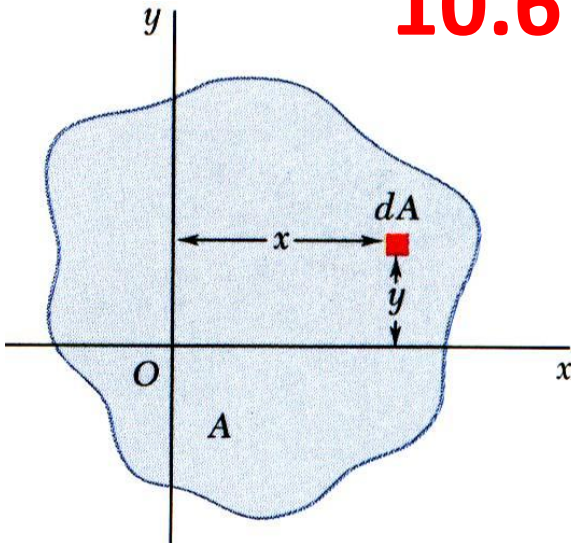


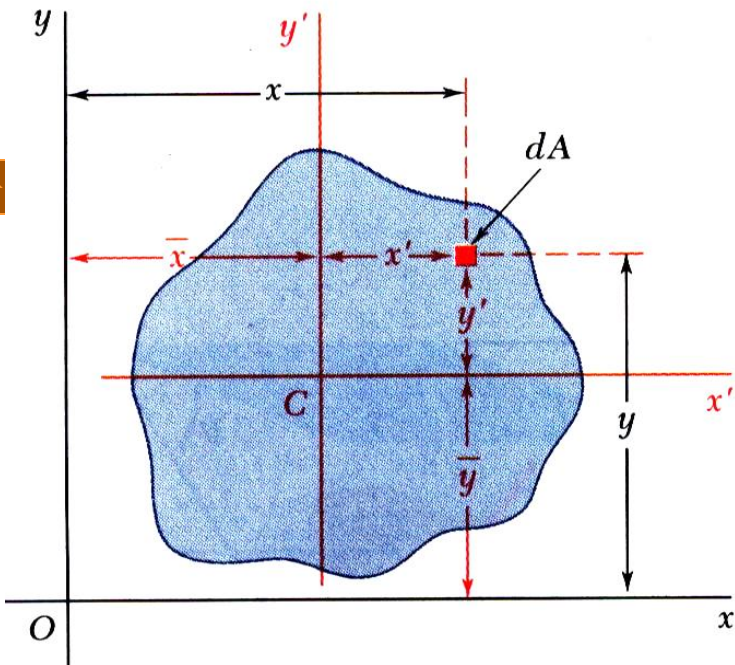
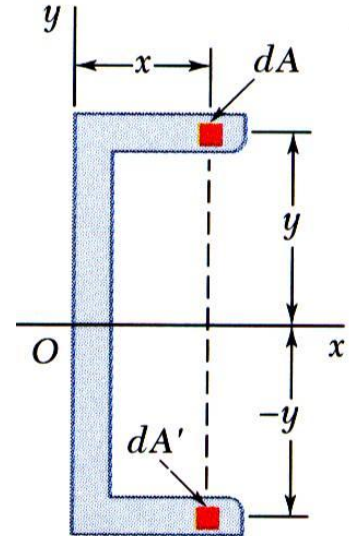
10.6 Product of Inertia



- *Product of Inertia:*

$$I_{xy} = \int xy \, dA$$

- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.



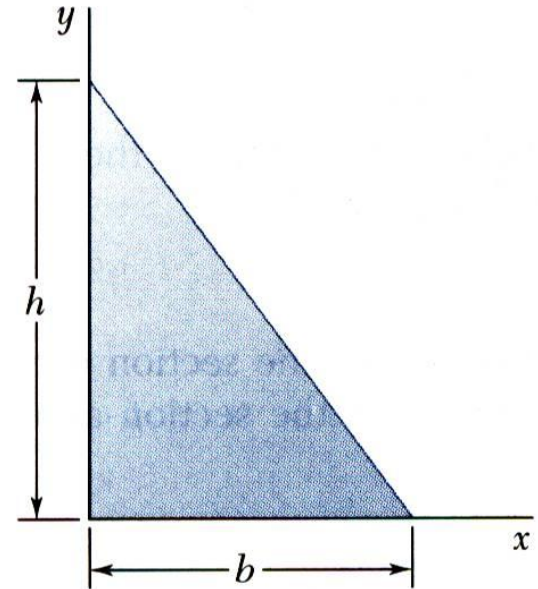
- Parallel axis theorem for products of inertia:

$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$

Sample Problem 10.6

Determine the product of inertia of the right triangle

- (a) with respect to the x and y axes and
- (b) with respect to centroidal axes parallel to the x and y axes.



SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

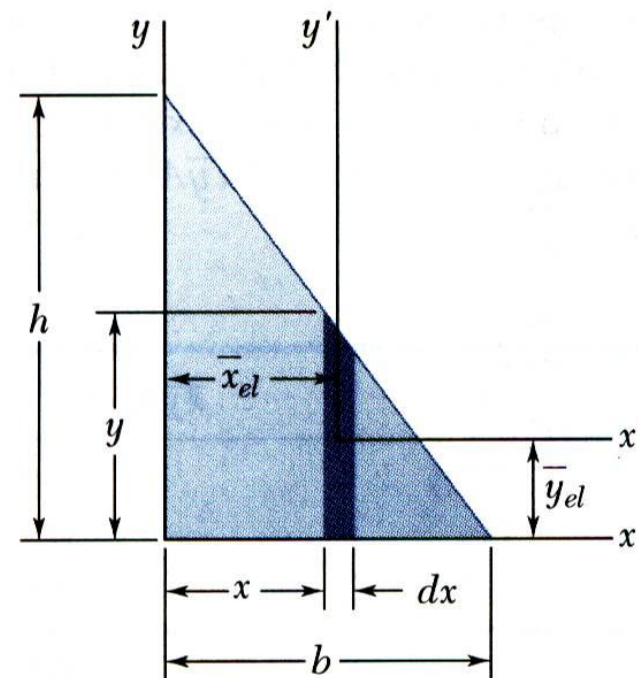
Sample Problem 10.6 (continue)

SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1 - \frac{x}{b}\right) \quad dA = y dx = h\left(1 - \frac{x}{b}\right) dx$$

$$\bar{x}_{el} = x \quad \bar{y}_{el} = \frac{1}{2}y = \frac{1}{2}h\left(1 - \frac{x}{b}\right)$$



Integrating dI_x from $x = 0$ to $x = b$,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el} \bar{y}_{el} dA = \int_0^b x \left(\frac{1}{2}\right) h^2 \left(1 - \frac{x}{b}\right)^2 dx$$

$$= h^2 \int_0^b \left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2} \right) dx = h^2 \left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2} \right]_0^b$$

$$I_{xy} = \frac{1}{24} b^2 h^2$$

Sample Problem 10.6 (continue)

- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

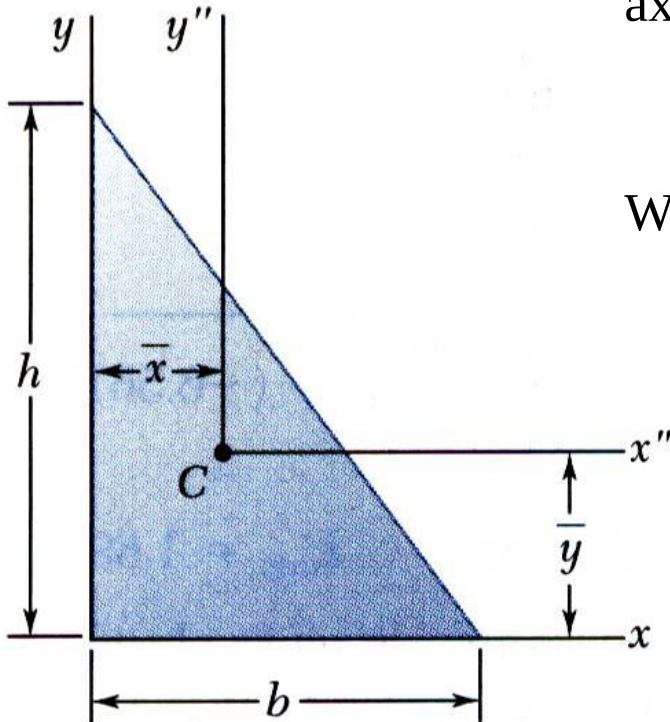
$$\bar{x} = \frac{1}{3}b \quad \bar{y} = \frac{1}{3}h$$

