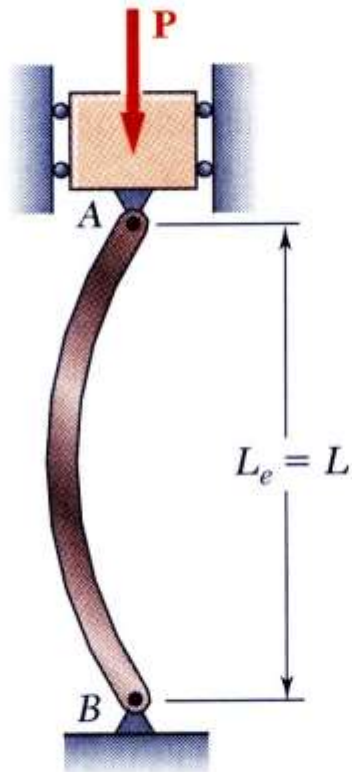
$$L_e = 2L = \text{equivalent length}$$

Extension of Euler's Formula

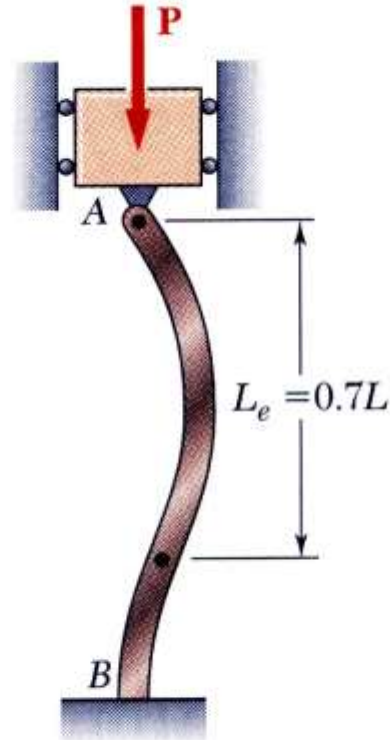
(a) One fixed end, one free end



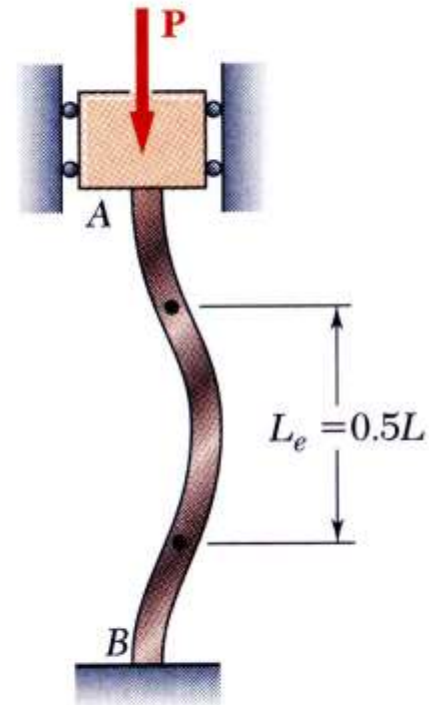
(b) Both ends pinned



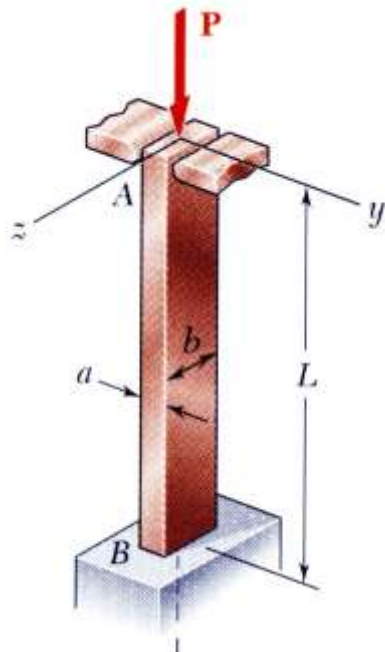
(c) One fixed end, one pinned end



(d) Both ends fixed



Sample Problem 10.1



An aluminum column of length L and rectangular cross-section has a fixed end at B and supports a centric load at A . Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

- Determine the ratio a/b of the two sides of the cross-section corresponding to the most efficient design against buckling.
- Design the most efficient cross-section for the column.

$$L = 20 \text{ in.}$$

$$E = 10.1 \times 10^6 \text{ psi}$$

$$P = 5 \text{ kips}$$

$$FS = 2.5$$

Sample Problem 10.1

SOLUTION:

The most efficient design occurs when the resistance to buckling is equal in both planes of symmetry. This occurs when the slenderness ratios are equal.

