

Chapter 2

First Order Differential Equations

This chapter deals with ordinary differential equations of first order,

$$\frac{dy}{dt} = f(t, y), \quad (1)$$

where f is a given function of two variables. Our object is to solve this equation; that is to find a differentiable function $y = \varphi(t)$ that satisfies this equation for all t in some interval. Unfortunately, there is no general method for solving the equation in terms of elementary functions for an arbitrary function f . Instead, we will describe several methods, each of which is applicable to a certain subclass of first order equations.

2.1 Linear Equations; Method of Integrating Factor

The general form of first-order linear ordinary differential equation is

$$P_1(t)y' + P_0(t)y = Q(t) \quad (2)$$

which can be written in the form

$$y' + p(t)y = g(t), \quad (3)$$

where $p(t)$ and $g(t)$ are given functions of t .

Example. Solve the differential equation

$$y' + \frac{1}{t}y = 1.$$

Solution.

We can't integrate the equation in this form.

Let us multiply both sides by t :

$$ty' + y = t$$

the new form can be written as

$$\frac{d}{dt}(ty) = t$$

Which can be integrated to give

$$\int \frac{d}{dt}(ty) dt = \int t dt \rightarrow ty = \frac{1}{2}t^2 + c \quad \text{or} \quad y = \frac{1}{2}t + \frac{c}{t}$$

Integrating Factor

Our main goal in this section is to solve the first-order linear equation

$$y' + p(t)y = g(t). \quad (4)$$

We will look for a function, say $\mu(t)$, such that when multiplying equation (4) by this function we can integrate the given equation. If we multiply equation (4) by the undetermined function $\mu(t)$, we obtain

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t). \quad (5)$$

The term $\mu(t)y'$ comes from $\frac{d}{dt}(\mu y)$:

$$\frac{d}{dt}(\mu y) = \mu y' + \mu' y.$$

Solving the last equation for $\mu(t)y'$ and substituting into (10) we obtain

$$\frac{d}{dt}(\mu y) - \mu' y + \mu p y = \mu g. \quad (6)$$

We can integrate $\frac{d}{dt}(\mu y)$ and μg , but we can not integrate $-\mu' y + \mu p y$.

Can we eliminate this terms?

Since μ is unknown, we can choose μ so that $-\mu' y + \mu p y = 0$. That is $(\mu' - p\mu)y = 0$. In general $y \neq 0$, so we require $\mu' - p\mu = 0$. This equation can be written as

$$\frac{\mu'}{\mu} = p,$$

provided that $\mu(t) \neq 0$. Thus, integrating both sides gives

$$\ln \mu = \int p(t)dt + k.$$

By choosing the arbitrary constant k to be zero, we obtain the simplest possible function for μ , namely,

$$\mu(t) = \exp \left[\int p(t)dt \right]. \quad (7)$$

The function $\mu(t)$ given by equation (7) is called the integrated factor of the first-order linear equation (4).

Coming back to the differential equation (11) we find

$$\frac{d}{dt}(\mu y) = \mu g$$

which can be integrated to give

$$\mu(t)y(t) = \int \mu(t)g(t)dt + c.$$

As a result we obtain the general solution of equation (4)

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)g(t)dt + c \right]. \quad (8)$$

Remark. To use the formulas (7) and (8) it is necessary to make sure that the differential equation is exactly in the form $y' + p(t)y = g(t)$.

Example 1. Find the general solution of the differential equation $y' + y = 1$.

Solution.

In this example we have a first order linear equation

$y' + py = g$ with $p(t) = g(t) = 1$

using formula $\mu(t) = \exp \left[\int p(t) dt \right]$ we obtain

$\mu(t) = \exp \left[\int p(t) dt \right] = \exp \left[\int dt \right] = e^t$

Multiplying the equation by $\mu(t)$ gives

$e^t [y' + y = 1] \implies e^t y' + e^t y = e^t$

The last equation can be written as

$$\frac{d}{dt}(e^t y) = e^t$$

Integrating both sides yields

$$e^t y = e^t + c$$

and hence the general solution is

$$y(t) = 1 + ce^{-t}$$

Example 2. Solve the differential equation $y' = t - 2y$.

Solution.

Firstly, we write the equation in the standard form

$y' + py = g$

Thus, the equation reads $y' + 2y = t$

It follows that $p(t) = 2$

And hence

$$\mu(t) = \exp \left[\int 2 dt \right] = e^{2t}$$

Multiplying the equation by $\mu(t)$ gives

$$e^{2t} [y' + 2y = t] \implies e^{2t} y' + 2e^{2t} y = te^{2t}$$

The last equation can be written as

$$\frac{d}{dt}(e^{2t} y) = te^{2t}$$

Integrating both sides yields

$$e^{2t} y = \int t e^{2t} dt + c$$

Derivatives of u	Integrals of (dv/dt)
t	e^{2t}
1	$\frac{1}{2} e^{2t}$
0	$\frac{1}{4} e^{2t}$

$$\text{Thus } e^{2t} y = \frac{1}{2} t e^{2t} - \frac{1}{4} e^{2t} + c$$

Therefore, the general solution is

$$y(t) = \frac{1}{2} t - \frac{1}{4} + ce^{-2t}$$

Example 3. Solve the differential equation $ty' - 4y = t^3$.

Solution.

The equation is standard form reads $y' - \frac{4}{t}y = t^2$

It follows that $p(t) = -\frac{4}{t}$

And hence

$$\mu(t) = \exp \left[\int -\frac{4}{t} dt \right] = e^{-4 \ln t} = t^{-4}$$

Multiplying the equation by $\mu(t)$ gives

$$t^{-4} [y' - \frac{4}{t}y = t^2] \implies t^{-4} y' - 4t^{-5} y = t^{-2}$$

The last equation can be written as

$$\frac{d}{dt}(t^{-4} y) = t^{-2}$$

Integrating both sides yields

$$t^{-4} y = -t^{-1} + c$$

As a result, the general solution is

$$y(t) = ct^4 - t^3$$

Initial Value Problems

The general solution of the first order linear equation $y' + p(t)y = g(t)$ is

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)g(t)dt + C \right],$$

where C is an arbitrary constant. Each value of C gives us a solution. Sometimes it is important to pick out one particular member of these solutions. To do so we identify a particular point (t_0, y_0) through which the graph of the solution is required to pass. This requirement is usually written as $y(t_0) = y_0$ and is referred to as an initial condition.

Definition. (Initial value problem)

A first order differential equation $y' = f(t, y)$ together with an initial condition $y(t_0) = y_0$ is called an initial value problem.

Example 1. Solve the initial value problem

$$y' - y = te^{-t}, \quad y(0) = 1.$$

Solution.

First of all we solve the differential equation $y' - y = te^{-t}$
 The integrating factor is $\mu(t) = e^{-t}$
 Multiplying both sides by $\mu(t)$ gives
 $e^{-t}[y' - y = te^{-t}] \implies e^{-t}y' - e^{-t}y = te^{-2t}$
 Or

$$\frac{d}{dt}(e^{-t}y) = te^{-2t}$$

 Integrating both sides gives

$$e^{-t}y = \int te^{-2t}dt + c$$

Derivatives of u	Integrals of (dv/dt)
t	e^{-2t}
1	$-\frac{1}{2}e^{-2t}$
0	$\frac{1}{4}e^{-2t}$

Thus

$$e^{-t}y = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + c$$

The general solution of the differential equation is

$$y(t) = \frac{-1}{2}te^{-t} - \frac{1}{4}e^{-t} + ce^t$$

Now we apply the initial condition $y(0) = 1$

$$1 = 0 - \frac{1}{4} + c$$

It follows that $c = \frac{5}{4}$

and hence the solution of the initial value problem is

$$y(t) = \frac{-1}{2}te^{-t} - \frac{1}{4}e^{-t} + \frac{5}{4}e^t$$

Example 2. Find the solution of the initial value problem

$$\frac{dy}{dt} + \left(\frac{2t+1}{t} \right) y = e^{-2t}, \quad y(1) = 0.$$

Solution.

The integrating factor is $\mu(t) = \exp \left[\int \frac{2t+1}{t} dt \right] = \exp \left[\int (2 + \frac{1}{t}) dt \right]$
 $= e^{2t + \ln t} = te^{2t}$

Multiplying both sides by $\mu(t)$ gives

$$te^{2t} \left[\frac{dy}{dt} + \left(\frac{2t+1}{t} \right) y \right] = te^{2t} e^{-2t} \implies te^{2t}y' + (2t+1)e^{2t}y = t$$

Or

$$\frac{d}{dt}(te^{2t}y) = t$$

Integrating both sides gives

$$te^{2t}y = \frac{1}{2}t^2 + c$$

and the general solution of the differential equation is

$$y(t) = \frac{1}{2}te^{-2t} + \frac{c}{t}e^{-2t}$$

Now we apply the initial condition $y(1) = 0$

$$0 = \frac{1}{2}e^{-2} + ce^{-2} \implies c = -\frac{1}{2}$$

It follows that the solution of the initial value problem is

$$y(t) = \left(\frac{t}{2} - \frac{1}{2t} \right) e^{-2t}$$

Example 3. Solve the initial value problem

$$ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}.$$

Solution.

