



## Chapter 3

# Second Order Linear Equations

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- In order to present the key ideas in the simplest possible context, we describe them in this chapter for second order equations.
- Another reason to study second order linear equations is that they are vital to any serious investigation of the classical areas of mathematical physics.
- One cannot go very far in the development of fluid mechanics, heat conduction, wave motion, or electromagnetic phenomena without finding it necessary to solve second order linear differential equations.

## Chapter 3

# Second Order Linear Equations

- This chapter deals with linear equations of **second order**.
- They are very important in the study of differential equations for two main reasons.
- The first is that linear equations in general have a rich theoretical structure that underlies a number of systematic methods of solution.
- Further, this structure and these methods can be understood at a fairly elementary mathematical level.

### 3.1 Homogeneous Equations with Constant Coefficients

- The general form of a second order ordinary differential equation is

$$y'' = f(t, y, y'),$$

where  $f$  is a function of three variables  $t, y, y'$ .

#### Definition

(Linear second order equations)

A second order differential equation is linear if it can be written as

$$P_2(t)y'' + P_1(t)y' + P_0(t)y = Q(t). \quad (1)$$

If a second order differential can not be written in the form (1), then it is nonlinear.

$$P_2(t)y'' + P_1(t)y' + P_0(t)y = Q(t) \quad (1)$$

#### Remark

Since  $P_2(t)$  is not identically zero, equation (1) can be written as

$$y'' + p(t)y' + q(t)y = g(t). \quad (2)$$

#### Example 1

The equation

$$ty'' + 6t^2y' - 2t^3y = e^t$$

is a linear second order differential equation.

#### Example 2

The equation

$$y'' + yy' = 0$$

is a nonlinear second order differential equation.

#### Example 2

The equation

$$y'' + y = e^t$$

is a nonhomogeneous second order linear differential equation.

## Homogeneous equations

#### Definition

A second order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

is said to be homogeneous if the term  $g(t)$  is zero for all  $t$ .

Otherwise, the equation is called nonhomogeneous. The term  $g(t)$  is called nonhomogeneous term.

#### Example 1

The equation

$$y'' - \sin t y' + 5y = 0$$

is a homogenous second order linear differential equation.

#### Definition

##### (Initial value problem)

An initial value problem consists of a second order differential equation  $y'' = f(t, y, y')$  together with a pair of initial conditions  $y(t_0) = y_0$ ,  $y'(t_0) = y'_0$ , where  $y_0$  and  $y'_0$  are given numbers.

- A second order linear equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is called an equation with **constant coefficients** if  $p(t)$  and  $q(t)$  are constant functions.

- Before studying linear second order equations in detail, we consider methods of solving some special classes of nonlinear second order equations.

## Equations with the dependent variable missing

- Consider a second order ordinary differential equation

$$y'' = f(t, y, y').$$

- If a second order differential equation has the form

$$y'' = f(t, y'),$$

then the equation is an equation with the dependent variable missing.

- The substitution  $v(t) = y'$  reduces this equation to

$$v' = f(t, v)$$

which is a first order equation.

- If we can solve this equation for  $v$ , then  $y$  can be obtained by solving the simple first order equation  $y' = v$ .

$$\frac{dv}{v^2} = t dt$$

•

$$-\frac{1}{v} = \frac{1}{2}t^2 + c_1.$$

- Thus we obtain

$$v = \frac{-2}{t^2 + 2c_1}.$$

- Having obtained  $v$ , we now use  $y' = v$  to obtain  $y$  as follows:

$$y = \int \frac{-2}{t^2 + 2c_1} dt \quad \boxed{u = \frac{t}{\sqrt{2c_1}} \Rightarrow dt = \sqrt{2c_1} du}$$

$$= \int \frac{-2\sqrt{2c_1} du}{2c_1(u^2 + 1)}$$

- It follows that, (provided  $c_1 \neq 0$ ),

$$y = -\frac{\sqrt{2}}{\sqrt{c_1}} \tan^{-1} \left( \frac{t}{\sqrt{2c_1}} \right) + c_2.$$

## Examples

### Example 1

Solve the equation

$$y'' = t(y')^2.$$

### Solution

- This is a second order equation with the dependent variable  $y$  missing.
- Let  $y' = v$ .
- Then the equation becomes

$$v' = tv^2$$

which is a separable first order equation.

- Writing it in the form

$$\frac{dv}{v^2} = t dt$$

and integrating yields

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## Examples

### Example 2

Solve the initial value problem

$$t^2 y'' + (y')^2 - 2ty' = 0, \quad y(2) = 5, \quad y'(2) = 2.$$

### Solution

- The given differential equation is a second order equation with the dependent variable  $y$  missing.
- Let  $y' = v$ .
- Then the equation becomes  $t^2 v' + v^2 - 2tv = 0$  or

$$t^2 v' - 2tv = -v^2$$

which is a Bernoulli first order equation with  $n = 2$ .

- Let  $w = v^{1-n} = v^{-1}$ .
- Then  $w' = -v^{-2}v'$ .

$$t^2 v' - 2tv = -v^2$$

- The Bernoulli equation can be written as

$$t^2 v^{-2} v' - 2tv^{-1} = -1.$$

- Substituting  $w = v^{-1}$ ,  $w' = -v^{-2}v'$  into this equation gives

$$-t^2 w' - 2tw = -1 \quad \text{or} \quad t^2 w' + 2tw = 1.$$

- OR  $\frac{d}{dt}(t^2 w) = 1$

- Integrating give

$$t^2 w = t + c_1$$

and hence

$$t^2 v^{-1} = t + c_1 \implies v = \frac{t^2}{t + c_1}.$$

- As a result we get  $y' = \frac{t^2}{t+c_1}$ .
- Using the initial condition  $y'(2) = 2$  we obtain

$$2 = \frac{4}{2 + c_1} \implies 2 + c_1 = 2 \implies c_1 = 0$$

$$y' = \frac{t^2}{t + c_1}$$

- and hence  $y' = t$ .

- Therefore,

$$y = \frac{1}{2}t^2 + c_2.$$

- The initial condition  $y(2) = 5$  implies

$$5 = 2 + c_2 \implies c_2 = 3$$

and hence the solution of the initial value problem is

$$y = \frac{1}{2}t^2 + 3.$$

## Equations with the independent variable missing

- If a second order differential equation has the form

$$y'' = f(y, y'),$$

then the equation is an equation with the independent variable missing.

- Using the substitution  $v(y) = y'$ , we get

$$y'' = \frac{d}{dt}v(y) = \frac{dv}{dy} \frac{dy}{dt} = \frac{dv}{dy} v$$

- Thus the substitution  $v(y) = y'$  reduces the given equation to

$$v \frac{dv}{dy} = f(y, v)$$

which is a first order equation.

- If we can solve this equation for  $v(y)$ , then  $y$  can be obtained by solving the simple first order equation  $y' = v(y)$ .

## Examples

### Example 1

Solve the equation

$$y'' = 2y(y')^3.$$

### Solution

- This is a second order equation with the independent variable  $t$  missing.

- Let  $y' = v(y)$ .

- Then  $y'' = v \frac{dv}{dy}$  and the equation becomes

$$v \frac{dv}{dy} = 2yv^3$$

or

$$\frac{dv}{dy} = 2yv^2$$

provided  $v \neq 0$ .

$$\frac{dv}{dy} = 2yv^2$$

- This is a separable first order equation.
- Let us solve this equation:

$$\begin{aligned}\frac{dv}{v^2} &= 2ydy \\ \Rightarrow -\frac{1}{v} &= y^2 + c_1\end{aligned}$$

- Thus we obtain the solution

$$v = \frac{-1}{y^2 + c_1}.$$

- Having obtained  $v$ , we now use  $y' = v$  to obtain  $y$ .

•

$$y' = \frac{-1}{y^2 + c_1}$$

a separable equation for  $y(t)$

$$y' = \frac{-1}{y^2 + c_1}$$

•

$$\begin{aligned}\Rightarrow (y^2 + c_1)dy &= -dt \\ \Rightarrow \frac{1}{3}y^3 + c_1y &= -t + c_2\end{aligned}$$

- Therefore, the solution is given by

$$\boxed{\frac{1}{3}y^3 + c_1y + t = c_2.}$$

## Examples

### Example 2

Solve

$$(y + 1)y'' = (y')^2.$$

### Solution

- The given differential equation is a second order equation with the independent variable  $t$  missing.
- Let  $y' = v(y)$ .
- Then the equation becomes a separable first order equation

$$(y + 1)v \frac{dv}{dy} = v^2$$

or

$$\frac{dv}{v} = \frac{dy}{y + 1},$$

$v \neq 0$

$$\frac{dv}{v} = \frac{dy}{y + 1}$$

- Integrating to obtain

$$\begin{aligned}\ln |v| &= \ln |y + 1| + c \\ \Rightarrow |v| &= e^c |y + 1| \Rightarrow v = \pm e^c (y + 1)\end{aligned}$$

- Thus the solution can be written as

$$v = c_1(y + 1).$$

- Next we solve  $y' = v = c_1(y + 1)$ .
- It is a first order linear equation with integrating factor  $\mu(t) = e^{-c_1 t}$ .

$$\begin{aligned}y' - c_1y &= c_1 \Rightarrow e^{-c_1 t}(y' - c_1y) = c_1 e^{-c_1 t} \\ \Rightarrow \frac{d}{dt}(e^{-c_1 t}y) &= c_1 e^{-c_1 t} \Rightarrow e^{-c_1 t}y = -e^{-c_1 t} + c_2\end{aligned}$$

- Therefore,

$$\boxed{y = c_2 e^{c_1 t} - 1.}$$

$$v \frac{dv}{dy} + e^{-2y} = 0$$

- It can be written as

$$v dv = -e^{-2y} dy.$$

- Integrating to get

$$\frac{1}{2}v^2 = \frac{1}{2}e^{-2y} + c_1$$

or

$$v^2 = e^{-2y} + 2c_1.$$

- Using the initial conditions  $y(3) = 0, y'(3) = 1 = v(3)$  we obtain

$$1 = 1 + 2c_1 \implies \boxed{c_1 = 0}$$

and hence  $(y')^2 = e^{-2y}.$

- Therefore,

$$y' = \pm e^{-y}.$$

- Since  $y'(3) = 1$ , we have

$$y' = e^{-y}.$$

## 3.2 Fundamental Solutions of Linear Homogeneous Equations

- It follows that

$$e^y dy = dt \implies e^y = t + c_2.$$

- The initial condition  $y(3) = 0$  implies that

$$1 = 3 + c_2 \implies \boxed{c_2 = -2}$$

- and hence the solution of the initial value problem is

$$e^y = t - 2$$

and hence

$$\boxed{y = \ln(t - 2)}.$$

### Example 3

Solve the initial value problem

$$y'' + e^{-2y} = 0, \quad y(3) = 0, \quad y'(3) = 1.$$

### Solution

- The given differential equation is a second order equation with the independent variable  $t$  missing.
- Let  $y' = v(y).$
- Then  $y'' = v \frac{dv}{dy}$  and the equation becomes

$$v \frac{dv}{dy} + e^{-2y} = 0$$

which is a separable first order equation.

- Then, for any function  $\varphi$  that is twice differentiable on  $I$ , we define the differential operator  $L$  by the equation

$$L[\varphi] = \varphi'' + p\varphi' + q\varphi.$$

- Note that  $L[\varphi]$  is a function on  $I$ . The value of  $L[\varphi]$  at a point  $t$  is

$$L[\varphi](t) = \varphi''(t) + p(t)\varphi'(t) + q(t)\varphi(t).$$

- In this chapter we study the second order linear equation  $L[\varphi](t) = g(t)$ .
- Since it is customary to use the symbol  $y$  to denote  $\varphi(t)$ , we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = g(t). \quad (3)$$

- With equation (3) we associate a set of initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad (4)$$

where  $t_0$  is any point in the interval  $I$ , and  $y_0$  and  $y'_0$  are given real numbers.

- Recall that a second order differential equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

is said to be homogeneous if the term  $g(t)$  is zero for all  $t$ .

- Otherwise, the equation is called nonhomogeneous.
- The term  $g(t)$  is called nonhomogeneous term.

- The following Theorem explains whether the initial value problem (3), (4) always has a solution, and whether it may have more than one solution.

#### Theorem

Let

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0$$

be an initial value problem. If the functions  $p$ ,  $q$  and  $g$  are continuous on an interval  $I = (\alpha, \beta)$  containing the initial point  $t_0$ , then the initial value has a unique solution  $y = \varphi(t)$  valid for all  $t \in I$ .