With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

$$\bar{I}_{x''y''} = \frac{1}{24}b^2h^2 - \left(\frac{1}{3}b\right)\left(\frac{1}{3}h\right)\left(\frac{1}{2}bh\right)$$

$$\bar{I}_{x''y''} = -\frac{1}{72}b^2h^2$$



10.7 Moments of Inertia about inclined axis

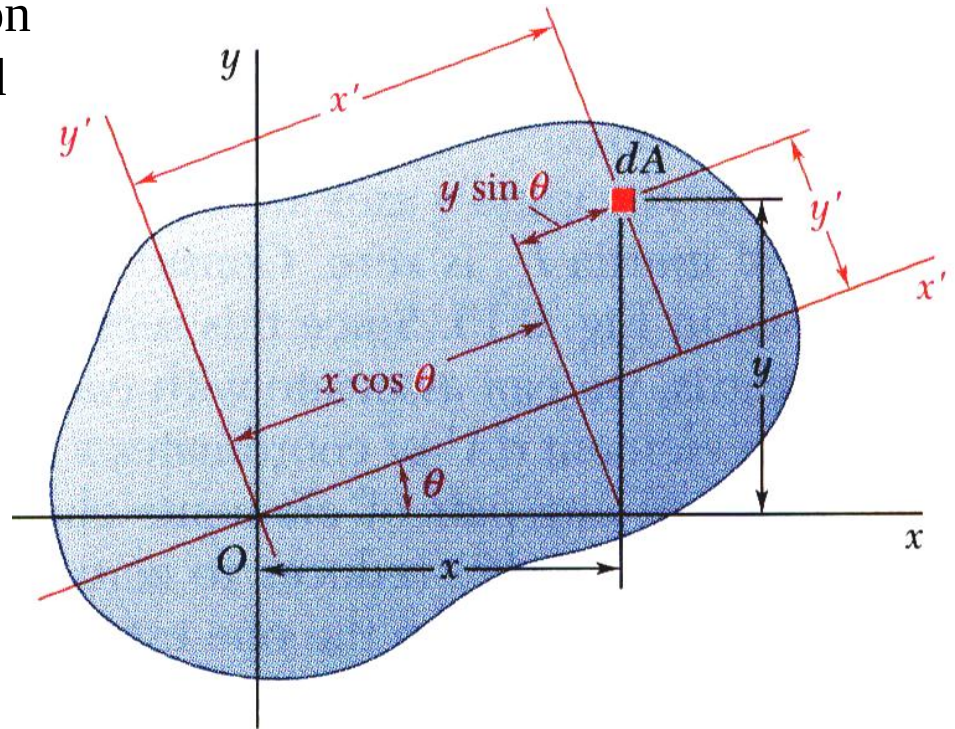
In structural and mechanical design, it is sometimes necessary to calculate the moment of inertia with respect to a set of inclined u, v , axes when the values of θ, I_x, I_y, I_{xy} are known.

To do this we will use transformation equations which relates the x, y , and x', y' coordinates.

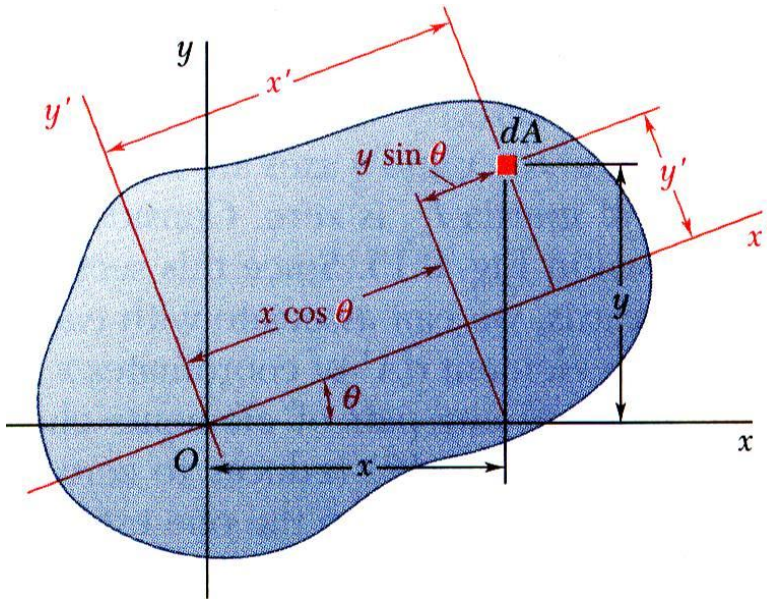
From Figure, these equations are:

Note: $x' = x \cos \theta + y \sin \theta$

$$y' = y \cos \theta - x \sin \theta$$



Moments of Inertia about inclined axis,, *continue*



- The change of axes yields

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

- By adding the equations for $I_{x'}$ and $I_{y'}$, we can show that the polar moment of inertia about z axis passing through point O is independent of the orientation of x' and y' ;

$$J_o = I_{x'} + I_{y'} = I_x + I_y$$

Given $I_x = \int y^2 dA$ $I_y = \int x^2 dA$
 $I_{xy} = \int xy dA$

we wish to determine moments and product of inertia with respect to new axes x' and y' .

- These equations show that $I_{x'}$, $I_{y'}$ and $I_{x'y'}$ depend on the angle of the inclination, θ , of the x' , y' axes.

Principal Axes and Principal Moments of Inertia

We will now determine the orientation of these axes about which $I_{x'}$, $I_{y'}$ are maximum and minimum. These particular axes are called principal axes

By differentiating the first of Eqs. 10-9 with respect to θ and setting the result to zero. Thus;

$$\frac{dI_{x'}}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right) \sin 2\theta - 2I_{xy} \cos 2\theta = 0$$

Therefore, at $\theta = \theta_p$;

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2}$$

By substituting for θ in $I_{x'}$, $I_{y'}$ and $I_{x'y'}$ equations and simplifying, we obtain;

$$I_{\max \min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

Summary,,,,,

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad \text{..... 10-9 (a)}$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad \text{..... 10-9 (a)}$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad \text{..... 10-9 (a)}$$

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2} \quad \text{..... 10-10}$$

$$I_{\max \min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad \text{..... 10-11}$$

Equations 10-9 to 10-11 have a graphical solution that is convenient to use and generally easy to remember.

Principal Axes and Principal Moments of Inertia

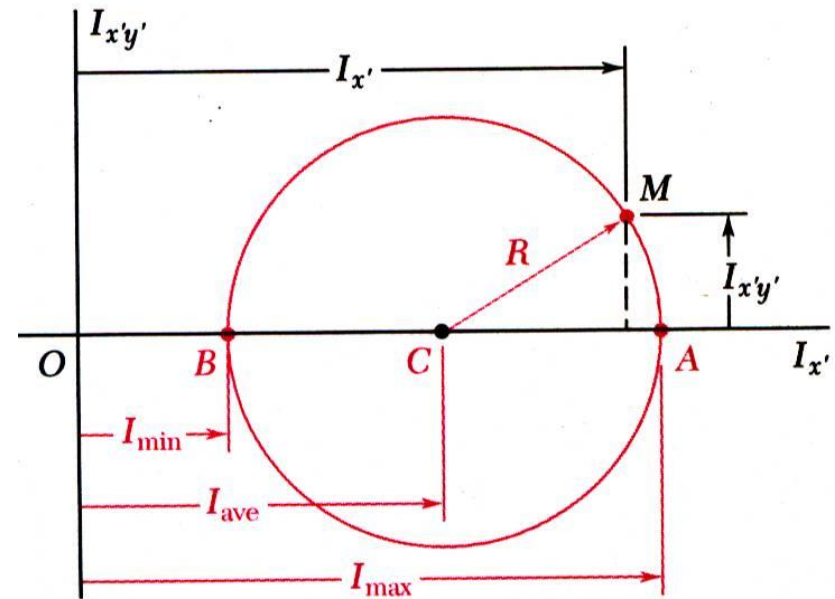
Squaring Eqs. 10-9a and 10-9c and adding, it is found that;

