

Chapter 10

MOMENTS of INERTIA for AREAS, RADIUS OF GYRATION

Today's Objectives:

Students will be able to:

- a) Define the moments of inertia (MoI) for an area.
- b) Determine the MoI for an area by integration.



READING QUIZ

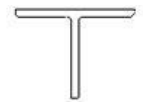
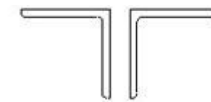
1. The definition of the Moment of Inertia for an area involves an integral of the form
- A) $\int x \, dA$. B) $\int x^2 \, dA$.
C) $\int x^2 \, dm$. D) $\int m \, dA$.
2. Select the SI units for the Moment of Inertia for an area.
- A) m^3 B) m^4
C) $kg \cdot m^2$ D) $kg \cdot m^3$





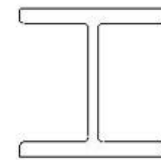
APPLICATIONS

Many structural members are made of tubes rather than solid squares or rounds. Why?



Angles

Tee



Channel

Column

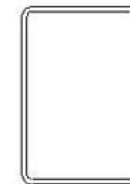
Beam

What parameters of the cross sectional area influence the designer's selection?

How can we determine the value of these parameters for a given area?



S.H.S

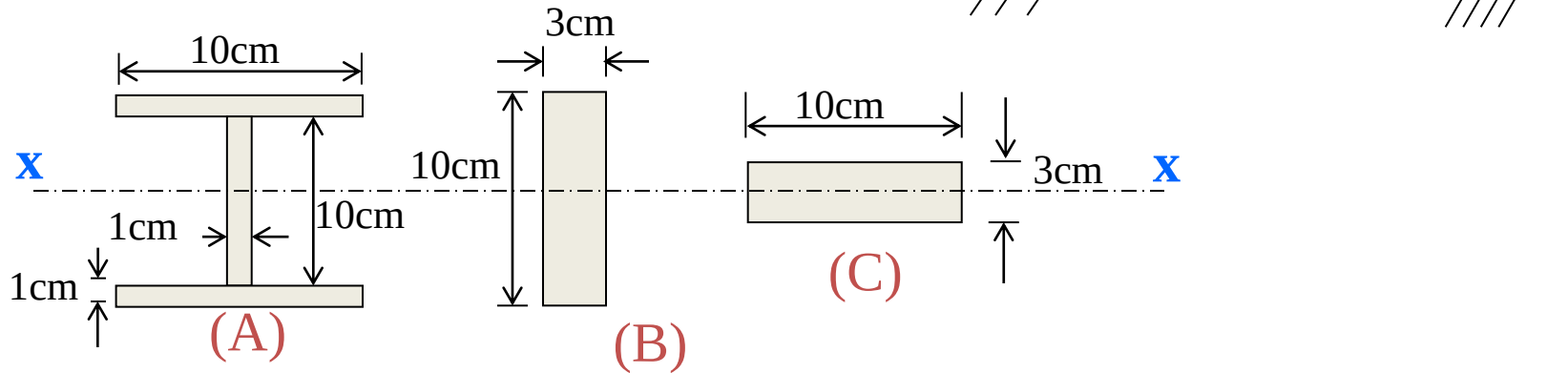


R.H.S



C.H.S

MOMENTS OF INERTIA

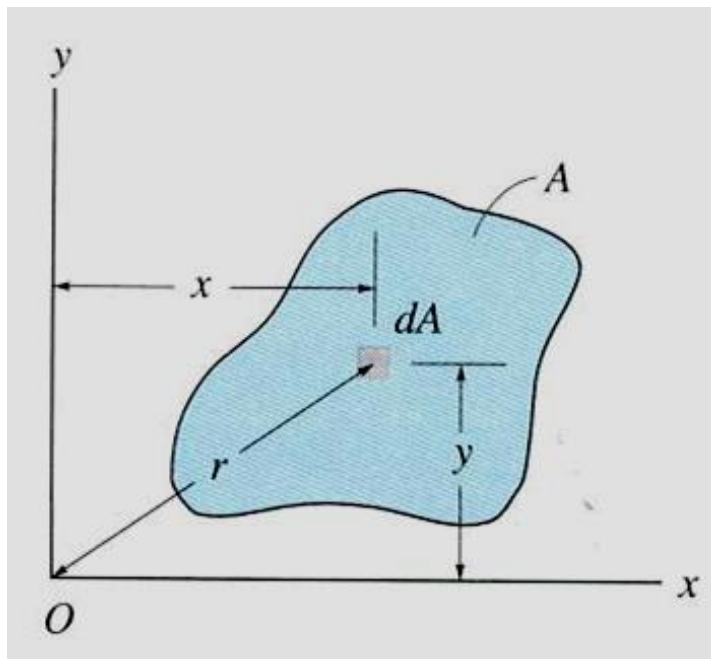


Consider three different possible cross sectional shapes and areas for the beam **RS**. All have the same total area and, assuming they are made of same material, they will have the same mass per unit length.

For the given vertical loading **P** on the beam, which shape will develop less internal stress and deflection? Why?