- Buckling in xy Plane:

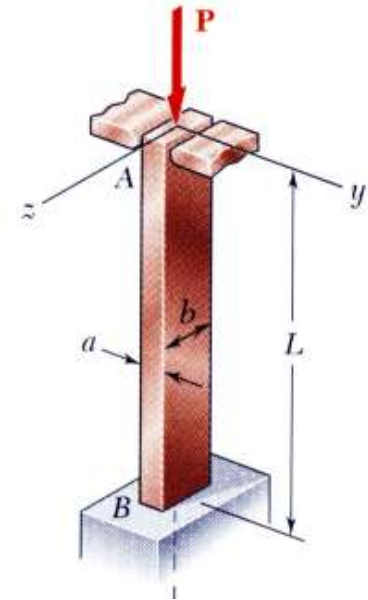
$$r_z^2 = \frac{I_z}{A} = \frac{\frac{1}{12}ba^3}{ab} = \frac{a^2}{12} \quad r_z = \frac{a}{\sqrt{12}}$$

$$\frac{L_{e,z}}{r_z} = \frac{0.7L}{a/\sqrt{12}}$$

- Buckling in xz Plane:

$$r_y^2 = \frac{I_y}{A} = \frac{\frac{1}{12}ab^3}{ab} = \frac{b^2}{12} \quad r_y = \frac{b}{\sqrt{12}}$$

$$\frac{L_{e,y}}{r_y} = \frac{2L}{b/\sqrt{12}}$$



- Most efficient design:

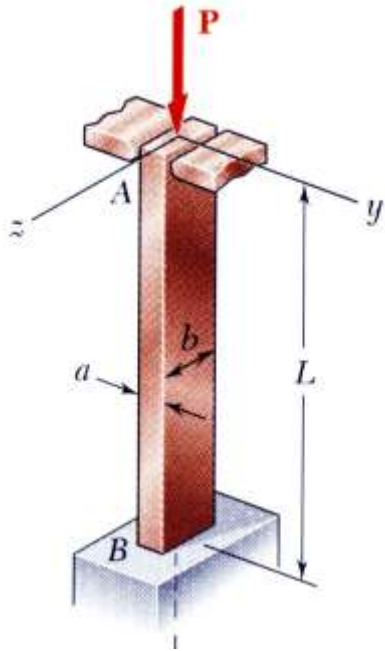
$$\frac{L_{e,z}}{r_z} = \frac{L_{e,y}}{r_y}$$

$$\frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}}$$

$$\frac{a}{b} = \frac{0.7}{2}$$

$$\boxed{\frac{a}{b} = 0.35}$$

Sample Problem 10.1



- Design:

$$\frac{L_e}{r_y} = \frac{2L}{b/\sqrt{12}} = \frac{2(20 \text{ in})}{b/\sqrt{12}} = \frac{138.6}{b}$$

$$P_{cr} = (FS)P = (2.5)(5 \text{ kips}) = 12.5 \text{ kips}$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{12500 \text{ lbs}}{(0.35b)b}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \frac{\pi^2 (10.1 \times 10^6 \text{ psi})}{(138.6/b)^2}$$

$$\frac{12500 \text{ lbs}}{(0.35b)b} = \frac{\pi^2 (10.1 \times 10^6 \text{ psi})}{(138.6/b)^2}$$