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

where;

$$I_{ave} = \frac{I_x + I_y}{2},$$

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$



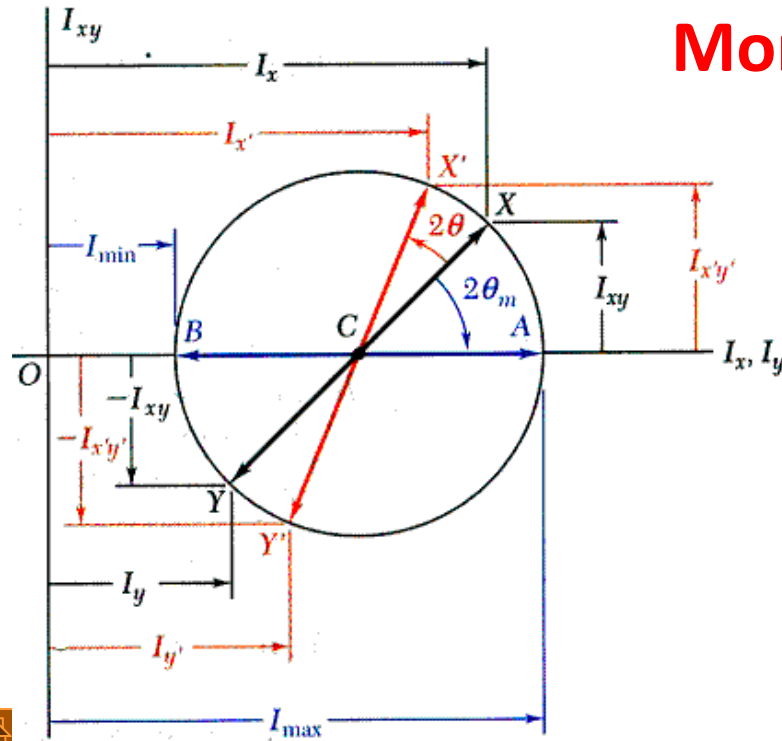
- At the points A and B , $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively.

$$I_{\max, \min} = I_{ave} \pm R$$

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

- The equation for θ_m defines two angles, 90° apart which correspond to the *principal axes* of the area about O .
- $I_{\max}$ and $I_{\min}$ are the *principal moments of inertia* of the area about O .

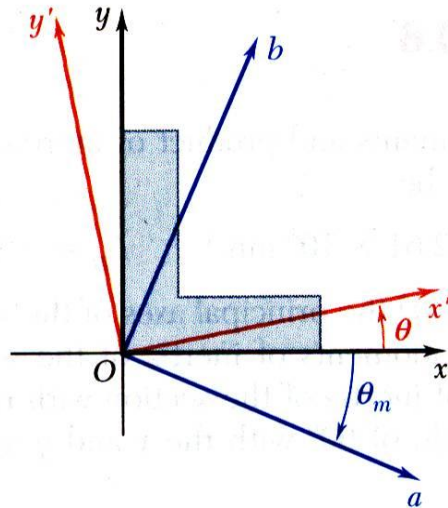
10.8 Mohr's Circle for Moments and Products of Inertia



- The moments and product of inertia for an area are plotted as shown and used to construct *Mohr's circle*,

$$I_{ave} = \frac{I_x + I_y}{2} \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

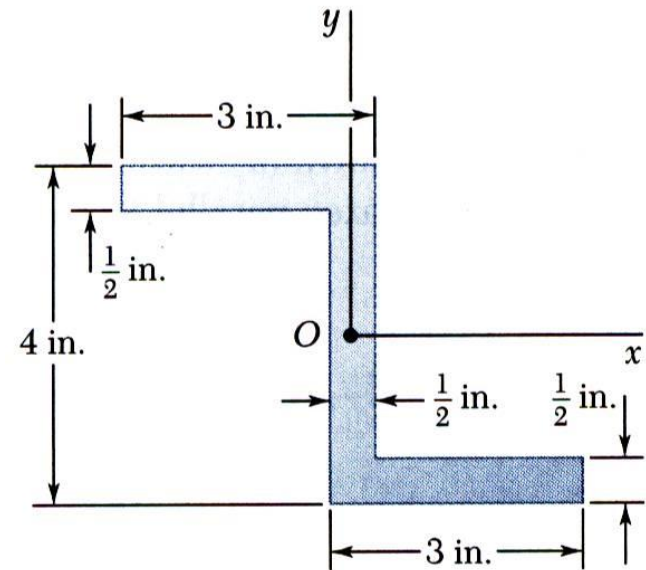
- Mohr's circle may be used to graphically or analytically determine the moments and product of inertia for any other rectangular axes including the principal axes and principal moments and products of inertia.



