

Chapter 1

Introduction

Differential equations began with Leibniz, the Bernoulli brothers, and others from the 1680s, not long after Newton's "fluxional equations" in the 1670s. One of the first examples of a first order equation treated by Newton was

$$y' = 1 - 3x + y + x^2 + xy.$$

A second access to differential equations is the consideration of geometrical problems such as inverse tangent problems. A particular example describes the path of a silver pocket watch and was proposed around 1674 by "Claudius Perraltus Medicus Parisinus" to Leibniz: a curve $y(x)$ is required whose tangent AB is given, say everywhere of constant length a . This leads to

$$y' = \frac{-y}{\sqrt{a^2 - y^2}},$$

(a first order differential equation.) The solution of these problems had to wait until Leibniz (1684) and above all until the famous paper of Jacob Bernoulli (1690).

In this chapter we give perspective to your study of differential equations. First, we indicate several ways of classifying equations, in order to provide organizational structure for the course. The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. Nevertheless, it remains a dynamic field of inquiry today, with many interesting open questions.

1.1 Some Basic Mathematical Models; Directional Fields

The subject of differential equations constitutes a large and very important branch of mathematics. Moreover differential equations has applications in many branches of science and engineering. From the early days of the calculus the subject has been an area of great theoretical research and practical applications, and it continues to be so in our day. This much stated, several questions naturally arise. Just what is a differential equation and what does it signify? Where and how do differential equations originate and of what use are they? Confronted with a differential equation, what does one do with it, how does one do it, and what are the results of such activity? These questions indicate three major aspects of the subject: theory, method, and application. The purpose of this chapter is to introduce the reader to the basic aspects of the subject and at the same time give a brief survey of the three aspects just mentioned. In the course of the chapter, we shall find answers to the general questions raised above, answers that will become more and more meaningful as we proceed with the study of differential equations in the following chapters.

Definition. (Differential equation)

A differential equation is a relation involving an unknown function and at least one of its derivatives. If the unknown function depends only on one variable, then the differential equation is an ordinary differential equation.

Example. The following are differential equations

$$(a) \frac{dy}{dx} + xy = 0.$$

$$(b) \frac{d^2x}{dt^2} + 5t = \sin t.$$

$$(c) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = xy.$$

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms the relations are equations and the rates are derivatives. Equations containing derivatives are differential equations. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a mathematical model of the process. In this section we give two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

Example 1. (A Falling Object)

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Solution. We begin by introducing letters to represent various quantities of possible interest in this problem. The motion takes place during a certain time interval, so let us use t to denote time. Also, let us use v to represent the velocity of the falling object. The velocity will presumably change with time, so we think of v as a function of t ; in other words, t is the independent variable and v is the dependent variable. The choice of units of measurement is somewhat arbitrary, and there is nothing in the statement of the problem to suggest appropriate units, so we are free to make any choice that seems reasonable. To be specific, let us measure time t in seconds and velocity v in meters/second. Further, we will assume that v is positive in the downward direction, that is, when the object is falling.

The physical law that governs the motion of objects is Newton's second law, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = ma. \quad (1)$$

In this equation m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. To keep our units consistent, we will measure m in kilograms, a in meters/second², and F in newtons. Of course, a is related to v by $a = dv/dt$, so we can rewrite Equation (1) in the form

$$F = m \frac{dv}{dt}. \quad (2)$$

Next, consider the forces that act on the object as it falls. Gravity exerts a force equal to the weight of the object, or mg , where g is the acceleration due to gravity. In the units we have chosen, g has been

determined experimentally to be approximately equal to 9.8 m/sec^2 near the earth's surface. There is also a force due to air resistance, or drag, which is more difficult to model. This is not the place for an extended discussion of the drag force; suffice it to say that it is often assumed that the drag is proportional to the velocity, and we will make that assumption here. Thus the drag force has the magnitude γv , where γ is a constant called the drag coefficient.

In writing an expression for the net force F we need to remember that gravity always acts in the downward (positive) direction, while drag acts in the upward (negative) direction. Thus

$$F = mg - \gamma v \quad (3)$$

and Equation (2) then becomes

$$m \frac{dv}{dt} = mg - \gamma v. \quad (4)$$

Equation (4) is a mathematical model of an object falling in the atmosphere near sea level. Note that the model contains the three constants m , g , and γ . The constants m and γ depend very much on the particular object that is falling, and usually will be different for different objects. It is common to refer to them as parameters, since they may take on a range of values during the course of an experiment. On the other hand, the value of g is the same for all objects.