$$b = 1.620 \text{ in.}$$

$$a = 0.35b = 0.567 \text{ in.}$$

$$L = 20 \text{ in.}$$

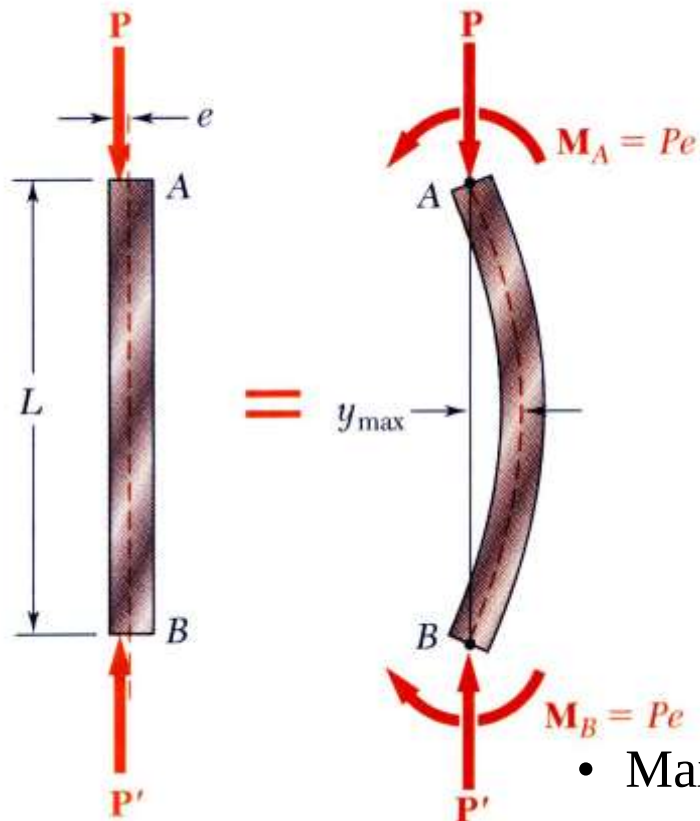
$$E = 10.1 \times 10^6 \text{ psi}$$

$$P = 5 \text{ kips}$$

$$FS = 2.5$$

$$a/b = 0.35$$

Eccentric Loading; The Secant Formula



- Eccentric loading is equivalent to a centric load and a couple.
- Bending occurs for any nonzero eccentricity. Question of buckling becomes whether the resulting deflection is excessive.
- The deflection become infinite when $P = P_{cr}$

$$\frac{d^2y}{dx^2} = \frac{-Py - Pe}{EI}$$

$$y_{\max} = e \left[\sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right] \quad P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

- Maximum stress

$$\begin{aligned} \sigma_{\max} &= \frac{P}{A} \left[1 + \frac{(y_{\max} + e)c}{r^2} \right] \\ &= \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L_e}{r} \right) \right] \end{aligned}$$

Eccentric Loading; The Secant Formula

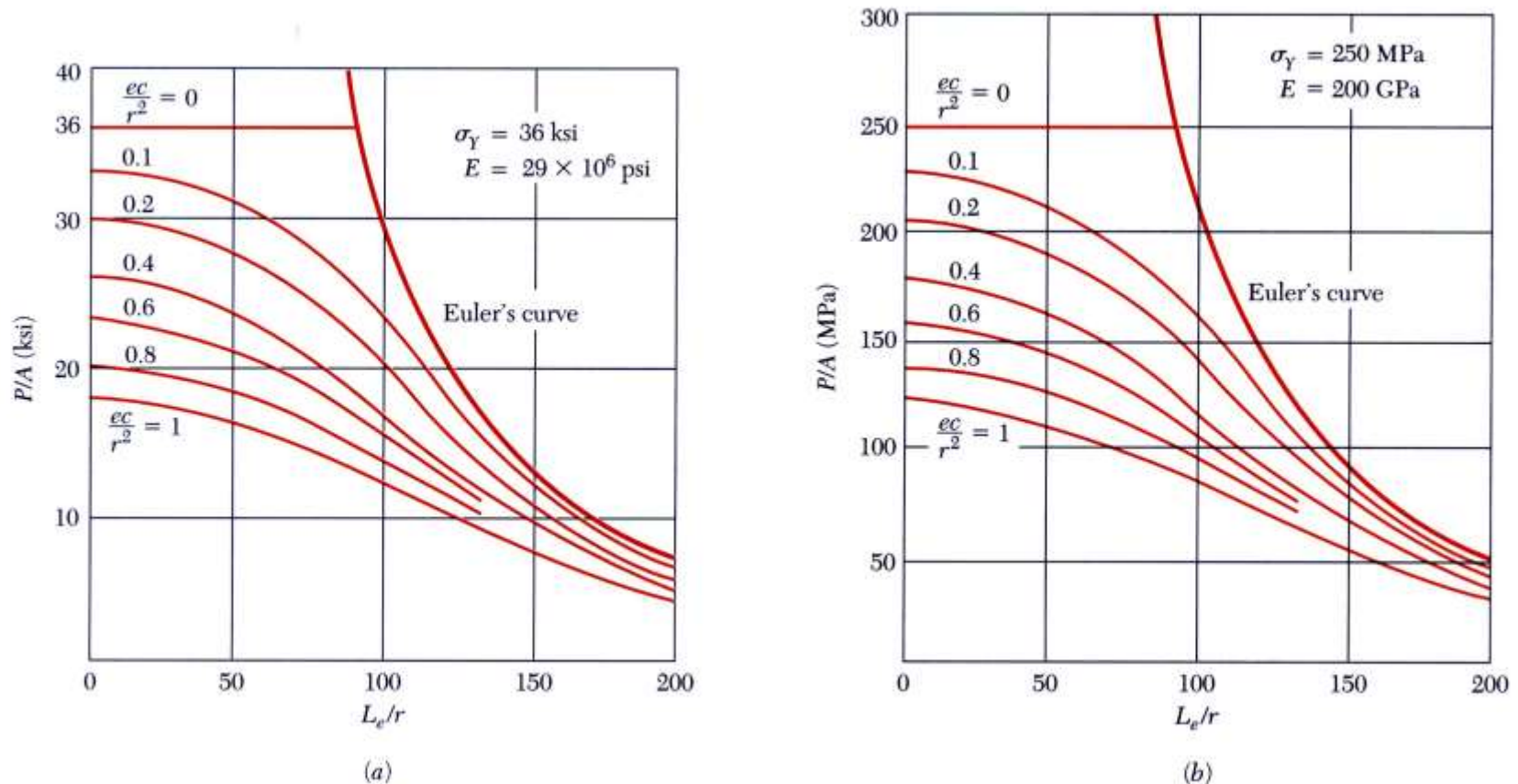
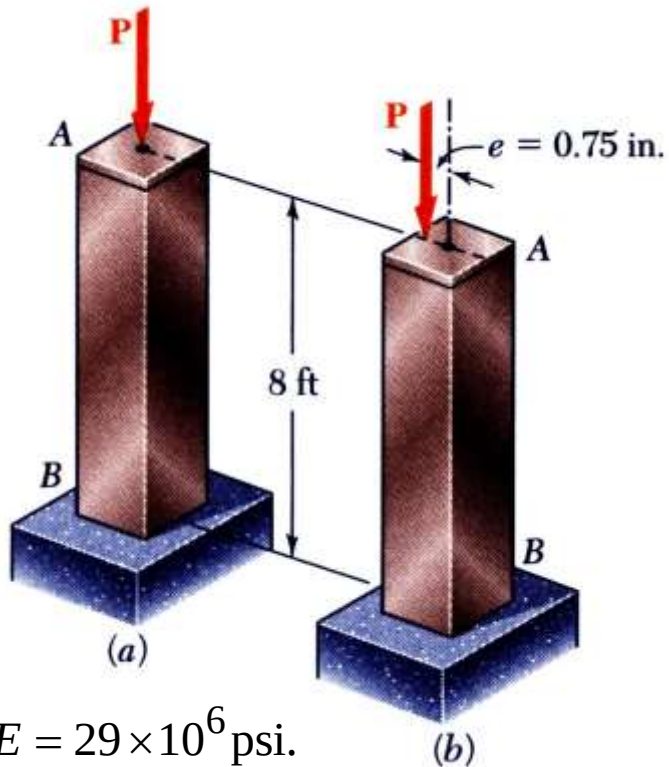