## Examples

### Example

Find the largest interval in which the solution of the initial value problem

$$t y'' + \sin t y' - \cos t y = 1, \quad y(-1) = 1, \quad y'(-1) = 2$$

is certain to exist.

### Solution

- In this example  $p(t) = \frac{\sin t}{t}$ ,  $q(t) = -\frac{\cos t}{t}$ ,  $g(t) = \frac{1}{t}$ .
- These function are continuous on  $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$ .
- Since  $t_0 = -1$ , the interval is  $(-\infty, 0)$ .

## Structure of solutions of linear homogeneous equations

- Unlike first order linear equations, there is no useful expression for the solutions of second order linear equations.
- Therefore, we need information about the form and structure of solutions that might be helpful in finding solutions of particular problems.

### Theorem

(Principle of Superposition)

If  $y_1$  and  $y_2$  are two solutions of the linear homogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination  $\varphi(t) = c_1y_1 + c_2y_2$  is also a solution for any values of the constants  $c_1$  and  $c_2$ .

- A special case of Theorem 5 occurs if either  $c_1$  or  $c_2$  is zero.
- Then we conclude that any multiple of a solution of  $L[y] = y'' + p(t)y' + q(t)y = 0$  is also a solution.

### Example

Consider the equation  $y'' - y = 0$ . Then  $y_1 = e^t$  and  $y_2 = e^{-t}$  are solutions. Moreover any linear combination  $c_1y_1 + c_2y_2 = c_1e^t + c_2e^{-t}$  is also a solution. For example,

$$y_3 = 5e^t, y_4 = -3e^{-t}, y_5 = 2e^t + 3e^{-t}$$

are solutions of the equation.

## General solution of linear homogeneous equation

- Let  $y_1$  and  $y_2$  be two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

- Is  $\varphi(t) = c_1y_1 + c_2y_2$  the general solution of the equation?
- To answer this question, we have to find  $c_1$  and  $c_2$  such that  $y(t_0) = y_0$ ,  $y'(t_0) = y'_0$  for any given real numbers  $y_0, y'_0$ .
- Applying these initial conditions gives

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y'_1(t_0) + c_2y'_2(t_0) &= y'_0. \end{aligned} \quad (5)$$

- The system (5) has solution if and only if

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} \neq 0.$$

## The Wronskian

### Definition

The determinant  $W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix} = y_1(t_0)y'_2(t_0) - y'_1(t_0)y_2(t_0)$  is called the Wronskian determinant of the functions  $y_1$  and  $y_2$ .

### Remark

Sometimes we write  $W = W(y_1, y_2)(t_0)$  to emphasize that the Wronskian depends on  $y_1$  and  $y_2$  and that it is evaluated at the point  $t_0$ .

## Examples

### Example 1

Find the Wronskian of  $y_1 = e^t$  and  $y_2 = e^{-t}$  at  $t = 0$ .

### Solution

- Using the definition we have

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}.$$

- Thus  $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$ .

### Example 2

Find the Wronskian of  $y_1 = t$  and  $y_2 = te^{-t}$ .

### Solution

$$\begin{aligned} \bullet W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ \bullet &= \begin{vmatrix} t & te^{-t} \\ 1 & e^{-t} - te^{-t} \end{vmatrix} \\ \bullet &= t(1-t)e^{-t} - te^{-t} = -t^2e^{-t}. \end{aligned}$$

## General solution of second order linear homogenous equation

### Theorem

Let  $y_1$  and  $y_2$  be two solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

If  $W(y_1, y_2) \neq 0$ , then

$$y = c_1y_1 + c_2y_2,$$

where  $c_1$  and  $c_2$  are arbitrary constants, is the general solution of  $L[y] = 0$ .

## Fundamental set of solutions

### Definition

If  $y_1$  and  $y_2$  are two solutions of  $L[y] = 0$  with  $W(y_1, y_2) \neq 0$ , then the set  $\{y_1, y_2\}$  is called a fundamental set of solutions of  $L[y] = 0$ .

## Examples

### Example 1

Show that  $y_1 = 1$  and  $y_2 = e^t$  form a fundamental set of solutions of the equation  $y'' - y' = 0$ .

#### Solution

(a)  $y_1 = 1$  and  $y_2 = e^t$  are solutions of the equation?

We have  $y_1 = 1$ ,  $y_1' = y_1'' = 0$ . Thus

$$y_1'' - y_1' = 0 - 0 = 0$$

and hence  $y_1$  is a solution.

We have  $y_2 = e^t$ ,  $y_2' = y_2'' = e^t$ . Thus

$$y_2'' - y_2' = e^t - e^t = 0$$

and hence  $y_2$  is a solution.

(b)  $W(y_1, y_2) \neq 0$ ?

$$W(y_1, y_2) = \begin{vmatrix} 1 & e^t \\ 0 & e^t \end{vmatrix} = e^t \neq 0.$$

Therefore, (a) and (b) implies that  $y_1$  and  $y_2$  form a fundamental set of solutions of the equation  $y'' - y' = 0$ .

### Example 2

Show that  $y_1 = t$  and  $y_2 = \frac{1}{t}$  form a fundamental set of solutions of the equation  $t^2 y'' + ty' - y = 0$ ,  $t > 0$ .

#### Solution

(a)  $y_1 = t$  and  $y_2 = \frac{1}{t}$  are solutions of the equation?

We have  $y_1 = t$ ,  $y_1' = 1$ ,  $y_1'' = 0$ . Thus

$$t^2 y_1'' + ty_1' - y_1 = 0 + t - t = 0$$

and hence  $y_1$  is a solution.

We have  $y_2 = t^{-1}$ ,  $y_2' = -t^{-2}$ ,  $y_2'' = 2t^{-3}$ . Thus

$$t^2 y_2'' + ty_2' - y_2 = 2t^{-1} - t^{-1} - t^{-1} = 0$$

and hence  $y_2$  is a solution.

(b)  $W(y_1, y_2) \neq 0$ ?

$$W(y_1, y_2) = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix} = -2t^{-1} \neq 0.$$

Therefore, (a) and (b) implies that  $y_1 = t$  and  $y_2 = \frac{1}{t}$  form a fundamental set of solutions of the equation  $t^2 y'' + ty' - y = 0$ ,  $t > 0$ .

### 3.3 Linear Independence and the Wronskian

- We have seen that the general solution of a second order linear homogeneous differential equation can be represented as a linear combination of two solutions  $y_1$  and  $y_2$  whose Wronskian  $W(y_1, y_2)$  is not zero.
- There is relation between this condition and the concept of linear independence of two functions.
- We briefly discuss this very important algebraic idea in this section.

### Linear dependence

#### Definition

Two functions  $f$  and  $g$  are said to be linearly dependent on an interval  $I$  if there exist two constants  $k_1$  and  $k_2$ , not both zero, such that

$$k_1 f(t) + k_2 g(t) = 0$$

for all  $t \in I$ . The functions  $f$  and  $g$  are said to be linearly independent on an interval  $I$  if they are not linearly dependent.

#### Example

Determine whether the given pair of functions is linearly independent or linearly dependent.

(a)  $f(t) = 3t - 5$ ,  $g(t) = 9t - 15$ .

(b)  $f(t) = t$ ,  $g(t) = t - 1$ .

#### Solution

(a)  $f(t) = 3t - 5$ ,  $g(t) = 9t - 15$ .

- To solve this problem we assume that  $k_1 f(t) + k_2 g(t) = k_1(3t - 5) + k_2(9t - 15) = 0$  for all  $t \in \mathbb{R}$ .
- Now we look for values of  $k_1$  and  $k_2$  that satisfy this equation.
- Since we have two unknowns  $k_1$  and  $k_2$ , we need two equations.
- We can choose any two values of  $t \in \mathbb{R}$ .
- Choosing  $t = 0$  gives

$$-5k_1 - 15k_2 = 0.$$

- Choosing  $t = 1$  gives

$$-2k_1 - 6k_2 = 0.$$

- It follows that  $k_1 = -3k_2$  and hence there are solutions other than  $k_1 = k_2 = 0$ .
- For example we can choose  $k_2 = 1$ ,  $k_1 = -3$  and hence we see that

$$g(t) = 3f(t).$$

- Therefore,  $f(t) = 3t - 5$ ,  $g(t) = 9t - 15$  are linearly dependent.

(b)  $f(t) = t, \quad g(t) = t - 1.$

- We assume that

$$k_1 f(t) + k_2 g(t) = k_1 t + k_2(t - 1) = 0$$

for all  $t \in \mathbb{R}.$

- Now we look for values of  $k_1$  and  $k_2$  that satisfy this equation.
- Since we have two unknowns  $k_1$  and  $k_2$ , we need two equations.
- We can choose any two values of  $t \in \mathbb{R}.$
- Choosing  $t = 0$  gives

$$-k_2 = 0 \quad \text{and hence} \quad k_2 = 0.$$

- Choosing  $t = 1$  gives

$$k_1 = 0.$$

- It follows that the only solution is  $k_1 = k_2 = 0.$
- Therefore  $f(t) = t, \quad g(t) = t - 1$  are linearly independent.

### Theorem

If  $f$  and  $g$  are differentiable functions on an open interval  $I$  and if  $W(f, g)(t_0) \neq 0$  for some  $t_0 \in I$ , then  $f$  and  $g$  are linearly independent on  $I$ .

### Example

Determine whether the given pair of functions is linearly independent or linearly dependent.

- (a)  $f(t) = t^2, \quad g(t) = t.$
- (b)  $f(t) = t^2, \quad g(t) = -2t^2.$
- (c)  $f(t) = \cos(2t), \quad g(t) = \sin(2t).$
- (d)  $f(t) = e^{1-2t}, \quad g(t) = e^{-2t}.$
- (e)  $f(t) = t^3, \quad g(t) = t^2|t|.$

### Solution

(a)  $f(t) = t^2, \quad g(t) = t.$

- $W(f, g) = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = -t^2.$
- Thus  $W(f, g)(1) = -1 \neq 0.$
- It follows that  $f$  and  $g$  are linearly independent on  $\mathbb{R}.$

(b)  $f(t) = t^2, \quad g(t) = -2t^2.$

- $W(f, g) = \begin{vmatrix} t^2 & -2t^2 \\ 2t & -4t \end{vmatrix} = 0, \text{ for all } t.$
- This does not imply that  $f$  and  $g$  are linearly dependent.
- To show that  $f$  and  $g$  are linearly dependent we use the definition.
- We note that  $g(t) = -2f(t)$ ; that is  $2f(t) + g(t) = 0.$
- This shows that  $f$  and  $g$  are linearly dependent.

(c)  $f(t) = \cos(2t), \quad g(t) = \sin(2t).$

- $W(f, g) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2 \neq 0 \text{ for all } t \in \mathbb{R}.$
- Thus  $f$  and  $g$  are linearly independent on  $\mathbb{R}.$

(d)  $f(t) = e^{1-2t}, \quad g(t) = e^{-2t}.$

- $W(f, g) = \begin{vmatrix} e^{1-2t} & e^{-2t} \\ -2e^{1-2t} & -2e^{-2t} \end{vmatrix} = 0, \text{ for all } t.$
- We note that  $f(t) = eg(t)$ ; that is  $f(t) - eg(t) = 0.$
- This shows that  $f$  and  $g$  are linearly dependent.

## Abel's Theorem

(e)  $f(t) = t^3, \quad g(t) = t^2|t|.$

- $W(f, g) = \begin{vmatrix} t^3 & t^2|t| \\ 3t^2 & 3t|t| \end{vmatrix} = 0$ , for all  $t$ .
- However, in any interval  $I$  containing 0,  $f$  and  $g$  are linearly independent.
- To see this we assume

$$k_1 f(t) + k_2 g(t) = k_1 t^3 + k_2 t^2|t| = 0$$

for all  $t \in I = (-2, 2)$ .

- Choosing  $t = 1$  gives  $k_1 + k_2 = 0$  and hence  $k_1 = -k_2$ .
- Choosing  $t = -1$  gives  $-k_1 + k_2 = 0$  and hence  $k_1 = k_2$ .
- Thus  $k_1 = k_2 = 0$ .

- The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two **solutions** of any second order linear homogeneous equation even if the solutions themselves are not known.
- The theorem was derived by the Norwegian mathematician Niels Henrik Abel (1802-1829) in 1827 and is known as Abel's formula.