Let us write the equation in standard form

$$y' + \frac{2}{t}y = \frac{t^2 - t + 1}{t}$$

it follows that $p(t) = \frac{2}{t}$

and hence the integrating factor is $\mu(t) = \exp \left[\int \left(\frac{2}{t} \right) dt \right] = \exp [2 \ln t] = e^{\ln t^2}$

Thus $\mu(t) = t^2$

Multiplying both sides of the equation by $\mu(t)$ gives

$$t^2 \left[y' + \frac{2}{t}y \right] = \frac{t^2 - t + 1}{t} \implies t^2 y' + 2ty = t(t^2 - t + 1)$$

Or $\frac{d}{dt}(t^2 y) = t^3 - t^2 + t$

Integrating both side gives

$$t^2 y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$

And the general solution of the differential equation is

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$$

Apply the initial condition $y(1) = \frac{1}{2}$, we obtain

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \implies c = \frac{1}{12}$$

It follows that the solution of the initial value problem is

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

2.2 Separable Equations

Consider the first order differential equation

$$y' = f(t, y).$$

We can write it as

$$y' = -\frac{M(t, y)}{N(t, y)} \quad \text{or} \quad N(t, y) \frac{dy}{dt} + M(t, y) = 0.$$

One way do this is by setting $M(t, y) = -f(t, y)$ and $N(t, y) = 1$, but there may be other ways as well.

Definition. (Separable Equation)

A first order differential $y' = -\frac{M(t, y)}{N(t, y)}$ is separable if M is a function of t only and N is a function of y only.

Solving separable equations

Separable equations are the simplest first-order differential equations. Consider a separable first order differential equation written in differential form

$$N(y)dy + M(t)dt = 0. \quad (9)$$

We can integrate this equation directly as

$$\int N(y)dy = - \int M(t)dt.$$

Example 1. Show that the equation $(t^2 + 1)y' + 4ty = 0$ is separable and solve it.

Solution.

The equation can be written as

$$(t^2 + 1) \frac{dy}{dt} = -4ty \implies \frac{1}{y} \frac{dy}{dt} = \frac{-4t}{t^2 + 1}$$

Thus, the equation is in the form $M(y) \frac{dy}{dt} + N(t) = 0$ with $M(y) = \frac{1}{y}$, $N(t) = \frac{-4t}{(t^2+1)}$ and so it is separable

To solve the equation, we write it in the differential form as

$$\frac{1}{y} dy = \frac{-4t}{t^2 + 1} dt$$

Integrating both sides gives

$$\int \frac{1}{y} dy = \int \frac{-4t}{t^2 + 1} dt$$

$$u = t^2 + 1 \implies du = 2t dt$$

$$\implies \ln |y| = \int \frac{-2 du}{u} \implies \ln |y| = -2 \ln |u| + c$$

Thus, we obtain

$$\ln |y| = -2 \ln |t^2 + 1| + c$$

Solving for y gives

$$e^{\ln |y|} = e^{-2 \ln |t^2 + 1| + c}$$

$$|y| = e^{-2 \ln |t^2 + 1|} e^c$$

$$|y| = \frac{e^c}{(t^2 + 1)^2}$$

$$y = \frac{\pm e^c}{(t^2 + 1)^2}$$

Thus, y can be written as

$$y = \frac{c^*}{(t^2 + 1)^2}$$

where $c^* = \pm e^c$

Example 2. Solve the differential equation $e^y(\sin x + 1) \frac{dy}{dx} + (e^y + 1) \cos x = 0$.

Solution.

$$e^y(\sin x + 1) \frac{dy}{dx} + (e^y + 1) \cos x = 0 \implies e^y(\sin x + 1) \frac{dy}{dx} = -(e^y + 1) \cos x$$

$$\frac{e^y}{e^y + 1} \frac{dy}{dx} = \frac{-\cos x}{1 + \sin x}$$

Thus, the differential equation is separable and it can be written as

$$\frac{e^y}{e^y + 1} dy = \frac{-\cos x}{1 + \sin x} dx$$

Integrating both sides yields

$$\int \frac{e^y}{e^y + 1} dy = \int \frac{-\cos x}{1 + \sin x} dx$$

$$\ln(e^y + 1) = -\ln |1 + \sin x| + \ln c$$

Solving for y one obtains

$$e^{\ln(e^y + 1)} = e^{-\ln |1 + \sin x| + \ln c}$$

$$e^y + 1 = e^{-\ln |1 + \sin x|} e^{\ln c}$$

$$e^y + 1 = \frac{c}{|1 + \sin x|}$$

$$e^y = \frac{c}{|1 + \sin x|} - 1$$

Therefore, we have

$$y = \ln \left[\frac{c}{|1 + \sin x|} - 1 \right]$$

Example 3. Solve the initial value problem $4y(x^2 + 5x + 6)y' = (3x + 8)(y^2 + 4)$, $y(1) = 2$.

Solution.

The given differential equation is separable since it can be written as

$$\frac{4y}{y^2 + 4} dy = \frac{3x + 8}{x^2 + 5x + 6} dx$$

Integrating both sides yields

$$\begin{aligned} \int \frac{4y}{y^2 + 4} dy &= \int \frac{3x + 8}{x^2 + 5x + 6} dx \\ \implies 2\ln(y^2 + 4) &= \int \frac{3x + 8}{x^2 + 5x + 6} dx \end{aligned}$$

To integrate the right hand side we will use partial fraction

$$\begin{aligned} \frac{3x + 8}{x^2 + 5x + 6} &= \frac{3x + 8}{(x + 2)(x + 3)} = \frac{A}{x + 2} + \frac{B}{x + 3} \\ 3x + 8 &= A(x + 3) + B(x + 2) \\ x = -2: &\implies A = 2 \\ x = -3: &\implies B = 1 \end{aligned}$$

It follows that

$$\frac{3x + 8}{x^2 + 5x + 6} = \frac{2}{x + 2} + \frac{1}{x + 3}$$

Hence

$$\begin{aligned} 2\ln(y^2 + 4) &= \int \left[\frac{2}{x + 2} + \frac{1}{x + 3} \right] dx \\ &= 2\ln|x + 2| + \ln|x + 3| + c \\ &= \ln(|x + 2|^2 |x + 3|) + c \end{aligned}$$

Solving for y one obtains

$$(y^2 + 4)^2 = e^c (x + 2)^2 |x + 3|$$

Now we apply the initial condition $x = 1, y = 2$:

$$\begin{aligned} (4 + 4)^2 &= e^c (1 + 2)^2 |1 + 3| \\ 64 &= 36e^c \\ e^c &= \frac{16}{9} \end{aligned}$$

Therefore, the solution of the initial value problem is

$$(y^2 + 4)^2 = \frac{16}{9} (x + 2)^2 |x + 3|$$

Example 4. Solve the initial value problem $8 \cos^2 y + \csc^2 x \frac{dy}{dx} = 0$, $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$.

Solution.

The equation is separable

$$\csc^2 x \frac{dy}{dx} = -8 \cos^2 y$$

$$\frac{dy}{\cos^2 y} = \frac{-8 dx}{\csc^2 x}$$

$$\sec^2 y dy = -8 \sin^2 x dx$$

Integrating both sides yields

$$\begin{aligned} \int \sec^2 y dy &= \int -8 \sin^2 x dx \\ \implies \tan y &= -8 \int \frac{1 - \cos(2x)}{2} dx \\ \implies \tan y &= -4 \left[x - \frac{1}{2} \sin(2x) \right] + c \end{aligned}$$

Thus, the solution of the differential equation is

$$\tan y = 2 \sin(2x) - 4x + c$$

Next we apply the initial condition $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

$$\begin{aligned} \tan\left(\frac{\pi}{4}\right) &= 2 \sin\left(\frac{\pi}{6}\right) - \frac{\pi}{3} + c \\ 1 &= 1 - \frac{\pi}{3} + c \implies c = \frac{\pi}{3} \end{aligned}$$

Therefore, the solution of the initial value problem is

$$\tan y = 2 \sin(2x) - 4x + \frac{\pi}{3}$$

Homogeneous Equations

Definition. A first order differential equation $y' = f(t, y)$ is said to be homogeneous if it can be written as

$$y' = F(y/t),$$

for some function F of one variable.

Example. Determine whether the following equations are homogeneous:

$$(1) \quad y' = \frac{y^2 - 3ty}{t^2}.$$

$$(2) \quad y' = \ln t - \ln y + \frac{t+y}{t-y}.$$

$$(3) \quad y' = \frac{t^3 + 4ty}{y^2}.$$

$$(1) \quad y' = \frac{y^2 - 3ty}{t^2} = \frac{y^2}{t^2} - \frac{3y}{t} = \left(\frac{y}{t}\right)^2 - 3\left(\frac{y}{t}\right) \leqslant \text{homogeneous equation}$$

$$(2) \quad y' = \ln t - \ln y + \frac{t+y}{t-y} = \ln\left(\frac{t}{y}\right) + \frac{t(1+\frac{y}{t})}{t(1-\frac{y}{t})} = \ln\left(\frac{t}{y}\right) + \frac{(1+\frac{y}{t})}{(1-\frac{y}{t})} \leqslant \text{homogeneous equation}$$

$$(3) \quad y' = \frac{t^3 + 4ty}{y^2} = \frac{t^3}{y^2} + \frac{4t}{y} = t\left(\frac{t}{y}\right)^2 + 4\left(\frac{t}{y}\right) \leqslant \text{isn't homogeneous equation}$$

Solution.

Solving Homogeneous Equations

Let $y' = F(y/t)$ be a homogeneous equation. The transformation $y = tv$ transforms it to the separable equation

$$tv' = F(v) - v.$$

Example 1. Show the the equation $(x^2 - y^2)y' = xy$ is homogeneous and solve it.

Solution.

Firstly, let us show that the equation is homogeneous

$$(x^2 - y^2)y' = xy \implies y' = \frac{xy}{x^2 - y^2} \implies y' = \frac{xy}{x^2 \left(1 - \frac{y^2}{x^2}\right)} = \frac{\frac{y}{x}}{\left(1 - \frac{y^2}{x^2}\right)}$$

This shows that equation is homogeneous.