Fig. 10.24 Load per unit area, P/A , causing yield in column.

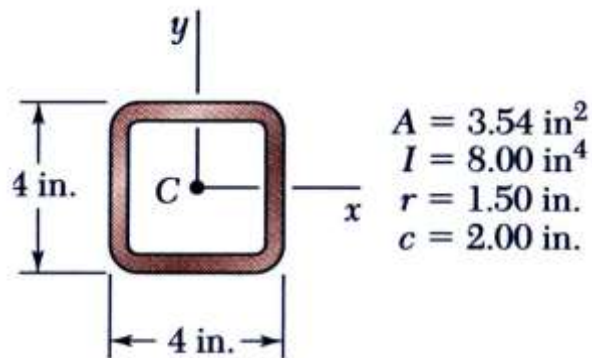
$$\sigma_{\max} = \sigma_Y = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{1}{2} \sqrt{\frac{P}{EA}} \frac{L_e}{r} \right) \right]$$

Sample Problem 10.2

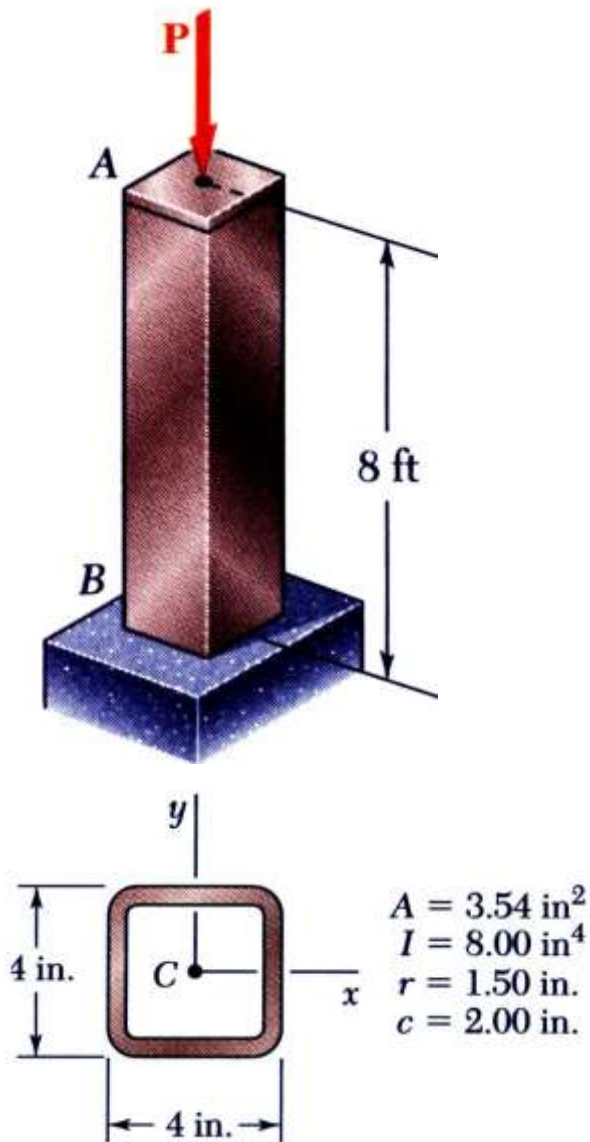


The uniform column consists of an 8-ft section of structural tubing having the cross-section shown.