The answer depends on the MoI of the beam about the x-axis. It turns out that Section **(A)** has the highest MoI because most of the area is farthest from the x axis. Hence, it has the least stress and deflection.

10.1 MOMENTS OF INERTIA FOR AREAS



For the differential area dA , shown in the figure:

$$d I_x = y^2 dA ,$$

$$d I_y = x^2 dA , \text{ and,}$$

$$d J_O = r^2 dA ,$$

where

J_O is the **polar moment of inertia** about the pole O or z axis.

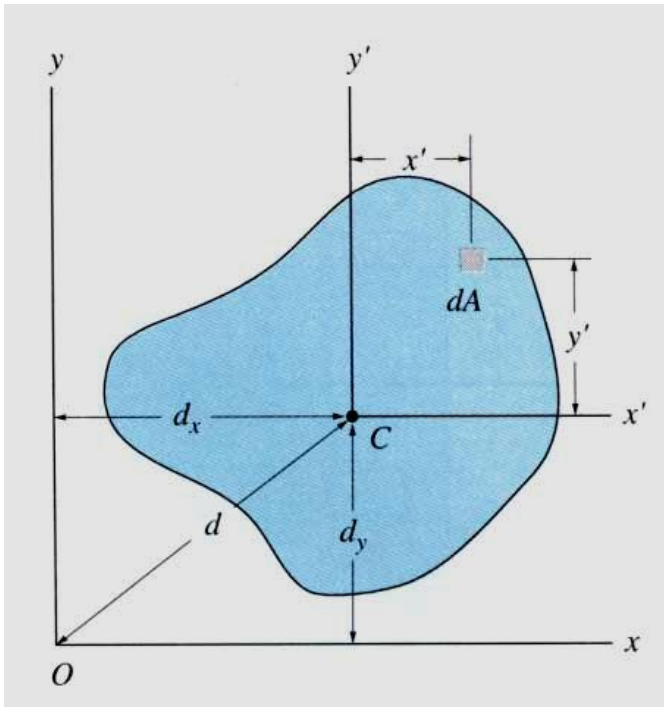
Moments of inertia for the entire area are obtained by integration.

$$I_x = \int_A y^2 dA ; \quad I_y = \int_A x^2 dA$$

$$J_O = \int_A r^2 dA = \int_A (x^2 + y^2) dA = I_x + I_y$$

The MoI is also referred to as the [second moment of an area](#) and has [units of length to the fourth power \(\$m^4\$ or \$in^4\$ \)](#).

10.2 PARALLEL-AXIS THEOREM FOR AN AREA



This theorem relates the moment of inertia (MoI) of an area about an axis passing through the area's centroid to the MoI of the area about a corresponding parallel axis.

This theorem has many practical applications, especially when working with composite areas.

Consider an area with centroid C. The x' and y' axes pass through C. The MoI about the x -axis, which is parallel to, and located at distance d_y from x' axis, is found by using the parallel-axis theorem.