Now we will solve the equation.

Let $y = xv$.

Then we have $y' = xv' + v$

Substituting into the equation gives

$$xv' + v = \frac{v}{(1 - v^2)}$$

Simplifying the equation, we get the separable differential equation

$$xv' = \frac{v}{(1 - v^2)} - v = \frac{v - v(1 - v^2)}{(1 - v^2)} = \frac{v^3}{(1 - v^2)}$$

Thus, we have

$$\frac{1 - v^2}{v^3} dv = \frac{dx}{x}$$

Integrating both sides, we obtain

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x} \implies \int (v^{-3} - v^{-1}) dv = \ln|x| + c$$

$$-\frac{1}{2v^2} - \ln|v| = \ln|x| + c$$

Substituting $v = y/x$ we obtain

$$-\frac{x^2}{2y^2} - \ln\left|\frac{y}{x}\right| = \ln|x| + c$$

$$\implies \ln\left|\frac{y}{x}\right| + \ln|x| = -c - \frac{x^2}{2y^2}$$

$$\implies \ln|y| = -c - \frac{x^2}{2y^2}$$

Thus, the solution is $\ln|y| + \frac{x^2}{2y^2} + c = 0$

Example 2. Solve the equation $y + \sqrt{t^2 + y^2} - t \frac{dy}{dt} = 0, \quad t > 0.$

Solution.

The equation can be written as

$$y' = \frac{y + \sqrt{t^2 + y^2}}{t} = \frac{y}{t} + \frac{|t| \sqrt{1 + \frac{y^2}{t^2}}}{t} = \frac{y}{t} + \sqrt{1 + \frac{y^2}{t^2}} \quad (\text{since } t > 0)$$

This shows that equation is homogeneous.

To solve the equation, we use the transformation $y = tv$

Then we have $y' = tv' + v$

Substituting into the equation gives

$$tv' + v = v + \sqrt{v^2 + 1}$$

Simplifying the equation, we get the separable differential equation

$$tv' = \sqrt{v^2 + 1}$$

Thus, we have

$$\frac{dv}{\sqrt{v^2 + 1}} = \frac{dt}{t}$$

Integrating both sides, we obtain

$$\int \frac{dv}{\sqrt{v^2 + 1}} = \int \frac{dt}{t}$$

We will use trigonometric substitutions to integrate the left-hand side.

$$\begin{aligned} v &= \tan \theta, & \theta &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ dv &= \sec^2 \theta d\theta, \\ \sqrt{v^2 + 1} &= \sec \theta \end{aligned}$$

Thus, the integration becomes

$$\int \frac{\sec^2 \theta d\theta}{\sec \theta} = \ln|t| + c \implies \int \sec \theta d\theta = \ln t + c \quad (\text{since } t > 0)$$

$$\ln|\sec \theta + \tan \theta| = \ln t + c$$

Substituting $\sec \theta = \sqrt{v^2 + 1}$ and $\tan \theta = v$, we get

$$\ln|\sqrt{v^2 + 1} + v| = \ln t + c$$

Substituting $v = y/t$, we obtain

$$\ln \left| \sqrt{\left(\frac{y^2}{t^2}\right) + 1} + \left(\frac{y}{t}\right) \right| = \ln t + c$$

Taking the exponential of both sides gives

$$\exp \left(\ln \left| \sqrt{\left(\frac{y^2}{t^2}\right) + 1} + \left(\frac{y}{t}\right) \right| \right) = \exp(\ln t + c)$$

$$\left| \sqrt{\left(\frac{y^2}{t^2}\right) + 1} + \left(\frac{y}{t}\right) \right| = e^c t$$

$$\left| \sqrt{\left(\frac{y^2 + t^2}{t^2}\right) + \left(\frac{y}{t}\right) \right| = e^c t$$

$$\left| \frac{\sqrt{y^2 + t^2}}{t} + \left(\frac{y}{t}\right) \right| = e^c t$$

$$\left| \frac{\sqrt{y^2 + t^2} + y}{t} \right| = e^c t$$

Thus, the solution of the differential equation is given by

$$y + \sqrt{t^2 + y^2} = c^* t^2$$

Example 3. Solve the equation $y' = \frac{2ty}{t^2 - 3y^2}.$

Solution.

The equation can be written as

$$y' = \frac{2ty}{t^2 - 3y^2} = \frac{2ty}{t^2 \left(1 - \frac{3y^2}{t^2}\right)} = \frac{\frac{2y}{t}}{1 - \frac{3y^2}{t^2}}$$

This shows that equation is homogeneous.

To solve the equation, we use the transformation $y = tv$.

Then we have $y' = tv' + v$

Substituting into the equation gives

$$tv' + v = \frac{2v}{1 - 3v^2}$$

Simplifying the equation, we get the separable differential equation

$$tv' = \frac{2v}{1 - 3v^2} - v = \frac{2v - v(1 - 3v^2)}{1 - 3v^2} = \frac{v + 3v^3}{1 - 3v^2}$$

Thus, we have

$$\frac{(1 - 3v^2)dv}{v(1 + 3v^2)} = \frac{dt}{t}$$

Integrating both sides, we obtain

$$\int \frac{(1 - 3v^2)dv}{v(1 + 3v^2)} = \int \frac{dt}{t}$$

We will use integration by partial fraction to integrate the left-hand side

$$\frac{(1 - 3v^2)dv}{v(1 + 3v^2)} = \frac{A}{v} + \frac{Bv + C}{3v^2 + 1} \implies 1 - 3v^2 = A(3v^2 + 1) + v(Bv + C)$$

$$v = 0 \implies A = 1$$

$$\implies -6v = 6Av + 2Bv + C$$

$$v = 0 \implies C = 0$$

$$\implies -6 = 6A + 2B \implies B = -3 - 3A \implies B = -6$$

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Thus, the integration becomes

$$\int \left(\frac{1}{v} - \frac{6v}{3v^2 + 1} \right) dv = \ln|t| + c$$

$$\implies \ln|v| - \ln|3v^2 + 1| = \ln|t| + c$$

$$\ln \left| \frac{v}{3v^2 + 1} \right| = \ln|t| + c$$

$$\left| \frac{v}{3v^2 + 1} \right| = e^c |t|$$

Substituting $v = y/t$, we obtain

$$\left| \frac{\frac{y}{t}}{\frac{3y^2}{t^2} + 1} \right| = e^c |t|$$

$$\left| \frac{ty}{3y^2 + t^2} \right| = e^c |t|$$

$$\left| \frac{|ty|}{3y^2 + t^2} \right| = e^c |t|$$

Thus, the solution of the differential equation is given by

$$|y| = c_1 (3y^2 + t^2)$$

Example 4. Solve the initial value problem $t^2 y' = y^2 + ty - t^2$, $y(1) = 2$.

Solution.

First of all, we write the equation in the form

$$y' = \frac{y^2 + ty - t^2}{t^2} = \left(\frac{y}{t}\right)^2 + \frac{y}{t} - 1$$

Thus, we see that the equation is homogeneous

We let $y = tv$ and obtain

$$tv' + v = v^2 + v - 1$$

Or

$$tv' = v^2 - 1$$

Separating variables, we find

$$\frac{dv}{v^2 - 1} = \frac{dt}{t}$$

Since

$$\frac{1}{v^2 - 1} = \frac{1}{2} \frac{1}{v - 1} - \frac{1}{2} \frac{1}{v + 1}$$

Integrating both sides of the equation to obtain

$$\frac{1}{2} \ln|v - 1| - \frac{1}{2} \ln|v + 1| = \ln|t| + c$$

Or

$$\ln \left| \frac{v - 1}{v + 1} \right| = 2 \ln|t| + 2c$$

Taking the exponential of both sides to get

$$\left| \frac{v - 1}{v + 1} \right| = e^{2c} |t|^2$$

Now replacing v by y/t , we obtain the general solution of the differential equation the form

$$\left| \frac{y - t}{y + t} \right| = e^{2c} |t|^2$$

Solving for y :

$$\frac{y - t}{y + t} = \pm e^{2c} |t|^2 = c_1 t^2$$

The initial condition requires that $y = 2$ when $t = 1$

This gives $c_1 = \frac{1}{3}$

It follows that

$$\frac{y - t}{y + t} = \frac{1}{3} t^2 \implies 3(y - t) = t^2(y + t) \implies (t^2 - 3)y = -3t - t^3$$

$$y = \frac{-t(3 + t^2)}{t^2 - 3}$$

Thus, the solution of the initial value problem is

$$y = \frac{t(3 + t^2)}{3 - t^2}$$

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analysis may suggest the most promising directions to explore experimentally, and may indicate fairly precisely what experimental data will be most helpful.

Example 1. (Mixing)

At time $t = 0$ a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering the tank at a rate of r gal/min, and that the well-stirred mixture is draining from the tank at the same rate. Set up the initial value problem that describes this flow process. Find the amount of salt $Q(t)$ in the tank at any time, and also find the limiting amount Q_L that is present after a very long time. If $r = 3$ and $Q_0 = 2Q_L$, find the time T after which the salt level is within 2% of Q_L . Also find the flow rate that is required if the value of T is not to exceed 45 min.