## Abel's Theorem

### Theorem

If  $y_1$  and  $y_2$  are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where  $p$  and  $q$  are continuous on an open interval  $I$ , then the Wronskian  $W(y_1, y_2)(t)$  is given by

$$W(y_1, y_2)(t) = c \exp \left[ - \int p(t) dt \right],$$

where  $c$  is a certain constant that depends on  $y_1$  and  $y_2$ , but not on  $t$ . Further,  $W(y_1, y_2)(t)$  is either zero for all  $t \in I$  (if  $c = 0$ ) or else is never zero in  $I$  (if  $c \neq 0$ ).

## Proof

### Proof.

- Assume  $y_1$  and  $y_2$  are solutions of the differential equation  $L[y] = y'' + p(t)y' + q(t)y = 0$ .
- This implies that

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0, \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{aligned} \tag{6}$$

- Multiplying the first equation by  $-y_2$ , the second by  $y_1$ , and adding the resulting equations, we obtain

□

- $$y_1 y_2'' - y_2 y_1'' + p(y_1 y_2' - y_2 y_1') = 0. \quad (7)$$

- Let  $W(t) = W(y_1, y_2)(t)$ .

- Then  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

- It follows that  $W' = y_1 y_2'' - y_2 y_1''$ .

- Thus equation (7) becomes

$$W' + pW = 0.$$

- Now we can solve this first order linear differential equation for  $W$ :

$$\left( \exp \left[ \int p(t) dt \right] W \right)' = 0.$$

## Example

### Example

Find the Wronskian of two solutions of the equation  $t^2 y'' - t(t+2)y' + (t+1)y = 0$ .

### Solution

- In this example  $p(t) = \frac{-t(t+2)}{t^2} = \frac{-(t+2)}{t}$ .

- By Abel's formula

$$W = c \exp \left[ - \int p(t) dt \right]$$

- $$= c \exp \left[ \int \frac{(t+2)}{t} dt \right] = c \exp \left[ \int \left( 1 + \frac{2}{t} \right) dt \right].$$

- $$W = c e^{t+2 \ln |t|} = c t^2 e^t.$$

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## Theorem

- Abel's formula establishes a stronger version of Theorem 10 for two functions that are solutions of a second order linear homogeneous differential equation.

### Theorem

Let  $y_1$  and  $y_2$  be solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where  $p$  and  $q$  are continuous on an open interval  $I$ . Then

- (1)  $y_1$  and  $y_2$  are linearly dependent on  $I$  if and only if  $W(y_1, y_2)(t) = 0$  for all  $t \in I$ ,
- (2)  $y_1$  and  $y_2$  are linearly independent on  $I$  if and only if  $W(y_1, y_2)(t) \neq 0$  for all  $t \in I$ .

## Example

### Example

Let  $y_1$  and  $y_2$  be two linearly independent solutions of the equation  $y'' + 2y' + ty = 0$ . If  $W(y_1, y_2)(0) = 2$ , then find  $W(y_1, y_2)(1)$ .

### Solution

- By Abel's formula

$$W = c \exp \left[ - \int p(t) dt \right] = c \exp \left[ - \int 2 dt \right] = c e^{-2t}.$$

- Using  $W(0) = 2$  gives  $c = 2$
- Thus  $W(t) = 2e^{-2t}$ .
- It follows that  $W(1) = 2e^{-2}$ .

### 3.4 Complex Roots of the Characteristic Equation

- In this section we study a method to find the general solution to the special case of the second order homogeneous linear differential equation in which all of the coefficients are real constants.

- As we have seen before, the equation  $y'' + p(t)y' + q(t)y = g(t)$  can be written as

$$P_2(t)y'' + P_1(t)y' + P_0(t)y = Q(t), \quad (8)$$

where  $P_2(t) \neq 0$ .

- If  $P_2(t)y'' + P_1(t)y' + P_0(t)y = Q(t)$  is an equation with constant coefficients, then  $P_2(t) = a$ ,  $P_1(t) = b$ , and  $P_0(t) = c$  are constant functions.
- Thus a second order linear equation with constant coefficients has the general form

$$ay'' + by' + cy = Q(t), \quad (9)$$

where  $a$ ,  $b$ , and  $c$  are given constants.

### Linear Equations with Constant Coefficients

#### Definition

A second order differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is called an equation with constant coefficients if  $p(t)$  and  $q(t)$  are constant functions.

### Examples

#### Example 1

The equation  $y'' + 5y' - 4y = 0$  is a second order differential equation with constant coefficients.

#### Example 2

The equation  $y'' + ty = 0$  is a second order differential equation with nonconstant coefficients.

## Solving homogeneous equation with constant coefficients

### Example 3

Solve the equation  $ay' + by = 0$ .

### Solution

- Integrating factor  $\mu = \exp \left[ \int \frac{b}{a} dt \right] = e^{bt/a}$ .
- Thus the solution is

$$y(t) = ce^{-bt/a}.$$

## Solving homogeneous equation with constant coefficients

- Consider a second order linear homogeneous equation

$$ay'' + by' + cy = 0, \quad (10)$$

where  $a$ ,  $b$ , and  $c$  are given constants.

- Based on our experience with the last example, we look for solutions in the form  $y = e^{rt}$ .
- Substituting into equation (10), we get

$$(ar^2 + br + c)e^{rt} = 0.$$

- Since  $e^{rt} \neq 0$  for all  $t$ , we can divide the last equation by  $e^{rt}$  to obtain

$$ar^2 + br + c = 0. \quad (11)$$

- Thus, equation (10) has solution in the form  $y = e^{rt}$  if and only if  $r$  is a solution of the polynomial equation (11).

## Characteristic equation

### Definition

The equation

$$ar^2 + br + c = 0$$

is called the characteristic equation for the differential equation

$$ay'' + by' + cy = 0$$

### Example

Find the characteristic equation for each of the following differential equations and solve it.

(1)  $5y'' - 3y' + 7y = 0$ .

(2)  $-4y'' + y' = 0$ .

### Solution

- (1) The characteristic equation is  $5r^2 - 3r + 7 = 0$ .  
The solutions of it are

$$r_{1,2} = \frac{3 \pm \sqrt{9 - 140}}{10} = \frac{3 \pm i\sqrt{131}}{10}.$$

- (2) The characteristic equation is  $-4r^2 + r = 0$  or  $r(4r - 1) = 0$ .  
Thus the solutions are  $r_1 = 0$ ,  $r_2 = \frac{1}{4}$ .

- Since the characteristic equation

$$ar^2 + br + c = 0$$

is a quadratic equation, it has two roots, which would provide us with the two independent solutions we need to form the general solution.

- However, the two roots of the characteristic equation may be real and distinct, real and repeated, or complex conjugate.
- These three cases lead to different types of solutions for

$$ay'' + by' + cy = 0$$

and we consider each case in turn.

## Real distinct roots

### Proposition

If the characteristic equation  $ar^2 + br + c = 0$  has real and distinct roots  $r_1$  and  $r_2$ , then the general solution of the differential equation  $ay'' + by' + cy = 0$  is given by

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where  $c_1$  and  $c_2$  are arbitrary constants.

### Proof.

- We know that  $y_1 = e^{r_1 t}$  and  $y_2 = e^{r_2 t}$  are solutions of  $ay'' + by' + cy = 0$ .
- So it remains to show that  $W(y_1, y_2) \neq 0$ .
- But  $W(y_1, y_2) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1 + r_2)t} \neq 0$ .

□

## Examples

### Example 1

Find the general solution of the equation

$$2y'' - 3y' + y = 0.$$

### Solution

- The characteristic equation is  $2r^2 - 3r + 1 = 0$
- and its solutions are  $r_1 = \frac{1}{2}$ ,  $r_2 = 1$ .
- Thus the general solution of the equation  $2y'' - 3y' + y = 0$  is  $y = c_1 e^{t/2} + c_2 e^t$ .

### Example 2

Solve the initial value problem

$$6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0.$$

### Solution

- The characteristic equation is  $6r^2 - 5r + 1 = 0$  and its solutions are  $r_1 = \frac{1}{2}$ ,  $r_2 = \frac{1}{3}$ .
- Thus the general solution of the equation  $6y'' - 5y' + y = 0$  is  $y = c_1 e^{t/2} + c_2 e^{t/3}$ .

- Initial conditions  $y(0) = 4, \quad y'(0) = 0$

$$y = c_1 e^{t/2} + c_2 e^{t/3}$$

$$y' = \frac{1}{2}c_1 e^{t/2} + \frac{1}{3}c_2 e^{t/3}$$

- Using the initial conditions we find

$$\begin{aligned} c_1 + c_2 &= 4, \\ \frac{1}{2}c_1 + \frac{1}{3}c_2 &= 0. \end{aligned}$$

•

$$\begin{aligned} c_1 + c_2 &= 4, \\ c_1 + \frac{2}{3}c_2 &= 0. \end{aligned}$$

- It follows that the solution of the initial value problem is

$$y = -8e^{t/2} + 12e^{t/3}.$$

## Complex roots

- The characteristic equation  $ar^2 + br + c = 0$  has two solutions

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- If  $b^2 - 4ac < 0$ , then the roots of the characteristic equation has the roots

$$r_{1,2} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \lambda \pm i\mu.$$

- It follows that  $y_1 = e^{(\lambda+i\mu)t}$  and  $y_2 = e^{(\lambda-i\mu)t}$  are solutions of the differential equation  $ay'' + by' + cy = 0$ .
- These two solutions are complex but we started with only real numbers in our differential equation and we would like our solutions to be real.
- The following formula we help us in finding the real solution from the complex solutions.

## Euler's Formula

### Proposition

For any real number  $t$ ,

$$e^{i\mu t} = \cos(\mu t) + i \sin(\mu t).$$

### Proof.

By Taylor series. □

## Real Solutions

- Consider the two solutions  $y_1 = e^{(\lambda+i\mu)t}$  and  $y_2 = e^{(\lambda-i\mu)t}$ .
- Using Euler's formula we can write these solutions as

$$y_1 = e^{\lambda t} [\cos(\mu t) + i \sin(\mu t)], \quad y_2 = e^{\lambda t} [\cos(\mu t) - i \sin(\mu t)].$$

- Since  $y_1$  and  $y_2$  are solutions, any linear combination

$$u = c_1 y_1 + c_2 y_2$$

is also a solution.

- Choose  $c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$ .
- Then we obtain the real solution

$$u_1 = e^{\lambda t} \cos(\mu t).$$

- Choose  $c_1 = \frac{1}{2i}, \quad c_2 = -\frac{1}{2i}$ .
- we obtain the real solution

$$u_2 = e^{\lambda t} \sin(\mu t).$$

### Proposition

If the characteristic equation  $ar^2 + br + c = 0$  has complex roots  $r_1 = \lambda + i\mu$  and  $r_2 = \lambda - i\mu$ , then the general solution of the differential equation

$$ay'' + by' + cy = 0$$

is given by

$$y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t),$$

where  $c_1$  and  $c_2$  are arbitrary constants.

### Proof.

We know that  $u_1$  and  $u_2$  are solutions of  $ay'' + by' + cy = 0$ . So it remains to show that  $W(u_1, u_2) \neq 0$ .  $\square$

• But

$$\begin{aligned} W(u_1, u_2) &= \begin{vmatrix} e^{\lambda t} \cos(\mu t) & e^{\lambda t} \sin(\mu t) \\ e^{\lambda t} [\lambda \cos(\mu t) - \mu \sin(\mu t)] & e^{\lambda t} [\lambda \sin(\mu t) + \mu \cos(\mu t)] \end{vmatrix} \\ &= e^{2\lambda t} \left( \cos(\mu t) [\lambda \sin(\mu t) + \mu \cos(\mu t)] \right. \\ &\quad \left. - \sin(\mu t) [\lambda \cos(\mu t) - \mu \sin(\mu t)] \right) \\ &= 2\mu e^{2\lambda t} \neq 0. \end{aligned}$$

## Examples

### Example 1

Find the general solution of the equation  $y'' - 2y' + 6y = 0$ .

### Solution

- The characteristic equation is  $r^2 - 2r + 6 = 0$
- and its solutions are

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 24}}{2} = 1 \pm i\sqrt{5}.$$

- It follows that the general solution of the given differential equation is

$$y = e^t [c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)].$$

### Example 2

Find the solution of the initial value problem

$$y'' + 4y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

### Solution

- The characteristic equation is  $r^2 + 4r + 5 = 0$  and its solutions are

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

- It follows that the general solution of the given differential equation is

$$y = e^{-2t} [c_1 \cos(t) + c_2 \sin(t)].$$

- Next we apply the initial conditions.  
 $y(0) = 0$  gives  $c_1 = 0$ .