READING QUIZ

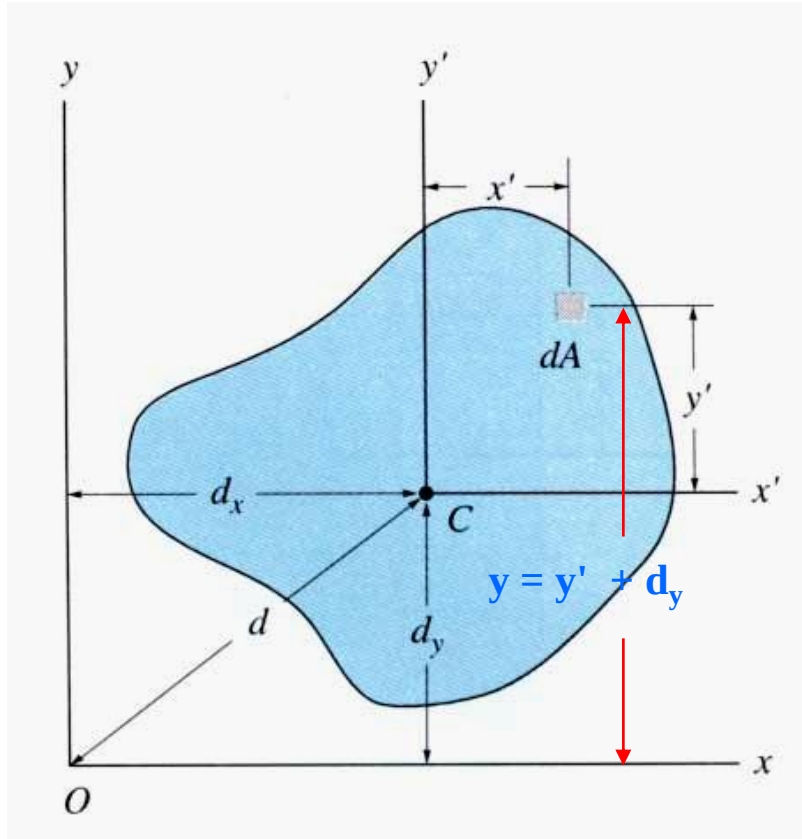
1. The parallel-axis theorem for an area is applied between
 - A) an axis passing through its centroid and any corresponding parallel axis.
 - B) any two parallel axis.
 - C) two horizontal axes only.
 - D) two vertical axes only.

2. The moment of inertia of a composite area equals the _____ of the MoI of all of its parts.
 - A) vector sum
 - B) algebraic sum (addition or subtraction)
 - C) addition
 - D) product



PARALLEL-AXIS THEOREM

(continued)



$$\begin{aligned} I_x &= \int_A (y' + d_y)^2 dA \\ &= \int_A y'^2 dA + 2d_y \int_A y' dA + d_y^2 \int_A dA \end{aligned}$$

Using the definition of the centroid:

$$\bar{y}' = (\int_A y' dA) / (\int_A dA) . \text{ Now}$$

since C is at the origin of the $x' - y'$ axes,

$$\bar{y}' = 0 , \text{ and hence } \int_A y' dA = 0 .$$

Thus,

$$I_x = \bar{I}_{x'} + Ad_y^2$$

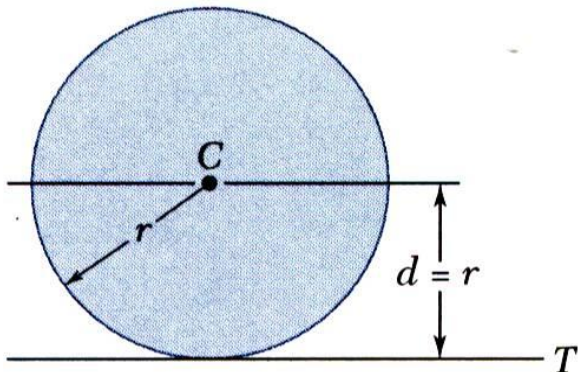
Similarly,

$$I_y = \bar{I}_{y'} + Ad_x^2$$

$$J_O = \bar{J}_C + Ad^2$$



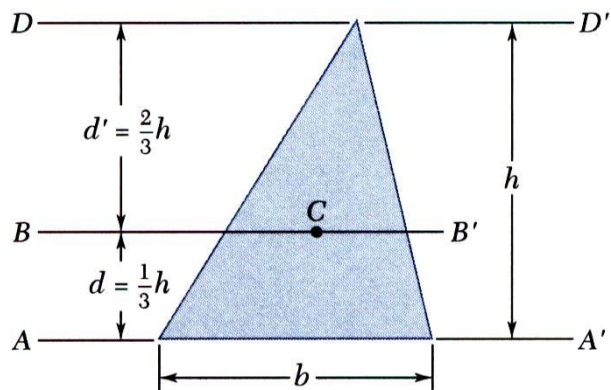
PARALLEL-AXIS THEOREM (examples)