Sample Problem 10.7

For the section shown, the moments of inertia with respect to the x and y axes are $I_x = 10.38 \text{ in}^4$ and $I_y = 6.97 \text{ in}^4$.

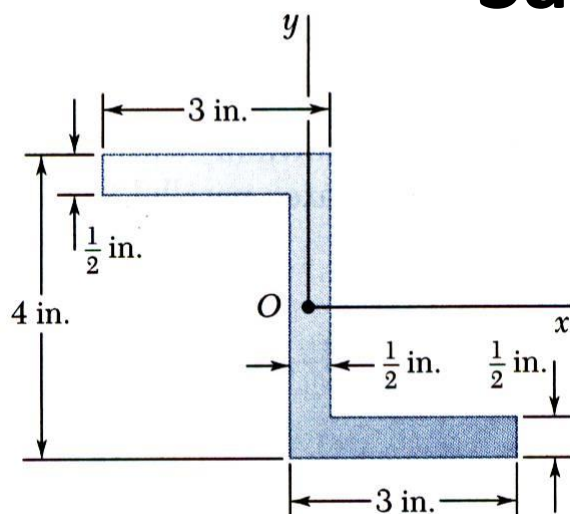
Determine (a) the orientation of the principal axes of the section about O , and (b) the values of the principal moments of inertia about O .



SOLUTION:

- Compute the product of inertia with respect to the xy axes by dividing the section into three rectangles and applying the parallel axis theorem to each.
- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9.27).

Sample Problem 10.7



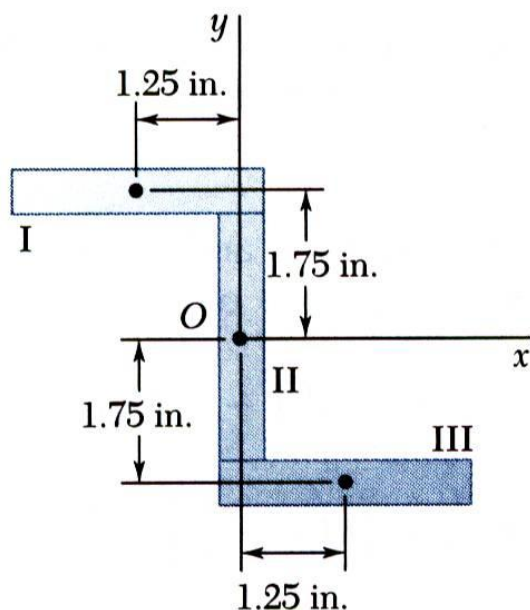
SOLUTION:

- Compute the product of inertia with respect to the xy axes by dividing the section into three rectangles.

Apply the parallel axis theorem to each rectangle,

$$I_{xy} = \sum (\bar{I}_{x'y'} + \bar{x}\bar{y}A)$$

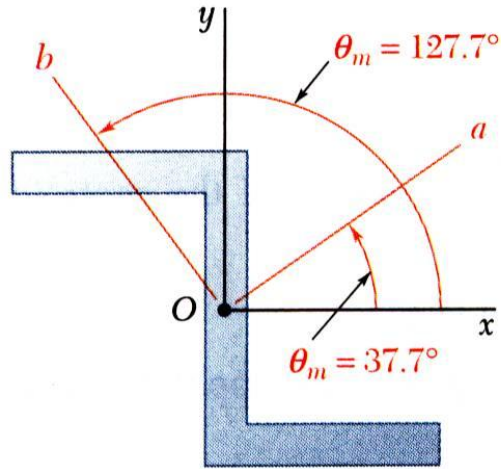
Note that the product of inertia with respect to centroidal axes parallel to the xy axes is zero for each rectangle.



