

Chapter 4

Higher Order Linear Equations

In this chapter we generalize the structure and methods of solutions of second order linear equations to third and higher order linear equations.

4.1 General Theory of n th Order Linear Equations

An n th order linear equation is an equation of the form

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = g(t),$$

where $p_1, p_2, \dots, p_n$, and g are continuous functions on some interval $I = (\alpha, \beta)$. The linear differential operator L of order n defined above is similar to the second order operator introduced in Chapter 3.

Example 1. The equation $y''' + ty'' - y = \sin t$ is a third order linear equation.

Example 2. The equation $5y^{(4)} + \sin ty' + e^t y = t$ is a fourth order linear equation.

Theorem 1. *Let*

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

be an initial value problem. If the functions $p_1, p_2, \dots, p_n$, and g are continuous on an interval $I = (\alpha, \beta)$ containing the initial point t_0 , then the initial value has a unique solution $y = \varphi(t)$ valid for all $t \in I$.

Example. Find the largest interval in which the solution of the initial value problem

$$(t-1)y''' + \tan ty'' + 3y = \cos t, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$

is certain to exist.

Solution.

The Homogeneous Equation

Firstly, we discuss the homogeneous equation

$$L[y] = 0.$$

Theorem 2. (Principle of Superposition)

If $y_1, y_2, \dots, y_n$ are n solutions of the linear homogeneous equation

$$L[y] = 0,$$

then the linear combination $\varphi(t) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is also a solution for any values of the constants $c_1, c_2, \dots, c_n$.

Theorem (2) helps us to construct the general solution of the linear homogeneous equation

$$L[y] = 0.$$

A linear combination $\varphi(t) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the general solution of $L[y] = 0$ if and only if it is possible to choose $c_1, c_2, \dots, c_n$ such that $\varphi(t) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ satisfies any set of initial conditions

$$y(t_0) = y_0, y'(t_0) = y'_0, \dots, y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

This required the condition

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1(t) & \dots & y_n(t) \\ y'_1(t) & \dots & y'_n(t) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{vmatrix} \neq 0.$$

Definition. (The Wronskian)

the Wronskian determinant of the functions $f_1, f_2, \dots, f_n$ is the determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1(t) & \dots & f_n(t) \\ f'_1(t) & \dots & f'_n(t) \\ \vdots & & \vdots \\ f_1^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{vmatrix}.$$

Example. Find the Wronskian of the functions $f_1(t) = 1, f_2(t) = t, f_3(t) = \cos t, f_4(t) = \sin t$.

Solution.

Theorem 3. Assume that $p_1, p_2, \dots, p_n$ are continuous on the open interval I . Let $y_1, y_2, \dots, y_n$ be solutions of the differential equation

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = 0.$$

If $W(y_1, y_2, \dots, y_n)(t_0) \neq 0$ for some point $t_0 \in I$, then $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$, where $c_1, c_2, \dots, c_n$, are arbitrary constants, is the general solution of $L[y] = 0$.

Definition. (Fundamental set of solutions)

If $y_1, y_2, \dots, y_n$ are solutions of $L[y] = 0$ with $W(y_1, y_2, \dots, y_n)(t_0) \neq 0$, then the set $\{y_1, y_2, \dots, y_n\}$ is called a fundamental set of solutions of $L[y] = 0$.

Example. Show that $y_1 = 1$, $y_2 = t$ and $y_3 = t^3$ form a fundamental set of solutions of the equation $ty''' - y'' = 0$.

Solution.

The Nonhomogeneous Equation

Now we return to the nonhomogeneous equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \dots + p_{n-1}(t)y' + p_n(t)y = g(t).$$

Theorem 4. The general solution of the nonhomogeneous equation $L[y] = g(t)$ is given by

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p,$$

where $y_1, y_2, \dots, y_n$ are fundamental solutions of the corresponding homogeneous equation $L[y] = 0$ and y_p is some particular solution of $L[y] = g(t)$.