To solve Equation (4) we need to find a function $v = v(t)$ that satisfies the equation. It is not hard to do this and we will show you how in the next section. For the present, however, let us see what we can learn about solutions without actually finding any of them. Our task is simplified slightly if we assign numerical values to m and γ , but the procedure is the same regardless of which values we choose. So, let us suppose that $m = 10\text{kg}$ and $\gamma = 2\text{kg/sec}$. If the units for γ seem peculiar, remember that γv must have the units of force, namely, kg-m/sec^2 . Then Equation (4) can be rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \quad (5)$$

Example 2. (Field Mice and Owls)

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. This assumption is not a well-established physical law (as Newton's law of motion is in Example 1), but it is a common initial hypothesis in a study of population growth. If we denote time by t and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \quad (6)$$

where the proportionality factor r is called the rate constant or growth rate. To be specific, suppose that time is measured in months and that the rate constant r has the value $0.5/\text{month}$. Then each term in Equation (6) has the units of mice/month.

Now let us add to the problem by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day. To incorporate this information into the model, we must add another term to the differential equation (6), so that it becomes

$$\frac{dp}{dt} = 0.5p - 450. \quad (7)$$

A more general version of Equation (7) is

$$\frac{dp}{dt} = rp - k, \quad (8)$$

where the growth rate r and the predation rate k are unspecified.

Example 3. (A Falling Object (continued))

Investigate the behavior of solutions of the equation

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \quad (9)$$

without actually finding the solutions in question.

Solution. We will proceed by looking at Equation (9) from a geometrical viewpoint. Suppose that v has a certain value. Then, by evaluating the right side of Equation (9), we can find the corresponding value of dv/dt . For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv -plane by drawing short line segments, or arrows, with slope 1.8 at several points on the line $v = 40$. Similarly, if $v = 50$, then $dv/dt = -0.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$. We obtain Figure 1 by proceeding in the same way with other values of v . Figure 1 is an example of what is called a direction field or sometimes a slope field.

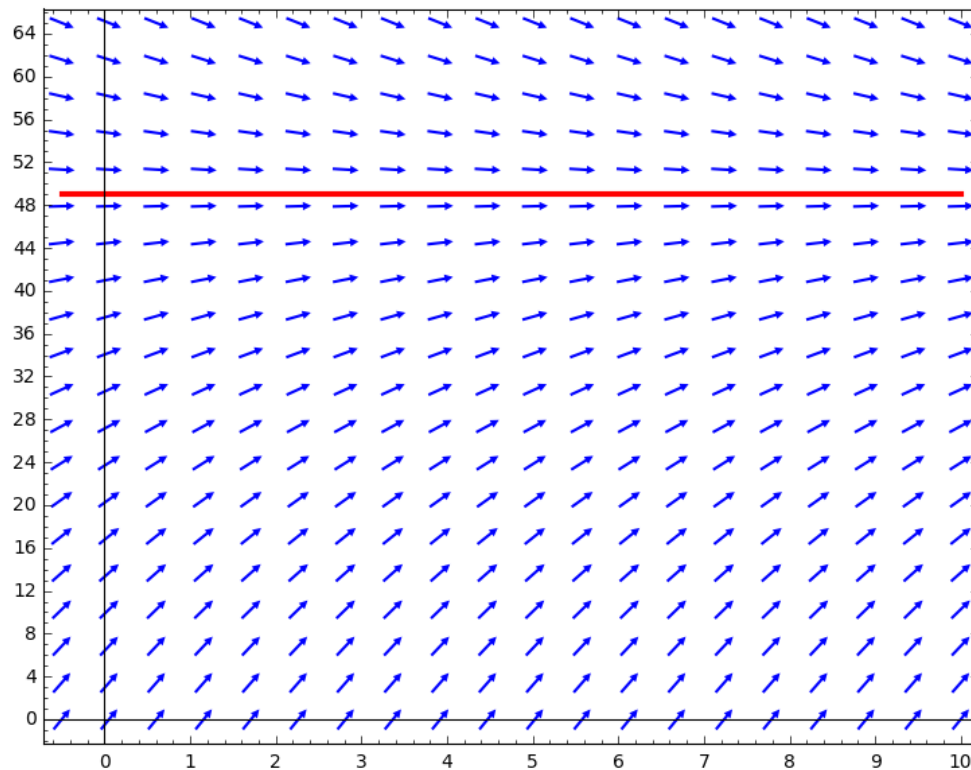


Figure 1: Directional field for equation (9)

The importance of Figure 1 is that each line segment is a tangent line to the graph of a solution of (9). Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of solutions. For instance, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing? Referring again to (9), we ask what value of v will cause dv/dt to be zero? The answer is $v = (5)(9.8) = 49\text{m/sec}$.

In fact, the constant function $v(t) = 49$ is a solution of Equation (9). To verify this statement, substitute $v(t) = 49$ into Equation (9) and observe that each side of the equation is zero. Because it

does not change with time, the solution $v(t) = 49$ is called an equilibrium solution. It is the solution that corresponds to a balance between gravity and drag. In Figure 1 we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From this figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increases.