Rectangle	Area, in ²	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\bar{y}A$, in ⁴
I	1.5	-1.25	+1.75	-3.28
II	1.5	0	0	0
III	1.5	+1.25	-1.75	-3.28
				$\sum \bar{x}\bar{y}A = -6.56$

$$I_{xy} = \sum \bar{x}\bar{y}A = -6.56 \text{ in}^4$$

Sample Problem 10.7



- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9.27).

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-6.56)}{10.38 - 6.97} = +3.85$$

$$2\theta_m = 75.4^\circ \text{ and } 255.4^\circ$$

$$\theta_m = 37.7^\circ \text{ and } \theta_m = 127.7^\circ$$

$$I_x = 10.38 \text{ in}^4$$

$$I_y = 6.97 \text{ in}^4$$

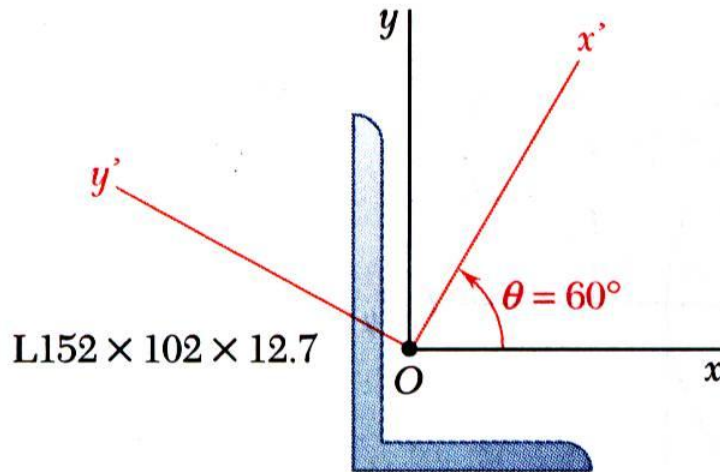
$$I_{xy} = -6.56 \text{ in}^4$$

$$\begin{aligned} I_{\max, \min} &= \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \\ &= \frac{10.38 + 6.97}{2} \pm \sqrt{\left(\frac{10.38 - 6.97}{2}\right)^2 + (-6.56)^2} \end{aligned}$$

$$I_a = I_{\max} = 15.45 \text{ in}^4$$

$$I_b = I_{\min} = 1.897 \text{ in}^4$$

Sample Problem 10.8



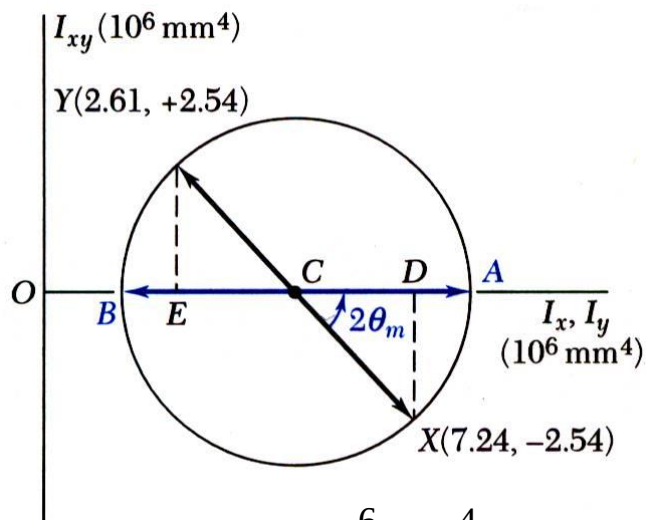
The moments and product of inertia with respect to the x and y axes are $I_x = 7.24 \times 10^6 \text{ mm}^4$, $I_y = 2.61 \times 10^6 \text{ mm}^4$, and $I_{xy} = -2.54 \times 10^6 \text{ mm}^4$.

Using Mohr's circle, determine (a) the principal axes about O , (b) the values of the principal moments about O , and (c) the values of the moments and product of inertia about the x' and y' axes

SOLUTION:

- Plot the points (I_x, I_{xy}) and $(I_y, -I_{xy})$. Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.