- Using Euler's formula and a factor of safety of two, determine the allowable centric load for the column and the corresponding normal stress.
- Assuming that the allowable load, found in part *a*, is applied at a point 0.75 in. from the geometric axis of the column, determine the horizontal deflection of the top of the column and the maximum normal stress in the column.



Sample Problem 10.2



SOLUTION:

- Maximum allowable centric load:

- Effective length,

$$L_e = 2(8 \text{ ft}) = 16 \text{ ft} = 192 \text{ in.}$$

- Critical load,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (29 \times 10^6 \text{ psi})(8.0 \text{ in}^4)}{(192 \text{ in})^2} = 62.1 \text{ kips}$$

- Allowable load,

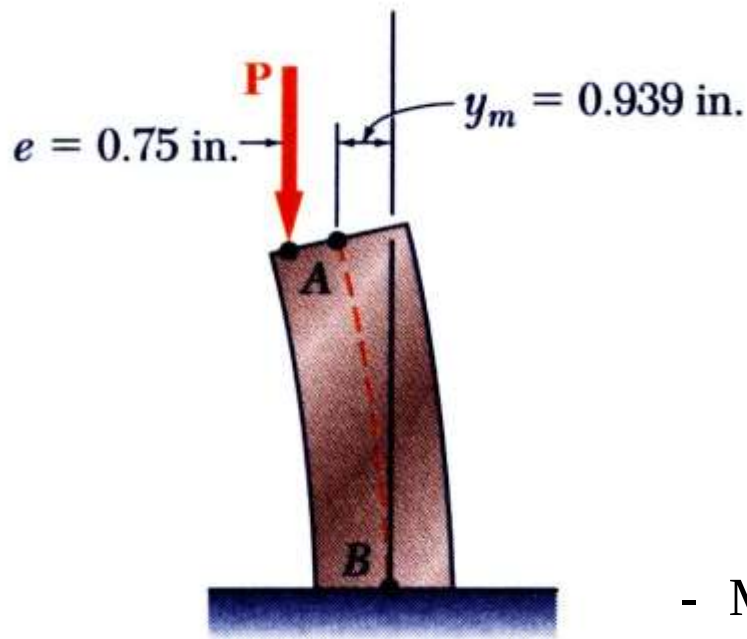
$$P_{all} = \frac{P_{cr}}{FS} = \frac{62.1 \text{ kips}}{2}$$

$$P_{all} = 31.1 \text{ kips}$$

$$\sigma = \frac{P_{all}}{A} = \frac{31.1 \text{ kips}}{3.54 \text{ in}^2}$$

$$\sigma = 8.79 \text{ ksi}$$

Sample Problem 10.2



- Eccentric load:

- End deflection,

$$y_m = e \left[\sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right]$$

$$= (0.75 \text{ in}) \left[\sec \left(\frac{\pi}{2\sqrt{2}} \right) - 1 \right]$$