- Thus  $y = c_2 e^{-2t} \sin(t)$ .
- It follows that  $y' = c_2 (-2e^{-2t} \sin(t) + e^{-2t} \cos(t))$ .
- Thus  $y'(0) = 1$  gives

$$c_2 = 1.$$

- Therefore, the solution of the initial value problem is

$$y = e^{-2t} \sin(t).$$

### Example 3

Find the general solution of the equation  $y'' + 16y = 0$ .

### Solution

- The characteristic equation is  $r^2 + 16 = 0$  and its solutions are

$$r_{1,2} = \pm 4i.$$

- It follows that the general solution of the given differential equation is

$$y = c_1 \cos(4t) + c_2 \sin(4t).$$

## 3.5 Repeated Roots; Reduction of Order

- In this section we consider the third possibility of the solutions of the characteristic equation

$$ar^2 + br + c = 0$$

of the second order linear homogeneous equation

$$ay'' + by' + cy = 0,$$

namely, that the two roots  $r_1$  and  $r_2$  are equal.

- Assume that  $b^2 - 4ac = 0$ .
- Then the characteristic equation has two equal roots  $r_1 = r_2 = \frac{-b}{2a}$ .
- As a result we can obtain only one solution  $y_1 = e^{-bt/2a}$  of the differential equation  $ay'' + by' + cy = 0$ .

- Thus we have to find a second solution  $y_2$  with  $W(y_1, y_2) \neq 0$ .
- There are several methods to obtain such a solution  $y_2$ .
- We will use a method originated by Jean d'Alembert (1717-1783), a French mathematician, and is known primarily for his work in mechanics and differential equations.
- We know that since  $y_1(t)$  is a solution, so is  $cy_1(t)$  for any constant  $c$ .
- We generalize this idea by replacing  $c$  by a function  $v(t)$  and trying to determine  $v(t)$  so that the product  $v(t)y_1(t)$  is a solution.

- Starting with  $y = vy_1$ , we obtain

$$y' = vy_1' + v'y_1, \quad y'' = vy_1'' + 2v'y_1' + v''y_1.$$

- Substituting the expressions in the equation

$$ay'' + by' + cy = 0 \quad \text{we get}$$

$$a(vy_1'' + 2v'y_1' + v''y_1) + b(vy_1' + v'y_1) + cvy_1 = 0.$$

- Arranging terms yields

$$v(ay_1'' + by_1' + cy_1) + 2av'y_1' + av''y_1 + bv'y_1 = 0.$$

- Since  $y_1 = e^{-bt/2a}$ , we have

$$y_1' = -\frac{b}{2a}e^{-bt/2a},$$

$$y_1'' = \frac{b^2}{4a^2}e^{-bt/2a}.$$

- It follows that

$$ay_1'' + by_1' + cy_1 = 0$$

- and the equation for  $v$  becomes

$$-be^{-bt/2a}v' + ae^{-bt/2a}v'' + be^{-bt/2a}v = 0.$$

- Thus the equation for  $v$  reduces to

$$v'' = 0.$$

- So we can find  $v(t) = c_2t + c_1$ , where  $c_1$  and  $c_2$  are arbitrary constants.
- Therefore,  $y(t) = e^{-bt/2a}(c_2t + c_1) = c_2te^{-bt/2a} + c_1e^{-bt/2a}$ .
- We note that  $y$  is a linear combination of two solutions:

$$y_1 = e^{-bt/2a}, \quad \text{and} \quad y_2 = te^{-bt/2a}$$

$$\text{with } W(y_1, y_2) = e^{-bt/2a} \neq 0.$$

## Examples

### Proposition

If the characteristic equation  $ar^2 + br + c = 0$  has two equal roots  $r_1 = r_2 = -\frac{b}{2a}$ , then the general solution of the differential equation  $ay'' + by' + cy = 0$  is given by

$$y(t) = e^{-bt/2a}(c_2t + c_1) = c_2te^{-bt/2a} + c_1e^{-bt/2a},$$

where  $c_1$  and  $c_2$  are arbitrary constants.

### Remark

In other words, in this case, there is one exponential solution corresponding to the repeated root, while a second solution is obtained by multiplying the exponential solution by  $t$ .

### Example 1

Find the general solution of the equation  $4y'' + 4y' + y = 0$ .

### Solution

- The characteristic equation is  $4r^2 + 4r + 1 = 0$  or

$$(2r + 1)^2 = 0.$$

- Its solutions are

$$r_1 = r_2 = -\frac{1}{2}.$$

- It follows that the general solution of the given differential equation is

$$y = e^{-t/2}[c_1t + c_2].$$

### Example 2

Find the solution of the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(-1) = 2, \quad y'(-1) = 1.$$

### Solution

- The characteristic equation is

$$r^2 + 6r + 9 = 0$$

or

$$(r + 3)^2 = 0.$$

- Thus its solutions are

$$r_1 = r_2 = -3.$$

- It follows that the general solution of the given differential equation is

$$y = e^{-3t}[c_1 t + c_2].$$

- Next we apply the initial conditions  $y(-1) = 2$ ,  $y'(-1) = 1$ .
- $y' = -3e^{-3t}[c_1 t + c_2] + c_1 e^{-3t} = e^{-3t}[-3c_1 t - 3c_2 + c_1]$ .
- Thus the initial conditions give

$$\begin{aligned} e^3[-c_1 + c_2] &= 2, \\ e^3[4c_1 - 3c_2] &= 1. \end{aligned}$$

•

$$\begin{aligned} -4c_1 + 4c_2 &= 8e^{-3}, \\ 4c_1 - 3c_2 &= e^{-3}. \end{aligned}$$

•

$$c_2 = 9e^{-3}, \quad c_1 = 7e^{-3}.$$

- Therefore, the solution of the initial value problem is

$$y = e^{-3t-3}(7t + 9).$$

## Reduction of Order

- The method we have used to obtain a second solution of  $ay'' + by' + cy = 0$  when  $r_1 = r_2$  can be applied to other equations.
- Suppose we know a solution  $y_1(t)$  of

$$y'' + p(t)y' + q(t)y = 0 \quad (12)$$

such that  $y_1(t)$  is not the zero function.

- To find a second solution we take  $y = v(t)y_1(t)$ .
- Substituting for  $y$ ,  $y'$ , and  $y''$  in (12) gives

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + q)y_1 v = 0.$$

- Since  $y_1$  is a solution of (12), we have  $y_1'' + py_1' + q = 0$  and hence the equation for  $v$  becomes

$$y_1 v'' + (2y_1' + py_1)v' = 0.$$

- This is a second order equation for  $v$  with dependent variable  $v$  missing.
- Thus, the substitution  $w = v'$  transforms it to a first order equation for  $w$ :

$$y_1 w' + (2y_1' + py_1)w = 0.$$

- Solving the last equation for  $w$ , we then can obtain  $v$  by integration.
- Finally,  $y$  can be found by the formula  $y = v(t)y_1(t)$ .
- This method is called the method of reduction of order since it reduces the problem from solving a second order equation to solving a first order equation.

## Examples

### Example 1

Use the method of reduction of order to find the general solution of the equation  $ty'' - 2(t-1)y' + (t-2)y = 0$ , if  $y_1(t) = e^t$  is a solution.

### Solution

- We use the transformation  $y = vy_1 = e^t v$ .
- It follows that

$$\begin{aligned} y' &= e^t v' + e^t v, \\ y'' &= e^t v'' + 2e^t v' + e^t v. \end{aligned}$$

- Substituting into the equation we obtain

$$te^t(v'' + 2v' + v) - 2(t-1)e^t(v' + v) + (t-2)e^t v = 0.$$

$$te^t(v'' + 2v' + v) - 2(t-1)e^t(v' + v) + (t-2)e^t v = 0.$$

- Dividing the equation by  $e^t$  and collecting similar terms, we obtain

$$tv'' + (2t-2t+2)v' + (t-2t+2+t-2)v = 0 \quad \text{OR} \quad tv'' + 2v' = 0.$$

- This is a second order equation for  $v$  with dependent variable  $v$  missing.
- Thus, the substitution  $w = v'$  transforms it to a first order linear equation for  $w$ :

$$tw' + 2w = 0.$$

- This equation can be integrated after multiplying it by  $t$ .

- $t^2 w' + 2tw = 0 \Rightarrow (t^2 w)' = 0$

- $t^2 w = c_1$

- Thus, we obtain

$$w = c_1/t^2.$$

- Next we integrate

$$v' = w = c_1/t^2$$

to find

$$v = -\frac{c_1}{t} + c_2.$$

- Finally, we use the formula  $y = e^t v$  to obtain the general solution of the given equation

$$y = e^t [c_2 - c_1/t].$$

### Example 2

Given that  $y_1(t) = t$  is a solution of  $2t^2 y'' + ty' - y = 0$ . Find the general solution of the equation.

### Solution

- We use the transformation  $y = vy_1 = tv$ .
- It follows that

$$\begin{aligned} y' &= tv' + v, \\ y'' &= tv'' + 2v'. \end{aligned}$$

- Substituting into the equation we obtain

$$2t^2(tv'' + 2v') + t(tv' + v) - tv = 0.$$

- Collecting similar terms, we obtain

$$2t^3 v'' + (4t^2 + t^2)v' + (t - t)v = 0 \quad \text{OR} \quad 2t^3 v'' + 5t^2 v' = 0.$$

- This is a second order equation for  $v$  with dependent variable  $v$  missing.
- Thus, the substitution  $w = v'$  transforms it to a first order linear equation for  $w$ :

$$2tw' + 5w = 0.$$

- This equation can be integrated after multiplying it by  $\frac{1}{2}t^{3/2}$ .

$$t^{5/2}w' + \frac{5}{2}t^{3/2}w = 0 \Rightarrow (t^{5/2}w)' = 0$$

•

$$t^{5/2}w = c_1$$

- Thus, we obtain

$$w = c_1 t^{-5/2}.$$

- Next we integrate to find

$$v = -\frac{2}{3}c_1 t^{-3/2} + c_2.$$

•

$$v = -\frac{2}{3}c_1 t^{-3/2} + c_2$$

- Finally, we use the formula  $y = tv$  to obtain the general solution of the given equation

$$y = -\frac{2}{3}c_1 t^{-1/2} + c_2 t.$$

### Example 3

Find the general solution of the equation  $t^2 y'' + ty' - 4y = 0$ ,  $t > 0$ , if  $y_1(t) = t^2$  is a solution.

### Solution

- We use the transformation  $y = vy_1 = t^2 v$ .

- It follows that

$$\begin{aligned} y' &= t^2 v' + 2tv, \\ y'' &= t^2 v'' + 4tv' + 2v. \end{aligned}$$

- Substituting into the equation we obtain

$$t^2(t^2 v'' + 4tv' + 2v) + t(t^2 v' + 2tv) - 4t^2 v = 0.$$

- Collecting similar terms, we obtain

$$t^4 v'' + (4t^3 + t^3)v' + (2t^2 + 2t^2 - 4t^2)v = 0 \quad \text{OR} \quad t^4 v'' + 5t^3 v' = 0.$$

•

$$t^4 v'' + 5t^3 v' = 0.$$

- This is a second order equation for  $v$  with dependent variable  $v$  missing.
- Thus, the substitution  $w = v'$  transforms it to a first order linear equation for  $w$ :

$$tw' + 5w = 0.$$

- This equation can be integrated after multiplying it by  $t^4$ :

$$t^5 w' + 5t^4 w = 0.$$

- Thus, we obtain

$$t^5 w = c_1 \quad \text{OR} \quad w = c_1 t^{-5}.$$

- Next we integrate to find

$$v = -\frac{1}{4}c_1 t^{-4} + c_2.$$

$$v = -\frac{1}{4}c_1 t^{-4} + c_2$$

- Finally, we use the formula  $y = t^2 v$  to obtain the general solution of the given equation

$$y = -\frac{1}{4}c_1 t^{-2} + c_2 t^2.$$

#### Example 4

Use the method of reduction of order to find the general solution of the equation  $t^2 y'' - t(t+2)y' + (t+2)y = 0$ ,  $t > 0$ , if  $y_1(t) = t$  is a solution.

#### Solution

- We use the transformation  $y = v y_1 = t v$ .
- It follows that

$$\begin{aligned} y' &= t v' + v, \\ y'' &= t v'' + 2 v'. \end{aligned}$$

- Substituting into the equation we obtain

$$t^2(t v'' + 2 v') - t(t+2)(t v' + v) + (t+2)t v = 0.$$

- $t^2(t v'' + 2 v') - t(t+2)(t v' + v) + (t+2)t v = 0$
- Collecting similar terms, we obtain  
 $t^3 v'' + (2t^2 - t^3 - 2t^2) v' + (-t^2 - 2t + t^2 + 2t) v = 0$  OR  $t^3 v'' - t^3 v' = 0$ .