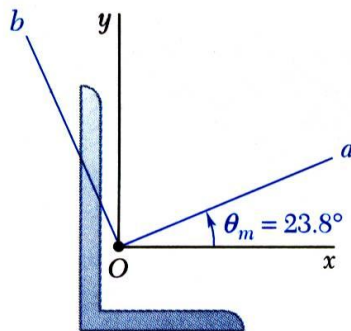
Sample Problem 10.8



$$I_x = 7.24 \times 10^6 \text{ mm}^4$$

$$I_y = 2.61 \times 10^6 \text{ mm}^4$$

$$I_{xy} = -2.54 \times 10^6 \text{ mm}^4$$



SOLUTION:

- Plot the points (I_x, I_{xy}) and $(I_y, -I_{xy})$. Construct Mohr's circle based on the circle diameter between the points.

$$OC = I_{ave} = \frac{1}{2}(I_x + I_y) = 4.925 \times 10^6 \text{ mm}^4$$

$$CD = \frac{1}{2}(I_x - I_y) = 2.315 \times 10^6 \text{ mm}^4$$

$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \text{ mm}^4$$

- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6^\circ \quad \theta_m = 23.8^\circ$$

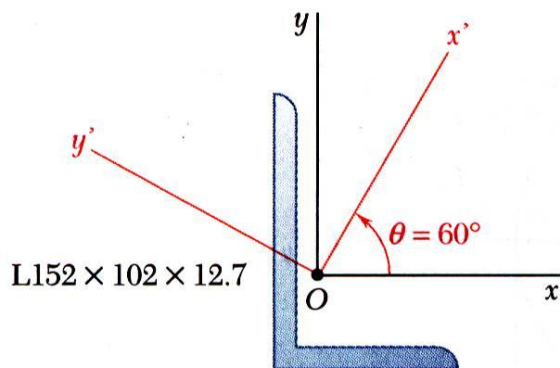
$$I_{\max} = OA = I_{ave} + R$$

$$I_{\max} = 8.36 \times 10^6 \text{ mm}^4$$

$$I_{\min} = OB = I_{ave} - R$$

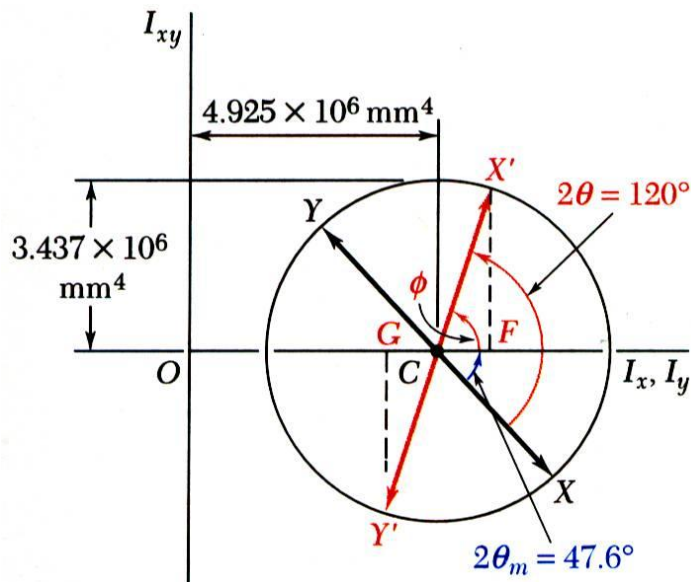
$$I_{\min} = 1.49 \times 10^6 \text{ mm}^4$$

Sample Problem 10.8



- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.

The points X' and Y' corresponding to the x' and y' axes are obtained by rotating CX and CY counterclockwise through an angle $\Theta = 2(60^\circ) = 120^\circ$. The angle that CX' forms with the x' axes is $\phi = 120^\circ - 47.6^\circ = 72.4^\circ$.



$$OC = I_{ave} = 4.925 \times 10^6 \text{ mm}^4$$

$$R = 3.437 \times 10^6 \text{ mm}^4$$

$$I_{x'} = OF = OC + CX' \cos \phi = I_{ave} + R \cos 72.4^\circ$$

$$I_{x'} = 5.96 \times 10^6 \text{ mm}^4$$

$$I_{y'} = OG = OC - CY' \cos \phi = I_{ave} - R \cos 72.4^\circ$$

$$I_{y'} = 3.89 \times 10^6 \text{ mm}^4$$

$$I_{x'y'} = FX' = CY' \sin \phi = R \sin 72.4^\circ$$

$$I_{x'y'} = 3.28 \times 10^6 \text{ mm}^4$$