$$y_m = 0.75 * (2.252 - 1) = 0.939 \text{ in.}$$

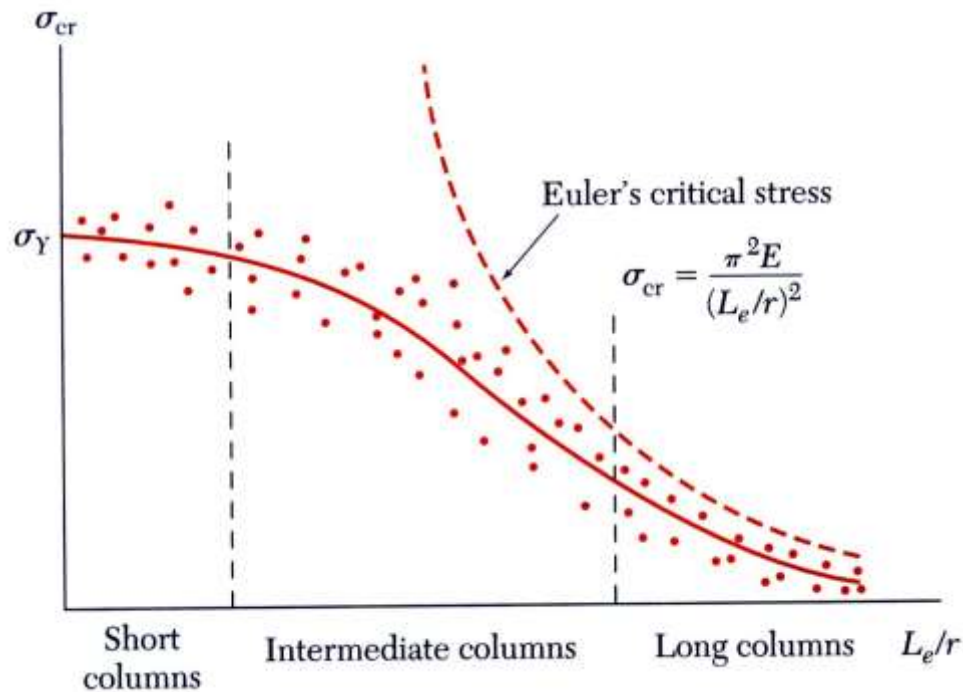
- Maximum normal stress,

$$\sigma_m = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) \right]$$

$$= \frac{31.1 \text{ kips}}{3.54 \text{ in}^2} \left[1 + \frac{(0.75 \text{ in})(2 \text{ in})}{(1.50 \text{ in})^2} \sec \left(\frac{\pi}{2\sqrt{2}} \right) \right]$$

$$\sigma_m = 8.79 * (1 + 0.667(2.252)) = 22.0 \text{ ksi}$$

Design of Columns Under Centric Load

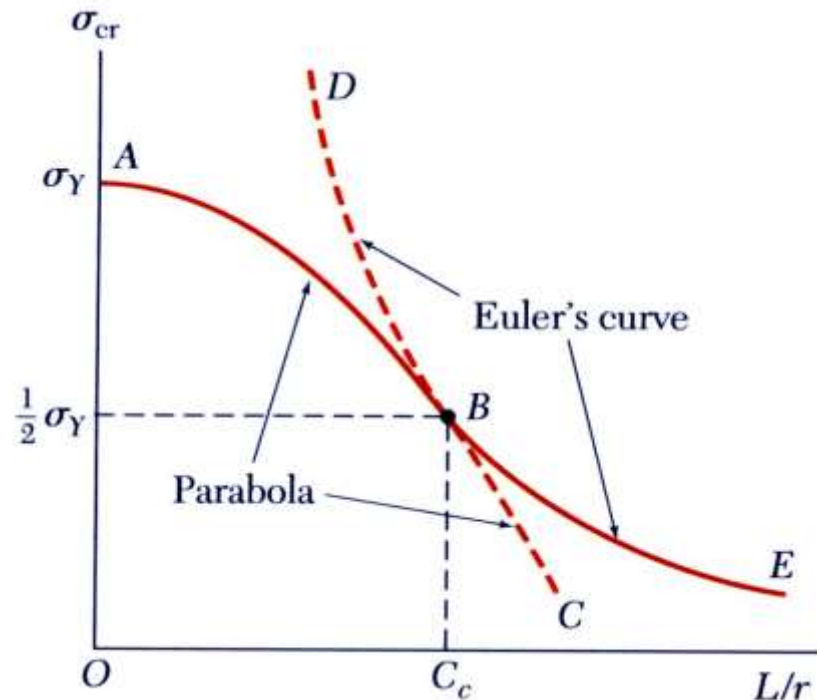


- Previous analyses assumed stresses below the proportional limit and initially straight, homogeneous columns
- Experimental data demonstrate
 - for large L_e/r , σ_{cr} follows Euler's formula and depends upon E but not σ_Y .
 - for small L_e/r , σ_{cr} is determined by the yield strength σ_Y and not E .
 - for intermediate L_e/r , σ_{cr} depends on both σ_Y and E .

Design of Columns Under Centric Load

Structural Steel

American Inst. of Steel Construction



- For $L_e/r > C_c$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \quad \sigma_{all} = \frac{\sigma_{cr}}{FS}$$

$$FS = 1.92$$

- For $L_e/r < C_c$

$$\sigma_{cr} = \sigma_Y \left[1 - \frac{(L_e/r)^2}{2C_c^2} \right] \quad \sigma_{all} = \frac{\sigma_{cr}}{FS}$$

$$FS = \frac{5}{3} + \frac{3}{8} \frac{L_e/r}{C_c} - \frac{1}{8} \left(\frac{L_e/r}{C_c} \right)^3$$