Solution. We assume that salt is neither created nor destroyed in the tank. Therefore variations in the amount of salt are due solely to the flows in and out of the tank. More precisely, the rate of change of salt in the tank, $\frac{dQ}{dt}$, is equal to the rate at which salt is flowing in minus the rate at which it is flowing out. In symbols,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out.} \quad (10)$$

The rate at which salt enters the tank is $\frac{1}{4}$ lb/gal times the flow rate, or $\frac{r}{4}$ lb/min. To find the rate at which salt leaves the tank we need to multiply the concentration of salt in the tank by the rate of outflow, r gal/min. Since the rates of flow in and out are equal, the volume of water in the tank remains constant at 100 gal, and since the mixture is “well-stirred,” the concentration throughout the tank is the same, namely, $[Q(t)/100]$ lb/gal. Therefore the rate at which salt leaves the tank is $[rQ(t)/100]$ lb/min. Thus the differential equation governing this process is

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}. \quad (11)$$

The initial condition is

$$Q(0) = Q_0. \quad (12)$$

Upon thinking about the problem physically, we might anticipate that eventually the mixture originally in the tank will be essentially replaced by the mixture flowing in, whose concentration is $\frac{1}{4}$ lb/gal. Consequently, we might expect that ultimately the amount of salt in the tank would be very close to 25 lb. We can reach the same conclusion from a geometrical point of view by drawing a direction field for Eq. (11) for any positive value of r .

To solve the problem analytically note that Eq. (11) is both linear and separable. The integrating factor is $e^{rt/100}$ and the general solution is

$$Q(t) = 25 + ce^{-rt/100}, \quad (13)$$

where c is an arbitrary constant. To satisfy the initial condition (12) we must choose $c = Q_0 - 25$. Therefore the solution of the initial value problem is

$$Q(t) = 25(1 - e^{-rt/100}) + Q_0 e^{-rt/100}. \quad (14)$$

From Eq. (14), you can see that $Q(t) \rightarrow 25$ (lb) as $t \rightarrow \infty$, so the limiting value Q_L is 25, confirming our physical intuition. In interpreting the solution (14), note that the second term on the right side is the portion of the original salt that remains at time t , while the first term gives the amount of salt in the tank due to the action of the flow processes. Plots of the solution for $r = 3$ and for several values of Q_0 are shown in Figure

Now suppose that $r = 3$ and $Q_0 = 2Q_L = 50$; then Eq. (14) becomes

$$Q(t) = 25 + 25e^{-0.03t}. \quad (15)$$

Since 2% of 25 is 0.5, we wish to find the time T at which $Q(t)$ has the value 25.5. Substituting $t = T$ and $Q = 25.5$ in Eq. (15) and solving for T , we obtain

$$T = (\ln 50)/0.03 = 130.4 \text{ (min)}. \quad (16)$$

To determine r so that $T = 45$, return to Eq. (14), set $t = 45$, $Q_0 = 50$, $Q(t) = 25.5$, and solve for r . The result is

$$r = \frac{100}{45} \ln 50 = 8.69 \text{ (gal/min)}. \quad (17)$$

Since this example is hypothetical, the validity of the model is not in question. If the flow rates are as stated, and if the concentration of salt in the tank is uniform, then the differential equation (10) is an accurate description of the flow process. While this particular example has no special significance, it is important that models of this kind are often used in problems involving a pollutant in a lake, or a drug in an organ of the body, for example, rather than a tank of salt water. In such cases the flow rates may not be easy to determine, or may vary with time. Similarly, the concentration may be far from uniform in some cases. Finally, the rates of inflow and outflow may be different, which means that the variation of the amount of liquid in the problem must also be taken into account.

Example 2. (Compound Interest)

Suppose that a sum of money is deposited in a bank or money fund that pays interest at an annual rate r . The value $S(t)$ of the investment at any time t depends on the frequency with which interest is compounded as well as the interest rate. Financial institutions have various policies concerning compounding: some compound monthly, some weekly, some even daily. If we assume that compounding takes place continuously, then we can set up a simple initial value problem that describes the growth of the investment.

The rate of change of the value of the investment is $\frac{dS}{dt}$, and this quantity is equal to the rate at which interest accrues, which is the interest rate r times the current value of the investment $S(t)$. Thus

$$\frac{dS}{dt} = rS \quad (18)$$

is the differential equation that governs the process. Suppose that we also know the value of the investment at some particular time, say,

$$S(0) = S_0. \quad (19)$$

Then the solution of the initial value problem (18), (19) gives the balance $S(t)$ in the account at any time t . This initial value problem is readily solved, since the differential equation (18) is both linear and separable. Consequently, by solving Eqs. (11) and (12), we find that

$$S(t) = S_0 e^{rt}. \quad (20)$$

Thus a bank account with continuously compounding interest grows exponentially.

Let us now compare the results from this continuous model with the situation in which compounding occurs at finite time intervals. If interest is compounded once a year, then after t years

$$S(t) = S_0(1 + r)^t.$$

If interest is compounded twice a year, then at the end of 6 months the value of the investment is $S_0[1 + (r/2)]$, and at the end of 1 year it is $S_0[1 + (r/2)]^2$. Thus, after t years we have

$$S(t) = S_0 \left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded m times per year, then

$$S(t) = S_0 \left(1 + \frac{r}{m}\right)^{mt}. \quad (21)$$

The relation between formulas (20) and (21) is clarified if we recall from calculus that

$$\lim_{m \rightarrow \infty} S_0 \left(1 + \frac{r}{m}\right)^{mt} = S_0 e^{rt}.$$

The same model applies equally well to more general investments in which dividends and perhaps capital gains can also accumulate, as well as interest. In recognition of this fact, we will from now on refer to r as the rate of return.

Table ?? shows the effect of changing the frequency of compounding for a return rate r of 8%. The second and third columns are calculated from Eq. (??) for quarterly and daily compounding, respectively, and the fourth column is calculated from Eq. (20) for continuous compounding. The results show that the frequency of compounding is not particularly important in most cases. For example, during a 10-year period the difference between quarterly and continuous compounding is \$17.50 per \$1000 invested, or less than \$2/year. The difference would be somewhat greater for higher rates of return and less for lower rates. From the first row in the table, we see that for the return rate $r = 8\%$, the annual yield for quarterly compounding is 8.24% and for daily or continuous compounding it is 8.33%.

Returning now to the case of continuous compounding, let us suppose that there may be deposits or withdrawals in addition to the accrual of interest, dividends, or capital gains. If we assume that the deposits or withdrawals take place at a constant rate k , then Eq. (??) is replaced by

$$\frac{dS}{dt} = rS + k, \quad (22)$$

where k is positive for deposits and negative for withdrawals. Equation (22) is linear with the integrating factor e^{-rt} , so its general solution is

$$S(t) = ce^{rt} - \frac{k}{r},$$

where c is an arbitrary constant. To satisfy the initial condition (19) we must choose $c = S_0 + (k/r)$. Thus the solution of the initial value problem (22), (19) is

$$S(t) = S_0 e^{rt} - \frac{k}{r}(e^{rt} - 1). \quad (23)$$

The first term in expression (23) is the part of $S(t)$ that is due to the return accumulated on the initial amount S_0 , while the second term is the part that is due to the deposit or withdrawal rate k .

The advantage of stating the problem in this general way without specific values for S_0 , r , or k lies in the generality of the resulting formula (23) for $S(t)$. With this formula we can readily compare the results of different investment programs or different rates of return.

For instance, suppose that one opens an individual retirement account (IRA) at age 25 and makes annual investments of \$2000 thereafter in a continuous manner. Assuming a rate of return of 8%, what will be the balance in the IRA at age 65? We have $S_0 = 0$, $r = 0.08$, and $k = \$2000$, and we wish to determine $S(40)$. From Eq. (23) we have

$$S(40) = (25,000)(e^{3.2} - 1) = \$588,313.$$

It is interesting to note that the total amount invested is \$80,000, so the remaining amount of \$508,313 results from the accumulated return on the investment. The balance after 40 years is also fairly sensitive to the assumed rate. For instance, $S(40) = \$508,948$ if $r = 0.075$ and $S(40) = \$681,508$ if $r = 0.085$.

Let us now examine the assumptions that have gone into the model. First, we have assumed that the return is compounded continuously and that additional capital is invested continuously. Neither of these is true in an actual financial situation. We have also assumed that the return rate r is constant for the entire period involved, whereas in fact it is likely to fluctuate considerably. Although we cannot reliably predict future rates, we can use expression (23) to determine the approximate effect of different rate projections. It is also possible to consider r and k in Eq. (22) to be functions of t rather than constants; in that case, of course, the solution may be much more complicated than Eq. (23).

The initial value problem (22), (19) and the solution (23) can also be used to analyze a number of other financial situations, including annuities, mortgages, and automobile loans among others.

2.4 Difference Between Linear and Nonlinear Equations

In this section we will try to answer the following questions:

1. Does an initial value problem always have a solution?
2. Can an initial value problem have more than one solution?
3. Is the solution valid for all t , or only for some restricted interval?

The following theorem answers these questions for a first order linear differential equation.

Theorem 1. Let $y' + p(t)y = g(t)$, $y(t_0) = y_0$ be an initial value problem. If the functions p and g are continuous on an interval $I = (\alpha, \beta)$ containing the initial point t_0 , then the initial value has a unique solution $y = \varphi(t)$ valid for all $t \in I$.

Remark. Theorem 1 says that the initial value problem $y' + p(t)y = g(t)$, $y(t_0) = y_0$ has a solution and this solution is unique. In addition the solution is valid for the interval where both p and g are continuous.

Example 1. Solve the initial value problem

$$(t^2 + 1)y' + 4ty = t, \quad y(2) = 1$$

and find the interval in which the solution is valid.

Solution.