- This is a second order equation for  $v$  with dependent variable  $v$  missing.
- Thus, the substitution  $w = v'$  transforms it to a first order linear equation for  $w$ :

$$w' - w = 0.$$

- This equation can be integrated after multiplying it by  $e^{-t}$ :

$$e^{-t} w' - e^{-t} w = 0 \quad \text{OR} \quad (e^{-t} w)' = 0$$

- Thus, we obtain

$$e^{-t} w = c_1 \quad \text{OR} \quad w = c_1 e^t.$$

- $v' = w = c_1 e^t$

- Next we integrate to find

$$v = c_1 e^t + c_2.$$

- Finally, we use the formula  $y = t v$  to obtain the general solution of the given equation

$$y = t(c_1 e^t + c_2).$$

## 3.6 Nonhomogeneous Equations; Method of Undetermined Coefficients

- In this section we consider the nonhomogeneous linear equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t). \quad (13)$$

- The following two theorems describe the structure of solutions of the nonhomogeneous equation (13) and provide a basis for constructing its general solution.

### Theorem 7

#### Theorem

If  $u_1$  and  $u_2$  are two solutions of the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t),$$

their difference  $u_1 - u_2$  is a solution of the corresponding homogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

#### Proof.

- Let  $u_1$  and  $u_2$  be two solutions of the nonhomogeneous equation  $L[y] = g(t)$  and let  $u = u_1 - u_2$ .
- Then  $L[u] = L[u_1] - L[u_2] = g(t) - g(t) = 0$ .

□

### Theorem 8

#### Theorem

The general solution of the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

is given by

$$y = c_1y_1 + c_2y_2 + y_p,$$

where  $y_1$  and  $y_2$  are fundamental solutions of the corresponding homogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and  $y_p$  is some particular solution of

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

#### Proof.

- Let  $y$  be any solution of  $L[y] = g(t)$  and let  $y_p$  be some particular solution of  $L[y] = g(t)$ .
- Then  $u = y - y_p$  is a solution of  $L[y] = 0$ .
- Thus  $u = y - y_p = c_1y_1 + c_2y_2$ .

□

#### Remark

*Theorem 8 states that to solve a nonhomogeneous linear equation  $L[y] = g(t)$ , there are two main steps:*

- (1) Find the general solution  $c_1y_1(t) + c_2y_2(t)$  of the corresponding homogeneous equation. This solution is frequently called the complementary solution and may be denoted by  $y_c(t)$ .*
- (2) Find some particular solution  $y_p$  of the nonhomogeneous equation.*

- We have already discussed how to find  $y_c(t)$ , at least when the homogeneous equation  $L[y] = 0$  has constant coefficients.
- Therefore, in the remainder of this section and in the next, we will focus on finding a particular solution  $y_p(t)$  of the nonhomogeneous equation  $L[y] = g(t)$ .
- There are two methods that we wish to discuss.
- They are known as the method of undetermined coefficients and the method of variation of parameters, respectively.
- Each has some advantages and some possible shortcomings.

## The Method of Undetermined Coefficients

- The method of undetermined coefficients consists of making an initial assumption about the form of the particular solution  $y_p$ , but with the coefficients left unspecified.
- We then substitute the assumed expression into  $L[y] = g(t)$  and attempt to determine the coefficients so as to satisfy that equation.
- If we are successful, then we have found a solution of the differential equation  $L[y] = g(t)$  and can use it for the particular solution  $y_p$ .
- If we cannot determine the coefficients, then this means that there is no solution of the form that we assumed.

- In this case we may modify the initial assumption and try again.
- The main advantage of the method of undetermined coefficients is that it is straightforward to execute once the assumption is made as to the form of  $y_p$ .
- Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance.
- For this reason, this method is usually used only for problems in which the homogeneous equation has constant coefficients and the nonhomogeneous term is restricted to a relatively small class of functions.

- In particular, we consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines.
- Despite this limitation, the method of undetermined coefficients is useful for solving many problems that have important applications.
- However, the algebraic details may become tedious and a computer algebra system can be very helpful in practical applications.
- We will illustrate the method of undetermined coefficients by several simple examples and then summarize some rules for using it.

#### Example

Find a particular solution of the equation  $y'' - y = e^{2t}$ .

#### Solution

- We want a function that its second order derivative minus itself is  $e^{2t}$ .

- Since  $g(t)$  is an exponential and we know that exponentials never just appear or disappear in the differentiation process it seems that a likely form of the particular solution would be

$$y_p = Ae^{2t}.$$

•

$$\begin{aligned} y_p' &= 2Ae^{2t}, \\ y_p'' &= 4Ae^{2t}. \end{aligned}$$

- Substituting for  $y_p$  and  $y_p''$  into the equation  $y'' - y = e^{2t}$  yields

$$4Ae^{2t} - Ae^{2t} = e^{2t}.$$

- This gives  $A = 1/3$  and hence we obtain a particular solution

$$y_p = \frac{1}{3}e^{2t}.$$

#### How to find $y_p$

- Consider

$$ay'' + by' + cy = g(t),$$

where  $a$ ,  $b$ , and  $c$  are constants.

- Let  $r_1$  and  $r_2$  be the roots of the characteristic equation

$$ar^2 + br + c = 0.$$

- (1) If  $g(t) = e^{\alpha t}(a_n t^n + \cdots + a_1 t + a_0)$  with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t} (A_n t^n + \cdots + A_1 t + A_0),$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha$ .

## Examples

### Example 1

Determine a suitable form for particular solutions of the following equations. Do not evaluate the constants.

- (a)  $y'' - y = e^t(2t + 5)$ .
- (b)  $y'' - y = e^{2t}t^2$ .
- (c)  $y'' + 2y' + y = e^{-t}(t^3 - 2t)$ .
- (d)  $y'' + y = e^{3t}(\sin t + 3 \cos t)(3t^4 - t^2 + 7)$ .
- (e)  $y'' + 2y' + 5y = -5te^{-t} \cos(2t)$ .
- (f)  $4y'' + 4y' + 5y = (5t^2 - 4t - 1) \sin(3t)$ .
- (g)  $y'' + y = \tan t$ .
- (h)  $y'' + y = e^t t^{-2}$ .

- (2) If  $g(t) = e^{\alpha t}[A \cos(\beta t) + B \sin(\beta t)](a_n t^n + \dots + a_1 t + a_0)$  with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t} \left[ \cos(\beta t)(A_n t^n + \dots + A_1 t + A_0) + \sin(\beta t)(B_n t^n + \dots + B_1 t + B_0) \right],$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha + i\beta$ .

### Example 1 (a)

- (a)  $y'' - y = e^t(2t + 5)$ .

- In this example  $g(t) = e^t(2t + 5)$ .

(1) If  $g(t) = e^{\alpha t}(a_n t^n + \dots + a_1 t + a_0)$  with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form  $y_p = t^s e^{\alpha t}(A_n t^n + \dots + A_1 t + A_0)$ ,

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha$ .

- Here  $\alpha = 1$ ,  $\beta = 0$  and  $n = 1$ .
- It follows that

$$y_p = t^s e^t(A_1 t + A_0).$$

### Example 1 (a)

- Now  $\alpha = 1$  and we have

$$s = \begin{cases} 0, & r_1 \neq \alpha, r_2 \neq \alpha, \\ 1, & r_1 = \alpha, r_2 \neq \alpha, \\ 2, & r_1 = r_2 = \alpha. \end{cases}$$

- The characteristic equation of the equation reads  $r^2 - 1 = 0$  and hence  $r_{1,2} = \pm 1$ .
- Thus  $r_1 = \alpha$ ,  $r_2 \neq \alpha$  and hence  $s = 1$ ; that is

$$y_p = t e^t(A_1 t + A_0)$$

## Example 1 (b)

(b)  $y'' - y = e^{2t}t^2$ .

- In this example  $g(t) = e^{2t}t^2$ .

(1) If  $g(t) = e^{\alpha t}(a_n t^n + \dots + a_1 t + a_0)$

with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t}(A_n t^n + \dots + A_1 t + A_0),$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha$ .

- Here  $\alpha = 2$ ,  $\beta = 0$  and  $n = 2$ .

- It follows that

$$y_p = t^s e^{2t}(A_2 t^2 + A_1 t + A_0).$$

- The characteristic equation of the equation reads  $r^2 - 1 = 0$  and hence  $r_{1,2} = \pm 1$ .

- Thus  $s = 0$  and

$$y_p = e^{2t}(A_2 t^2 + A_1 t + A_0)$$

## Example 1 (c)

(c)  $y'' + 2y' + y = e^{-t}(t^3 - 2t)$ .

- In this example  $g(t) = e^{-t}(t^3 - 2t)$ .

- Here  $\alpha = -1$ ,  $\beta = 0$  and  $n = 3$ .

- It follows that

$$y_p = t^s e^{-t}(A_3 t^3 + A_2 t^2 + A_1 t + A_0).$$

- The characteristic equation of the equation reads  $r^2 + 2r + 1 = 0$  and hence  $r_{1,2} = -1$ .

- Thus  $s = 2$  and

$$y_p = t^2 e^{-t}(A_3 t^3 + A_2 t^2 + A_1 t + A_0)$$

## Example 1 (d)

(d)  $y'' + y = e^{3t}(\sin t + 3 \cos t)(3t^4 - t^2 + 7)$ .

- In this example  $g(t) = e^{3t}(\sin t + 3 \cos t)(3t^4 - t^2 + 7)$ .

If  $g(t) = e^{\alpha t}[A \cos(\beta t) + B \sin(\beta t)](a_n t^n + \dots + a_1 t + a_0)$

with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t} [\cos(\beta t)(A_n t^n + \dots + A_1 t + A_0) + \sin(\beta t)(B_n t^n + \dots + B_1 t + B_0)],$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha + i\beta$ .

- Here  $\alpha = 3$ ,  $\beta = 1$  and  $n = 4$ .

- It follows that

$$y_p = t^s e^{3t} [\cos(t)(A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0) + \sin(t)(B_4 t^4 + B_3 t^3 + B_2 t^2 + B_1 t + B_0)].$$

- Now  $\alpha + i\beta = 3 + i$  and we have

$$s = \begin{cases} 0, & r_1 \neq \alpha + i\beta, r_2 \neq \alpha + i\beta, \\ 1, & r_1 = \alpha + i\beta, r_2 \neq \alpha + i\beta, \\ 2, & r_1 = r_2 = \alpha + i\beta? \end{cases}$$

- The characteristic equation of the equation reads  $r^2 + 1 = 0$  and hence  $r_{1,2} = \pm i \neq 3 + i$ .

- Thus  $s = 0$  and hence

$$y_p = e^{3t} [\cos(t)(A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0) + \sin(t)(B_4 t^4 + B_3 t^3 + B_2 t^2 + B_1 t + B_0)].$$

## Example 1 (e)

(e)  $y'' + 2y' + 5y = -5te^{-t} \cos(2t).$

- In this example  $g(t) = -5te^{-t} \cos(2t).$

If  $g(t) = e^{\alpha t}[A \cos(\beta t) + B \sin(\beta t)](a_n t^n + \dots + a_1 t + a_0)$  with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t} [\cos(\beta t)(A_n t^n + \dots + A_1 t + A_0) + \sin(\beta t)(B_n t^n + \dots + B_1 t + B_0)],$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha + i\beta$ .

- Here  $\alpha = -1, \beta = 2$  and  $n = 1$ .