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \frac{1}{4}\pi r^4 + (\pi r^2)r^2$$

$$= \frac{5}{4}\pi r^4$$



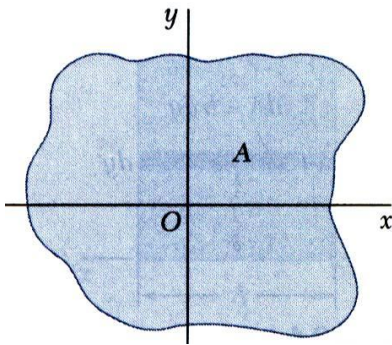
- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$

$$I_{BB'} = I_{AA'} - Ad^2 = \frac{1}{12}bh^3 - \frac{1}{2}bh\left(\frac{1}{3}h\right)^2$$

$$= \frac{1}{36}bh^3$$

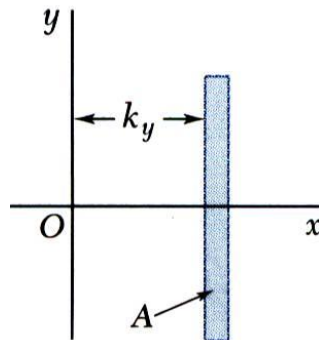
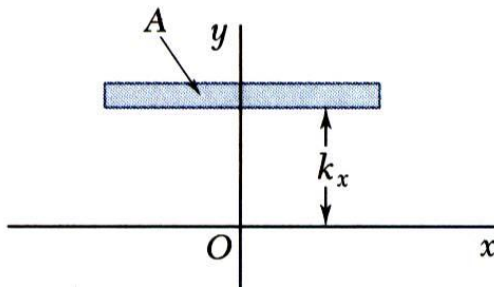
10.3 - RADIUS OF GYRATION OF AN AREA



- Consider area A with moment of inertia I_x . Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x .

$$I_x = k_x^2 A \quad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

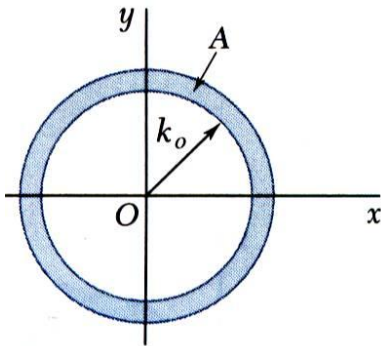


- Similarly,

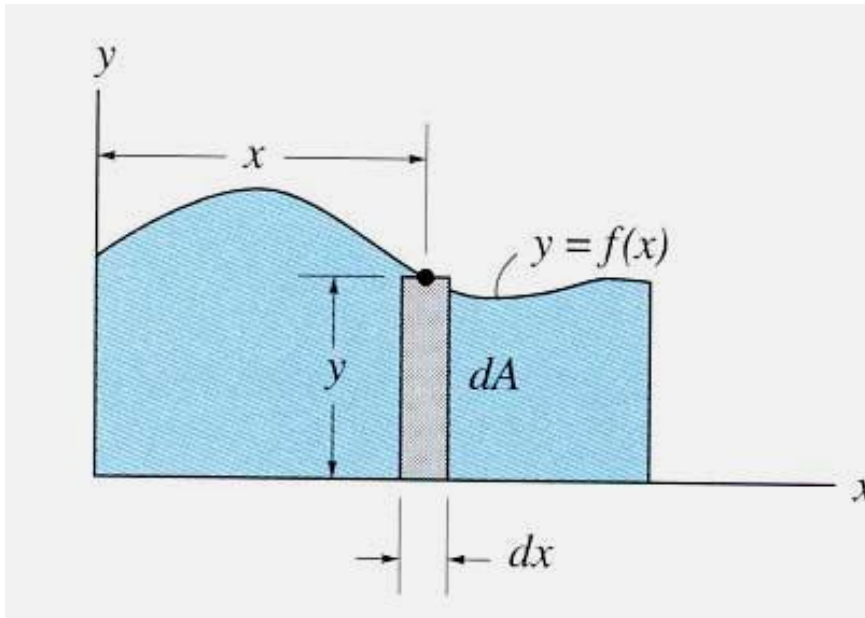
$$I_y = k_y^2 A \quad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \quad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$