In this example the differential equation is a first order linear differential equation with $p(t) = \frac{4t}{t^2+1}$, $g(t) = \frac{t}{t^2+1}$

Thus, p and g are continuous for all $t \in \mathbb{R}$ and hence initial value problem has a unique solution and the solution is valid for all $t \in \mathbb{R}$

The integrating factor for this linear equation is

$\mu(t) = (t^2 + 1)^2$ and the solution to the differential equation is

$$y(t) = \frac{t^4 + 2t^2 + 4c}{4(t^2 + 1)^2}$$

$$(t^2 + 1)^2 \left[y' + \frac{4t}{(t^2 + 1)} y = \frac{t}{(t^2 + 1)} \right]$$

$$(t^2 + 1)^2 y' + 4t(t^2 + 1)y = t(t^2 + 1)$$

$$(t^2 + 1)^2 y = \frac{1}{4}t^4 + \frac{1}{2}t^2 + c$$

Thus, the solution to the differential equation is

$$y(t) = \frac{t^4 + 2t^2 + 4c}{4(t^2 + 1)^2}$$

The initial condition gives $c = 19$ and hence the solution to the initial value problem is

$$y(t) = \frac{t^4 + 2t^2 + 76}{4(t^2 + 1)^2}, \quad t \in \mathbb{R}$$

Example 2. Solve the initial value problem

$$(t^2 - 1)y' + (t - 1)y = t + 1, \quad y(0) = 2020$$

and find the interval in which the solution is valid.

Solution.

In this example $p(t) = \frac{1}{t+1}$, $g(t) = \frac{1}{t-1}$

Thus, p and g are continuous for all

$$t \in \mathbb{R} - \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

Since $t_0 = 0 \in (-1, 1)$, the initial value problem has a unique Solution and the solution is valid for all $t = 0 \in (-1, 1)$

The solution to the initial value problem is

$$y(t) = \frac{t + 2\ln|t - 1| + 2020}{(t + 1)}, \quad t \in (-1, 1)$$

Nonlinear Equations

If the differential equation is nonlinear, we must replace Theorem 1 by the following general theorem.

Theorem 2. Let $y' = f(t, y)$, $y(t_0) = y_0$ be an initial value problem. If the functions f and $\partial f / \partial y$ are continuous on some rectangle $\{(t, y) : \alpha < t < \beta, \gamma < y < \delta\}$ containing the initial point (t_0, y_0) , then the initial value has a unique solution $y = \varphi(t)$ valid for all $t \in (t_0 - h, t_0 + h) \subseteq I$ for some positive number h .

Observe that the hypotheses in Theorem 2 reduce to those in Theorem 1 if the differential equation is linear. For then $f(t, y) = -p(t)y + g(t)$ and $\partial f(t, y) / \partial y = -p(t)$, so the continuity of f and $\partial f / \partial y$ is equivalent to the continuity of p and g in this case. The proof of Theorem 1 is comparatively simple because it could be based on the expression that gives the solution of an arbitrary linear equation. There is no corresponding expression for the solution of the differential equation $y' = f(t, y)$, so the proof of Theorem 2 is much more difficult. It is discussed in depth in more advanced books on differential equations.

Interval of Definition

According to Theorem 1, the solution of a linear equation, $y' + p(t)y = g(t)$, subject to the initial condition $y(t_0) = y_0$, exists throughout any interval about $t = t_0$ in which the functions p and g are continuous.

On the other hand, for a nonlinear initial value problem satisfying the hypotheses of Theorem 2, the interval in which a solution exists may be difficult to determine. The solution $y = \varphi(t)$ is certain to exist as long as the point $(t, \varphi(t))$ remains within a region in which the hypotheses of Theorem 2 are satisfied. This is what determines the value of h in that theorem. However, since $\varphi(t)$ is usually not known, it may be impossible to locate the point $(t, \varphi(t))$ with respect to this region. In any case, the interval in which a solution exists may have no simple relationship to the function f in the differential equation $y' = f(t, y)$.

General Solution

Another way in which linear and nonlinear equations differ is in connection with the concept of a general solution. For a first order linear equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant. For nonlinear equations this may not be the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant. For instance, for the differential equation $y' = y^2$, the expression $y = \frac{-1}{t+c}$ contains an arbitrary constant, but does not include all solutions of the differential equation. To show this, observe that the function $y = 0$ for all t is certainly a solution of the differential equation, but it cannot be obtained from the solution $y = \frac{-1}{t+c}$ by assigning a value to c . The existence of “additional” solutions is common for nonlinear equations. Thus we will use the term “general solution” only when discussing linear equations.

Implicit Solution

Recall again that, for an initial value problem for a first order linear equation, the formula

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)g(t)dt + c \right].$$

provides an explicit formula for the solution $y = \varphi(t)$. As long as the necessary antiderivatives can be found, the value of the solution at any point can be determined merely by substituting the appropriate

value of t into the equation. The situation for nonlinear equations is much less satisfactory. Usually, the best that we can hope for is to find an equation

$$F(t, y) = 0 \quad (24)$$

involving t and y that is satisfied by the solution $y = \varphi(t)$. Even this can be done only for differential equations of certain particular types, of which separable equations are the most important. The equation (24) is called an integral, or first integral, of the differential equation, and (as we have already noted) its graph is an integral curve, or perhaps a family of integral curves. Equation (24), assuming it can be found, defines the solution implicitly; that is, for each value of t we must solve equation (24) to find the corresponding value of y . If equation (24) is simple enough, it may be possible to solve it for y by analytical means and thereby obtain an explicit formula for the solution. However, more frequently this will not be possible, and you will have to resort to a numerical calculation to determine the value of y for a given value of t . Once several pairs of values of t and y have been calculated, it is often helpful to plot them and then to sketch the integral curve that passes through them. One can use a computer to do this.

In summary, linear equations have several nice properties that can be summarized in the following statements:

1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.
2. There is an expression for the solution. Moreover, although it involves two integrations, the expression is an explicit one for the solution $y = \varphi(t)$ rather than an equation that defines φ implicitly.
3. The possible points of discontinuity, or singularities, of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all t , then the solution also exists and is continuous for all t .

None of these statements is true, in general, of nonlinear equations. While a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as the differential equation.

Bernoulli Equations

The equation

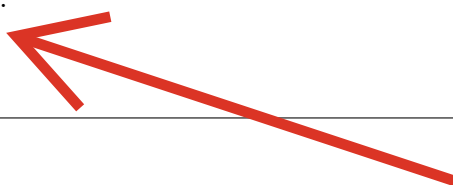
$$y' + p(t)y = q(t)y^n, \quad n \in \mathbb{R} - \{0, 1\}, \quad (25)$$

is called a Bernoulli equation after Jakob Bernoulli. In order to solve the Bernoulli equation we use the transformation (substitution) $v = y^{1-n}$ to reduce it to a linear equation for v :

$$v' + (1-n)p(t)v = (1-n)q(t).$$

Example 1. Solve the equation $y' + y = ty^3$.

Solution.



Example 2. Solve the initial value problem $y' + \frac{y}{2t} = \frac{t}{y^2}$, $y(1) = 2$.

Solution

- The given equation is a Bernoulli equation with $n = -2$.
- So we use the transformation $v = y^3$.
- It follows that $v' = 3y^2y'$.
- The Bernoulli equation can be written as

$$y^2y' + \frac{1}{2t}y^3 = t.$$

- The linear equation for v reads

$$\frac{1}{3}v' + \frac{1}{2t}v = t$$

- or
 - Thus the integrating factor for this equation reads
 - Multiplying the linear equation by $\mu(t)$, we find
 - or
 - Integrating, we find
 - Thus the solution of the linear equation is
 - and hence the solution of the given Bernoulli equation as
 - The initial condition $y(1) = 2$ gives
 - Therefore, the solution of the initial value problem is
- $$y^3 = \frac{6}{7}t^2 + \frac{50}{7}t^{-3/2}.$$

Solution

- The given equation is a Bernoulli equation with $n = 3$.
- So we use the transformation $v = y^{1-n} = y^{-2}$.
- Differentiating to obtain $v' = -2y^{-3}y'$.
- Now multiply the equation through by y^{-3} , thereby expressing it in the equivalent form

$$y^{-3}y' + y^{-2} = t.$$

- Using the transformation, the given Bernoulli equation transforms into the linear equation for v

$$-\frac{1}{2}v' + v = t$$

or

$$v' - 2v = -2t.$$

- Thus the integrating factor for this equation reads
 - Multiplying the linear equation by $\mu(t) = e^{-2t}$, we find
 - or
 - Integrating, we find
 - Thus the solution of the linear equation is
 - and hence we obtain the solution of the given Bernoulli equation as
- $$y^2 = \frac{2}{2t + 1 + 2ce^{2t}}.$$

Discontinuous Coefficients

Sometimes linear differential equations occur with one or both of the functions p and g have jump discontinuities. If t_1 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_1$ and $t > t_1$. Afterward, the two solutions are matched so that y is continuous at t_1 ; this is accomplished by a proper choice of the arbitrary constants.

Example. Solve the initial value problem $y' + y = g(t)$, $y(0) = 6$, where $g(t) = \begin{cases} 5, & 0 \leq t < 10 \\ 1, & t \geq 10. \end{cases}$

Solution.

Solution

- For $0 \leq t < 10$:
- The equation reads $y' + y = 5$ and hence the solution is $y(t) = 5 + c_1e^{-t}$.
- For $t \geq 10$:
- The equation reads $y' + y = 1$ and hence the solution is $y(t) = 1 + c_2e^{-t}$.