Direction Fields

Direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \quad (10)$$

where f is a given function of the two variables t and y , sometimes referred to as the rate function. A useful direction field for equations of the general form (10) can be constructed by evaluating f at each point of a rectangular grid consisting of at least a few hundred points. Then, at each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation. The construction of a direction field is often a useful first step in the investigation of a differential equation.

Two observations are worth particular mention. First, in constructing a direction field, we do not have to solve Equation (10), but merely evaluate the given function $f(t, y)$ many times. Thus direction fields can be readily constructed even for equations that may be quite difficult to solve. Second, repeated evaluation of a given function is a task for which a computer is well suited and you should usually use a computer to draw a direction field.

1.3 Classification of Differential Equations

Definition. (Ordinary differential equation)

A differential equation involving ordinary derivatives of one or more dependent variable with respect to a single independent variable is called an ordinary differential equation.

Example. The following differential equations are ordinary differential equations:

$$\begin{aligned} \text{(a)} \quad & \frac{d^3 y}{dt^3} + (t - 3) \left(\frac{dy}{dt} \right)^3 + y^2 \frac{dy}{dt} = 0. \\ \text{(b)} \quad & \frac{d^2 x}{dt^2} + \frac{dy}{dt} + \frac{dy}{dt} \frac{dx}{dt} + t^2 \left(\frac{dx}{dt} \right)^2 = \cos t. \end{aligned}$$

Definition. (Order of a differential equation)

The order of a differential equation is the order of the highest derivative that appears in the equation.

Example. Find the order of each of the following differential equations.

$$\begin{aligned} \text{(a)} \quad & y' + 4y^2 = t. \\ \text{(b)} \quad & ty - 2yy' + y'' + (y')^2 = 3ty'. \\ \text{(c)} \quad & -\frac{d^5 y}{dt^5} + e^t y = \ln y. \end{aligned}$$

Solution.

- a) First order
- b) Second order
- c) Fifth order

General form of n -th order ordinary differential equation

The general form of an ordinary differential equation of n -th order is

$$F(t, y, y', \dots, y^{(n-1)}, y^{(n)}) = 0, \quad (11)$$

where $y' = \frac{dy}{dt}$, $y'' = \frac{d^2y}{dt^2}$, $\dots$, $y^{(n)} = \frac{d^ny}{dt^n}$. We assume that equation (11) can be written in normal form

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}). \quad (12)$$

Solution of an ordinary differential equation

Definition. A solution of the differential equation $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$ on the interval $\alpha < t < \beta$ is a function $y = \phi(t)$, such that ϕ' , ϕ'' , $\dots$, $\phi^{(n)}$ exist and satisfy $\phi^{(n)} = f(t, \phi, \phi', \dots, \phi^{(n-1)})$ for every $t \in (\alpha, \beta)$.

Example 1. Show that $y = e^{-t}$ is a solution of the differential equation $y'' + 2y' + y = 0$ on $\mathbb{R}$.

Solution.

$$y = e^{-t}, y' = -e^{-t}, y'' = e^{-t}$$

$$y'' + 2y' + y = e^{-t} - 2e^{-t} + e^{-t} = 0$$

Thus $y = e^{-t}$ is a solution

Example 2. Let $y = \frac{1}{t+c}$, where c is a constant. Is y a solution of the differential equation $y' + y^2 = 0$?

Solution.

$$y = \frac{1}{t+c}, y' = \frac{-1}{(t+c)^2}$$

$$y' + y^2 = \frac{-1}{(t+c)^2} + \frac{1}{(t+c)^2} = 0$$

Thus $y = \frac{1}{t+c}$ is a solution of the equation

Definition. (Linear and nonlinear equations)

- (1) An ordinary differential equation of order n is linear if it can be written in the form

$$P_n(t) \frac{d^ny}{dt^n} + P_{n-1}(t) \frac{d^{n-1}y}{dt^{n-1}} + \dots + P_1(t) \frac{dy}{dt} + P_0(t)y = Q(t),$$

where $P_n(t)$ is not identically zero.

- (2) An ordinary differential equation is nonlinear if it is not linear.

Example. Determine whether the given equation is linear or nonlinear.

(a) $\frac{d^3y}{dt^3} - 5\frac{dy}{dt} + 6ty = 0.$

(b) $ty^{(4)} + t^2y''' - 3t^3y' = 3te^t.$

(c) $y'' + 4y' + 6y^2 = 0.$

(d) $\frac{d^2y}{dt^2} + 7\left(\frac{dy}{dt}\right)^3 + y = \sin t.$

(e) $yy^{(3)} + ty' + y = \tan t.$

(f) $y' = \sin y.$

(g) $y'' + (t - 1)y' + \cos(t + 1)y = \sin t.$

Solution.

In the following sections we will study methods of solving ordinary differential equations. The mathematical theory and the techniques for solving linear equations are highly developed. In contrast, for nonlinear equations the situation is not as satisfactory. There is no general method for solving nonlinear equations and the theory associated with such equations is also more complicated than the theory of linear equations.