10.4 MoI FOR AN AREA BY INTEGRATION

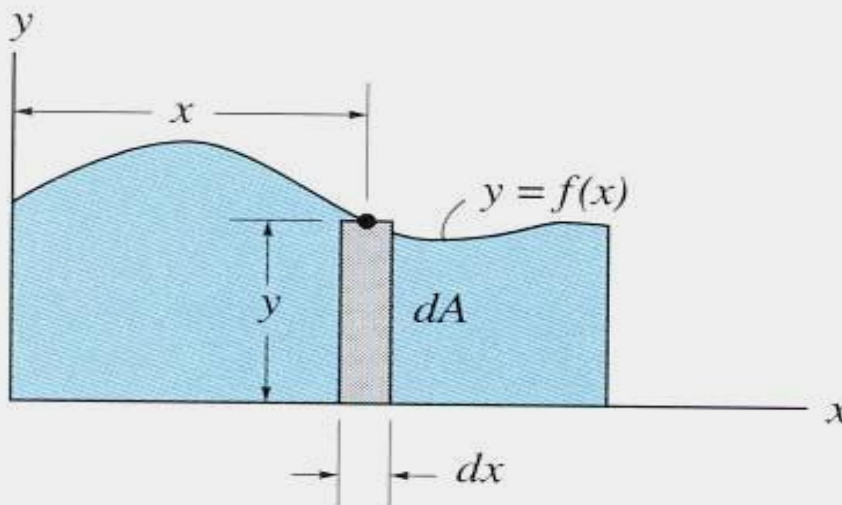


For simplicity, the area element used has a differential size in only one direction (dx or dy). This results in a single integration and is usually simpler than doing a double integration with two differentials, $dx \cdot dy$.

The step-by-step procedure is:

1. Choose the element dA : There are two choices: a vertical strip or a horizontal strip. Some considerations about this choice are:
 - a) The element parallel to the axis about which the MoI is to be determined usually results in an easier solution. For example, we typically choose a horizontal strip for determining I_x and a vertical strip for determining I_y .

MoI FOR AN AREA BY INTEGRATION, (*continue*)



b) If y is easily expressed in terms of x (e.g., $y = x^2 + 1$), then choosing a vertical strip with a differential element dx wide may be advantageous.

2. Integrate to find the MoI. For example, given the element shown in the figure above:

$$I_y = \int x^2 dA = \int x^2 y dx \quad \text{and}$$

$$I_x = \int dI_x = \int (1/3) y^3 dx \quad (\text{using the information for a rectangle about its base from the inside back cover of the textbook}).$$

Since in this case the differential element is dx , [y needs to be expressed in terms of x and the integral limit must also be in terms of x](#). As you can see, choosing the element and integrating can be challenging. It may require a trial and error approach plus experience.

EXAMPLE 10.1

Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the x' – y' plane and passing through the centroid C .

Solution (Case 1)

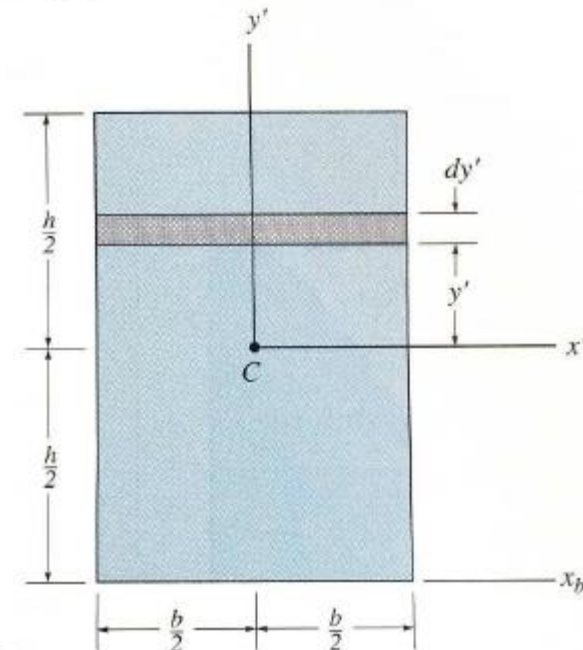
Part (a). The differential element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b \, dy'$, then

$$\begin{aligned}\bar{I}_{x'} &= \int_A y'^2 \, dA = \int_{-h/2}^{h/2} y'^2 (b \, dy') = b \int_{-h/2}^{h/2} y'^2 \, dy' \\ &= \frac{1}{12} bh^3\end{aligned}$$

Ans.