- It follows that

$$y_p = t^s e^{-t} [\cos(2t)(A_1 t + A_0) + \sin(2t)(B_1 t + B_0)]$$

- In this example  $\alpha + i\beta = -1 + 2i$ .
- The characteristic equation of the equation reads  $r^2 + 2r + 5 = 0$  and hence  $r_{1,2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$ .

$$\alpha + i\beta = -1 + 2i$$

- Thus  $r_1 = -1 + 2i = \alpha + i\beta, \quad r_2 = -1 - 2i \neq \alpha + i\beta$  and hence  $s = 1$ .
- Therefore,

$$y_p = t e^{-t} [\cos(2t)(A_1 t + A_0) + \sin(2t)(B_1 t + B_0)]$$

## Example 1 (f)

(f)  $4y'' + 4y' + 5y = (5t^2 - 4t - 1) \sin(3t).$

- In this example  $g(t) = (5t^2 - 4t - 1) \sin(3t).$
- Here  $\alpha = 0, \beta = 3$  and  $n = 2$ .
- It follows that

$$y_p = t^s [\cos(3t)(A_2 t^2 + A_1 t + A_0) + \sin(3t)(B_2 t^2 + B_1 t + B_0)].$$

- The characteristic equation of the equation reads  $4r^2 + 4r + 5 = 0$  and hence  $r_{1,2} = \frac{-4 \pm 8i}{8} = -\frac{1}{2} \pm i \neq \alpha + i\beta = 3i$ .
- Thus  $s = 0 \implies$

$$y_p = \cos(3t)(A_2 t^2 + A_1 t + A_0) + \sin(3t)(B_2 t^2 + B_1 t + B_0)$$

## Example 1 (g) and (h)

(g)  $y'' + y = \tan t.$

- In this example  $g(t) = \tan t.$
- We can not use the method of undetermined coefficients.

(h)  $y'' + y = e^t t^{-2}.$

- In this example  $g(t) = e^t t^{-2}.$
- We can not use the method of undetermined coefficients.

## 3.6 Nonhomogeneous Equations; Method of Undetermined Coefficients (part2)

To solve a nonhomogeneous linear equation  $L[y] = g(t)$ , there are two main steps:

- (1) Find the general solution (complementary solution)  
 $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$  of the corresponding homogeneous equation.
- (2) Find some particular solution  $y_p$  of the nonhomogeneous equation.

### Example 2

Find a particular solution of the equation  $y'' + y' = t^2 - t + 1$ .

### Solution

- In this example  $g(t) = t^2 - t + 1$ .
- It follows that

$$y_p = t^s(A_2 t^2 + A_1 t + A_0).$$

- The characteristic equation of the equation reads  $r^2 + r = 0$  and hence  $r_1 = 0$ ,  $r_2 = -1$ .
- Thus  $s = 1$ .
- Therefore,

$$y_p = t(A_2 t^2 + A_1 t + A_0) = A_2 t^3 + A_1 t^2 + A_0 t.$$

## How to find $y_p$

- Consider

$$ay'' + by' + cy = g(t),$$

where  $a$ ,  $b$ , and  $c$  are constants.

- (1) If  $g(t) = e^{\alpha t}(a_n t^n + \cdots + a_1 t + a_0)$  with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t} (A_n t^n + \cdots + A_1 t + A_0),$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha$ .

- (2) If  $g(t) = e^{\alpha t}[A \cos(\beta t) + B \sin(\beta t)](a_n t^n + \cdots + a_1 t + a_0)$  with  $n = 0, 1, 2, \dots$ , then  $y_p$  has the form

$$y_p = t^s e^{\alpha t} [\cos(\beta t)(A_n t^n + \cdots + A_1 t + A_0) + \sin(\beta t)(B_n t^n + \cdots + B_1 t + B_0)],$$

where  $s$  is the number of the roots  $r_1, r_2$  that are equal to  $\alpha + i\beta$ .

- In order to determine the coefficients  $A_2, A_1, A_0$  we substitute  $y_p$  into the equation

$$y'' + y' = t^2 - t + 1.$$

- $y_p = A_2 t^3 + A_1 t^2 + A_0 t$
- $y'_p = 3A_2 t^2 + 2A_1 t + A_0$   
 $y''_p = 6A_2 t + 2A_1$
- $(6A_2 t + 2A_1) + (3A_2 t^2 + 2A_1 t + A_0) = t^2 - t + 1$ .
- Now we compare the coefficients of different powers of  $t$ :

$$\begin{array}{lll} t^2 : & 3A_2 = 1 & \Rightarrow A_2 = \frac{1}{3} \\ t : & 2A_1 + 6A_2 = -1 & \Rightarrow A_1 = -\frac{3}{2} \\ 1 : & A_0 + 2A_1 = 1 & \Rightarrow A_0 = 4 \end{array}$$

- As a result we find

$$y_p = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t.$$

### Example 3

Find the general solution of  $y'' + 2y' + y = e^{-t}(2t - 5)$ .

### Solution

- The general solution of the nonhomogeneous equation  $L[y] = g(t)$  is given by

$$y = c_1 y_1 + c_2 y_2 + y_p = y_c + y_p,$$

where  $y_c = c_1 y_1 + c_2 y_2$  is the general solution of the corresponding homogeneous equation  $L[y] = 0$  and  $y_p$  is some particular solution of  $L[y] = g(t)$ .

- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation

$$y'' + 2y' + y = e^{-t}(2t - 5).$$

- In this example  $g(t) = e^{-t}(2t - 5)$ .
- It follows that

$$y_p = t^s e^{-t}(A_1 t + A_0).$$

- Since  $r_1 = r_2 = -1$ , we have  $s = 2$ .
- Therefore,

$$y_p = t^2 e^{-t}(A_1 t + A_0) = e^{-t}(A_1 t^3 + A_0 t^2).$$

- In order to determine the coefficients  $A_1, A_0$  we substitute  $y_p$  into the equation.

- First of all we find the general solution  $y_c$  of the homogeneous equation

$$y'' + 2y' + y = 0$$

(the complementary solution).

- The characteristic equation of the equation reads  $r^2 + 2r + 1 = 0$  and hence  $r_1 = r_2 = -1$ .
- It follows that the complementary solution is

$$y_c = c_1 e^{-t} + c_2 t e^{-t}.$$

- $$\begin{aligned} y_p &= e^{-t}(A_1 t^3 + A_0 t^2) \\ y_p' &= -e^{-t}(A_1 t^3 + A_0 t^2) + e^{-t}(3A_1 t^2 + 2A_0 t) \\ &= e^{-t}(-A_1 t^3 - A_0 t^2 + 3A_1 t^2 + 2A_0 t) \\ y_p'' &= -e^{-t}(-A_1 t^3 - A_0 t^2 + 3A_1 t^2 + 2A_0 t) \\ &\quad + e^{-t}(-3A_1 t^2 - 2A_0 t + 6A_1 t + 2A_0) \\ &= e^{-t}(A_1 t^3 + A_0 t^2 - 6A_1 t^2 - 4A_0 t + 6A_1 t + 2A_0) \\ y_p'' + 2y_p' + y_p &= e^{-t}(2t - 5) \Rightarrow \\ &\quad e^{-t}(A_1 t^3 + A_0 t^2 - 6A_1 t^2 + 6A_1 t - 4A_0 t + 2A_0) \\ &\quad + 2e^{-t}(-A_1 t^3 - A_0 t^2 + 3A_1 t^2 + 2A_0 t) \\ &\quad + e^{-t}(A_1 t^3 + A_0 t^2) = e^{-t}(2t - 5). \end{aligned}$$
- Now we divide both sides by  $e^{-t}$  and then compare the coefficients of different powers of  $t$ :

$$\begin{array}{llll} t^3 : & A_1 - 2A_1 + A_1 = 0 & \checkmark \\ t^2 : & A_0 - 6A_1 - 2A_0 + 6A_1 + A_0 = 0 & \checkmark \\ t : & -4A_0 + 6A_1 + 4A_0 = 2 & \Rightarrow A_1 = \frac{1}{3} \\ 1 : & 2A_0 = -5 & \Rightarrow A_0 = -\frac{5}{2} \end{array}$$

- As a result we find

$$y_p = e^{-t} \left( \frac{1}{3}t^3 - \frac{5}{2}t^2 \right)$$

- and the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = c_1 e^{-t} + c_2 t e^{-t} + e^{-t} \left( \frac{1}{3}t^3 - \frac{5}{2}t^2 \right).$$

- It follows that the general solution of the homogeneous equation is  $y_c = c_1 \sin(\sqrt{2} t) + c_2 \cos(\sqrt{2} t)$ .
- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $y'' + 2y = e^t \sin(3t)$ .
- In this example  $g(t) = e^t \sin(3t)$ .
- It follows that

$$y_p = e^t [A_0 \sin(3t) + B_0 \cos(3t)].$$

- Since  $r_{1,2} = \pm i\sqrt{2}$  and  $\alpha + i\beta = 1 + 3i$ , we have  $s = 0$ .
- Therefore,

$$y_p = e^t [A_0 \sin(3t) + B_0 \cos(3t)].$$

#### Example 4

Solve the equation  $y'' + 2y = e^t \sin(3t)$ .

#### Solution

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $y'' + 2y = 0$  and  $y_p$  is some particular solution of  $y'' + 2y = e^t \sin(3t)$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation  $y'' + 2y = 0$ .
- The characteristic equation of the equation reads  $r^2 + 2 = 0$  and hence  $r_{1,2} = \pm i\sqrt{2}$ .

- In order to determine the coefficients  $A_0, B_0$  we substitute  $y_p$  into the equation:

$$\begin{aligned} y_p &= e^t [A_0 \sin(3t) + B_0 \cos(3t)] \\ y_p' &= e^t [A_0 \sin(3t) + B_0 \cos(3t)] + e^t [3A_0 \cos(3t) - 3B_0 \sin(3t)] \\ &= e^t [(A_0 - 3B_0) \sin(3t) + (B_0 + 3A_0) \cos(3t)] \\ y_p'' &= e^t [(A_0 - 3B_0) \sin(3t) + (B_0 + 3A_0) \cos(3t)] \\ &\quad + e^t [3(A_0 - 3B_0) \cos(3t) - 3(B_0 + 3A_0) \sin(3t)] \\ &= e^t [(-8A_0 - 6B_0) \sin(3t) + (-8B_0 + 6A_0) \cos(3t)] \end{aligned}$$

- Now  $y'' + 2y = e^t \sin(3t)$  implies

$$\begin{aligned} &e^t [(-8A_0 - 6B_0) \sin(3t) + (-8B_0 + 6A_0) \cos(3t)] \\ &+ 2e^t [A_0 \sin(3t) + B_0 \cos(3t)] = e^t \sin(3t). \end{aligned}$$

- Now we divide both sides by  $e^t$  and then compare the coefficients of  $\sin(3t)$  and  $\cos(3t)$ :

$$\begin{aligned} \sin(3t) : \quad &(-8A_0 - 6B_0) + 2A_0 = 1 \quad \Rightarrow -6A_0 - 6B_0 = 1 \\ \cos(3t) : \quad &(-8B_0 + 6A_0) + 2B_0 = 0 \quad \Rightarrow A_0 = B_0 \end{aligned}$$

- It follows that

$$A_0 = B_0 = -\frac{1}{12}.$$

- As a result we find

$$y_p = -\frac{1}{12}e^t [\sin(3t) + \cos(3t)]$$

- and the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = c_1 \sin(\sqrt{2} t) + c_2 \cos(\sqrt{2} t) - \frac{1}{12}e^t [\sin(3t) + \cos(3t)].$$

### Example 5

Solve the initial value problem

$$y'' - y = e^{2t}, \quad y(0) = 1, \quad y'(0) = 0.$$

### Solution

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $y'' - y = 0$  and  $y_p$  is some particular solution of  $y'' - y = e^{2t}$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation  $y'' - y = 0$ .
- The characteristic equation of the equation reads  $r^2 - 1 = 0$  and hence  $r_{1,2} = \pm 1$ .

- It follows that the general solution of the homogeneous equation is  $y_c = c_1 e^t + c_2 e^{-t}$ .

- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $y'' - y = e^{2t}$ .

- In this example  $g(t) = e^{2t}$ .

- It follows that

$$y_p = A_0 t^s e^{2t}.$$

- Since  $r_{1,2} = \pm 1$  and  $\alpha = 2$ , we have  $s = 0$ .

- Therefore,

$$y_p = A_0 e^{2t}.$$

- In order to determine the coefficient  $A_0$  we substitute  $y_p$  into the equation:

$$\begin{aligned} y_p &= A_0 e^{2t} \\ y_p' &= 2A_0 e^{2t} \\ y_p'' &= 4A_0 e^{2t} \end{aligned}$$

- Now  $y'' - y = e^{2t}$  implies

$$4A_0 e^{2t} - A_0 e^{2t} = e^{2t}.$$

- Now we compare the coefficients to find

$$A_0 = \frac{1}{3}.$$

- As a result we find

$$y_p = \frac{1}{3}e^{2t}$$

- and the general solution of the nonhomogeneous equation is

$$y = y_c + y_p = c_1 e^t + c_2 e^{-t} + \frac{1}{3}e^{2t}.$$

## Generalization

- Lastly we apply the initial conditions  $y(0) = 1, y'(0) = 0$ :

$$y' = c_1 e^t - c_2 e^{-t} + \frac{2}{3} e^{2t}.$$

- As a result we get

$$\begin{aligned} c_1 + c_2 + \frac{1}{3} &= 1, \\ c_1 - c_2 + \frac{2}{3} &= 0. \end{aligned}$$

or

$$\begin{aligned} c_1 + c_2 &= \frac{2}{3}, \\ c_1 - c_2 &= -\frac{2}{3}. \end{aligned}$$

- Thus

$$c_1 = 0, \quad c_2 = \frac{2}{3}.$$

- Therefore

$$y = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

### Theorem

Suppose that  $g(t)$  is the sum of two terms,

$$g(t) = g_1(t) + g_2(t),$$

and let  $u_1$  be a solution of  $L[y] = g_1(t)$  and  $u_2$  a solution of  $L[y] = g_2(t)$ . Then  $y_p = u_1 + u_2$  is a solution of  $L[y] = g_1(t) + g_2(t)$ .