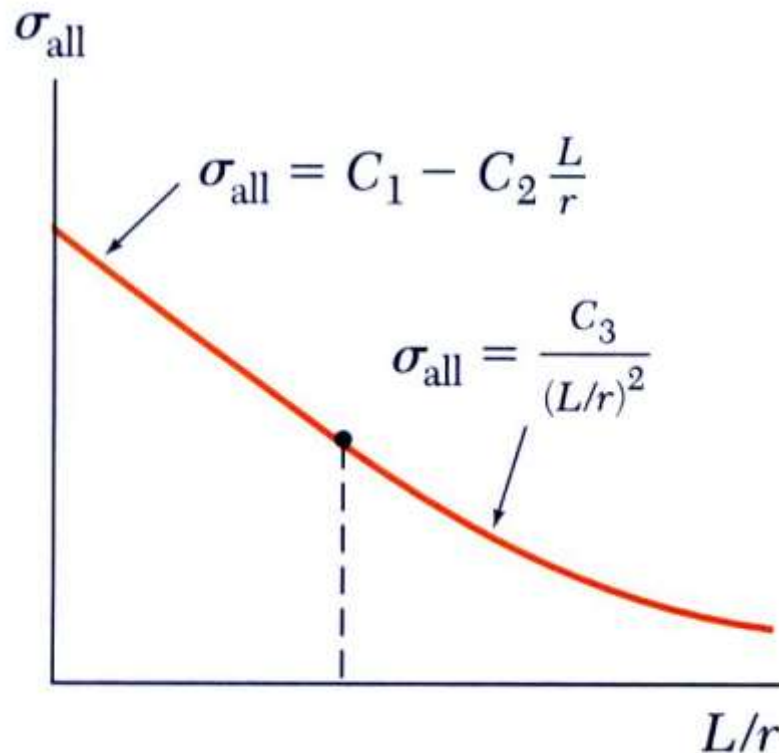
- At $L_e/r = C_c$

$$\sigma_{cr} = \frac{1}{2} \sigma_Y \quad C_c^2 = \frac{2\pi^2 E}{\sigma_Y}$$

Design of Columns Under Centric Load

Aluminum

Aluminum Association, Inc.



- Alloy 6061-T6

$$L_e/r < 66:$$

$$\begin{aligned}\sigma_{all} &= [20.2 - 0.126(L_e/r)] \text{ ksi} \\ &= [139 - 0.868(L_e/r)] \text{ MPa}\end{aligned}$$

$$L_e/r \geq 66:$$

$$\sigma_{all} = \frac{51000 \text{ ksi}}{(L_e/r)^2} = \frac{351 \times 10^3 \text{ MPa}}{(L_e/r)^2}$$

- Alloy 2014-T6

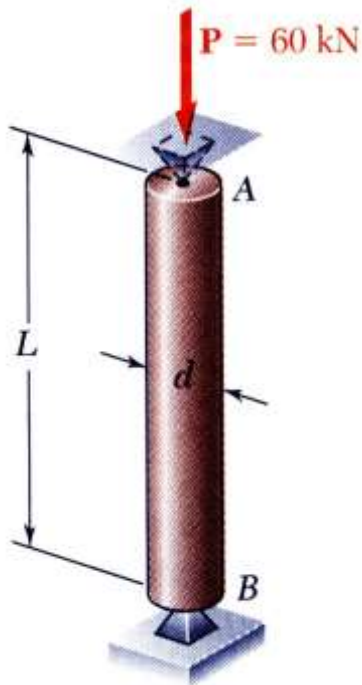
$$L_e/r < 55:$$

$$\begin{aligned}\sigma_{all} &= [30.7 - 0.23(L_e/r)] \text{ ksi} \\ &= [212 - 1.585(L_e/r)] \text{ MPa}\end{aligned}$$

$$L_e/r \geq 55:$$

$$\sigma_{all} = \frac{54000 \text{ ksi}}{(L_e/r)^2} = \frac{372 \times 10^3 \text{ MPa}}{(L_e/r)^2}$$

Sample Problem 10.4

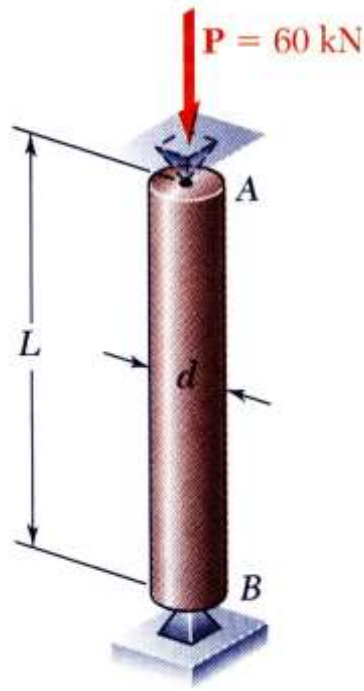


SOLUTION:

- With the diameter unknown, the slenderness ratio can not be evaluated. Must make an assumption on which slenderness ratio regime to utilize.
- Calculate required diameter for assumed slenderness ratio regime.
- Evaluate slenderness ratio and verify initial assumption. Repeat if necessary.