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the result of part (a) and applying the parallel-axis theorem, Eq. 10–3.

$$\begin{aligned}I_{x_b} &= \bar{I}_{x'} + Ad_y^2 \\ &= \frac{1}{12} bh^3 + bh \left(\frac{h}{2} \right)^2 = \frac{1}{3} bh^3\end{aligned}$$



This result can also be obtained by direct integration (as in part a) but with y varies from (0 to h)

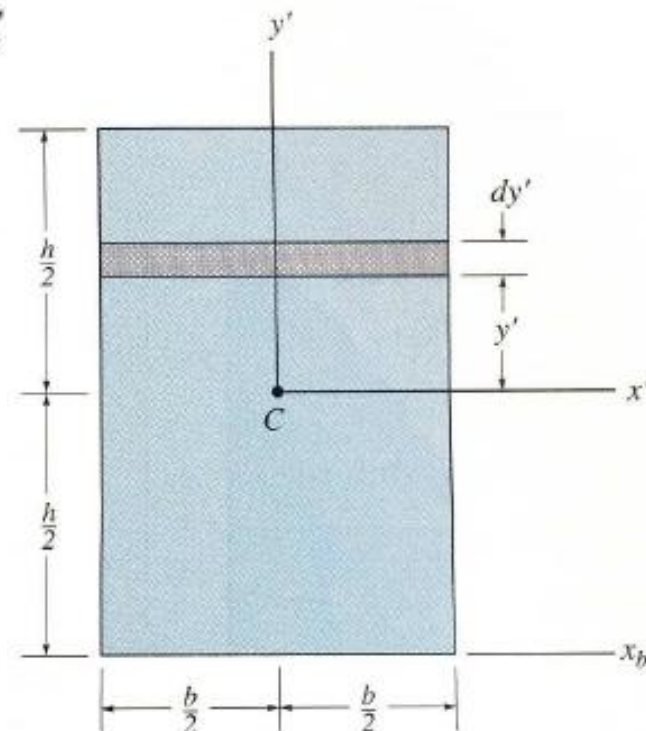
EXAMPLE 10.1, (*continue*)

Part (c). To obtain the polar moment of inertia about point C , we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

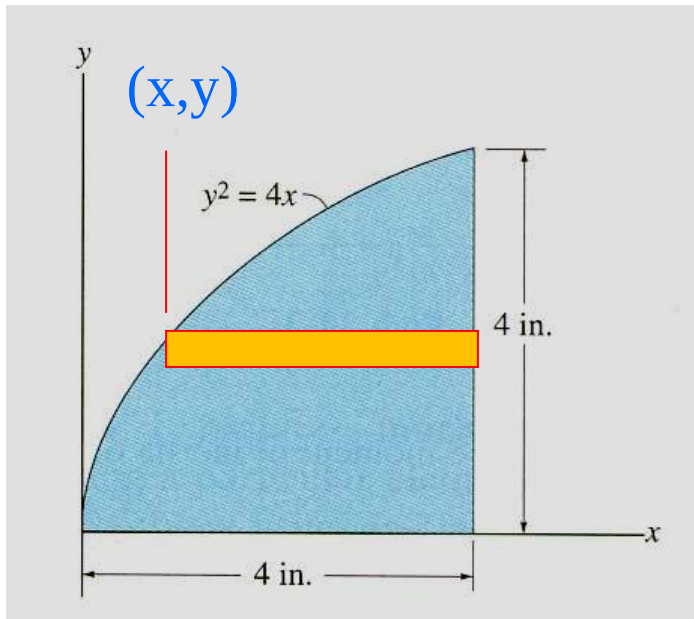
$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10-2, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2)$$



EXAMPLE 10.2



Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.

Plan: Follow the steps given earlier.