### Proof.

By direct substitution. □

### Example 1

Determine a suitable form for  $y_p$  of

$$y'' + y' - 6y = 10e^{2t} - 18e^{3t} - 6t - 11.$$

### Solution

- In this example

$$g(t) = 10e^{2t} - 18e^{3t} - 6t - 11 = g_1(t) + g_2(t) + g_3(t), \text{ where } g_1(t) = 10e^{2t}, \quad g_2(t) = -18e^{3t}, \quad g_3(t) = -6t - 11.$$

- It follows that  $y_p = u_1 + u_2 + u_3$ , where

$$\begin{aligned} g_1(t) = 10e^{2t} &\Rightarrow u_1 = t^{s_1} A_0 e^{2t}, \\ g_2(t) = -18e^{3t} &\Rightarrow u_2 = B_0 t^{s_2} e^{3t}, \\ g_3(t) = -6t - 11 &\Rightarrow u_3 = t^{s_3} (C_1 t + C_0). \end{aligned}$$

- $y_p = u_1 + u_2 + u_3$ , where

$$\begin{aligned} g_1(t) = 10e^{2t} &\Rightarrow u_1 = t^{s_1} A_0 e^{2t}, \\ g_2(t) = -18e^{3t} &\Rightarrow u_2 = B_0 t^{s_2} e^{3t}, \\ g_3(t) = -6t - 11 &\Rightarrow u_3 = t^{s_3} (C_1 t + C_0). \end{aligned}$$

- The characteristic equation of the equation reads  $r^2 + r - 6 = 0$  and hence  $r_{1,2} = 2, -3$ .
- Thus  $s_1 = 1, s_2 = s_3 = 0$  and hence

$$y_p = A_0 t e^{2t} + B_0 e^{3t} + (C_1 t + C_0).$$

### Example 2

Solve the equation  $y'' - 2y' - 8y = 4e^{2t} - 21e^{3t}$ .

#### Solution

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $y'' - 2y' - 8y = 0$  and  $y_p$  is some particular solution of  $y'' - 2y' - 8y = 4e^{2t} - 21e^{3t}$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation  $y'' - 2y' - 8y = 0$ .
- The characteristic equation of the equation reads  $r^2 - 2r - 8 = 0$  and hence  $r_1 = 4$ ,  $r_2 = -2$ .

- It follows that the general solution of the homogeneous equation is  $y_c = c_1 e^{4t} + c_2 e^{-2t}$ .
- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $y'' - 2y' - 8y = 4e^{2t} - 21e^{3t}$ .
- In this example  $g(t) = 4e^{2t} - 21e^{3t} = g_1(t) + g_2(t)$ , where  $g_1(t) = 4e^{2t}$ ,  $g_2(t) = -21e^{3t}$ .
- It follows that  $y_p = u_1 + u_2$ , where

$$\begin{aligned} g_1(t) = 4e^{2t} &\Rightarrow u_1 = t^{s_1} A_0 e^{2t}, \\ g_2(t) = -21e^{3t} &\Rightarrow u_2 = B_0 t^{s_2} e^{3t}. \end{aligned}$$

- Since  $r_1 = 4$ ,  $r_2 = -2$ ,  $s_1 = s_2 = 0$  and hence

$$y_p = A_0 e^{2t} + B_0 e^{3t}.$$

- In order to determine the coefficients  $A_0, B_0$  we substitute  $y_p$  into the equation:

$$\begin{aligned} y_p &= A_0 e^{2t} + B_0 e^{3t} \\ y_p' &= 2A_0 e^{2t} + 3B_0 e^{3t} \\ y_p'' &= 4A_0 e^{2t} + 9B_0 e^{3t} \end{aligned}$$

- Now  $y'' - 2y' - 8y = e^{2t} - 21e^{3t}$  implies  $4A_0 e^{2t} + 9B_0 e^{3t} - 2(2A_0 e^{2t} + 3B_0 e^{3t}) - 8(A_0 e^{2t} + B_0 e^{3t}) = 4e^{2t} - 21e^{3t}$ .

- Now we compare the coefficients of  $e^{2t}$  and  $e^{3t}$ :

$$\begin{aligned} e^{2t}: \quad 4A_0 - 4A_0 - 8A_0 &= 4 &\Rightarrow A_0 &= -\frac{1}{2} \\ e^{3t}: \quad 9B_0 - 6B_0 - 8B_0 &= -21 &\Rightarrow B_0 &= \frac{21}{5} \end{aligned}$$

- As a result we find

$$y_p = \frac{21}{5} e^{3t} - \frac{1}{2} e^{2t}$$

- and the general solution of the nonhomogeneous equation is

$$y = c_1 e^{4t} + c_2 e^{-2t} + \frac{21}{5} e^{3t} - \frac{1}{2} e^{2t}.$$

### Example 3

Find the general solution of the equation  
 $y'' + 9y = e^{3t} + e^{-3t} + e^{3t} \sin(3t)$ .

### Solution

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $y'' + 9y = 0$  and  $y_p$  is some particular solution of  $y'' + 9y = e^{3t} + e^{-3t} + e^{3t} \sin(3t)$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation  $y'' + 9y = 0$ .
- The characteristic equation of the equation reads  $r^2 + 9 = 0$  and hence  $r_1 = 3i$ ,  $r_2 = -3i$ .

- It follows that the general solution of the homogeneous equation is  $y_c = c_1 \sin(3t) + c_2 \cos(3t)$ .
- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $y'' + 9y = e^{3t} + e^{-3t} + e^{3t} \sin(3t)$ .
- In this example  
 $g(t) = e^{3t} + e^{-3t} + e^{3t} \sin(3t) = g_1(t) + g_2(t) + g_3(t)$ , where  
 $g_1(t) = e^{3t}$ ,  $g_2(t) = e^{-3t}$ ,  $g_3(t) = e^{3t} \sin(3t)$ .
- It follows that  $y_p = u_1 + u_2 + u_3$ , where

$$\begin{aligned} g_1(t) = e^{3t} &\Rightarrow u_1 = t^{s_1} A_0 e^{3t}, \\ g_2(t) = e^{-3t} &\Rightarrow u_2 = B_0 t^{s_2} e^{-3t}, \\ g_3(t) = e^{3t} \sin(3t) &\Rightarrow u_3 = t^{s_3} e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)]. \end{aligned}$$

- Since  $r_1 = 3i$ ,  $r_2 = -3i$ ,  $s_1 = s_2 = s_3 = 0$ , and hence

$$y_p = A_0 e^{3t} + B_0 e^{-3t} + e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)].$$

- In order to determine the coefficients  $A_0$ ,  $B_0$ ,  $C_0$ ,  $D_0$  we substitute  $y_p$  into the equation:

$$\begin{aligned} y_p &= A_0 e^{3t} + B_0 e^{-3t} + e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)] \\ y_p' &= 3A_0 e^{3t} - 3B_0 e^{-3t} + 3e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)] \\ &\quad + e^{3t} [3C_0 \cos(3t) - 3D_0 \sin(3t)] \\ y_p'' &= 9A_0 e^{3t} + 9B_0 e^{-3t} + 9e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)] \\ &\quad + 6e^{3t} [3C_0 \cos(3t) - 3D_0 \sin(3t)] \\ &\quad + e^{3t} [-9C_0 \sin(3t) - 9D_0 \cos(3t)] \end{aligned}$$

- Now  $y'' + 9y = e^{3t} + e^{-3t} + e^{3t} \sin(3t)$  implies

$$\begin{aligned} &9A_0 e^{3t} + 9B_0 e^{-3t} + 9e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)] \\ &+ 6e^{3t} [3C_0 \cos(3t) - 3D_0 \sin(3t)] + e^{3t} [-9C_0 \sin(3t) - 9D_0 \cos(3t)] \\ &+ 9(A_0 e^{3t} + B_0 e^{-3t} + e^{3t} [C_0 \sin(3t) + D_0 \cos(3t)]) \\ &= e^{3t} + e^{-3t} + e^{3t} \sin(3t) \end{aligned}$$

- Now we compare the coefficients of  $e^{3t}$ ,  $e^{-3t}$ ,  $e^{3t} \sin(3t)$ ,  $e^{3t} \cos(3t)$ :

$$\begin{aligned} e^{3t} : & \quad 9A_0 + 9A_0 = 1 & \Rightarrow A_0 = \frac{1}{18} \\ e^{-3t} : & \quad 9B_0 + 9B_0 = 1 & \Rightarrow B_0 = \frac{1}{18} \\ e^{3t} \sin(3t) : & \quad 9C_0 - 18D_0 - 9C_0 + 9C_0 = 1 & \Rightarrow 9C_0 - 18D_0 = 1 \\ e^{3t} \cos(3t) : & \quad 9D_0 + 18C_0 - 9D_0 + 9D_0 = 0 & \Rightarrow 18C_0 + 9D_0 = 0 \end{aligned}$$

- The linear system for  $C_0$ ,  $D_0$  can be written as

$$\begin{aligned} 9C_0 - 18D_0 &= 1 \\ 36C_0 + 18D_0 &= 0 \end{aligned}$$

- It follows that

$$C_0 = \frac{1}{45}, \quad D_0 = -2C_0 = -\frac{2}{45}.$$

- As a result we find

$$y_p = \frac{1}{18} e^{3t} + \frac{1}{18} e^{-3t} + e^{3t} \left[ \frac{1}{45} \sin(3t) - \frac{2}{45} \cos(3t) \right]$$

- and the general solution of the nonhomogeneous equation is

$$y = c_1 \sin(3t) + c_2 \cos(3t) + \frac{1}{18}e^{3t} + \frac{1}{18}e^{-3t} + e^{3t} \left[ \frac{1}{45} \sin(3t) - \frac{2}{45} \cos(3t) \right].$$

- Variation of parameters is a method for finding particular solutions of a nonhomogeneous linear equations.
- It was introduced by Lagrange.
- The method of variation of parameters is a general method that can be applied to any linear differential equation.
- The disadvantage of the method of variation of parameters is that it requires evaluation of certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties.

### Description of the method

- Consider a linear nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad (14)$$

and let  $y_c = c_1 y_1 + c_2 y_2$  be the general solution of the corresponding homogeneous equation  $L[y] = 0$ .

- We assume that a particular solution has the form

$$y = u_1 y_1 + u_2 y_2, \quad (15)$$

where  $u_1$  and  $u_2$  are functions to be determined.

- Differentiating  $y$ , we obtain

$$y' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2. \quad (16)$$

- Since there are two unknown functions to be determined, and only one condition to be satisfied, we can impose the following second condition that produces a convenient simplification:

$$u_1' y_1 + u_2' y_2 = 0. \quad (17)$$

- The condition (22) enable us to determine  $u_1$  and  $u_2$  and help us to avoid terms involving  $u_1''$  and  $u_2''$  so keep the differential equations for  $u_1$  and  $u_2$  at first order.
- As a result equation (16) becomes

$$y' = u_1 y_1' + u_2 y_2'. \quad (18)$$

- Differentiating (18) yields

$$y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'. \quad (19)$$

- Using (15), (18), and (19) into (14) gives  $[u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'] + p[u_1 y_1' + u_2 y_2'] + q[u_1 y_1 + u_2 y_2] = g(t)$ .
- Collecting the coefficients of  $u_1$  and  $u_2$  yields  $u_1[y_1'' + p y_1' + q y_1] + u_2[y_2'' + p y_2' + q y_2] + u_1' y_1' + u_2' y_2' = g(t)$ . (20)
- Since  $y_1$  and  $y_2$  are solutions of  $L[y] = 0$ , the coefficients of  $u_1$  and  $u_2$  are both zero.