- It follows the the solution of the differential equation is

$$y(t) = \begin{cases} 5 + c_1e^{-t}, & 0 \leq t < 10 \\ 1 + c_2e^{-t}, & t \geq 10. \end{cases}$$

- Next we impose the continuity condition at $t = 10$.
- It follows that $5 + c_1e^{-10} = 1 + c_2e^{-10}$.
- So we can solve for c_2 as $c_2 = 4e^{10} + c_1$
- and hence the solution can be written as

$$y(t) = \begin{cases} 5 + c_1e^{-t}, & 0 \leq t < 10 \\ 1 + c_1e^{-t} + 4e^{10-t}, & t \geq 10. \end{cases}$$

- Lastly we use the initial condition to find c_1 .
- We have $6 = y(0) = 5 + c_1$ and hence $c_1 = 1$.

2.5 Autonomous Equations and Population Dynamics

An important class of first order equations are those in which the independent variable does not appear explicitly. Such equations are called autonomous and have the form

$$\frac{dy}{dt} = f(y). \quad (26)$$

We will discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Section 1.1 we have already considered the special case of Equation (26) in which $f(y) = ay + b$. Equation (26) is separable, so the discussion in Section 2.2 is applicable to it, but our main object now is to show how geometric methods can be used to obtain important qualitative information directly from the differential equation, without solving the equation. Of fundamental importance in this effort are the concepts of stability and instability of solutions of differential equations. These ideas were introduced in Chapter 1, are discussed further here, and are examined in greater depth and in a more general setting in Chapter 9.

Exponential Growth

Let $y = \phi(t)$ be the population of the given species at time t . The simplest hypothesis concerning the variation of population is that the rate of change of y is proportional to the current value of y , that is,

$$\frac{dy}{dt} = ry, \quad (27)$$

where the constant of proportionality r is called the rate of growth or decline, depending on whether it is positive or negative. Here, we assume that $r > 0$, so the population is growing.

Solving Equation (27) subject to the initial condition

$$y(0) = y_0, \quad (28)$$

we obtain

$$y(t) = y_0 e^{rt}. \quad (29)$$

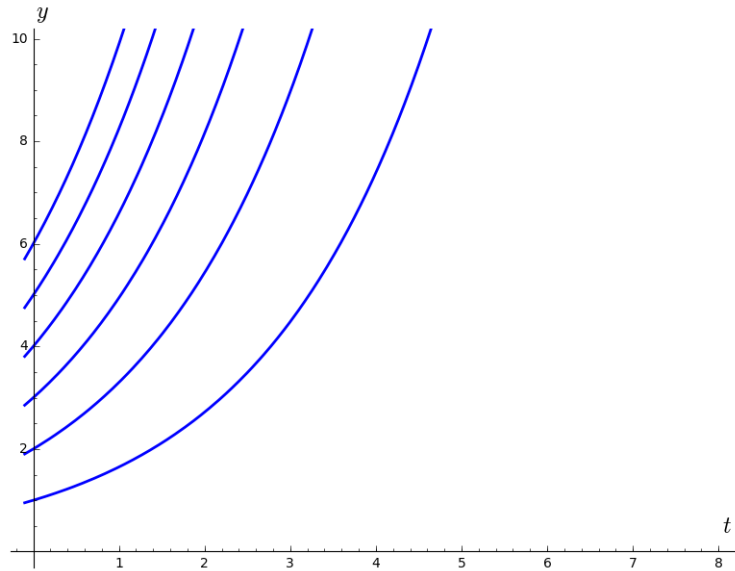
Thus the mathematical model consisting of the initial value problem (27), (28) with $r > 0$ predicts that the population will grow exponentially for all time, as shown in Figure ???. Under ideal conditions Equation (29) has been observed to be reasonably accurate for many populations, at least for limited periods of time. However, it is clear that such ideal conditions cannot continue indefinitely; eventually, limitations on space, food supply, or other resources will reduce the growth rate and bring an end to uninhibited exponential growth.

Logistic Growth

To take account of the fact that the growth rate actually depends on the population, we replace the constant r in Equation (27) by a function $h(y)$ and thereby obtain the modified equation

$$\frac{dy}{dt} = h(y)y. \quad (30)$$

We now want to choose $h(y)$ so that $h(y) \cong r > 0$ when y is small, $h(y)$ decreases as y grows larger, and $h(y) < 0$ when y is sufficiently large. The simplest function having these properties is $h(y) = r - ay$,

Figure 1: Exponential growth: y versus t for $dy/dt = ry$

where a is also a positive constant. Using this function in Equation (30), we obtain

$$\frac{dy}{dt} = (r - ay)y. \quad (31)$$

Equation (31) is known as the Verhulst equation or the logistic equation. It is often convenient to write the logistic equation in the equivalent form

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K}\right) y, \quad (32)$$

where $K = r/a$. The constant r is called the intrinsic growth rate, that is, the growth rate in the absence of any limiting factors. The interpretation of K will become clear shortly.

We first seek solutions of Equation (32) of the simplest possible type, that is, constant functions. For such a solution $\frac{dy}{dt} = 0$ for all t , so any constant solution of Equation (32) must satisfy the algebraic equation

$$r \left(1 - \frac{y}{K}\right) y = 0.$$

Thus the constant solutions are $y = \phi_1(t) = 0$ and $y = \phi_2(t) = K$. These solutions are called equilibrium solutions of Equation (32) because they correspond to no change or variation in the value of y as t increases. In the same way, any equilibrium solutions of the more general Equation (26) can be found by locating the roots of $f(y) = 0$. The zeros of $f(y)$ are also called critical points.

In many situations it is sufficient to have the qualitative information about the solution $y = \phi(t)$ of Equation (32). This information can be obtained entirely from the graph of $f(y)$ versus y , and without solving the differential equation (32). However, if we wish to have a more detailed description of logistic growth—for example, if we wish to know the value of the population at some particular time—then we must solve Equation (32) subject to the initial condition (28). Provided that $y \neq 0$ and $y \neq K$, we can write Equation (32) in the form

$$\frac{dy}{\left(1 - \frac{y}{K}\right) y} = r dt. \quad (33)$$

Using a partial fraction expansion on the left side and integrating both sides we obtain

$$\ln |y| - \ln \left|1 - \frac{K}{y}\right| = rt + c, \quad (34)$$

where c is an arbitrary constant of integration to be determined from the initial condition $y(0) = y_0$. We have already noted that if $0 < y_0 < K$, then y remains in this interval for all time. Thus in this case we can remove the absolute value bars in Equation (34), and by taking the exponential of both sides, we find that

$$\frac{y}{1 - \frac{K}{y}} = Ce^{rt}, \quad (35)$$

where $C = e^c$. To satisfy the initial condition $y(0) = y_0$ we must choose $C = y_0/[1 - (y_0/K)]$. Using this value for C in Equation (35) and solving for y , we obtain

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}. \quad (36)$$

We have derived the solution (36) under the assumption that $0 < y_0 < K$. If $y_0 > K$, then the details of dealing with Equation (34) are only slightly different, and we leave it to you to show that Equation (36) is also valid in this case. Finally, note that Equation (36) also contains the equilibrium solutions $y = \phi_1(t) = 0$ and $y = \phi_2(t) = K$ corresponding to the initial conditions $y_0 = 0$ and $y_0 = K$, respectively.

All the qualitative information about the solution can be obtained by examining the solution (36). In particular, if $y_0 = 0$, then Equation (36) requires that $y(t) = 0$ for all t . If $y_0 > 0$, and if we let $t \rightarrow \infty$ in Equation (36), then we obtain

$$\lim_{t \rightarrow \infty} y(t) = K.$$

Thus for each $y_0 > 0$ the solution approaches the equilibrium solution $y = \phi_2(t) = K$ asymptotically (in fact, exponentially) as $t \rightarrow \infty$. Therefore we say that the constant solution $\phi_2(t) = K$ is an asymptotically stable solution of Equation (32), or that the point $y = K$ is an asymptotically stable equilibrium or critical point. This means that after a long time the population is close to the saturation level K regardless of the initial population size, as long as it is positive.

On the other hand, the situation for the equilibrium solution $y = \phi_1(t) = 0$ is quite different. Even solutions that start very near zero grow as t increases and, as we have seen, approach K as $t \rightarrow \infty$. We say that $\phi_1(t) = 0$ is an unstable equilibrium solution or that $y = 0$ is an unstable equilibrium or critical point. This means that the only way to guarantee that the solution remains near zero is to make sure its initial value is exactly equal to zero.

Example. The logistic model has been applied to the natural growth of the halibut population in certain areas of the Pacific Ocean. Let y , measured in kilograms, be the total mass, or biomass, of the halibut population at time t . The parameters in the logistic equation are estimated to have the values $r = 0.71/\text{year}$ and $K = 80.5 \times 10^6$ kg. If the initial biomass is $y_0 = 0.25K$, find the biomass 2 years later. Also find the time τ for which $y(\tau) = 0.75K$.

Solution. It is convenient to scale the solution (36) to the carrying capacity K ; thus we write Equation (36) in the form

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + [1 - (y_0/K)]e^{-rt}}. \quad (37)$$

Using the data given in the problem, we find that

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75e^{-1.42}} \cong 0.5797.$$

Consequently, $y(2) \cong 46.7 \times 10^6$ kg.