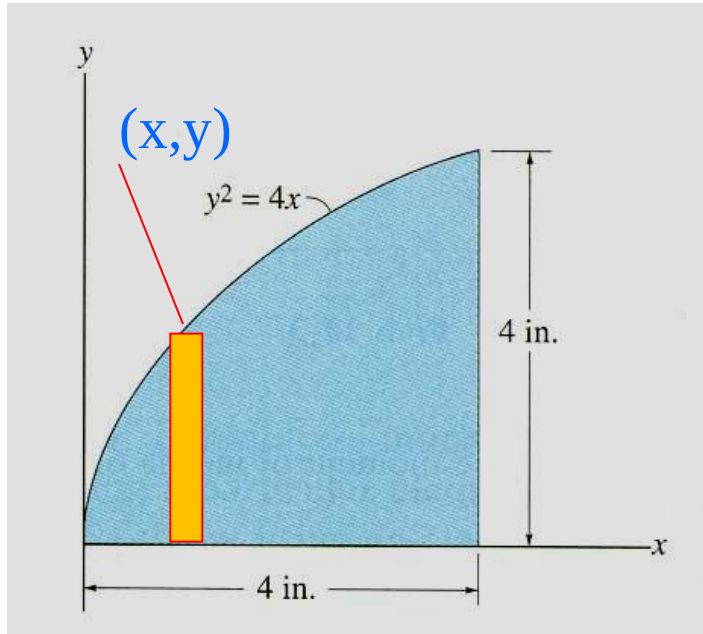
Solution

$$I_x = \int y^2 dA$$

$$dA = (4 - x) dy = (4 - y^2/4) dy$$

$$\begin{aligned} I_x &= \int_0^4 y^2 (4 - y^2/4) dy \\ &= \left[(4/3) y^3 - (1/20) y^5 \right]_0^4 = 34.1 \text{ in}^4 \end{aligned}$$

EXAMPLE 10.2 (continued)



$$\begin{aligned} I_y &= \int x^2 dA = \int x^2 y dx \\ &= \int x^2 (2\sqrt{x}) dx \\ &= 2 \int_0^4 x^{2.5} dx \\ &= \left[(2/3.5) x^{3.5} \right]_0^4 \\ &= 73.1 \text{ in}^4 \end{aligned}$$

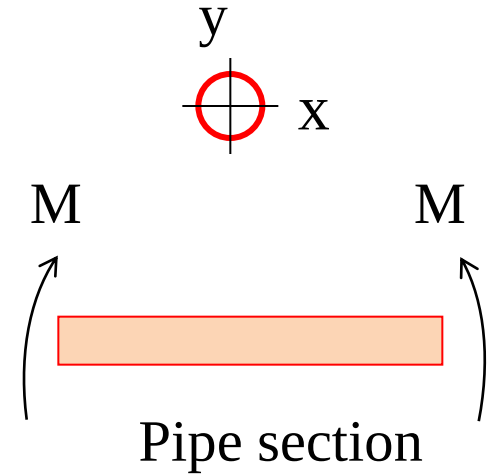
In the above example, it will be difficult to determine I_y using a horizontal strip. However, I_x in this example can be determined using a vertical strip. So,

$$I_x = \int (1/3) y^3 dx = \int (1/3) (2\sqrt{x})^3 dx .$$

CONCEPT QUIZ

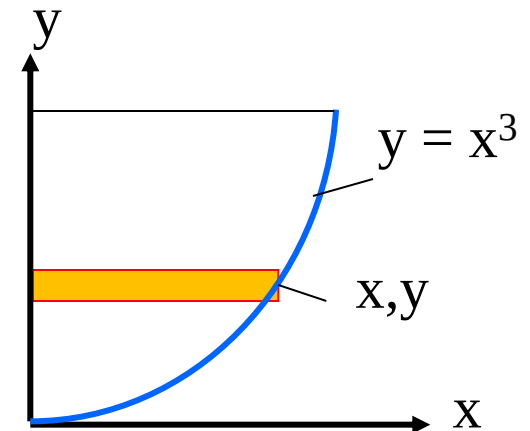
1. A pipe is subjected to a bending moment as shown. Which property of the pipe will result in lower stress (assuming a constant cross-sectional area)?

- A) Smaller I_x B) Smaller I_y
C) Larger I_x D) Larger I_y



2. In the figure to the right, what is the differential moment of inertia of the element with respect to the y-axis (dI_y)?