Using the aluminum alloy 2014-T6, determine the smallest diameter rod which can be used to support the centric load $P = 60 \text{ kN}$ if a) $L = 750 \text{ mm}$,
b) $L = 300 \text{ mm}$

Sample Problem 10.4



c = cylinder radius

r = radius of gyration

$$= \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi c^4/4}{\pi c^2}} = \frac{c}{2}$$

- For $L = 750$ mm, assume $L/r > 55$
- Determine cylinder radius:

$$\sigma_{all} = \frac{P}{A} = \frac{372 \times 10^3 \text{ MPa}}{(L/r)^2}$$

$$\frac{60 \times 10^3 \text{ N}}{\pi c^2} = \frac{372 \times 10^3 \text{ MPa}}{\left(\frac{0.750 \text{ m}}{c/2}\right)^2} \quad c = 18.44 \text{ mm}$$

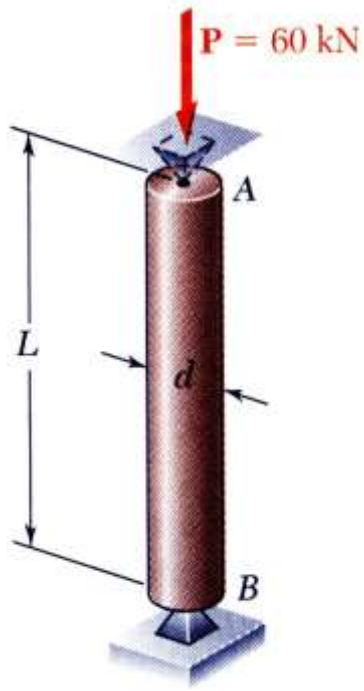
- Check slenderness ratio assumption:

$$\frac{L}{r} = \frac{L}{c/2} = \frac{750 \text{ mm}}{(18.44 \text{ mm})} = 81.3 > 55$$

assumption was correct

$$d = 2c = 36.9 \text{ mm}$$

Sample Problem 10.4



- For $L = 300$ mm, assume $L/r < 55$
- Determine cylinder radius:

$$\sigma_{all} = \frac{P}{A} = \left[212 - 1.585 \left(\frac{L}{r} \right) \right] \text{MPa}$$

$$\frac{60 \times 10^3 \text{ N}}{\pi c^2} = \left[212 - 1.585 \left(\frac{0.3 \text{ m}}{c/2} \right) \right] \times 10^6 \text{ Pa}$$

$$c = 12.00 \text{ mm}$$

- Check slenderness ratio assumption:

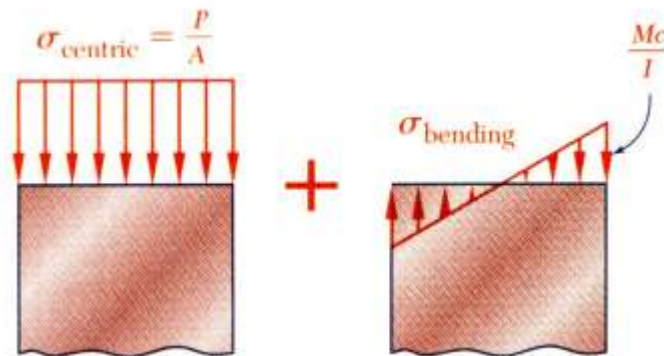
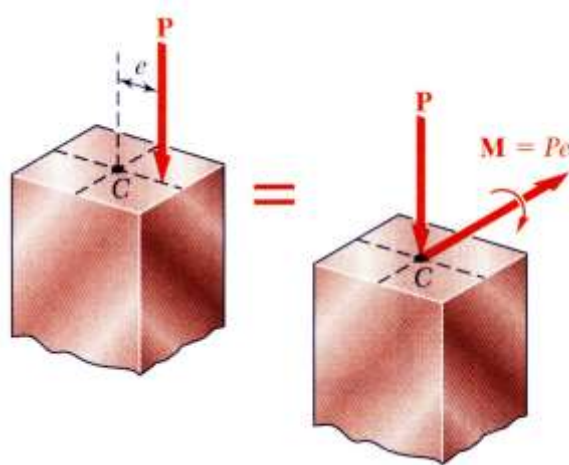
$$\frac{L}{r} = \frac{L}{c/2} = \frac{300 \text{ mm}}{(12.00 \text{ mm})} = 50 < 55$$

assumption was correct

$$d = 2c = 24.0 \text{ mm}$$



Design of Columns Under an Eccentric Load



- An eccentric load P can be replaced by a centric load P and a couple $M = Pe$.
- Normal stresses can be found from superposing the stresses due to the centric load and couple,

$$\sigma = \sigma_{centric} + \sigma_{bending}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I}$$

- Allowable stress method:

$$\frac{P}{A} + \frac{Mc}{I} \leq \sigma_{all}$$

- Interaction method:

$$\frac{P/A}{(\sigma_{all})_{centric}} + \frac{Mc/I}{(\sigma_{all})_{bending}} \leq 1$$



Good Luck

Eng./ Baraa Shuhaibar