To find τ we can first solve Equation (37) for t . We obtain

$$e^{-rt} = \frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]},$$

hence

$$t = -\frac{1}{r} \ln \left(\frac{(y_0/K)[1 - (y/K)]}{(y/K)[1 - (y_0/K)]} \right).$$

Using the given values of r and y_0/K and setting $y/K = 0.75$, we find that

$$\tau = -\frac{1}{0.71} \ln \left(\frac{(0.25)(0.25)}{(0.75)(0.75)} \right) = \frac{1}{0.71} \ln 9 \cong 3.095 \text{ years.}$$

2.6 Exact Equations and Integrating Factors

Given a first order differential equation there are several methods to solve it according to its class. The most important classes are linear equations and separable equations, which we have discussed previously. In this section we consider the class of exact equations. We will write the equation in the form $M(x, y) + N(x, y) \frac{dy}{dx} = 0$.

Definition. (Exact differential equations)

A differential equation $M(x, y) + N(x, y) \frac{dy}{dx} = 0$ is exact in a domain R if there is a function $\psi(x, y)$ such that

$$\frac{\partial}{\partial x} \psi(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial}{\partial y} \psi(x, y) = N(x, y)$$

for all $(x, y) \in R$.

Example. The equation $y^2 dx + 2xy dy = 0$ is exact in $\mathbb{R}^2$ since $\psi(x, y) = xy^2$ satisfies

$$\frac{\partial}{\partial x} \psi(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial}{\partial y} \psi(x, y) = N(x, y).$$

The following theorem gives a systematic method to determine whether a given differential equation is exact.

Theorem 3. Assume that the functions M , N , M_y , and N_x are continuous in a domain R . Then the equation $M(x, y) + N(x, y) \frac{dy}{dx} = 0$ is exact in R if and only if

$$M_y(x, y) = N_x(x, y)$$

for all $(x, y) \in R$.

Example. Determine whether the given equation is exact.

(a) $y + 2x \frac{dy}{dx} = 0$.

(b) $(2x \sin y + y^3 e^x) + (x^2 \cos y + 3y^2 e^x) \frac{dy}{dx} = 0$.

Solution.

Solution

- (a) In this example $M = y$ and $N = 2x$.
 • Thus $M_y = 1 \neq N_x = 2$
 • and hence the equation is not exact.
- (b) In this example
 $M = 2x \sin y + y^3 e^x$ and $N = x^2 \cos y + 3y^2 e^x$.
 • So $M_y = 2x \cos y + 3y^2 e^x$
 • and $N_x = 2x \cos y + 3y^2 e^x = M_y$
 • and hence the equation is exact.

The method of solving exact equations

Let $M(x, y) + N(x, y) \frac{dy}{dx} = 0$ be an exact equation. Then, by definition, there is a function $\psi(x, y)$ such that

$$\psi_x = M \quad \text{and} \quad \psi_y = N. \quad (38)$$

Using (38) to replace M and N , we can write the differential equation as

$$\psi_x + \psi_y \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{d}{dx} \psi(x, y) = 0.$$

Integrating we find that the solution of the differential equation is given by

$$\psi(x, y) = c.$$

We will use equation (38) to find $\psi(x, y)$ and the solution of the given exact equation.

We may start by any one of the two equations. For example, starting by $\psi_x = M$ and integrating with respect to x , holding y constant, we find

$$\psi(x, y) = \int M(x, y) dx + h(y).$$

The function h is an arbitrary function of y , playing the role of the arbitrary constant. Next we choose $h(y)$ such that the second equation $\psi_y = N$ is satisfied. As a result we obtain

$$h'(y) = N(x, y) - \int M_y(x, y) dx. \quad (39)$$

Remark. It is essential that the right hand side of equation (39) is a function of y only. This can be established by differentiating this quantity with respect to x , obtaining

$$N_x(x, y) - M_y(x, y)$$

which is zero.

Example 1. Show that the equation $(3x^2 + 4xy) + (2x^2 + 2y) \frac{dy}{dx} = 0$ is exact and solve it.

Solution.

Examples

Example 1

Show that the equation $(3x^2 + 4xy) + (2x^2 + 2y) \frac{dy}{dx} = 0$ is exact and solve it.

Solution

- In this example $M = 3x^2 + 4xy$ and $N = 2x^2 + 2y$.
- It follows that

$$M_y = 4x, \quad N_x = 4x$$
- $\implies M_y = N_x$
- Thus the equation is exact.

$$M = 3x^2 + 4xy \quad \text{and} \quad N = 2x^2 + 2y$$

- The solution is given by $\psi(x, y) = c$, where

$$\boxed{\psi_x = M} \quad \text{and} \quad \boxed{\psi_y = N}$$
- Now using $\psi_x = M$ gives

$$\psi(x, y) = \int (3x^2 + 4xy) dx = x^3 + 2x^2y + h(y).$$
- Using $\psi_y = N$ gives

$$2x^2 + h'(y) = 2x^2 + 2y.$$
- This implies that $h'(y) = 2y$
- and hence $h(y) = y^2$.
- Therefore, $\psi(x, y) = x^3 + 2x^2y + y^2$ and the solution of the differential equation is given by

$$\boxed{x^3 + 2x^2y + y^2 = c.}$$

Example 2. Solve the equation $2xydx + (x^2 + 5)dy = 0$.

Solution.

Examples

Example 2

Solve the equation $2xydx + (x^2 + 5)dy = 0$.

Solution

- In this example $M = 2xy$ and $N = x^2 + 5$.
- It follows that

$$M_y = 2x = N_x.$$

- Thus the equation is exact.
- The solution is given by $\psi(x, y) = c$, where

$$\boxed{\psi_x = M} \quad \text{and} \quad \boxed{\psi_y = N}$$

- Now using $\psi_y = N$ gives

$$\psi(x, y) = \int (x^2 + 5)dy = (x^2 + 5)y + h(x).$$

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$$\psi(x, y) = (x^2 + 5)y + h(x).$$

- Using $\psi_x = M$ gives

$$2xy + h'(x) = 2xy.$$

- This implies that $h'(x) = 0$ and hence $h(x) = 0$.
- Therefore $\psi(x, y) = (x^2 + 5)y$ and the solution of the differential equation is given by $(x^2 + 5)y = c$ or

$$\boxed{y = \frac{c}{x^2 + 5}}$$

Example 3. Solve the equation $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0$, $x > 0$.

Solution.

Examples

Example 3

Solve the equation $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0$, $x > 0$.

Solution

- In this example $M = \left(\frac{y}{x} + 6x\right)$ and $N = (\ln x - 2)$.
- It follows that

$$M_y = \frac{1}{x} = N_x.$$

- Thus the equation is exact.
- The solution is given by $\psi(x, y) = c$, where

$$\boxed{\psi_x = M} \quad \text{and} \quad \boxed{\psi_y = N}$$

- Now using $\psi_x = M$ gives

$$\psi(x, y) = \int \left(\frac{y}{x} + 6x\right)dx = y \ln x + 3x^2 + h(y)$$

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$$\psi(x, y) = y \ln x + 3x^2 + h(y)$$

- Using $\psi_y = N$ gives

$$\ln x + h'(y) = \ln x - 2.$$

- This implies that $h'(y) = -2$ and hence $h(y) = -2y$.
- Therefore $\psi(x, y) = y \ln x + 3x^2 - 2y$ and the solution is given by

$$y \ln x + 3x^2 - 2y = c$$

or

$$\boxed{y = \frac{c - 3x^2}{\ln x - 2}}$$

Example 4. Solve the equation $\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0, \quad x > 0.$

Solution.

Examples

Example 4

Solve the equation $\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0, \quad x > 0.$

Solution

- In this example

$$M = \frac{x}{(x^2 + y^2)^{3/2}} \quad \text{and} \quad N = \frac{y}{(x^2 + y^2)^{3/2}}.$$

- $M = x(x^2 + y^2)^{-3/2}$ and $N = y(x^2 + y^2)^{-3/2}$
- It follows that

$$M_y = \frac{-3xy}{(x^2 + y^2)^{5/2}} = N_x.$$

- Thus the equation is exact.

- Now using $\psi_x = M$ gives

$$\psi(x, y) = \int \frac{x dx}{(x^2 + y^2)^{3/2}} = \int \frac{du}{2u^{3/2}} = -u^{-1/2} + h(y)$$

- Now using $\psi_x = M$ gives

$$\psi(x, y) = \frac{-1}{(x^2 + y^2)^{1/2}} + h(y).$$

- Using $\psi_y = N$ gives

$$\frac{y}{(x^2 + y^2)^{3/2}} + h'(y) = \frac{y}{(x^2 + y^2)^{3/2}}.$$

- This implies that $h'(y) = 0$ and hence $h(y) = 0$.

- Therefore, $\psi(x, y) = \frac{-1}{(x^2 + y^2)^{1/2}}$ and the solution is given

$$\text{by } \frac{-1}{(x^2 + y^2)^{1/2}} = c \Rightarrow \frac{-1}{c} = (x^2 + y^2)^{1/2} \text{ or } \boxed{x^2 + y^2 = \bar{c}}$$

Integrating Factors

Given an non-exact equations can be converted to exact equations by multiply the them by suitable integrating factors. Let us investigate this idea by considering a general first order differential equation

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0. \quad (40)$$

We multiply equation (40) by a function $\mu(x, y)$ and we look for choosing μ so that the resulting equation

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y) \frac{dy}{dx} = 0 \quad (41)$$

is exact. That is

$$(\mu M)_y = (\mu N)_x \quad \text{or} \quad M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \quad (42)$$

Equation (42) is a first order partial differential equation. If we find any solution μ of (42), then equation (41) will be exact and then can be solved by the method described above.

In general, equation (42) is as difficult to solve as the original equation (40). Therefore, while in principle integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations are as follows.