- A) $x^2 y \, dx$ B) $(1/12) x^3 \, dy$
C) $y^2 x \, dy$ D) $(1/3) y \, dy$



GROUP PROBLEM SOLVING

Given: The shaded area shown.

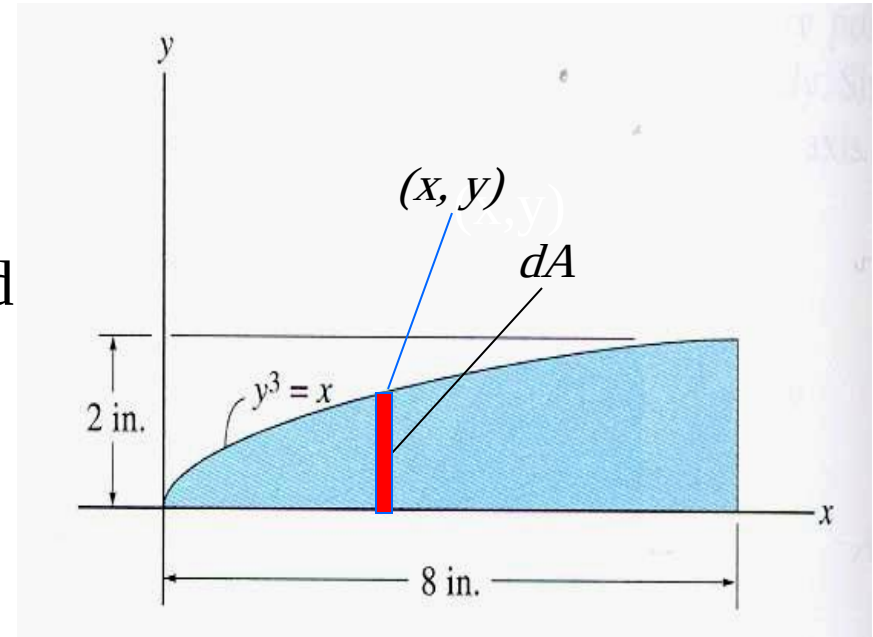
Find: I_x and I_y of the area.

Plan: Follow the steps described earlier.

Solution

$$\begin{aligned} I_x &= \int_0^8 (1/3) y^3 dx \\ &= \int_0^8 (1/3) x dx = \left[x^2 / 6 \right]_0^8 = 10.7 \text{ in}^4 \end{aligned}$$

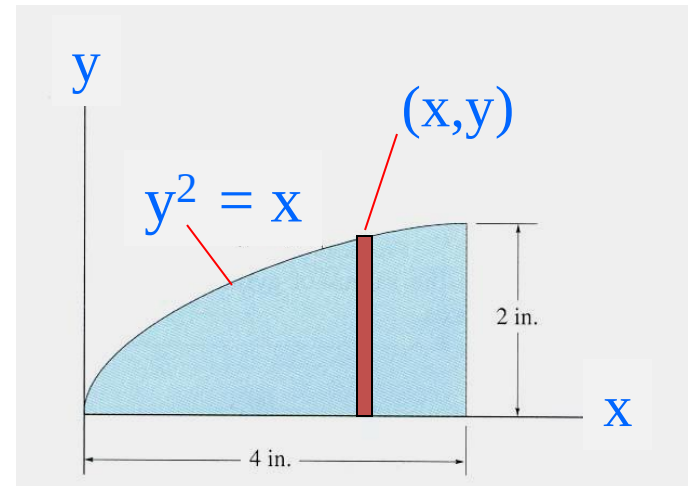
$$\begin{aligned} I_y &= \int x^2 dA = \int x^2 y dx \\ &= \int x^2 (x^{1/3}) dx \\ &= \int_0^8 x^{7/3} dx = \left[(3/10) x^{10/3} \right]_0^8 = 307 \text{ in}^4 \end{aligned}$$



ATTENTION QUIZ

1. When determining the MoI of the element in the figure, dI_y equals

- A) $x^2 dy$ B) $x^2 dx$
C) $(1/3) y^3 dx$ D) $x^{2.5} dx$



2. Similarly, dI_x equals

- A) $(1/3) x^{1.5} dx$ B) $y^2 dA$
C) $(1/12) x^3 dy$ D) $(1/3) x^3 dx$