4.2 Homogeneous Equations with Constant Coefficients

In this section we consider the homogeneous constant coefficient equation

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0, \quad (1)$$

where $a_n, \dots, a_0$ are constants and $a_n \neq 0$. As in the case of second order, we look for solutions in the form $y = e^{rt}$. Substituting y into equation (1) yields

$$L[y] = e^{rt} Z(r) = 0.$$

Since $e^{rt} \neq 0$, it follows that $y = e^{rt}$ is a solution of (1) if and only if r is a root of the polynomial $Z(r)$.

Definition. (Characteristic polynomial)

The polynomial $Z(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$ is called the characteristic polynomial of the differential equation $L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$.

An important question in mathematics for more than 200 years was whether every polynomial equation has at least one root. The affirmative answer to this question, the fundamental theorem of algebra, was given by Carl Friedrich Gauss in his doctoral dissertation in 1799, although his proof does not meet modern standards of rigor. Several other proofs have been discovered since, including three by Gauss himself. Today, students often meet the fundamental theorem of algebra in a first course on complex variables, where it can be established as a consequence of some of the basic properties of complex analytic functions.

Theorem 5. (Fundamental Theorem of Algebra)

A polynomial $Z(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$ of degree n has exactly n roots, say $r_1, \dots, r_n$, and it can be written as

$$Z(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n).$$

We have to distinguish between three case:

- (1) All the roots are real and distinct.
- (2) Roots are distinct but not all them are real.
- (3) Roots are not distinct.

Real and Distinct Roots

If the roots $r_1, \dots, r_n$ of the characteristic polynomial $Z(r)$ are real and no two of them are equal, then the general solution of $L[y] = 0$ is given by

$$y = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}.$$

Finding Roots of a Polynomial

Consider a polynomial $Z(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0$. For third and fourth degree polynomials there are formulas, analogous to the formula for quadratic equations but more complicated, that give exact expressions for the roots. Root-finding algorithms are readily available on calculators and

computers. Sometimes they are included in the differential equation solver, so that the process of factoring the characteristic polynomial is hidden and the solution of the differential equation is produced automatically.

If we need to factor the characteristic polynomial by hand, here is one result that is sometimes helpful. Assume that $a_n, \dots, a_0$ are integers.

- (1) The polynomial equation $Z(r) = 0$ has a solution $r \in \mathbb{Z}$ only if r divides a_0 .
- (2) The polynomial equation $Z(r) = 0$ has a solution $r = \frac{p}{q} \in \mathbb{Q}$, where p and q have no common factors, only if p divides a_0 and q divides a_n .

Example 1. Find the roots of $Z(r) = 12r^4 + 31r^3 + 75r^2 + 37r + 5 = 0$.

Solution.

Example 2. Find the solution of the equation $y''' - 5y'' + 6y' = 0$.

Solution.

Example 3. Find the general solution of the equation $y''' - 2y'' - y' + 2y = 0$.

Solution.

Example 4. Find the general solution of the equation $y^{(3)} - 5y'' - 22y' + 56y = 0$.

Solution.

Complex Roots

In the case where the coefficients $a_0, \dots, a_n$ of $Z(r)$ are real, then complex roots of $Z(r)$ occur in conjugate pairs. If $r = \lambda + i\mu$ is a root, so is $\bar{r} = \lambda - i\mu$, and the corresponding solutions are

$$u_1(t) = e^{(\lambda+i\mu)t} = e^{\lambda t}(\cos(\mu t) + i \sin(\mu t))$$

and

$$u_2(t) = e^{(\lambda-i\mu)t} = e^{\lambda t}(\cos(\mu t) - i \sin(\mu t)).$$

Since a linear combination of solutions of is also a solution, this implies that the real-valued functions

$$y_1 = e^{\lambda t} \cos(\mu t) = \frac{1}{2}(u_1 + u_2), \quad y_2 = e^{\lambda t} \sin(\mu t) = \frac{1}{2i}(u_1 - u_2)$$

are solutions. Therefore, if all the roots of $Z(r)$ are distinct, then the general solution of $L[y] = 0$ is still of the form

$$y = c_1 e^{r_1 t} + \dots + c_n e^{r_n t}.$$

However we can replace the complex functions $e^{(\lambda \pm i\mu)t}$ by the real-valued functions $e^{\lambda t} \cos(\mu t)$ and $e^{\lambda t} \sin(\mu t)$.