- Hence, equation (20) reduces to

$$u_1' y_1' + u_2' y_2' = g(t). \quad (21)$$

- 

$$u_1' y_1 + u_2' y_2 = 0. \quad (22)$$

- Equations (22) and (21) form a system of two linear algebraic equations for the derivatives  $u_1'$  and  $u_2'$  of the unknown functions.
- Multiplying equation (22) by  $y_2'$  and (21) by  $y_2$  gives

$$\begin{aligned} u_1' y_1 y_2' + u_2' y_2 y_2' &= 0, \\ u_1' y_1' y_2 + u_2' y_2 y_2' &= y_2 g(t). \end{aligned}$$

- Subtracting the second equation from the first, we obtain

$$(y_1 y_2' - y_1' y_2) u_1' = -y_2 g(t).$$

- Since  $\{y_1, y_2\}$  is a fundamental set of solutions of  $L[y] = 0$ , the Wronskian  $W(y_1, y_2) = y_1 y_2' - y_1' y_2 \neq 0$ .
- Thus we obtain

$$u_1' = \frac{-y_2 g(t)}{W(y_1, y_2)}. \quad (23)$$

- Substituting this solution into (21) we can solve for  $u_2'$  as

$$u_2' = \frac{y_1 g(t)}{W(y_1, y_2)}. \quad (24)$$

- Finally we integrate (23) and (24) to obtain  $u_1$  and  $u_2$ .
- The constants of integration can be taken to be zero, since we are looking for a particular solution.

## Examples

### Example 1

Find the general solution of  $y'' + y = \frac{\sin t}{\cos^2 t}$

### Solution

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $y'' + y = 0$  and  $y_p$  is some particular solution of  $y'' + y = \frac{\sin t}{\cos^2 t}$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation  $y'' + y = 0$ .

- The characteristic equation of the equation reads  $r^2 + 1 = 0$  and hence  $r_1 = i$ ,  $r_2 = -i$ .
- It follows that the general solution of the homogeneous equation is  $y_c = c_1 \sin(t) + c_2 \cos(t)$ .
- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $y'' + y = \frac{\sin t}{\cos^2 t}$ .
- We must use the method of variation of parameters.
- So we assume

$$y_p = u_1 y_1 + u_2 y_2 = u_1 \sin(t) + u_2 \cos(t),$$

where  $u_1$  and  $u_2$  satisfy

$$\begin{aligned} u_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)}, \\ u_2' &= \frac{y_1 g(t)}{W(y_1, y_2)}. \end{aligned}$$

- Thus

$$y_p = -\sin(t) \ln |\cos t| - \cos(t) [\tan(t) - t]$$

- and the general solution is

$$y = c_1 \sin(t) + c_2 \cos(t) - \sin(t) \ln |\cos t| - [\tan(t) - t] \cos(t).$$

- Here  $g(t) = \frac{\sin t}{\cos^2 t}$ ,  $y_1(t) = \sin(t)$ ,  $y_2(t) = \cos(t)$ .
- Let us calculate  $W(y_1, y_2)$ :

$$W(y_1, y_2) = \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = -1.$$

- It follows that

$$\begin{aligned} u_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)}, \\ u_2' &= \frac{y_1 g(t)}{W(y_1, y_2)}. \end{aligned}$$

- 

$$\begin{aligned} u_1' &= \cos t \frac{\sin t}{\cos^2 t} = \tan t, \\ u_2' &= -\sin t \frac{\sin t}{\cos^2 t} = -\tan^2 t = -(\sec^2 t - 1) \end{aligned}$$

- and hence

$$u_1 = -\ln |\cos t|, \quad u_2 = -[\tan t - t].$$

### Example 2

Find the general solution of  $y'' + 3y' + 2y = \frac{1}{e^t + 1}$

### Solution

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $y'' + 3y' + 2y = 0$  and  $y_p$  is some particular solution of  $y'' + 3y' + 2y = \frac{1}{e^t + 1}$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation  $y'' + 3y' + 2y = 0$ .

- The characteristic equation of the equation reads  $r^2 + 3r + 2 = (r+1)(r+2) = 0$  and hence  $r_1 = -1$ ,  $r_2 = -2$ .
- It follows that the general solution of the homogeneous equation is  $y_c = c_1 e^{-t} + c_2 e^{-2t}$ .
- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $y'' + 3y' + 2y = \frac{1}{e^t + 1}$ .
- We have to use the method of variation of parameters.
- So we assume

$$y_p = u_1 y_1 + u_2 y_2 = u_1 e^{-t} + u_2 e^{-2t},$$

where  $u_1$  and  $u_2$  satisfy

$$\begin{aligned} u_1' &= \frac{-y_2 g(t)}{W(y_1, y_2)}, \\ u_2' &= \frac{y_1 g(t)}{W(y_1, y_2)}. \end{aligned}$$

- Thus

$$\begin{aligned} y_p &= e^{-t} \ln(e^t + 1) + e^{-2t} [\ln(e^t + 1) - 1 - e^t] \\ &= (e^{-t} + e^{-2t}) \ln(1 + e^t) - e^{-2t} - e^{-t} \end{aligned}$$

- and the general solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} + (e^{-t} + e^{-2t}) \ln(1 + e^t) - e^{-2t} - e^{-t}$$

or

$$y = c_1 e^{-t} + c_2 e^{-2t} + (e^{-t} + e^{-2t}) \ln(1 + e^t).$$

- Here  $g(t) = \frac{1}{e^t + 1}$ ,  $y_1(t) = e^{-t}$ ,  $y_2(t) = e^{-2t}$ .
- Let us calculate  $W(y_1, y_2)$ :

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -2e^{-3t} + e^{-3t} = -e^{-3t}.$$

- It follows that

$$\begin{aligned} u_1' &= \frac{-e^{-2t}}{-e^{-3t}} \frac{1}{e^t + 1} = \frac{e^t}{e^t + 1}, \\ u_2' &= \frac{e^{-t}}{-e^{-3t}} \frac{1}{e^t + 1} = -\frac{e^{2t}}{e^t + 1} \end{aligned}$$

- and hence

$$\begin{aligned} u_1 &= \int \frac{e^t}{e^t + 1} dt = \ln(e^t + 1), \\ u_2 &= - \int \frac{e^{2t}}{e^t + 1} dt = -e^t - 1 + \ln(e^t + 1). \end{aligned}$$

### Example 3

Consider the equation  $L[y] = t^2 y'' - 2y = 3t^2 - 1$ ,  $t > 0$ .

- Show that  $y_1 = t^2$  and  $y_2 = t^{-1}$  are fundamental solutions of  $L[y] = 0$ .
- Find the general solution of the equation  $L[y] = 3t^2 - 1$ .

### Solution

- Show that  $y_1 = t^2$  and  $y_2 = t^{-1}$  are fundamental solutions of  $L[y] = 0$ .
- $y_1 = t^2$  and  $y_2 = t^{-1}$  are solutions of the equation?
  - We have  $y_1 = t^2$ ,  $y_1' = 2t$ ,  $y_1'' = 2$ . Thus

$$t^2 y_1'' - 2y_1 = 2t^2 - 2t^2 = 0$$

and hence  $y_1$  is a solution.

- We have  $y_2 = t^{-1}$ ,  $y_2' = -t^{-2}$ ,  $y_2'' = 2t^{-3}$ .

- Thus

$$t^2 y_2'' - 2y_2 = 2t^{-1} - 2t^{-1} = 0$$

and hence  $y_2$  is a solution.

- (ii)  $W(y_1, y_2) \neq 0$ ?

$$W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0.$$

- Therefore, (i) and (ii) implies that  $y_1$  and  $y_2$  form a fundamental set of solutions of the equation  $t^2 y'' - 2y' = 0$ .

- So we assume

$$y_p = u_1 y_1 + u_2 y_2 = u_1 t^2 + u_2 t^{-1},$$

where  $u_1$  and  $u_2$  satisfy

$$u_1' = \frac{-y_2 g(t)}{W(y_1, y_2)},$$

$$u_2' = \frac{y_1 g(t)}{W(y_1, y_2)}.$$

- Here  $g(t) = 3 - t^{-2}$ ,  $y_1(t) = t^2$ ,  $y_2(t) = t^{-1}$ .
- We have calculate  $W(y_1, y_2) = -3$ .
- It follows that

$$u_1' = \frac{-t^{-1}}{-3}(3 - t^{-2}) = \frac{3 - t^{-2}}{3t} = t^{-1} - \frac{1}{3}t^{-3},$$

$$u_2' = \frac{t^2}{-3}(3 - t^{-2}) = -\frac{1}{3}t^2(3 - t^{-2}) = -t^2 + \frac{1}{3}$$

- (b) Find the general solution of the equation  $L[y] = 3t^2 - 1$ .

- The general solution is given by

$$y = y_c + y_p,$$

where  $y_c$  is the general solution of the corresponding homogeneous equation  $t^2 y'' - 2y' = 0$  and  $y_p$  is some particular solution of  $t^2 y'' - 2y' = 3t^2 - 1$ .

- First of all we find the general solution  $y_c$  of the homogeneous equation.
- Since  $y_1 = t^2$  and  $y_2 = t^{-1}$  are fundamental solutions of  $L[y] = 0$ , we have  $y_c = c_1 t^2 + c_2 t^{-1}$ .
- Next we look for a particular solution  $y_p$  of the nonhomogeneous equation  $t^2 y'' - 2y' = 3t^2 - 1$ .
- We have to use the method of variation of parameters.
- Note that  $g(t) = \frac{3t^2 - 1}{t^2}$ .

- and hence

$$u_1 = \int \left[ t^{-1} - \frac{1}{3}t^{-3} \right] dt = \ln|t| + \frac{1}{6}t^{-2},$$

$$u_2 = \int \left[ -t^2 + \frac{1}{3} \right] dt = -\frac{1}{3}t^3 + \frac{1}{3}t.$$

- Thus

$$y_p = t^2 \left[ \ln|t| + \frac{1}{6}t^{-2} \right] + t^{-1} \left[ -\frac{1}{3}t^3 + \frac{1}{3}t \right]$$

$$= t^2 \ln|t| - \frac{1}{3}t^2 + \frac{1}{2}$$

- and the general solution is

$$y = c_1 t^2 + c_2 t^{-1} + t^2 \ln|t| + \frac{1}{3}t^2 + \frac{1}{2}$$

or

$$y = c_1 t^2 + c_2 t^{-1} + t^2 \ln|t| + \frac{1}{2}.$$

## Reduction of Order

- The method of reduction of order can also be used for the nonhomogeneous equation.

### Example

Use the method of reduction of order to find the general solution of the equation  $(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}$ ,  $0 < t < 1$ , if  $y_1(t) = e^t$  is a solution of  $(1-t)y'' + ty' - y = 0$ .

### Solution

- We use the transformation  $y = vy_1 = e^t v$ .
- It follows that

$$\begin{aligned}y' &= e^t v' + e^t v, \\y'' &= e^t v'' + 2e^t v' + e^t v.\end{aligned}$$

- Thus, the substitution  $w = v'$  transforms it to a first order linear equation for  $w$ :

$$(t-1)w' + (t-2)w = -2(t-1)^2e^{-2t}$$

OR

$$w' + \frac{t-2}{t-1}w = -2(t-1)e^{-2t}$$

- This equation has the integrating factor  $\mu(t) = \frac{e^t}{t-1}$ .
- Thus, we obtain

$$\frac{e^t}{t-1}w' + \frac{e^t(t-2)}{(t-1)^2}w = -2e^{-t}$$

and hence

$$\frac{e^t}{t-1}w = 2e^{-t} + c_1.$$

- Substituting into the equation we obtain

$$(1-t)e^t(v'' + 2v' + v) + te^t(v' + v) - e^t v = 2(t-1)^2e^{-t}.$$

- Dividing the equation by  $e^t$  and collecting similar terms, we obtain

$$-(t-1)v'' + (2-2t+t)v' + (1-t+t-1)v = 2(t-1)^2e^{-2t}$$

OR

$$(t-1)v'' + (t-2)v' = -2(t-1)^2e^{-2t}.$$

- This is a second order equation for  $v$  with dependent variable  $v$  missing.

- It follows that

$$v' = w = 2e^{-2t}(t-1) + c_1(t-1)e^{-t}.$$

- Next we integrate to find

$$v = -e^{-2t}\left(t - \frac{1}{2}\right) - c_1te^{-t} + c_2.$$

- Finally, we use the formula  $y = e^t v$  to obtain the general solution of the given equation

$$y = c_2e^t - c_1t - e^{-t}\left(t - \frac{1}{2}\right).$$

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