(1) If $\frac{M_y - N_x}{N}$ is a function of x only, then μ is a function of x only and is given by

$$\mu(x) = \exp \left[\int \frac{M_y - N_x}{N} dx \right].$$

(2) If $\frac{N_x - M_y}{M}$ is a function of y only, then μ is a function of y only and is given by

$$\mu(y) = \exp \left[\int \frac{N_x - M_y}{M} dy \right].$$

Example 1. Solve the equation $y^2 - (xy + 2)y' = 0$.

Solution.

Solution

- In this example $M(x, y) = y^2$ and $N(x, y) = -(xy + 2)$.
- So $M_y - N_x = 2y + y = 3y \neq 0$ for $y \neq 0$.
- Thus the equation is not exact.
- Let us look for an integrating factor μ .
- Note that $\frac{N_x - M_y}{M} = \frac{-3y}{y^2} = \frac{-3}{y}$ is a function of y only.
- It follows that

$$\begin{aligned} \mu(y) &= \exp \left[\int \frac{N_x - M_y}{M} dy \right] \\ &= \exp \left[\int -\frac{3}{y} dy \right] = \exp(-3 \ln y) = y^{-3} \end{aligned}$$

$$y^2 - (xy + 2)y' = 0$$

- Multiplying the given equation by $\mu = y^{-3}$ it becomes

$$\frac{1}{y} - \frac{(xy + 2)}{y^3} y' = 0$$

which is an exact solution.

- Solving $\psi_x = M = \frac{1}{y}$ we find

$$\psi(x, y) = \frac{x}{y} + h(y).$$

- The second equation $\psi_y = N = -\frac{(xy + 2)}{y^3}$ gives

$$-\frac{x}{y^2} + h'(y) = -\frac{(xy + 2)}{y^3}$$

- and hence $h'(y) = \frac{-2}{y^3}$.
- As a result we obtain $h(y) = y^{-2}$.

$$\psi(x, y) = \frac{x}{y} + h(y)$$

- Therefore,

$$\psi(x, y) = \frac{x}{y} + \frac{1}{y^2} = \frac{yx + 1}{y^2}$$

and the solution of the differential equation is given by

$$\boxed{\frac{yx + 1}{y^2} = c.}$$

Example 2. Solve the equation $ydx + (2xy - e^{-2y})dy = 0$.

Solution.

Examples

Example 2

Solve the equation $ydx + (2xy - e^{-2y})dy = 0$.

Solution

- In this example $M(x, y) = y$ and $N(x, y) = 2xy - e^{-2y}$.
- So $M_y - N_x = 1 - 2y \neq 0$ for $y \neq 1/2$.
- Thus the equation is not exact.
- Let us look for an integrating factor μ .
- Note that $\frac{N_x - M_y}{M} = \frac{2y - 1}{y}$ is a function of y only
- It follows that

$$\begin{aligned}\mu(y) &= \exp \left[\int \frac{N_x - M_y}{M} dy \right] = \exp \left[\int \frac{2y - 1}{y} dy \right] \\ &= \exp(2y - \ln y) = \frac{e^{2y}}{y}.\end{aligned}$$

$$ydx + (2xy - e^{-2y})dy = 0$$

- Multiplying the given equation by $\mu = \frac{e^{2y}}{y}$ it becomes

$$e^{2y} dx + (2xe^{2y} - y^{-1})dy = 0$$

which is an exact solution.

- Solving $\psi_x = M = e^{2y}$ we find

$$\psi(x, y) = xe^{2y} + h(y).$$

- The second equation $\psi_y = N = 2xe^{2y} - y^{-1}$ gives

$$2xe^{2y} + h'(y) = 2xe^{2y} - y^{-1}$$

and hence $h'(y) = -1/y$.

- As a result we obtain $h(y) = -\ln |y|$.

$$\psi(x, y) = xe^{2y} + h(y)$$

- Therefore, $\psi(x, y) = xe^{2y} - \ln |y|$ and the solution of the differential equation is given by

$$xe^{2y} - \ln |y| = c.$$

Example 3. Solve the equation $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$.

Solution.

Solution

- In this example, $M = 4xy + 3y^2 - x$ and $N = x(x + 2y)$.
- Thus $M_y = 4x + 6y \neq N_x = 2x + 2y$ and hence the equation is not exact.
- Let us look for an integrating factor μ .
- Note that $\frac{N_x - M_y}{M} = \frac{2x + 2y - 4x - 6y}{4xy + 3y^2 - x} = \frac{-2x - 4y}{4xy + 3y^2 - x}$ is a NOT function of y only.

$$(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$$

- But $\frac{M_y - N_x}{N} = \frac{2x + 4y}{x(x + 2y)} = \frac{2}{x}$ is a function of x only.
- It follows that
$$\mu(x) = \exp \left[\int \frac{M_y - N_x}{N} dx \right] = \exp \left[\int \frac{2}{x} dx \right] = \exp(2 \ln x) = x^2.$$
- Multiplying the given equation by $\mu = x^2$ it becomes
$$x^2(4xy + 3y^2 - x)dx + x^3(x + 2y)dy = 0$$
 which is an exact solution.
- Solving $\psi_y = N = x^4 + 2x^3y$ we find
$$\psi(x, y) = x^4y + x^3y^2 + h(x).$$
- The second equation $\psi_x = M = 4x^3y + 3x^2y^2 - x^3$ gives
$$4x^3y + 3x^2y^2 + h'(x) = 4x^3y + 3x^2y^2 - x^3$$
- and hence $h'(x) = -x^3$.
- As a result we obtain $h(x) = -\frac{1}{4}x^4$.

$$\psi(x, y) = x^4y + x^3y^2 + h(x)$$

- Therefore, $\psi(x, y) = x^4y + x^3y^2 - \frac{1}{4}x^4$ and the solution of the differential equation is given by

$$x^4y + x^3y^2 - \frac{1}{4}x^4 = c.$$

Example 4. Solve the equation $(x^2 - y^2)y' = xy$.

Solution.

Example 4

Solve the equation $(x^2 - y^2)y' = xy$.

Solution

- In this example $M = -xy$, $N = x^2 - y^2$.
- It follows that $M_y = -x \neq N_x = 2x$.
- Thus the equation is not exact.
- Looking for an integrating factor, we note that $M_y - N_x = -3x$ and hence $(N_x - M_y)/M = -3/y$ is a function of y only.
- As a result we obtain $\mu(y) = y^{-3}$.

$$-xy + (x^2 - y^2)y' = 0$$

- Multiplying the given equation by $\mu = y^{-3}$ it becomes an exact equation $-xy^{-2} + y^{-3}(x^2 - y^2)y' = 0$.
- Using the first equation $\psi_x = M = -xy^{-2}$ to obtain $\psi(x, y) = -\frac{1}{2}x^2y^{-2} + h(y)$.
- The second equation $\psi_y = N = y^{-3}(x^2 - y^2)$ gives

$$x^2y^{-3} + h'(y) = y^{-3}(x^2 - y^2)$$

- and so $h'(y) = -y^{-1}$.
- Thus $h(y) = -\ln|y|$ and hence $\psi(x, y) = -\frac{1}{2}x^2y^{-2} - \ln|y|$.
- Therefore, the solution of the given equation is given by

$$-\frac{1}{2}x^2y^{-2} - \ln|y| = c.$$

Example 5. Solve the equation $3x^2y + 2xy + y^3 + (x^2 + y^2)y' = 0$.

Solution.

Example 5

Solve the equation $3x^2y + 2xy + y^3 + (x^2 + y^2)y' = 0$.

Solution

- Here $M = 3x^2y + 2xy + y^3$ and $N = x^2 + y^2$.
- Thus $M_y = 3x^2 + 2x + 3y^2 \neq N_x = 2x$ and so the equation is not exact.
- Looking for integration factors, we find $\frac{N_x - M_y}{M} = \frac{-3x^2 - 3y^2}{3x^2y + 2xy + y^3}$ is NOT a function of y only.
- But $\frac{M_y - N_x}{N} = \frac{3x^2 + 3y^2}{x^2 + y^2} = 3$ is a function of x only.
- It follows that we have an integrating factor $\mu(x) = e^{3x}$.
- As a result, the equation can be written in exact form as

$$e^{3x}(3x^2y + 2xy + y^3) + e^{3x}(x^2 + y^2)y' = 0.$$

$$e^{3x}(3x^2y + 2xy + y^3) + e^{3x}(x^2 + y^2)y' = 0$$

- It is better to start by $\psi_y = N = e^{3x}(x^2 + y^2)$.
- Thus $\psi(x, y) = e^{3x}x^2y + \frac{1}{3}y^3e^{3x} + h(x)$.
- Now using $\psi_x = M$ gives $3e^{3x}x^2y + 2e^{3x}xy + y^3e^{3x} + h'(x) = e^{3x}(3x^2y + 2xy + y^3)$
- $\Rightarrow h'(x) = 0$ and hence $h(x) = 0$.
- Therefore, $\psi(x, y) = e^{3x}(x^2y + \frac{1}{3}y^3)$ and the solution is

$$e^{3x}(x^2y + \frac{1}{3}y^3) = c.$$

Additional Examples

Example 1. Solve $y' = \frac{t^2 y - y^3}{t^3}$.

Solution.

Example 2. Solve $y \ln y \, dx + (x - \ln y) \, dy = 0$.

Solution.

Example 3. Solve $y' \sin x + y \cos x = 3x^2$, $0 < x < \pi$.

Solution.

Example 4. Solve the initial value problem $y' = \frac{y^3}{1 - 2ty^2}$, $y(0) = 1$.

Solution.

Example 5. Solve $y' = \frac{t}{t^2y + y^3}$. Hint: let $u = t^2$.

Solution.