Example 1. Solve $y^{(4)} - 16y = 0$.

Solution.

Example 2. Find the general solution of the equation $y^{(4)} + 5y'' + 4y = 0$.

Solution.

Example 3. Find the general solution of the equation $y^{(4)} + y^{(3)} + 2y'' + y' + y = 0$.

Solution.

The equation $r^n + 1 = 0$

To find the n roots of the equation $r^n + 1 = 0$ we write $-1 = e^{i\pi(2m+1)}$. As a result we obtain

$$r_n = e^{i\pi(2m+1)/n}, \quad m = 0, 1, \dots, n-1.$$

Example 1. Solve $r^3 + 1 = 0$.

Solution.

Example 2. Solve the differential equation $y^{(5)} + y = 0$.

Solution.

Repeated Roots

If the roots of the characteristic equation are not distinct, that is, if $r_1 = r_2 = \cdots = r_s$, then the fundamental solutions corresponding to $r_1, r_2, \dots, r_s$ are

$$e^{r_1 t}, te^{r_1 t}, \dots, t^{s-1}e^{r_1 t}.$$

Similarly, if a complex root $r = \lambda + i\mu$ is repeated s times, then the complex conjugate $\bar{r} = \lambda - i\mu$ is also repeated s times. The corresponding solutions are

$$e^{\lambda t} \cos(\mu t), e^{\lambda t} \sin(\mu t), te^{\lambda t} \cos(\mu t), te^{\lambda t} \sin(\mu t), \dots, t^{s-1}e^{\lambda t} \cos(\mu t), t^{s-1}e^{\lambda t} \sin(\mu t).$$

Example 1. Find the general solution of $y^{(4)} + 3y^{(3)} + 3y'' + y' = 0$.

Solution.

Example 2. Find the solution of the initial value problem

$$y''' - y'' + y' - y = 0, \quad y(\pi/2) = 2, \quad y'(\pi/2) = 1, \quad y''(\pi/2) = 0.$$

Solution.

Example 3. Find the general solution of the equation $2y^{(4)} + 11y^{(3)} + 18y'' + 4y' - 8y = 0$.

Solution.

Example 4. Find the general solution of the equation $y^{iv} + 2y'' + y = 0$.

Solution.

4.3 The Method of Undetermined Coefficients

The method of undetermined coefficients can be used to obtain a particular solution of the nonhomogeneous n th order linear equation with constant coefficients

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(t) \quad (2)$$

provided that $g(t)$ is a sum of polynomials, exponentials, sines, and cosines functions.

Just as for the second order linear equation, we choose the suitable form for the particular solution according to the following rules.

(1) If $g(t) = e^{\alpha t} (a_n t^n + \cdots + a_1 t + a_0)$ with $n = 0, 1, 2, \dots$, then y_p has the form

$$y_p = t^s e^{\alpha t} (A_n t^n + \cdots + A_1 t + A_0),$$

where s is the number of the roots of $Z(r)$ that are equal to α .

(2) If $g(t) = e^{\alpha t} [A \cos(\beta t) + B \sin(\beta t)] (a_n t^n + \cdots + a_1 t + a_0)$ with $n = 0, 1, 2, \dots$, then y_p has the form

$$y_p = t^s e^{\alpha t} [\cos(\beta t)(A_n t^n + \cdots + A_1 t + A_0) + \sin(\beta t)(B_n t^n + \cdots + B_1 t + B_0)],$$

where s is the number of the roots of $Z(r)$ that are equal to $\alpha + i\beta$.

Example 1. Find the general solution of $y^{iv} - 4y'' = t^2 + e^t$.

Solution.

Example 2. Solve the initial value problem $y^{iv} + 2y''' + y'' + 8y' - 12y = 12 \sin t - 20e^{-t}$,
 $y(0) = 3$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 2$.

Solution.

Example 3. Find a suitable form for a particular solution of the differential equation

$$y^{(4)} - y''' - y'' + y' = t^2 + 4e^t + t \sin t.$$

Do not evaluate the constants.

Solution.

Example 4. Find a suitable form for a particular solution of the differential equation

$$y''' + y'' - 4y' - 4y = 8t + 8 + 6e^{-t}.$$

Do not evaluate the constants.

Solution.

Example 5. Find a suitable form for a particular solution of the differential equation

$$y^{iv} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t} \sin t.$$

Do not evaluate the constants.

Solution.

4.4 The Method of Variation of Parameters

The method of variation of parameters is a general method to find a particular solution of the nonhomogeneous n th order linear differential equation

$$L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = g(t).$$

In this method we start by finding the general solution of the corresponding homogeneous equation $L[y] = 0$, say

$$y_c = c_1 y_1 + \cdots + c_n y_n.$$

Then we look for a particular solution of the nonhomogeneous equation $L[y] = g(t)$ in the form

$$y_p = u_1 y_1 + \cdots + u_n y_n,$$

where $u_1, \dots, u_n$ are functions to be determine. Since we have n functions to determine, we will have to specify n conditions. One of these is clearly that y_p satisfy $L[y] = g(t)$. The other $n - 1$ conditions are

chosen so as to make the calculations as simple as possible. Since we can hardly expect a simplification in determining y_p if we must solve high order differential equations for $u_1, \dots, u_n$, it is natural to impose conditions to suppress the terms that lead to higher derivatives of $u_1, \dots, u_n$. These n equations are

$$\begin{aligned} u'_1 y_1 + u'_2 y_2 + \dots + u'_n y_n &= 0, \\ u'_1 y'_1 + u'_2 y'_2 + \dots + u'_n y'_n &= 0, \\ &\vdots \\ u'_1 y_1^{(n-1)} + u'_2 y_2^{(n-1)} + \dots + u'_n y_n^{(n-1)} &= g(t). \end{aligned} \tag{3}$$

The system (3) is a linear algebraic system for the unknown quantities $u'_1, \dots, u'_n$. By solving this system and then integrating the resulting expressions, you can obtain the coefficients $u_1, \dots, u_n$. The system of equations (3) has a solution if the determinant of coefficients is nonzero for each value of t . However, the determinant of coefficients is precisely $W(y_1, y_2, \dots, y_n)$, and it is nowhere zero since $y_1, \dots, y_n$ are linearly independent solutions of the homogeneous equation $L[y] = 0$. Hence it is possible to determine $u'_1, \dots, u'_n$. Using Cramer's rule, we find that the solution of the system of equations (3) is

$$u'_m(t) = \frac{g(t)W_m(t)}{W(t)}, \quad m = 1, 2, \dots, n, \tag{4}$$

where $W(t) = W(y_1, y_2, \dots, y_n)(t)$ and W_m is the determinant obtained from W by replacing the m th column by the column $(0, 0, \dots, 0, 1)$. Integrating we obtain

$$u_m(t) = \int \frac{g(t)W_m(t)}{W(t)} dt, \quad m = 1, 2, \dots, n. \tag{5}$$

Example 1. Find the general solution of $y''' + y' = \tan t$, $0 < t < \frac{\pi}{2}$.

Solution.

Example 2. Solve $y''' - 2y'' - y' + 2y = e^{4t}$.

Solution.

Example 3. Consider the equation $L[y] = ty''' + 3y'' = t^2 + 1, \quad t > 0$.

- (a) Show that $y_1 = 1$, $y_2 = t$, and $y_3 = t^{-1}$ are fundamental solutions of $L[y] = 0$.
- (b) Find the general solution of the equation $L[y] = t^2 + 1$.

Solution.

Example 4. Find a particular solution

$$t^3 y''' - t^2(t+3)y'' + 2t(t+3)y' - 2(t+3)y = -4t^4$$

given the fundamental set of solutions $\{t, t^2, te^t\}$.

Solution.